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Circle Criteria, Small-gain Conditions and Internal Stability for Infinite-dimensional Systems*

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For a large class of infinite-dimensional systems, the small-gain condition and the circle criterion both ensure global exponential stability. The small-gain condition is sharp in the sense that there exist real destabilizing perturbations with gain equal to the critical value.

Key Words—Feedback; nonlinear control systems; distributed parameter systems; time lag systems; stability criteria; robustness; frequency domain.

Abstract—An internal version of the small-gain theorem is given for a class of plants including time-delay and distributed systems. More precisely: it is shown that for a large class of infinite dimensional state-space systems the small-gain condition in L^2 is sufficient for *internal* stability. Furthermore it is proved that there exist *unstable* feedback systems of loop gain equal to one. The internal version of the small-gain theorem is used in order to establish that the circle criterion is sufficient for *global exponential* stability.

1. INTRODUCTION

THE CIRCLE criterion is a well-known graphical stability test for feedback systems consisting of a linear time-invariant system in the forward-loop and a sector bounded, possibly time-varying, memoryless nonlinearity in the feedback-loop. One of its most appealing aspects is that it generalizes the sufficiency portion of the Nyquist criterion: the critical point is replaced by a critical disk whose size and location (in the complex plane) is determined by the slopes of the lines which form the boundaries of the sector. There is a vast literature on the subject—we only mention the original papers by Narendra and Goldwyn (1964), Sandberg (1964), Zames (1966) and Freedman *et al.* (1969); the books by Willems (1971), Narendra and Taylor (1973), Desoer and Vidyasagar (1975); Vidyasagar (1978) and Mees (1981); and the collection of original contributions edited by MacFarlane (1979). Basically, there are two approaches to the circle criterion:

- The input-output approach, resulting in a criterion checking *input-output stability* (see e.g. Sandberg, 1964; Zames, 1966; Desoer and Vidyasagar, 1975; Vidyasagar, 1978); and
- The state-space approach, giving a criterion for *internal* stability in the sense of Lyapunov (see e.g. Narendra and Goldwyn, 1964; Narendra and Taylor, 1973; Vidyasagar, 1978).

Those versions of the circle criterion which cover infinite-dimensional systems are formulated in input-output terms (cf. Sandberg, 1964; Zames, 1966; Freedman *et al.*, 1969; Willems, 1971; Desoer and Vidyasagar, 1975; Vidyasagar, 1978; Mees, 1981). Up till now it has not been investigated if an infinite-dimensional feedback system satisfying the conditions of the circle criterion will be *internally* stable. It is the purpose of this paper to show that for a large class of infinite-dimensional systems the circle criterion will ensure *global exponential* stability. This will be done by combining frequency-domain methods for distributed parameter systems (see Callier and Desoer, 1978, 1980; Desoer and Wang, 1980; Logemann, 1986) and recent results in the state-space theory of infinite-dimensional systems (see Pritchard and Salamon, 1987; Curtain, 1988, 1989). We present a proof which does not use Lyapunov techniques and which seems to be new for finite-dimensional systems as well. It is based on an internal version of the small-gain theorem (see Section 3 of this paper), exponential weighting (see e.g. Desoer and Vidyasagar, 1975) and recent results on the relationship between input-output stability and internal stability for infinite-dimensional systems (see Curtain, 1988). The circle criterion given in this paper applies to infinite-dimensional systems with unbounded control action, hence it covers retarded systems with point-delays in the input

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or the output and distributed systems with boundary control.

In Section 2 we introduce a fairly large class of infinite-dimensional state-space systems which allows for unbounded control and observation. Our set-up is closely related to the one considered in Salamon (1984), Pritchard and Salamon (1987) and Curtain (1988, 1989). We prove a number of technical results which will be needed in the sequel. In particular it is shown that the class of systems under consideration is closed with respect to perturbations induced by static output feedback. Section 3 is devoted to the well-known small-gain condition, which will play a key-role in the proof of the circle criterion. We show that a feedback system consisting of an exponentially stable infinite-dimensional system in the forward loop and a sector-bounded memoryless nonlinearity (locally Lipschitz continuous operator of finite gain) in the feedback loop is globally exponentially stable (globally asymptotically stable) if the product of the L^2-gains is smaller than one. Moreover it is shown that there exist *unstable* feedback systems of loop-gain equal to one. More precisely: for a given exponentially stable infinite-dimensional *real* plant we construct a linear finite-dimensional *real* compensator making the L^2-loop gain equal to one and causing destabilization. The compensator can be chosen to be constant if we allow for *complex* coefficients. This result bears a close resemblance to recent work on stability radii of linear finite-dimensionals systems (cf. Hinrichsen and Pritchard, 1986, 1989, 1990). Finally, in Section 4, we show that for the class of infinite-dimensional systems under consideration the circle criterion is sufficient for global exponential stability.

Notation

Let X and Y be Banach spaces. Then we define

$L^q(0, t; X)$	q times integrable functions from $[0, t]$ to X
$LL^q(0, \infty; X)$	q times locally integrable functions form $[0, \infty)$ to X
$\mathscr{L}(X, Y)$	bounded linear operators from X to Y
$\mathscr{L}(X)$	bounded linear operators from X into itself.

The truncation operator π_t: $LL^q(0, \infty; X) \to L^q(0, \infty; X)$ is defined by

$$(\pi_t f)(\tau) = \begin{cases} f(\tau), & 0 \le \tau \le t \\ 0, & \tau > t. \end{cases}$$

Let T be an operator; then

$D(T)$	domain of T
$\sigma(T)$	spectrum of T.

Let M be a matrix; then

$\bar{\sigma}(M)$	largest singular value of M.

For $\alpha \in \mathbb{R}$ let $\mathbb{C}_\alpha$ denote the open right-half plane given by $\operatorname{Re}(s) > \alpha$.

Let $H^\infty(\mathbb{C}_\alpha, \mathbb{C}^{m\times n})$ and $H^2(\mathbb{C}_\alpha, \mathbb{C}^{m\times n})$ denote the usual Hardy spaces of functions defined on $\mathbb{C}_\alpha$ with values in $\mathbb{C}^{m\times n}$. For $H \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m\times n})$ define

$$\|H\|_\infty := \sup_{s\in\mathbb{C}_0} \bar{\sigma}(H(s))$$

$\mathbb{L}$ denotes the unilateral Laplace transform.

Let f be a closed curve in $\mathbb{C}$ and $a \in \mathbb{C}$; then

$\operatorname{ind}(f, a)$	winding number of f with respect to a.

Algebras of transfer functions:

For $t \in \mathbb{R}$ let δ_t denote the Dirac distributions with support $\{t\}$. The convolution algebra $\mathscr{A}_-$ consists of all distributions f with support in $[0, \infty)$ of the form

$$f = \sum_{i=0}^{\infty} f_i\, \delta_{t_i} + f_a$$

where $0 = t_0 < t_1 < \cdots$, $f_i \in \mathbb{C}$, f_a is a measurable complex-valued function and in addition

$$\sum_{i=0}^{\infty} |f_i|\, e^{\varepsilon t_i} + \int_0^\infty |f_a(t)|\, e^{\varepsilon t}\, dt < \infty$$

for some $\varepsilon = \varepsilon(f) > 0$. Moreover we define

$$\mathscr{W}_- := \{f \in \mathscr{A}_- \mid f_i = 0\ \forall i \ge 1\},$$

$$\hat{\mathscr{A}}_- := \{\mathbb{L}f \mid f \in \mathscr{A}_-\} \quad \text{and} \quad \hat{\mathscr{W}}_- := \{\mathbb{L}f \mid f \in \mathscr{W}_-\}.$$

The algebra $\hat{\mathscr{B}}$ (see Callier and Desoer, 1978, 1980) consists of all transfer functions g of the form $g = g_1 + g_2$, where $g_1 \in \hat{\mathscr{A}}_-$ and g_2 is a strictly proper rational function (with complex coefficients) which has all its poles in $\bar{\mathbb{C}}_0$.

