IMA Journal of Mathematical Control & Information (1992) **9**, 255–263

Stability and stabilizability of linear infinite-dimensional discrete-time systems

HARTMUT LOGEMANN
Institut für Dynamische Systeme, Universität Bremen, Postfach 330 440, 2800 Bremen 33, Germany

[Received November 1991]

This paper contains three results on stability and stabilizability of linear time-invariant infinite-dimensional discrete-time systems. (1) Power stability is characterized in transfer-function terms using the concepts of stabilizability and detectability. (2) Under the assumption that the input operator is compact, we present a necessary and sufficient condition for stabilizability involving spectral properties of the system operator and a projection of the infinite-dimensional system onto a certain finite-dimensional subspace of the state space. (3) It is shown that, if the input and output spaces are finite-dimensional, then stabilization by finite-dimensional dynamic output feedback is possible if and only if the system is detectable and stabilizable.

1. Introduction

Although stability properties of linear time-invariant infinite-dimensional discrete-time systems have received some attention in the literature in the last 20 years (e.g. Fuhrmann 1973; Przyłuski 1980, 1988; Weiss 1989), the notions of stabilizability and detectability have not been much investigated so far. This is very different from the situation in the continuous-time case, where the concept of stabilizability has been studied in some detail (see Pritchard & Zabczyk (1981) for a survey of the literature published in the 70's), and where recently a well-known sufficient condition for stabilizability (Triggiani 1975) was shown to be necessary as well, provided the input space is finite-dimensional; see Nevedev & Sholokhovich (1986) and Jacobson & Nett (1988). This result is closely related to recent work on stabilization of infinite-dimensional continuous-time systems by finite-dimensional dynamic output feedback; see e.g. Kamen *et al.* (1985), Logemann (1986a,b), and Jacobson & Nett (1988), where it was shown for various classes of infinite-dimensional systems that finite-dimensional stabilization is possible if and only if the plant is stabilizable and detectable.

In this paper, we want to prove (among other things) the above-mentioned results for discrete-time systems defined on arbitrary Banach spaces. The content of the paper is as follows. In Section 2 we show that, if the system is stabilizable and detectable, then power-stability is equivalent to input–output stability in the sense that the transfer function of the system is holomorphic and bounded outside the closed unit disc. We mention that this result does not require the input and output spaces to be finite-dimensional. Moreover, under the assumption that the input operator is compact, we present a necessary and sufficient condition for stabilizability involving spectral properties of the system operator and a projection of the infinite-dimensional system onto a certain finite-dimensional subspace of the state space. The results in Section 2 are used in Section 3 in order to prove that an infinite-dimensional discrete-time plant with finitely many inputs and outputs can be stabilized by finite-dimensional output feedback if and only if the plant is stabilizable and detectable.

2. Power stability, input–output stability, stabilizability, and detectability

Let $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ be Banach spaces, with $A \in \mathfrak{B}(\mathcal{X}, \mathcal{X})$, $B \in \mathfrak{B}(\mathcal{U}, \mathcal{X})$, $C \in \mathfrak{B}(\mathcal{X}, \mathcal{Y})$, and $D \in \mathfrak{B}(\mathcal{U}, \mathcal{Y})$, where $\mathfrak{B}(\mathcal{V}, \mathcal{W})$ is the Banach space of bounded linear operators from $\mathcal{V}$ to $\mathcal{W}$. We shall investigate discrete-time systems of the form

$$\left.\begin{aligned} x(j+1) &= Ax(j) + Bu(j) \\ y(j) &= Cx(j) + Du(j) \end{aligned}\right\} \quad (j = 0, 1, \ldots), \qquad x(0) = x_0. \tag{1}$$