Finally, let $\mathscr{A}_{-,r}$ and $\mathscr{W}_{-,r}$ denote the real-valued counterparts of $\mathscr{A}_-$ and $\mathscr{W}_-$, respectively. The function algebras $\hat{\mathscr{A}}_{-,r}$ and $\hat{\mathscr{W}}_{-,r}$ consist by definition of the Laplace transformed elements of $\mathscr{A}_{-,r}$ and $\mathscr{W}_{-,r}$ respectively.

2. SYSTEM DESCRIPTION

In a formal sense our basic model is

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{2.1a}$$

$$y(t) = Cx(t), \quad t \ge 0 \tag{2.1b}$$

where $u \in LL^2(0, \infty; \mathbb{R}^p)$, A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on a real Hilbert space W, $C \in \mathscr{L}(W, \mathbb{R}^p)$ and $B \in \mathscr{L}(\mathbb{R}^p, V)$, where V is a real Hilbert space satisfying $V \supset W$. We are interested in the mild solution of (2.1a), i.e. in the trajectory given by the variation-of-constants formula

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)Bu(\tau)\, d\tau. \tag{2.2}$$

In order to make the expression under the integral in (2.2) meaningful we assume that $S(t)$

extends to a strongly continuous semigroup on V. Let us introduce the following assumptions.

Assumption 1. The map $W \to V$, $x \mapsto x$ is bounded and W is dense in V.

Assumption 2. There exist $t_1 > 0$ and $\alpha > 0$ such that $\|CS(\cdot)x\|_{L^2(0,t_1)} \le \alpha \|x\|_V$ for all $x \in W$.

Assumption 3. There exist $t_2 > 0$ and $\beta > 0$ such that

$$\int_0^{t_2} S(t_2 - \tau)Bu(\tau)\, d\tau \in W$$

and

$$\left\| \int_0^{t_2} S(t_2 - \tau)Bu(\tau)\, d\tau \right\|_W \le \beta \|u\|_{L^2(0,t_2)}$$

for all $u \in L^2(0, t_2; \mathbb{R}^p)$.

Assumption 4. There exists $\varepsilon > 0$ such that

$$\int_0^t CS(\tau)Bu\, d\tau = C \int_0^t S(\tau)Bu\, d\tau \qquad (2.3)$$

for all $u \in \mathbb{R}^p$ and $t \in [0, \varepsilon]$.

Remark 1. (i) Suppose that assumptions 1 and 2 are satisfied. Then the bounded linear operator $\mathcal{O}_W : W \to L^2(0, t_1; \mathbb{R}^p)$, $x \mapsto CS(\cdot)x$ can be uniquely extended to a bounded linear operator $\mathcal{O}_V : V \to L^2(0, t_1; \mathbb{R}^p)$. For $x \in V$ we define $CS(\cdot)x := \mathcal{Q}_V(x)$.

(ii) Assumption 3 implies that for every $x_0 \in W$ and every $u \in L^2(0, t_2; \mathbb{R}^p)$ formula (2.2) defines a continuous function $x(\cdot)$ on $[0, t_2]$ with values in W. Of course, we define the output by

$$y(t) = CS(t)x_0 + C \int_0^t S(t - \tau)Bu(\tau)\, d\tau \qquad (2.4)$$

for $0 \le t \le t_2$.

(iii) If assumption 2 holds for one particular $t_1 > 0$, then it can be shown that it is satisfied for all $t_1 > 0$, where α will depend on t_1. Moreover if $S(t)$ is exponentially stable on V then we can choose a constant α which does not depend on t_1.

(iv) The statement (iii) remains valid if we replace assumption 2 by assumption 3, t_1 by t_2, α by β and exponential stability on V by exponential stability on W.

(v) Notice that the l.h.s. of (2.3) has to be interpreted via assumptions 1 and 2 (cf. statements (i) and (iii) of this remark) while the r.h.s. makes sense because of assumption 3 and statement (iv).

(vi) Let A_V denote the infinitesimal generator of $S(t)$ on V. Suppose that assumptions 1–3 and

Assumption 5. $D(A_V) \subset W$ with continuous dense injection, where $D(A_V)$ is endowed with the graph norm of A_V

are satisfied. Then it can be shown that assumption 4 holds (cf. Pritchard and Salamon, 1987).

(vii) For $t \ge 0$ define

$$\mathscr{C}_t : L^2(0, t; \mathbb{R})^p \to V,$$
$$u \mapsto \int_0^t S(t - \tau)Bu(\tau)\, d\tau.$$

Assumption 3 means that there exists $t_2 > 0$ such that

$$\text{Range}\,(\mathscr{C}_{t_2}) \subset W \qquad (2.5)$$

and

$$\mathscr{C}_{t_2} \in \mathscr{L}(L^2(0, t_2; \mathbb{R}^p), W). \qquad (2.6)$$

If assumptions 1 and 5 are satisfied it can be shown as in Weiss (1989a) that (2.6) is implied by (2.5).

(viii) It is easy to show that

$$\int_t^T CS(\tau)Bu\, d\tau = C \int_t^T S(\tau)Bu\, d\tau$$

for all $T \ge t \ge 0$ and $u \in \mathbb{R}^p$ provided that assumptions 1–4 hold.

The above set-up and various modifications of it have been introduced and investigated in Salamon (1984), Curtain and Salamon (1986), Pritchard and Salamon (1987), Bontsema and Curtain (1988) and Curtain (1988, 1989). Related work has been done by Weiss (1989a, b). We remark that assumptions 1–3 and 5 are standard assumptions in the above references. Assumption 4 seems to be new. Remark 1(vi) shows that it is not stronger than assumption 5. While assumption 4 is stable under perturbations induced by static output feedback (cf. Lemma 2(ii)) we were not able to show that this is true for assumption 5.

If assumptions 1 and 2 are satisfied then it follows from Remark 1(i) and (iii) that for all $x \in V$ the expression $CS(\cdot)x$ makes sense as an element in $LL^2(0, \infty; \mathbb{R}^p)$. This leads us to the following definition.

Definition 1. Assume that assumptions 1 and 2 are satisfied and let $e_1, \ldots, e_p$ denote the canonical basis of $\mathbb{R}^p$. We can give a meaning to $CS(\cdot)B$ as an element in $LL^2(0, \infty; \mathbb{R}^{p \times p})$ by defining $CS(\cdot)B := (CS(\cdot)Be_1, \ldots, CS(\cdot)Be_p)$. We call $CS(\cdot)B$ the *impulse response matrix* of the system (2.1).

Remark 2. (i) Suppose that assumptions 1 and 2 hold. Then it is not difficult to show that for all $x \in V$ and $\tau \ge 0$

$$CS(\cdot + \tau)x = CS(\cdot)S(\tau)x,$$

and hence

$$CS(\cdot + \tau)B = CS(\cdot)S(\tau)B.$$

(ii) There is another way of making sense of the expression $CS(\cdot)B$ provided that assumptions 1 and 2 hold. It follows from assumption 1 that $\mathscr{L}(\mathbb{R}^p, W)$ is dense in $\mathscr{L}(\mathbb{R}^p, V)$ and the canonical injection $\mathscr{L}(\mathbb{R}^p, W) \to \mathscr{L}(\mathbb{R}^p, V)$, $R \mapsto R$ is bounded. From assumption 2 we obtain that there exists a constant $\gamma > 0$ such that for all $R \in \mathscr{L}(\mathbb{R}^p, W)$

$$\|CS(\cdot)R\|_{L^2(0,t_1;\mathbb{R}^{p\times p})} \le \gamma \|R\|_{\mathscr{L}(\mathbb{R}^p, V)}.$$

As in Remark 1(i) we conclude that the operator $\mathscr{L}(\mathbb{R}^p, W) \to L^2(0, t_1; \mathbb{R}^{p\times p})$, $R \mapsto CS(\cdot)R$ can be uniquely extended to a linear bounded operator $\mathscr{L}(\mathbb{R}^p, V) \to L^2(0, t_1; \mathbb{R}^{p\times p})$. The image of B under this operator is denoted by $CS(\cdot)B$. It is easy to see that the definition of $CS(\cdot)B$ given here and the one given above coincide.