Denote the spectrum and the spectral radius of A by $\sigma(A)$ and $\mathrm{r}(A)$, respectively. It is well known that

$$\mathrm{r}(A) = \lim_{n \to \infty} \|A^n\|^{1/n}. \tag{2}$$

The point spectrum of A is denoted by $\sigma_p(A)$. As usual the resolvent set of A is defined by $\rho(A) = \mathbb{C} \setminus \sigma(A)$. For $\alpha > 0$ and $\mathcal{Z}$ a Banach space, set

$$\mathbb{D}_\alpha := \{z \in \mathbb{C} : |z| < \alpha\}, \qquad \mathbb{E}_\alpha := \mathbb{C} \setminus \mathbb{D}_\alpha^{\mathrm{cl}},$$

$$\mathrm{H}_\alpha^\infty(\mathcal{Z}) := \{f : \mathbb{E}_\alpha \to \mathcal{Z} \colon f \text{ is holomorphic and bounded}\}.$$

It is well known that the mapping $z \mapsto C(zI - A)^{-1}B + D$ is a holomorphic $\mathfrak{B}(\mathcal{U}, \mathcal{Y})$-valued function on $\rho(A)$. In order to define precisely what we mean by the transfer function of system (1), some preparations are required. A transfer element at ∞ of system (1) is a pair $(\mathbb{E}_\alpha, G_\alpha)$, where G_α is a holomorphic function from $\mathbb{E}_\alpha$ to $\mathfrak{B}(\mathcal{U}, \mathcal{Y})$ such that

$$G_\alpha(z) = C(zI - A)^{-1}B + D$$

for all $z \in \rho(A) \cap \mathbb{E}_\alpha$. Let $\mathcal{T}$ denote the set of all transfer elements at ∞ of system (1). Notice that $\mathcal{T} \neq \varnothing$, since $\mathbb{E}_\alpha \subset \rho(A)$ for all sufficiently large α.

Definition 1 Setting $\alpha_0 := \inf\{\alpha > 0 : (\mathbb{E}_\alpha, G_\alpha) \in \mathcal{T}\}$, we define the *transfer function* $G : \mathbb{E}_{\alpha_0} \to \mathfrak{B}(\mathcal{U}, \mathcal{Y})$ of system (1) by

$$G(z) = G_\alpha(z) \quad \text{if } z \in \mathbb{E}_\alpha \text{ and } (\mathbb{E}_\alpha, G_\alpha) \in \mathcal{T},$$

where, in the case $\alpha_0 = 0$, we set $\mathbb{E}_0 := \mathbb{C} \setminus \{0\}$. □

It is clear that G is a well-defined $\mathfrak{B}(\mathcal{U}, \mathcal{Y})$-valued holomorphic function on $\mathbb{E}_{\alpha_0}$, and that G is in $\mathrm{H}_\alpha^\infty(\mathfrak{B}(\mathcal{U}, \mathcal{Y}))$ for all $\alpha > \mathrm{r}(A)$. If $\alpha_0 = 0$, then G can be continued holomorphically into 0 if and only if $G(z) = D$ for all $z \in \mathbb{E}_0$.

DEFINITION 2 System (1) is called (i) *power-stable* if A is power-stable, i.e. there exist $\mu \geqslant 1$ and $\gamma \in (0, 1)$ such that

$$\|A^n\| \leqslant \mu\gamma^n \quad \text{for all } n \in \mathbb{N}_0,$$

and (ii) *input–output stable* if $G \in \mathrm{H}_1^\infty(\mathfrak{B}(\mathcal{U}, \mathcal{Y}))$. □

The following lemma gives two simple characterizations of power stability.

LEMMA 1 The statements below are equivalent:

(i) system (1) is power-stable;
(ii) $\mathrm{r}(A) < 1$;
(iii) $[z \mapsto (zI - A)^{-1}] \in \mathrm{H}_1^\infty(\mathfrak{B}(\mathcal{X}, \mathcal{X}))$.

Proof. It follows from (2) that (i) $\Leftrightarrow$ (ii) and the implication (ii) $\Rightarrow$ (iii) holds trivially true. We show that (iii) $\Rightarrow$ (ii). Indeed, if $\mathrm{r}(A) = \alpha \geqslant 1$, then there exists $z_0 \in \sigma(A)$ with $|z_0| = \alpha$ and we can choose a sequence $z_n \in \mathbb{E}_\alpha \subset \rho(A)$ such that $\lim_{n\to\infty} z_n = z_0$. Since

$$\mathrm{r}((z_nI - A)^{-1}) = 1/\mathrm{dist}(z_n, \sigma(A))$$

(Kato 1976: Problem 6.16 on p. 177), we see that the sequence $\|(z_nI - A)^{-1}\|$ is unbounded, which shows that

$$[z \mapsto (zI - A)^{-1}] \notin \mathrm{H}_\alpha^\infty(\mathfrak{B}(\mathcal{X}, \mathcal{X})) \supseteq \mathrm{H}_1^\infty(\mathfrak{B}(\mathcal{X}, \mathcal{X})).$$

DEFINITION 3 (i) System (1) is called *stabilizable* if there exists $F \in \mathfrak{B}(\mathcal{X}, \mathcal{U})$ such that $A + BF$ is power-stable. (ii) System (1) is called *detectable* if there exists $H \in \mathfrak{B}(\mathcal{Y}, \mathcal{X})$ such that $A + HC$ is power-stable. □

The next theorem shows that the concepts of power-stability and input–output stability are equivalent, provided that the system is stabilizable and detectable.

THEOREM 2 The following statements are equivalent:

(i) system (1) is power-stable;
(ii) system (1) is input–output stable, stabilizable, and detectable.

Proof. The implication (i) $\Rightarrow$ (ii) is trivially true. In order to show that (ii) $\Rightarrow$ (i), pick $F \in \mathfrak{B}(\mathcal{X}, \mathcal{U})$ and $H \in \mathfrak{B}(\mathcal{Y}, \mathcal{X})$ such that $A + BF$ and $A + HC$ are power stable. Define

$$P(\cdot) \in H_1^\infty(\mathfrak{B}(\mathcal{X}, \mathcal{X})) : z \mapsto (zI - A - BF)^{-1}, \tag{3}$$

$$Q(\cdot) \in H_1^\infty(\mathfrak{B}(\mathcal{X}, \mathcal{U})) : z \mapsto -FP(z), \tag{4}$$

$$\tilde{P}(\cdot) \in H_1^\infty(\mathfrak{B}(\mathcal{X}, \mathcal{X})) : z \mapsto (zI - A - HC)^{-1}, \tag{5}$$

$$\tilde{Q}(\cdot) \in H_1^\infty(\mathfrak{B}(\mathcal{Y}, \mathcal{X})) : z \mapsto -\tilde{P}(z)H. \tag{6}$$

Then

$$(zI - A)P(z) + BQ(z) = I \quad \text{and} \quad \tilde{P}(z)(zI - A) + \tilde{Q}(z)C = I \quad \text{for all } z \in \mathbb{E}_1.$$

It follows, for $z \in \mathbb{E}_1 \cap \rho(A)$, that

$$CP(z) + C(zI - A)^{-1}BQ(z) = C(zI - A)^{-1}, \tag{7}$$

$$\tilde{P}(z) + \tilde{Q}(z)C(zI - A)^{-1} = (zI - A)^{-1}. \tag{8}$$

By assumption, $C(\bullet I - A)^{-1}B$ is bounded on $\mathbb{E}_1 \cap \rho(A)$, and so we obtain from (7) via (3) and (4) that $C(\bullet I - A)^{-1}$ is bounded on $\mathbb{E}_1 \cap \rho(A)$; hence, using (8), (5), and (6), we see that $(\bullet I - A)^{-1}$ is bounded on $\mathbb{E}_1 \cap \rho(A)$. By Lemma 1, it is sufficient to show that $\mathbb{E}_1 \subseteq \rho(A)$. Assume the contrary, i.e. $\mathbb{E}_1 \cap \sigma(A) \neq \emptyset$. Then there exist $z_0 \in \mathbb{E}_1 \cap \sigma(A)$ and $z_n \in \mathbb{E}_1 \cap \rho(A)$ such that $\lim_{n\to\infty} z_n = z_0$, and therefore the sequence $\|(z_n I - A)^{-1}\|$ is unbounded, which leads to a contradiction. □

COROLLARY 3 Suppose that system (1) is input–output stable. If (1) is stabilizable and detectable, then the transfer function G of (1) is in $H^\infty_\alpha(\mathfrak{B}(\mathcal{U}, \mathcal{Y}))$ for some $\alpha \in (0, 1)$. □

The next result gives a characterization of stabilizability, provided that the input operator B is compact. Hence it applies in particular to systems with finite-dimensional input space $\mathcal{U}$.