Lemma 1. Let μ_W and μ_V denote the exponential growth constants of $S(t)$ on W and V, pick $\mu > \max(\mu_W, \mu_V)$ and suppose that assumption 2 holds. Then there exist a number $\gamma(s) > 0$ for all $s \in \mathbb{C}_\mu$ such that

$$|C(sI - A)^{-1}x| \le \gamma(s)\|x\|_V \qquad (2.7)$$

for all $x \in W$ and $s \in \mathbb{C}_\mu$.

Remark 3. Suppose that assumptions 1 and 2 hold. Then it follows from Lemma 1 that the operator $C(sI - A)^{-1}: W \to \mathbb{C}^p$ can be extended uniquely to a bounded linear operator $V \to \mathbb{C}^p$ for any $s \in \mathbb{C}_\mu$. We denote this extension by $C(sI - A)^{-1}$ as well.

Proof of Lemma 1. Recall the following elementary property of a function $f \in H^2(\mathbb{C}_0, \mathbb{C}^p)$

$$|f(s)| \le \left(\frac{2}{\pi \operatorname{Re}(s)}\right)^{1/2} \|f\|_{H^2} \; \forall s \in \mathbb{C}_0 \qquad (2.8)$$

(see e.g. Koosis, 1980). Then using (2.8) and the fact that

$$\frac{1}{\sqrt{2\pi}} \mathbb{L}: L^2(0, \infty; \mathbb{R}^p) \to H^2(\mathbb{C}_0, \mathbb{C}^p)$$

is an isometry it follows from assumption 2 that (2.7) holds.

Proposition 1. Let μ_W and μ_V denote the exponential growth constants of $S(t)$ on W and V and let μ and ν be numbers satisfying $\mu > \max(\mu_W, \mu_V)$ and $\nu > \min(\mu_W, \mu_V)$. The following statements hold:

(i) Suppose that assumptions 1–3 are satisfied. Then

$$CS(\cdot)Be^{-\nu\cdot} \in L^1(0, \infty; \mathbb{R}^{p\times p}) \qquad (2.9)$$

and moreover

$$\mathbb{L}(CS(\cdot)B)(s) = C(sI - A)^{-1}B \qquad (2.10)$$

for all $s \in \mathbb{C}_\mu$.

(ii) Let assumptions 1–4 be satisfied. Then we have

$$C(sI - A)^{-1}B = C(sI - A_V)^{-1}B \qquad (2.11)$$

for all $s \in \mathbb{C}_\mu$.

Remark 4. Notice that the l.h.s. of (2.11) has to be interpreted in the sense of Remark 3 while the r.h.s. of (2.11) makes sense because of assumption 3, which ensures that $(sI - A_V)^{-1}B \in \mathscr{L}(\mathbb{R}^p, W)$ for all $s \in \mathbb{C}_\mu$.

Proof of Proposition 1. (i) (2.9) follows from Curtain (1988). In order to show that (2.10) holds it is useful to define $S_\mu(t) := S(t)e^{-\mu t}$. It is trivial that $S_\mu(t)$ is an exponentially stable strongly continuous semigroup on W and V. Moreover it is clear that C and $S_\mu(t)$ satisfy assumption 2. Hence the expression $CS_\mu(\cdot)B$ makes sense as an element in $LL^2(0, \infty; \mathbb{R}^{p\times p})$ and it is a routine exercise to show $CS_\mu(\cdot)B = CS(\cdot)Be^{-\mu}$.

For a given $u \in \mathbb{R}^p$ pick a sequence $x_n \in W$ such that $\lim_{n\to\infty} x_n = Bu$ (in V). It follows from assumption 2 (cf. Remark 1(i) and (iii)) that $CS_\mu(\cdot)x_n$ converges in $L^2(0, \infty; \mathbb{R}^p)$ and by definition $CS_\mu(\cdot)Bu = \lim_{n\to\infty} CS_\mu(\cdot)x_n$. Now the Laplace transform $\mathbb{L}: L_2(0, \infty; \mathbb{R}^p) \to H^2(\mathbb{C}_0, \mathbb{C}^p)$ is bounded and so

$$C((\cdot + \mu)I - A)^{-1}x_n \xrightarrow[H^2]{} \mathbb{L}(CS_\mu(\cdot)Bu)$$

as $n \to \infty$. Using (2.8) we obtain

$$C(sI - A)^{-1}x_n \xrightarrow[\mathbb{C}^p]{} \mathbb{L}(CS(\cdot)B)(s)u$$

for all $s \in \mathbb{C}_\mu$ as $n \to \infty$. On the other hand by definition (cf. Remark 3)

$$C(sI - A)^{-1}x_n \xrightarrow[\mathbb{C}^p]{} C(sI - A)^{-1}Bu$$

for all $s \in \mathbb{C}_\mu$ as $n \to \infty$. Hence

$$\mathbb{L}(CS(\cdot)B)(s)u = C(sI - A)^{-1}Bu$$

for all $s \in \mathbb{C}_\mu$, which is (2.10).

(ii) Let $s \in \mathbb{C}_\mu$, $s \neq 0$ and let $u \in \mathbb{R}^p$ be arbitrary. Using Remark 4, assumptions 1–4, Remark 1(viii) and (2.10) we obtain

$$\begin{aligned}\frac{1}{s}C(sI - A_V)^{-1}Bu &= C\frac{1}{s}(sI - A_V)^{-1}Bu \\ &= C\left(\mathbb{L}\left(\int_0^{\cdot} S(\tau)Bu\,d\tau\right)\right)(s) \\ &= \left(\mathbb{L}C\left(\int_0^{\cdot} S(\tau)Bu\,d\tau\right)\right)(s) \\ &= \left(\mathbb{L}\int_0^{\cdot} CS(\tau)Bu\,d\tau\right)(s) \\ &= \frac{1}{s}C(sI - A)^{-1}Bu\end{aligned}$$

for all $s \in \mathbb{C}_\mu$, and hence (2.11).

Remark 5. Suppose assumptions 1–3 hold. Then statement (ii) of Proposition 1 says that assumption 4 implies (2.11). It is possible to show that assumption 4 and (2.11) are actually equivalent.

Proposition 2. Let assumptions 1–4 be satisfied. Then

$$\int_0^t CS(t-\tau)Bu(\tau)\,d\tau = C\int_0^t S(t-\tau)Bu(\tau)\,d\tau \quad (2.12)$$

for all $u \in LL^2(0, \infty; \mathbb{R}^p)$ and $t \geq 0$.

Remark 6. Assume that assumptions 1–4 hold. Since the r.h.s. of (2.12) is the output of the system (2.1) under zero initial conditions Proposition 2 shows that it was justified to call $CS(\cdot)B$ the impulse response matrix of the system (2.1). (cf. Definition 1). Moreover—combining Proposition 2 and Proposition 1—it turns out that

$$C(sI-A)^{-1}B = C(sI-A_V)^{-1}B = \mathbb{L}(CS(\cdot)B)(s)$$

is the transfer matrix of the system (2.1) in the usual sense of the word.