THEOREM 4 Under the assumption that B is compact, system (1) is stabilizable if and only if the state space $\mathcal{X}$ admits a decomposition $\mathcal{X} = \mathcal{X}_s \oplus \mathcal{X}_u$ which satisfies the following conditions.

(i) $\dim \mathcal{X}_u < \infty$.
(ii) $A\mathcal{X}_s \subseteq \mathcal{X}_s$ and $A\mathcal{X}_u \subseteq \mathcal{X}_u$.
(iii) There exists $\alpha \in (0, 1)$ such that $\sigma(A) \cap \mathbb{E}^{cl}_\alpha$ consists of (at most) finitely many eigenvalues with finite algebraic multiplicities, $\sigma(A \restriction \mathcal{X}_u) = \sigma(A) \cap \mathbb{E}^{cl}_\alpha$, and $\sigma(A \restriction \mathcal{X}_s) = \sigma(A) \cap \mathbb{D}_\alpha$.
(iv) The finite-dimensional system $(A \restriction \mathcal{X}_u, PB)$ is controllable, where $P: \mathcal{X} \to \mathcal{X}$ is the projection onto $\mathcal{X}_u$ along $\mathcal{X}_s$.

Remarks. (i) The above theorem shows in particular that, if B is compact and system (1) is stabilizable, then it can be stabilized by a state-feedback operator of finite rank. In other words, system (1) can be stabilized (by state feedback) using finitely many input channels only.

(ii) It is clear that, if the output operator C is compact, an analogous result holds for detectability.

(iii) For infinite-dimensional continuous-time processes (defined by strongly continuous semigroups) with finite-dimensional input space, analogous results have been proved by Nemedov & Sholokhovich (1986) and Jacobson & Nett (1988) for bounded control action, and by Curtain (1988) for unbounded control action. Under the assumption that the state space is a Hilbert space and that the feedback operator is compact, Rebarber (1990) presented a generalization which covers a certain class of unbounded input operators of infinite rank.

Proof of Theorem 4. The sufficiency of the condition (i)–(iv) for stabilizability follows from well-known finite-dimensional results. In order to prove necessity, we shall proceed in two steps.

Step 1. We show first that there exists $\alpha < 1$ such that $\sigma(A) \cap \mathbb{E}_\alpha^{cl}$ consists of (at most) finitely many eigenvalues with finite algebraic multiplicities. First notice that, by assumption, there exists a stabilizing $F \in \mathfrak{B}(\mathcal{X}, \mathcal{U})$, i.e. $\mathrm{r}(A + BF) < 1$ by Lemma 1. Let $\alpha \in (\mathrm{r}(A + BF), 1)$ be fixed but arbitrary. Then $\rho(A + BF) \supseteq \mathbb{E}_\alpha^{cl}$ and, for $z \in \mathbb{E}_\alpha^{cl}$, the equality

$$zI - A = [I + BF(zI - A - BF)^{-1}](zI - A - BF)$$

holds. Hence, for $z \in \mathbb{E}_\alpha^{cl}$, we have

$$z \in \sigma(A) \quad \Leftrightarrow \quad -1 \in \sigma(BF(zI - A - BF)^{-1}) \quad \Leftrightarrow \quad -1 \in \sigma_p(BF(zI - A - BF)^{-1}).$$