Proof of Proposition 2. Let $T>0$ be given. It follows from assumption 3 that there exists $\gamma>0$ such that

$$\left\|\int_0^t S(t-\tau)Bu(\tau)\,d\tau\right\|_W \leq \gamma\,\|u\|_{L^2(0,t)} \quad (2.13)$$

for all $u \in L^2(0, t; \mathbb{R}^p)$ and $t \in [0, T]$. Using (2.13) we obtain

$$\left(\int_0^T \left|C\int_0^t S(t-\tau)Bu(\tau)\,d\tau\right|^2 dt\right)^{1/2} \leq \gamma\sqrt{T}\,\|C\|\,\|u\|_{L^2(0,T)} \quad (2.14)$$

for all $u \in L^2(0, T; \mathbb{R}^p)$. Define the operators

$$\mathcal{F}: LL^2(0, \infty; \mathbb{R}^p) \to LL^2(0, \infty; \mathbb{R}^p),$$

$$u \mapsto C\int_0^{\cdot} S(\cdot-\tau)Bu(\tau)\,d\tau$$

and

$$\mathcal{G}: LL^2(0, \infty; \mathbb{R}^p) \to LL^2(0, \infty; \mathbb{R}^p),$$

$$u \mapsto \int_0^{\cdot} CS(\cdot-\tau)Bu(\tau)\,d\tau.$$

It is sufficient to show that $\pi_t\mathcal{F}\pi_t = \pi_t\mathcal{G}\pi_t$ for all $t \geq 0$. Indeed, since $\mathcal{F}$ and $\mathcal{G}$ are causal, it then follows that $\mathcal{F} = \mathcal{G}$ and since $\mathcal{F}u$ and $\mathcal{G}u$ are continuous functions for all $u \in LL^2(0, \infty; \mathbb{R}^p)$ we obtain (2.12). It is clear that $\pi_t\mathcal{G}\pi_t$ is a bounded linear operator from $L^2(0, t; \mathbb{R}^p)$ into itself and it follows from (2.14) that the same is true for $\pi_t\mathcal{F}\pi_t$. Using assumption 4 it is a matter of routine to show that $\pi_t\mathcal{F}\pi_t u = \pi_t\mathcal{G}\pi_t u$ for all step-functions defined on $[0, t]$. Now the step-functions form a dense subset of $L^2(0, t; \mathbb{R}^p)$ and hence $\pi_t\mathcal{F}\pi_t = \pi_t\mathcal{G}\pi_t$.

We shall need the following perturbation results.

Lemma 2. (i) Suppose that assumptions 1 and 3 hold. Then for $F \in \mathcal{L}(W, \mathbb{R}^p)$ there exists a strongly continuous semigroup $S_F(t)$ on W which is the unique solution of

$$S_F(t)x = S(t)x + \int_0^t S(t-\tau)BFS_F(\tau)x\,d\tau \quad (2.15)$$

for all $x \in W$ and $t \geq 0$.

(ii) Suppose that assumptions 1–4 are satisfied. Then for $K \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^p)$ there exists a strongly continuous semigroup $S_K(t)$ on W and V which is the unique solution of

$$S_K(t)x = S(t)x + \int_0^t S(t-\tau)B_KCS_K(\tau)x\,d\tau \quad (2.16)$$

for all $x \in W$ and $t \geq 0$. Furthermore $S_K(t)$, B and C satisfy assumptions 2–4.

Proof. The lemma is well known (see e.g. Curtain, 1988) with the exception of the fact that assumption 4 remains valid for the perturbed semigroup $S_K(t)$. Let $T>0$ and $u \in \mathbb{R}^p$ be fixed, but arbitrary and let $x_n \in W$ be a sequence such that $Bu = \lim_{n\to\infty} x_n$ (in V). Now $S_K(t)$ and C satisfy assumption 2 and so we can define

$$CS_K(\cdot)Bu := \lim_{n\to\infty} CS_K(\cdot)x_n,$$

where the limit has to be understood in $L_2(0, T; \mathbb{R}^p)$. As a consequence there exists a subsequence x_{n_j} of x_n such that $CS_K(t)Bu = \lim_{j\to\infty} CS_K(t)x_{n_j}$ a.e. in $[0, T]$ and $CS(t)Bu = \lim_{j\to\infty} CS(t)x_{n_j}$ a.e. in $[0, T]$. Using (2.16) we obtain

$$\begin{aligned} CS_K(t)Bu &= \lim_{j\to\infty}\left\{CS(t)x_{n_j} + C\int_0^t S(t-\tau)BKCS_K(\tau)x_{n_j}\,d\tau\right\} \\ &= CS(t)Bu + C \times \int_0^t S(t-\tau)BKCS_K(\tau)Bu\,d\tau \quad (2.17) \end{aligned}$$

for almost all $t \in [0, T]$. In the following we shall use the fact that if $f \in L^1(0, t; W)$, then $f \in L^1(0, t; V)$ and ${}_V\int_0^t f(\tau)\,d\tau = {}_W\int_0^t f(\tau)\,d\tau$, where ${}_V\int$ and ${}_W\int$ denote integration in V and W, respectively. Integrating (2.17) gives

$$\begin{aligned} &\int_0^t CS_K(\lambda)Bu\,d\lambda \\ &= \int_0^t CS(\lambda)Bu\,d\lambda + \int_0^t C\,{}_V\!\int_0^\lambda S(\lambda-\tau)BKCS_K(\tau)Bu\,d\tau\,d\lambda \\ &= C\,{}_V\!\int_0^t S(\lambda)Bu\,d\lambda + C\,{}_W\!\int_0^t {}_V\!\int_0^\lambda S(\lambda-\tau)BKCS_K(\tau)Bu\,d\tau\,d\lambda \end{aligned}$$

$$= C \int_{V^0}^{t} \Big\{ S(\lambda)Bu + \int_{V^0}^{\lambda} S(\lambda - \tau)BKCS_K(\tau)Bu \, d\tau \Big\} \, d\lambda$$

$$= C \int_{V^0}^{t} \lim_{n\to\infty} \Big\{ S(\lambda)x_n + \int_{V^0}^{\lambda} S(\lambda - \tau)BKCS_K(\tau)x_n \, d\tau \Big\} \, d\lambda$$

$$= C \int_{V^0}^{t} \lim_{n\to\infty} S_K(\lambda)x_n \, d\lambda$$

$$= C \int_{V^0}^{t} S_K(\lambda)Bu \, d\lambda$$

for all $t \in [0, T]$, which proves the claim.

Remark 7. (i) Suppose that assumptions 1–4 are satisfied. Then it is easy to see that (2.16) actually holds for all $x \in V$. Let $x \in V$, choose a sequence $x_n \in W$ such that $\lim_{n\to\infty} x_n = x$ and define $CS_K(\cdot)x := \lim_{n\to\infty} CS_K(\cdot)x_n$, where the limit has to be understood in $L^2(0, t; \mathbb{R}^p)$. Then we have

$$S_K(t)x = \lim_{n\to\infty} S_K(t)x_n = \lim_{n\to\infty} S(t)x_n + \lim_{n\to\infty} \times \int_0^t S(t-\tau)BKCS_K(\tau)x_n \, d\tau$$

$$= S(t)x + \int_0^t S(t-\tau)BKCS_K(\tau)x \, d\tau.$$

(ii) We were not able to show that assumption 5 is stable under perturbations induced by static output feedback.

We now define exponential stabilizability with respect to unbounded perturbations.

Definition 2. Suppose that assumptions 1 and 3 hold. We say that the pair $(S(t), B)$ is unbounded exponentially stabilizable on W if there exists an operator $F \in \mathscr{L}(W, \mathbb{R}^p)$ such that the perturbed semigroup $S_F(t)$ defined by (2.15) is exponentially stable on W.

If B is bounded with respect to W, the above definition coincides with the usual notion of exponential stabilizability on W. Moreover under the extra assumption 5 it is equivalent to the existence of a feedback $F \in \mathscr{L}(V, \mathbb{R}^p)$ which stabilizes $(S(t), B)$ on both spaces W and V (see Curtain, 1988).

The following lemma follows from Curtain (1988).