The second equivalence is true because the operator $BF(zI - A - BF)^{-1}$ is compact. Now, for $z \in \mathbb{E}_\alpha^{cl}$, suppose that $-1 \in \sigma_p(BF(zI - A - BF)^{-1})$. Then there exists a corresponding eigenvector $x \in \mathcal{X}$, with $x \neq 0$, such that

$$[I + BF(zI - A - BF)^{-1}]x = 0$$

and hence $(zI - A)(zI - A - BF)^{-1}x = 0$, which shows that $z \in \sigma_p(A)$. It follows that

$$\sigma(A) \cap \mathbb{E}_\alpha^{cl} = \sigma_p(A) \cap \mathbb{E}_\alpha^{cl}.$$

Moreover, using the above equivalence, the compactness of B and Thm 1.9 on p. 370 of Kato's book (Kato 1976) we may conclude that $\sigma_p(A) \cap \mathbb{E}_\alpha^{cl}$ is finite. Let $\lambda \in \sigma_p(A) \cap \mathbb{E}_\alpha^{cl}$. Then, in order to show that the algebraic multiplicity of λ is finite, assume the contrary. It then follows from Thm 5.28 on p. 239 in Kato (1976) that λ belongs to the essential spectrum of A. This leads to a contradiction of the fact that $\sigma(A + BF) \cap \mathbb{E}_\alpha^{cl} = \varnothing$, since the essential spectrum of an operator is conserved under compact perturbations (Kato 1976: Thm 5.35 on p. 244).

Step 2. Step 1 enables us in particular to apply Thm 6.17 on p. 178 in Kato's book (Kato 1976). It follows that there exists a projection $P: \mathcal{X} \to \mathcal{X}$ such that $\dim P\mathcal{X} = n < \infty$, A and P commute, $\sigma(A \upharpoonright P\mathcal{X}) = \sigma(A) \cap \mathbb{E}_\alpha^{cl} = \sigma_P(A) \cap \mathbb{E}_\alpha^{cl}$, and $\sigma(A \upharpoonright (I - P)\mathcal{X}) = \sigma(A) \cap \mathbb{D}_\alpha$. By setting $\mathcal{X}_u := P\mathcal{X}$ and $\mathcal{X}_s := (I - P)\mathcal{X}$, it is clear that the conditions (i)–(iii) are satisfied. In order to show that (iv) is satisfied as well, observe that, for all $z \in \mathbb{E}_\alpha^{cl}$, we have

$$(zI - A)(zI - A - BF)^{-1} - BF(zI - A - BF)^{-1} = I.$$

It follows that

$$\mathrm{rk}(zI\mathcal{X}_u - A \upharpoonright \mathcal{X}_u, PB) = n \quad \text{for all } z \in \mathbb{E}_\alpha^{cl}.$$

Since $A \upharpoonright \mathcal{X}_u$ has no spectrum in $\mathbb{D}_\alpha$, we obtain that the above rank condition holds for all $z \in \mathbb{C}$, which means, by the Hautus test, that the pair $(A \upharpoonright \mathcal{X}_u, PB)$ is controllable. □

Under the assumption that B is compact, Corollary 3 can be strengthened.

COROLLARY 5 Suppose that B is compact and that system (1) is input–output stable. If (1) is stabilizable, then the transfer function G of (1) is in $\mathrm{H}_\alpha^\infty(\mathfrak{L}(\mathcal{U}, \mathcal{Y}))$ for some $\alpha \in (0, 1)$. □

3. Stabilization by finite-dimensional dynamic output feedback

In this section, it is assumed throughout that the input and output spaces $\mathcal{U}$ and $\mathcal{Y}$ are finite-dimensional. Let us turn our attention towards stabilizing system (1) by finite-dimensional strictly proper compensators of the form

$$\left.\begin{aligned} \xi(j+1) &= F\xi(j) + Ky(j) \\ u(j) &= -L\xi(j) \end{aligned}\right\} \quad (j = 0, 1, \ldots), \qquad \xi(0) = \xi_0. \tag{9}$$

The following lemma will be useful for the proof of Theorem 7, the main result of this section.

LEMMA 6 Let $H \in \mathrm{H}^\infty_\alpha(\mathfrak{B}(\mathcal{U}, \mathcal{Y}))$ and suppose that $\lim_{|z|\to\infty} H(z) = H_\infty \in \mathfrak{B}(\mathcal{U}, \mathcal{Y})$. Under these conditions, there exists a sequence of rational matrix functions $R_n \in \mathrm{H}^\infty_\alpha(\mathfrak{B}(\mathcal{U}, \mathcal{Y}))$ such that

$$\lim_{n\to\infty} \|H - R_n\|_\infty = 0,$$

where $\|M\|_\infty$ denotes the sup-norm in $\mathrm{H}^\infty_\alpha(\mathfrak{B}(\mathcal{U}, \mathcal{Y}))$, i.e. the supremum over $z \in \mathbb{E}_\alpha$ of the largest singular value of $M(z)$.

Proof. It is well-known that the polynomials form a dense subset of the so-called disc algebra, i.e. the set of all holomorphic functions on $\mathbb{D}_1$ which are continuous on $\mathbb{D}_1^{\mathrm{cl}}$ (Rudin 1974; p. 397). Of course, this result extends to matrix-valued functions defined on $\mathbb{D}_\alpha^{\mathrm{cl}}$ and an application of the transformation $z \to 1/z$ proves the claim. □

THEOREM 7 Suppose that $\dim \mathcal{U} < \infty$ and $\dim \mathcal{Y} < \infty$. Then the following statements are equivalent:

(i) system (1) is stabilizable by a finite-dimensional (strictly proper) compensator of the form (9);
(ii) system (1) is stabilizable and detectable.

Remarks. (i) Theorem 7 can be considered as a generalization of a well-known finite-dimensional result by Hautus (1970). (ii) See Nett (1984), Kamen *et al.* (1985), Logemann (1986a,b), and Jacobson & Nett (1988) for analogous continuous-time results.

Proof of Theorem 7. Suppose first that (i) is true, i.e. there exists a finite-dimensional compensator (F, K, L) which stabilizes system (1). Notice that the closed-loop system operator A_c can be written as

$$A_c = \begin{bmatrix} A & -BL \\ KC & F - KDL \end{bmatrix} = \mathrm{diag}(A, F) + \mathrm{diag}(B, K)\,\tilde{D}\,\mathrm{diag}(C, -L),$$

where $\tilde{D}$ is defined by

$$\tilde{D} := \begin{bmatrix} O & I \\ I & D \end{bmatrix}.$$

Since $r(A_c) < 1$, we see that $(\mathrm{diag}(A, F), \mathrm{diag}(B, K))$ is stabilizable and $(\mathrm{diag}(A, F), \mathrm{diag}(C, -L))$ is detectable. It follows from Theorem 4 that there exists $\alpha \in (r(A_c), 1)$ such that $\sigma(\mathrm{diag}(A, F)) \cap \mathbb{E}_\alpha^{cl}$ consists of at most finitely many eigenvalues with finite algebraic multiplicities. As a consequence, the same is true for $\sigma(\mathrm{diag}(A, 0)) \cap \mathbb{E}_\alpha^{cl}$ by Thm 6.2 on p. 247 in Kato (1976). Hence $\sigma(A) \cap \mathbb{E}_\alpha^{cl}$ consists of at most finitely many eigenvalues with finite algebraic multiplicities. Therefore there exists a projection $P: \mathcal{X} \to \mathcal{X}$ which satisfies the conditions (i)–(iii) in Theorem 4. In order to show that system (1) is stabilizable, it is sufficient to prove that condition (iv) in Theorem 4 holds true as well. To this end, notice that, for all $z \in \mathbb{E}_\alpha^{cl}$, we have

$$[zI - \mathrm{diag}(A, F)](zI - A_c)^{-1} - \mathrm{diag}(B, K)\,\tilde{D}\,\mathrm{diag}(C, -L)\,(zI - A_c)^{-1} = I,$$