Lemma 3. Suppose that assumptions 1–4 are satisfied, that $(S(t), B)$ is unbounded exponentially stabilizable on W and denote the transfer matrix of the system (2.1) by G, i.e. $G(s) = \mathbb{L}(CS(\cdot)B)(s)$. Then G can be decomposed as $G = G_i + G_f$, where G_f is a strictly proper rational matrix having all its poles in $\bar{\mathbb{C}}_0$ and G_i is stable in the sense that there exists a number $\varepsilon > 0$ such that $e^{\varepsilon \cdot}\mathbb{L}^{-1}(G_i) \in L^1(0, \infty; \mathbb{R}^{p\times p})$.

Notice that Lemma 3 implies in particular that the entries of the transfer matrix G of (2.1) belong to the Callier–Desoer algebra (see Callier and Desoer, 1978, 1980) provided that assumptions 1–4 hold and $(S(t), B)$ is unbounded exponentially stabilizable on W. Moreover we see that under these conditions G is strictly proper in the sense that $\lim_{|s|\to\infty, s\in\bar{\mathbb{C}}_0} G(s) = 0$.

Remark 8. For examples of systems satisfying assumptions 1–4 we refer the reader to Salamon (1984), Pritchard and Salamon (1987) and Curtain (1988, 1989). It is known that for a large class of neutral systems with delays in the input or the output assumptions 1–4 hold (cf. Salamon, 1984; Pritchard and Salamon, 1987). Furthermore, it has been shown in Pritchard and Salamon (1985) that assumptions 1–4 are satisfied for retarded systems with delays both in the control and the observation. For parabolic, hyperbolic and spectral systems sufficient conditions for assumptions 1–4 to be satisfied were given in Pritchard and Salamon (1987) and Curtain (1988). They were applied to partial differential equation models of flexible structures in Bontsema *et al.* (1988).

3. THE SMALL GAIN CONDITION AND INTERNAL STABILITY

If we apply the nonlinear output-feedback law $u(t) = \varphi(t, y(t))$ to (2.1) we obtain

$$\dot{x}(t) = Ax(t) + B\varphi(t, Cx(t)), \quad x(0) = x_0. \tag{3.1}$$

We are interested in mild solutions of (3.1), i.e. in trajectories satisfying

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)B\varphi(\tau, Cx(\tau)) \, d\tau. \tag{3.2}$$

The following lemma gives a sufficient condition for the existence of a unique solution of (3.2).

Lemma 4. Suppose assumptions 1–4 hold. Let $\varphi: \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^p$ be such that $\varphi(t, y)$ is continuous in t and locally Lipschitzian in y, uniformly in t on bounded intervals. For all $x_0 \in W$ there exists a unique continuous solution in W of (3.2) which can be extended to the right as long as it remains bounded.

Proof. The Volterra integral equation

$$z(t) = CS(t)x_0 + \int_0^t CS(t-\tau)B\varphi(\tau, z(\tau)) \, d\tau \tag{3.3}$$

admits for given $x_0 \in W$ a unique continuous solution which can be extended to the right as long as it remains bounded (see Miller, 1971). If this solution is denoted by y then it follows from Proposition 2 that

$$y(t) = CS(t)x_0 + C\int_0^t S(t-\tau)B\varphi(\tau, y(\tau))\,\mathrm{d}\tau \tag{3.4}$$

and hence

$$x(t) := S(t)x_0 + \int_0^t S(t-\tau)B\varphi(\tau, y(\tau))\,\mathrm{d}\tau$$

is a solution of (3.2) which is continuous in W. In order to show uniqueness, let $x'(t)$ be another solution of (3.2). Since $Cx(t)$ and $Cx'(t)$ are both solutions of (3.4) and hence of (3.3), it follows that $Cx(t) = Cx'(t)$ for all t where $x(\cdot)$ and $x'(\cdot)$ exist. Using (3.2) we obtain that $x(\cdot)$ and $x'(\cdot)$ coincide. The continuation property can be shown similarly.

The following theorem gives a small-gain condition for the exponential stability of (3.2).

Theorem 1. Suppose that the assumptions of Lemma 4 are satisfied and that φ is of finite gain, i.e. there exists $\gamma > 0$ such that $|\varphi(t, y)| \le \gamma |y|$ for all $t \ge 0$, $y \in \mathbb{R}^p$. Moreover assume that $S(t)$ is exponentially stable on W and

$$\gamma \|G\|_\infty < 1 \tag{3.5}$$

where G is the transfer matrix of (2.1). Under these conditions there exist numbers $M = M(\gamma) > 0$ and $\mu > 0$ not depending on t or x_0 such that the solution $x(t; x_0)$ of (3.2) is globally defined and satisfies

$$\|x(t; x_0)\|_W \le M\mathrm{e}^{-\mu t}\|x_0\|_W$$

for all $t \ge 0$, $x_0 \in W$.

Proof. By assumption there exist $\lambda > 0$ and $L > 0$ such that $\|S(t)\|_{\mathscr{L}(W)} \le L\mathrm{e}^{-\lambda t}$ for all $t \ge 0$. It follows from Proposition 1(i) that $G \in H^\infty(\mathbb{C}_\eta, \mathbb{C}^{p\times p})$ for all $\eta > -\lambda$ and hence G is uniformly continuous on any vertical strip in $\mathbb{C}_{-\lambda}$ (see Corduneanu, 1968, p. 72). Therefore, by (3.5), there exists a number $\lambda^* \in (0, \lambda)$ such that

$$\gamma \|G_a\|_\infty < 1 \tag{3.6}$$

for all $a \in [0, \lambda^*]$, where G_a is defined by $G_a(s) := G(s-a)$. Setting $x(t) := x(t; x_0)$, $H(t) := CS(t)B$, $H_a(t) := \mathrm{e}^{at}H(t)$ and $\varphi_a(t, z) := \mathrm{e}^{at}\varphi(t, \mathrm{e}^{-at}z)$, where $a \in [0, \lambda^*]$, and using Proposition 2 we obtain

$$Cx(t) = CS(t)x_0 + \int_0^t H(t-\tau)\varphi(\tau, Cx(\tau))\,\mathrm{d}\tau.$$

Multiplying both sides of this equation by e^{at}, where $a \in [0, \lambda^*]$, gives

$$\mathrm{e}^{at}Cx(t) = \mathrm{e}^{at}CS(t)x_0 + \int_0^t H_a(t-\tau)\varphi_a(\tau, \mathrm{e}^{a\tau}Cx(\tau))\,\mathrm{d}\tau.$$

Let $[0, T)$ be the maximal interval of existence of $x(\cdot)$. For all $t \in [0, T)$ we have

$$\|\pi_t\mathrm{e}^{a\cdot}Cx(\cdot)\|_2 \le \pi_t\mathrm{e}^{a\cdot}CS(\cdot)x_0\|_2 + \gamma\|G_a\|_\infty\|\pi_t\mathrm{e}^{a\cdot}Cx(\cdot)\|_2.$$

Hence, by (3.6)

$$\|\pi_t\mathrm{e}^{a\cdot}Cx(\cdot)\|_2 \le (1-\gamma\|G_a\|_\infty)^{-1} \times \|C\|\, L(2(\lambda-a))^{-1/2}\|x_0\|_W. \tag{3.7}$$