which implies that

$$\mathrm{rk}(zI - A\restriction\mathcal{X}_u, PB) = \dim \mathcal{X}_u \quad \text{for all } z \in \mathbb{E}_\alpha^{cl},$$

where $\mathcal{X}_u := P\mathcal{X}$. Since $A\restriction\mathcal{X}_u$ has no spectrum in $\mathbb{D}_\alpha$ it follows that the above rank condition is satisfied for all $z \in \mathbb{C}$. An application of the Hautus test shows that the finite-dimensional system $(A\restriction\mathcal{X}_u, PB)$ is controllable. Using Theorem 4, we see that system (1) is stabilizable. In the same way it can be shown that system (1) is detectable (cf. Remark (i) on Theorem 4).

Conversely let us now assume that (ii) holds true. It then follows from Theorem 4 that there exists $\alpha \in (0, 1)$ such that $\sigma(A) \cap \mathbb{E}_\alpha^{cl}$ consists of at most finitely many eigenvalues with finite algebraic multiplicities. Hence there exists $\beta \in (\alpha, 1)$ such that on $\mathbb{E}_\beta$ the transfer-function matrix $G(z) = C(zI - A)^{-1}B + D$ of system (1) can be decomposed as

$$G(z) = H(z) + R(z),$$

where $H \in \mathrm{H}_\beta^\infty(\mathfrak{B}(U, Y))$ and R is a strictly proper rational matrix function with all its poles in $\mathbb{E}_1^{cl}$. Set

$$\hat{R}(z) := z^{-1}R(z)$$

and pick a right-coprime rational factorization of $\hat{R}(z)$:

$$\hat{R} = \hat{N}\hat{M}^{-1}, \qquad \det \hat{M}(\infty) \neq 0$$

with

$$\hat{P}\hat{N} + \hat{Q}\hat{M} = I,$$

where $\hat{M}$, $\hat{N}$, $\hat{P}$, and $\hat{Q}$ are stable proper rational matrices. It is clear that $\hat{N}$ is strictly proper. Set

$$P(z) := z^{-1}\hat{P}(z), \qquad N(z) := z\hat{N}(z), \quad M(z) := \hat{M}(z), \qquad Q(z) := \hat{Q}(z);$$

then it follows that

$$G = (HM + N)M^{-1}, \qquad P(HM + N) + (Q - PH)M = I \tag{10a,b}$$

Since M, N, P, and Q are stable proper rational matrices, there exists $\gamma \in (0, 1)$ such that they are bounded on $\mathbb{E}_\gamma$. Defining $\delta := \max\{\beta, \gamma\}$, we obtain that $(HM + N)M^{-1}$ is a right-coprime factorization of G over $\mathrm{H}_\delta^\infty(\mathfrak{B}(\mathcal{U}, \mathcal{Y}))$. Moreover, since $\lim_{|z|\to\infty} H(z) = D$ exists, it follows from Lemma 6 that there exists a sequence of proper rational matrices H_n converging to H in the H^∞ norm. Therefore, bringing in (10b), we see that

$$V_n := P(HM + N) + (Q - PH_n)M$$

is a unimodular matrix in $\mathrm{H}^\infty_\delta(\mathfrak{B}(\mathcal{U}, \mathcal{U}))$ for all sufficiently large n. Hence, by fractional representation theory (Vidyasagar *et al.* 1982), the compensator

$$S_n := (Q - PH_n)^{-1}P$$

will stabilize G for all sufficiently large n, in the sense that

$$\begin{bmatrix} G(I + S_nG)^{-1} & -GS_n(I + GS_n)^{-1} \\ S_nG(I + S_nG)^{-1} & S_n(I + GS_n)^{-1} \end{bmatrix} \in \mathrm{H}^\infty_\delta(\mathfrak{B}(\mathcal{U} \times \mathcal{Y}, \mathcal{U} \times \mathcal{Y})). \tag{11}$$