Since $x(\cdot)$ solves (3.2) it follows from (3.7) (set $a = 0$), assumption 3 and Remark 1(iv) that $\|x(t)\|_W$ is bounded on $[0, T)$. As a consequence $x(\cdot)$ is globally defined (i.e. $T = \infty$) by Lemma 4. Setting $K := (1-\gamma\|G_a\|_\infty)^{-1}\|C\|\, L(2(\lambda-a))^{-1/2}$ we obtain

$$\|\mathrm{e}^{a\cdot}Cx(\cdot)\|_2 \le K\|x_0\|_W. \tag{3.8}$$

Moreover, by assumption 3 and Remark 1(iv) there exists $\tilde\beta = \tilde\beta(a)$ not depending on t such that

$$\left\|\int_0^t \mathrm{e}^{a(t-\tau)}S(t-\tau)Bu(\tau)\,\mathrm{d}\tau\right\|_W \le \tilde\beta\|u\|_{L^2(0,t)}$$

for all $t \ge 0$. Now, by (3.2)

$$\mathrm{e}^{at}x(t) = \mathrm{e}^{at}S(t)x_0 + \int_0^t \mathrm{e}^{a(t-\tau)}S(t-\tau)B\varphi_a(\tau, C\mathrm{e}^{a\tau}x(\tau))\,\mathrm{d}\tau.$$

Hence

$$\begin{aligned}\|\mathrm{e}^{at}x(t)\|_W &\le \|\mathrm{e}^{at}S(t)x_0\|_W + \tilde\beta\|\varphi_a(\cdot, C\mathrm{e}^{a\cdot}x(\cdot))\|_{L^2(0,t)} \\ &\le L\|x_0\|_W + \tilde\beta\gamma\|\mathrm{e}^{a\cdot}Cx(\cdot)\|_{L^2(0,t)} \\ &\le (L + \tilde\beta\gamma K)\|x_0\|_W\end{aligned}$$

for all $t \ge 0$.

Remark 9. Let t_0 be non-negative and denote the solution of

$$x(t) = S(t-t_0)x_0 + \int_{t_0}^t S(t-\tau)B\varphi(\tau, Cx(\tau))\,\mathrm{d}\tau$$

(where $t \ge t_0$ and $x_0 \in W$) by $x(t; t_0, x_0)$. Suppose that the assumptions of Theorem 1 hold and that (3.5) is satisfied. Then it can be shown as in the proof of Theorem 1 that there exists $M = M(\gamma)$ and $\mu > 0$ not depending on t, t_0 and x_0 such that

$$\|x(t; t_0, x_0)\|_W \le M\mathrm{e}^{-\mu(t-t_0)}\|x_0\|_W \quad \forall t \ge t_0$$

i.e. we have global *uniform* exponential stability.

We shall now turn our attention to a certain class of dynamical nonlinearities. Let us consider operators Φ from $LL^2(0, \infty; \mathbb{R}^p)$ into itself which satisfy the following conditions:

(N1) Φ is causal, i.e. $\pi_t\Phi = \pi_t\Phi\pi_t$ for all $t \ge 0$.

(N2) Φ is locally Lipschitz continuous, i.e. for all $t \ge 0$ there exists $\ell_t \ge 0$ such that $\|\pi_t(\Phi u - \Phi v)\|_2 \le \ell_t\|\pi_t(u-v)\|_2$ for all $u, v \in LL^2(0, \infty; \mathbb{R}^p)$ (cf. Willems, 1971).

(N3) Φ is unbiased, i.e. $\Phi(0) = 0$.

Remark 10. Let N_φ denote the operator induced by the function $\varphi: \mathbb{R}^p \to \mathbb{R}^p$, i.e. $(N_\varphi(u))(t) = \varphi(u(t))$. Then in general N_φ will not be locally

Lipschitz in the sense of (N2) unless φ satisfies a global Lipschitz condition.

Lemma 5. Suppose that assumption 3 holds and that the operator $\Phi: LL^2(0,\infty;\mathbb{R}^p)\to LL^2(0,\infty;\mathbb{R}^p)$ satisfies the conditions (N1)–(N3). Then the equation

$$x(t)=S(t)x_0+\int_0^t S(t-\tau)B\Phi(Cx(\cdot))(\tau)\,\mathrm{d}\tau \tag{3.9}$$

admits for all $x_0\in W$ a globally defined continuous solution $x(\cdot;x_0):[0,\infty)\to W$ which will be called the mild solution of

$$\dot{x}(t)=Ax(t)+B\Phi(Cx(\cdot))(t),\quad x(0)=x_0.$$

Proof. Notice that the operator

$$\mathscr{G}: LL^2(0,\infty;\mathbb{R}^p)\to LL^2(0,\infty;W),$$

$$u(\cdot)\mapsto\int_0^{\cdot} S(\cdot-\tau)Bu(\tau)\,\mathrm{d}\tau$$

is strongly causal (see Willems, 1971), locally Lipschitz continuous (this follows from assumption 3) and unbiased. Since $\mathscr{G}$ is linear it is strongly causal uniformly with respect to past inputs (see Willems, 1971). Moreover it is trivial that the map

$$\mathscr{F}: LL^2(0,\infty;W)\to LL^2(0,\infty;\mathbb{R}^p),$$

$$z(\cdot)\mapsto\Phi(Cz(\cdot))$$

satisfies the conditions (N1)–(N3). Hence it follows from Corollary 4.1.2 on p. 99 in Willems (1971) that (3.9) admits a unique solution x in $LL^2(0,\infty;W)$. Since the r.h.s. of (3.9) is continuous in t we see that $x(t)$ is continuous as well.

Definition 3. For (3.9), the origin is called globally asymptotically stable if the following conditions are satisfied:

(i) The origin is globally attractive, i.e. $\lim_{t\to\infty}\|x(t;x_0)\|_W=0$ for all $x_0\in W$.

(ii) For all $\varepsilon>0$ there exists $\delta>0$ such that $\|x_0\|_W\le\delta$ implies $\|x(t;x_0)\|_W\le\varepsilon$ for all $t\ge 0$.

The following theorem shows that the small-gain condition (3.5) is sufficient for the origin of (3.9) to be globally asymptotically stable.

Theorem 2. Assume that assumptions 1–4 hold, the semigroup $S(t)$ is exponentially stable on W, (N1)–(N3) are satisfied and the operator Φ is of finite gain, i.e. there exists a constant $\gamma>0$ such that $\|\Phi(u)\|_2\le\gamma\|u\|_2$ for all $u\in L^2(0,\infty;\mathbb{R}^p)$. Under these conditions the origin of (3.9) will be globally asymptotically stable if

$$\gamma\|G\|_\infty<1, \tag{3.10}$$

where G denotes the transfer matrix of (2.1).

Proof. Consider the equation

$$y(t)=CS(t)x_0+\int_0^t CS(t-\tau)B\Phi(y(\cdot))(\tau)\,\mathrm{d}\tau. \tag{3.11}$$

If $x(t):=x(t;x_0)$ is the solution of (3.9) then it follows from Proposition 2 that $Cx(t)$ is a solution of (3.11). Using Remark 6, (3.10), and the small-gain theorem we obtain $Cx(\cdot)$, $z(\cdot):=\Phi(Cx(\cdot))\in L^2(0,\infty;\mathbb{R}^p)$. We prove global attraction first. It is convenient to set $w(t):=\int_0^t S(t-\tau)Bz(\tau)\,\mathrm{d}\tau$. By the exponential stability of $S(t)$ on W, assumption 3 and Remark 1(iv), there exist positive constants L, λ and η such that

$$\|S(t)\|_{\mathscr{L}(W)}\le L\mathrm{e}^{-\lambda t} \tag{3.12}$$

and

$$\left\|\int_0^t S(t-\tau)Bu(\tau)\,\mathrm{d}\tau\right\|_W\le\eta\|u\|_{L^2(0,t)} \tag{3.13}$$

for all $t\ge 0$ and $u\in L^2(0,t;\mathbb{R}^p)$.

Equations (3.12) and (3.13) yield

$$\begin{aligned}\|w(t)\|_W&\le\left\|\int_0^{t/2}S(t-\tau)Bz(\tau))\,\mathrm{d}\tau\right\|_W\\&\quad+\left\|\int_{t/2}^{t}S(t-\tau)Bz(\tau))\,\mathrm{d}\tau\right\|_W\\&=\left\|S\left(\frac{t}{2}\right)\int_0^{t/2}S\left(\frac{t}{2}-\tau\right)Bz(\tau)\,\mathrm{d}\tau\right\|_W\\&\quad+\eta\|z\|_{L^2(t/2,t)}\\&\le L\mathrm{e}^{-(\lambda/2)t}\eta\|z\|_{L^2(0,t/2)}\\&\quad+\eta\|z\|_{L^2(t/2,\infty)}\\&\le\eta(L\|z\|_2\mathrm{e}^{-(\lambda/2)t}+\|z\|_{L^2(t/2,\infty)}).\end{aligned}$$

Since z is in $L^2(0,\infty;\mathbb{R}^p)$ we have $\lim_{t\to\infty}\|z\|_{L^2(t/2,\infty)}=0$ and hence $\lim_{t\to\infty}\|w(t)\|_W=0$. It follows that $\lim_{t\to\infty}\|x(t)\|_W=0$. In order to complete the proof, notice that by (3.10) and (3.11)

$$\|Cx(\cdot)\|_2\le(1-\gamma\|G\|_\infty)^{-1}M\|x_0\|_W, \tag{3.14}$$

where $M:=\|C\|L/\sqrt{2\lambda}$, and conclude using (3.9) and (3.12)–(3.14) that

$$\|x(t)\|_W\le(L\|C\|+\eta\gamma(1-\gamma\|G\|_\infty)^{-1})\|x_0\|_W$$

for all $t\ge 0$.

Finite-dimensional results which are related to Theorems 1 and 2 can be found in Hinrichsen and Pritchard (1986, 1990).

The above results show that destabilization can only occur if the gain of the perturbation induced by output feedback is greater or equal to $1/\|G\|_\infty$. We will show that the small-gain condition is sharp in the sense that there exist destabilizing perturbations of gain equal to $1/\|G\|_\infty$. We shall make contact with the work of Hinrichsen and Pritchard on stability radii of

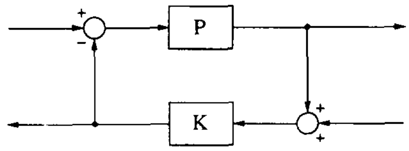

FIG. 1. Feedback system considered in Definition 4.

finite-dimensional linear systems (for a survey, see Hinrichsen and Pritchard, 1989).

The following concept of stability in the frequency domain is standard and will be used in the sequel.

Definition 4. Let P and K be in $\hat{\mathscr{B}}^{p\times p}$ such that $\det(I+PK)\not\equiv 0$ (well-posedness condition). Denote the feedback system shown in Fig. 1 by $\mathscr{F}[P, K]$ and set

$$H = H(P, K) := \begin{pmatrix} P(I+KP)^{-1} & -PK(I+PK)^{-1} \\ KP(I+KP)^{-1} & K(I+PK)^{-1} \end{pmatrix}$$

$\mathscr{F}[P, K]$ is called input-output stable if $H \in \hat{\mathscr{A}}_-^{2p\times 2p}$.

Remark 11. It is easy to show that

$$\inf_{s\in\bar{\mathbb{C}}_0} |\det(I+PK)(s)| = \inf_{s\in\bar{\mathbb{C}}_0} |\det(I+KP)(s)| > 0$$

is a necessary condition for the input output stability of $\mathscr{F}[P, K]$.

The following theorem shows that a stable plant P can be destabilized via output feedback with a compensator of norm $1/\|P\|_\infty$.

Theorem 3. Let P be in $\hat{\mathscr{W}}_{-,r}^{p\times p}$ and suppose $\bar{\sigma}(P(j\omega)) \not\equiv \|P\|_\infty$. Then the following statements hold:

(i) There exists a *complex* matrix $K\in\mathbb{C}^{p\times p}$ such that $\bar{\sigma}(K) = 1/\|P\|_\infty$ and $\mathscr{F}[P, K]$ is not input–output stable. The matrix K can be chosen *real* if there exists

$$\omega_0 \in \mathbb{R}\cup\{\infty\} \text{ satisfying } P(j\omega_0)\in\mathbb{R}^{p\times p}$$

and

$$\bar{\sigma}(P(j\omega_0)) = \|P\|_\infty.$$

(ii) There exists a *real* finite-dimensional proper compensator $K\in(\mathbb{R}(s))^{p\times p}\cap(H^\infty(\mathbb{C}_0))^{p\times p}$ such that $\|K\|_\infty = 1/\|P\|_\infty$ and $\mathscr{F}[P, K]$ is not input–output stable.

Proof. The proof is inspired by Vidyasagar's proof of the robustness criterion given in Doyle and Stein (1981); see Vidyasagar (1985), p. 273. Suppose the maximum of $\bar{\sigma}(P(j\omega))$ occurs at $\omega_0\in\mathbb{R}\cup\{\infty\}$. Let $\sigma_1, \ldots, \sigma_m$ denote the nonzero singular values of $P(j\omega_0)$, where $\sigma_1 = \|P\|_\infty$, and set $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_m)$. Select unitary (orthogonal, if $P(j\omega_0)$ is real) matrices U and V such that

$$UP(j\omega_0)V = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}.$$

(i) If we define

$$K := -V\begin{pmatrix} \frac{1}{\sigma_1} & 0 \\ 0 & 0 \end{pmatrix}U$$

then trivially $\bar{\sigma}(K) = 1/\|P\|_\infty$. Moreover, by assumption there exists $\omega_1\in\mathbb{R}\cup\{\infty\}$ such that $\bar{\sigma}(P(j\omega_1)) < \|P\|_\infty$. It follows that $\det(I+P(j\omega_1)K)\neq 0$, and hence the feedback system $\mathscr{F}[P, K]$ is well posed. By Remark 11 $\mathscr{F}[P, K]$ is not input–output stable, since for $e := U^{-1}(1\ 0\cdots 0)^T$ it holds that $(I+P(j\omega_0)K)e = 0$.

(ii) Let $v := (v_1, \ldots, v_p)^T$ and $u := (u_1, \ldots, u_p)$ denote the first column of v and the first row of U, respectively. Express the components v_i and u_i in the form

$$v_i = \bar{v}_i e^{j\varphi_i}, \quad u_i = \bar{u}_i e^{j\psi_i}, \quad i = 1, \ldots, p,$$

where $\bar{v}_i, \bar{u}_i\in\mathbb{R}$ and $\varphi_i, \psi_i\in[0, \pi)$. Define

$$K(s) := -\frac{1}{\sigma_1}\bar{v}(s)\bar{u}^T(s),$$

where

$$\bar{v}(s) := \begin{pmatrix} \bar{v}_1(s-a_1)/(s+a_1) \\ \vdots \\ \bar{v}_p(s-a_p)/(s+a_p) \end{pmatrix},$$

$$\bar{u}(s) := \begin{pmatrix} \bar{u}_1(s-b_1)/(s+b_1) \\ \vdots \\ \bar{u}_p(s-b_p)/(s+b_p) \end{pmatrix}$$

and the constants $a_i, b_i \geq 0$ are adjusted such that

$$\arg\left(\frac{j\omega_0 - a_i}{j\omega_0 + a_i}\right) = \varphi_i \quad\text{and}\quad \arg\left(\frac{j\omega_0 - b_i}{j\omega_0 + b_i}\right) = \psi_i.$$

By construction we have that $K\in\mathbb{R}(s)^{p\times p}\cap H^\infty(\mathbb{C}_0)^{p\times p}$,

$$K(j\omega_0) = -V\begin{pmatrix} \frac{1}{\sigma_1} & 0 \\ 0 & 0 \end{pmatrix}U$$

and $\|K\|_\infty = 1/\|P\|_\infty$. It follows as in the proof of (i) that $\mathscr{F}[P, K]$ is well posed and unstable.