Notice that, by construction, S_n is rational and that moreover S_n is strictly proper, since P is strictly proper and $Q(\infty) = M^{-1}(\infty)$ is nonsingular. Now let (F_n, K_n, L_n) be a stabilizable and detectable finite-dimensional realization of S_n and observe that the triple

$$\left(\begin{bmatrix} A & -BL_n \\ K_nC & F_n - K_nDL_n \end{bmatrix}, \begin{bmatrix} B & O \\ O & K_n \end{bmatrix}, \begin{bmatrix} C & O \\ O & L_n \end{bmatrix}\right)$$

is a stabilizable and detectable realization of the transfer-function matrix (11). An application of Theorem 2 shows that the closed-loop system operator

$$\begin{bmatrix} A & -BL_n \\ K_nC & F_n - K_nDL_n \end{bmatrix}$$

is power-stable for all sufficiently large n. □

Acknowledgements

The author would like to thank S. B. Townley (University of Exeter) and H. J. Zwart (University of Twente) for several helpful remarks and suggestions.

REFERENCES

CURTAIN, R. 1988 Equivalence of input–output stability and exponential stability for infinite-dimensional systems. *Math. Systems Theor.* **21**, 19–48.

FUHRMANN, P. 1973 On observability and stability in infinite-dimensional linear systems. *J. Optimiz. Theor. & Applics* **12**, 173–81.

HAUTUS, M. L. J. 1970 Stabilization, controllability and observability of linear autonomous systems. *Proc. Ned. Akad. Wetenschapen* Ser. A **73**, 448–55.

JACOBSON, C. A., & NETT, C. N. 1988 Linear state-space systems in infinite-dimensional space: the role and characterization of joint stabilizability/detectability. *IEEE Trans* AC **33**, 541–9.

KATO, T. 1976 *Perturbation Theory for Linear Operators* (2nd edn). Berlin: Springer-Verlag.

KAMEN, E. W., KHARGONEKAR, P. P., & TANNENBAUM, A. 1985 Stabilization of time-delay systems using finite-dimensional compensators. *IEEE Trans* AC **30**, 75–8.

LOGEMANN, H. 1986a On the existence of finite-dimensional compensators for retarded and neutral systems. *Int. J. Control* **43**, 109–21.

LOGEMANN, H. 1986b Funktionentheoretische Methoden in der Regelungstheorie unendlich-dimensionaler Systeme. PhD thesis, Institut für Dynamische Systeme, Universität Bremen (= Report 156).

NETT, C. N. 1984 The fractional representation approach to robust linear feedback design: a self-contained exposition. MSc Thesis, ESCS Dept., Rensselaer Polytech. Inst., Troy, N.Y.

NEVEDOV, S. A., & SHOLOKHOVICH, F. A. 1986 A criterion for the stabilizability of dynamical systems with finite-dimensional input. *Differential Equations* (*Translation of Differentsial'nye Uravneniya*) **22**, 163–6.

PRITCHARD, A. J., & ZABCZYK, J. 1981 Stability and stabilizability of infinite-dimensional systems. *SIAM Rev.* **23**, 25–52.

PRZYŁUSKI, K. M. 1980 The Lyapunov equation and the problem of stability for linear bounded discrete-time systems in Hilbert space. *Appl. Math. Optimiz.* **6**, 97–112.

PRZYŁUSKI, K. M. 1988 Stability of linear infinite-dimensional systems revisited. *Int. J. Control* **48**, 513–23.

REBARBER, R. 1990 Necessary conditions for exponential stabilizability of distributed parameter systems with infinite-dimensional unbounded feedback. *Systems Control Lett.* **14**, 241–8.

RUDIN, W. 1974 *Real and Complex Analysis* (2nd edn). New York: McGraw-Hill.

TRIGGIANI, R. 1975 On the stabilizability problem in Banach space. *J. Math. Anal. & Applics* **52**, 383–403.

VIDYASAGAR, M., SCHNEIDER, H., & FRANCIS, B. A. 1982 Algebraic and topological aspects of feedback stabilization. *IEEE Trans* AC **27**, 880–94.

WEISS, G. 1989 Weakly l^p-stable linear operators are power stable. *Int. J. Systems Sci.* **20**, 2323–8.