Remark 12. If P is in $\hat{\mathscr{A}}_{-,r}^{p\times p}$ there might fail to exist a $\omega_0\in\mathbb{R}\cup\{\infty\}$ such that $\bar{\sigma}(P(j\omega_0)) = \|P\|_\infty$. An inspection of the above proof shows that in this case there exists a sequence $K_n\in\mathbb{C}^{p\times p}(\mathbb{R}(s)^{p\times p}\cap H^\infty(\mathbb{C}_0)^{p\times p})$ such that $\bar{\sigma}(K_n)\searrow 1/\|P\|_\infty(\|K_n\|_\infty\searrow 1/\|P\|_\infty)$ as $n\to\infty$ and $\mathscr{F}[P, K_n]$ is not input-output stable for all $n\in\mathbb{N}$.

Corollary 1. Suppose that assumptions 1–4 hold true and that $S(t)$ is exponentially stable on W. Let G denote the transfer matrix of (2.1). The following statements hold true:

(i) There exists a matrix $K\in\mathbb{C}^{p\times p}$ satisfying $\bar{\sigma}(K) = 1/\|G\|_\infty$ such that the strongly continuous semigroup $S_K(t)$ which is the unique solution of (2.16) is not exponentially stable on

W. Under the extra assumption that $S(t)$ is exponentially stable on V it follows that $S_K(t)$ is not exponentially stable on V either.

(ii) Under the additional assumption that C is surjective there exists a matrix $K \in \mathbb{R}(s)^{p\times p} \cap H^\infty(\mathbb{C}_0)^{p\times p}$ of H^∞-norm equal to $1/\|G\|_\infty$ and an initial value $x_0 \in W$ such that the solution $x(t; x_0)$ of

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)B\mathcal{K}(Cx(\cdot))(\tau)\,d\tau, \tag{3.15}$$

is not in $L^1(0, \infty; W)$ and hence not exponentially decaying in W. In (3.15) $\mathcal{K}$ denotes the convolution operator on $L^2(0, \infty; \mathbb{R}^p)$ corresponding to K.

Remark 13. The result in statement (i) is similar to a result by Pritchard and Townley (1987) (and see also Pritchard and Townley, 1989). However their assumptions on the system are somewhat different from ours. A proof for the finite-dimensional case has been given in Hinrichsen and Pritchard (1986). A result in finite dimensions which is related to statement (ii) can be found in Hinrichsen and Pritchard (1990).

Proof of Corollary 1. (i) The claim follows from Theorem 3(i) and Curtain (1988).

(ii) By assumption there exist $w_1, \ldots, w_p \in D(A)$ such that $Cw_1, \ldots, Cw_p$ form a basis of $\mathbb{R}^p$. As in the proof of Theorem 3(ii) we can show that there exists $K \in \mathbb{R}(s)^{p\times p} \cap H^\infty(\mathbb{C}_0)^{p\times p}$ of H^∞-norm $1/\|G\|_\infty$ and a number $\omega_0 \in \mathbb{R}$ such that

$$\det(I - GK)(j\omega_0) = 0. \tag{3.16}$$

Applying C to both sides of (3.15), using Proposition 2, setting $x_i := (j\omega_0 I - A)w_i$ and taking Laplace transforms gives

$$C\hat{x}(s; x_i) = C(sI - A)^{-1}x_i + G(s)K(s)C\hat{x}(s; x_i). \tag{3.17}$$

Let us assume that $x(\cdot; x_0) \in L_1(0, \infty; W)$ for all $x_0 \in W$. This implies $x(\cdot; x_i) \in L^1(0, \infty; \tilde{W})$ for all $i = 1, \ldots, p$, where $\tilde{W}$ denotes the complexification of W. Hence (3.17) makes sense for $s = j\omega_0$ and we obtain

$$(I - G(j\omega_0)K(j\omega_0))C\hat{x}(j\omega_0; x_i) = Cw_i, \quad i = 1, \ldots, p.$$

By (3.16) the vectors Cw_i, $i = 1, \ldots, p$ do not span $\mathbb{R}^p$ which leads to a contradiction. For bounded control systems we can prove a result which is stronger than Corollary 1(ii).

Corollary 2. Assume that $B \in \mathcal{L}(\mathbb{R}^p, W)$, i.e. (2.1) is a bounded control system with state space W. Moreover suppose that $S(t)$ is exponentially stable. Then there exists a matrix $K \in \mathbb{R}(s)^{p\times p} \cap H^\infty(\mathbb{C}_0)^{p\times p}$ of H^∞-norm equal to $1/\|G\|_\infty$ such that the origin of (3.15) is not globally asymptotically stable.

Proof. Choose K as in the proof of Corollary 1(ii) and let $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ be a stabilizable and detectable realisation of K with n-dimensional space. Set

$$A_c := \begin{pmatrix} A + B\tilde{D}C & B\tilde{C} \\ \tilde{B}C & \tilde{A} \end{pmatrix}, \quad B_c := \begin{pmatrix} B & 0 \\ 0 & \tilde{B} \end{pmatrix}$$

and

$$C_c := \begin{pmatrix} C & 0 \\ 0 & \tilde{C} \end{pmatrix}.$$

It follows from (3.16) that

$$C_c(sI - A_c)^{-1}B_c = \begin{pmatrix} G(I - KG)^{-1} & GK(I - GK)^{-1} \\ KG(I - KG)^{-1} & K(I - GK)^{-1} \end{pmatrix}$$

has a pole in $j\omega_0$, and hence $j\omega_0 \in \sigma(A_c)$. Since (A_c, B_c) is exponentially stabilizable and (A_c, C_c) is exponentially detectable it follows (see Curtain, 1988 or Nett and Jacobson, 1988) that $j\omega_0$ is an eigenvalue of A_c. If $S_c(t)$ denotes the strongly continuous semigroup on $W \oplus \mathbb{R}^p$ generated by A_c then as in finite dimensions there exists $(w_0, z_0) \in W \oplus \mathbb{R}^p$ such that

$$\begin{pmatrix} w(t) \\ z(t) \end{pmatrix} := S_c(t)\begin{pmatrix} w_0 \\ z_0 \end{pmatrix}$$

does not converge to zero as $t \to \infty$. This implies in particular that

$$w(t) \not\to 0 \quad \text{as} \quad t \to \infty \tag{3.18}$$

because $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is exponentially stable. Moreover we have by construction

$$w(t) = S(t)w_0 + \int_0^t S(t-\tau)B\mathcal{K}(C(w(\cdot))(\tau)\,d\tau + f(t), \tag{3.19}$$

where $f(t) := \int_0^t S(t-\tau)B\tilde{C}e^{\tilde{A}\tau}z_0\,d\tau$ and $\mathcal{K}$ denotes the convolution operator on $L^2(0, \infty; \mathbb{R}^p)$ corresponding to K. We claim that there exists $x_0 \in W$ such that the solution $x(t, x_0)$ of (3.15) does not converge to zero as $t \to \infty$. Let us assume the contrary, i.e. $\lim_{t\to\infty} x(t; x_0) = 0$ for all $x_0 \in W$. Denoting convolution by $*$ and setting $H(t) := (\mathbb{L}^{-1}(K))(t)$ it follows in particular that the unique solution Y of

$$Y = CS(\cdot)B * H * Y - CS(\cdot)B$$

converges to zero as $t \to \infty$. For $R := Y * H$ we have then trivially that $\lim_{t\to\infty} R(t) = 0$. Notice that R is the resolvent of the kernel $CS(\cdot)B * H$, i.e. R is the unique solution of the Volterra integral equation

$$R = CS(\cdot)B * H * R - CS(\cdot)B * H.$$

This means in particular that the solution of

$$y = CS(\cdot)w_0 + CS(\cdot)B * H * y + Cf(\cdot)$$

is given by

$$y = (\delta_0 - R) * (C(S(\cdot)w_0 + f(\cdot)))$$

(see Miller, 1971). The function

$$\tilde{w}(t) = S(t)w_0 + f(t) + \int_0^t S(t-\tau)B\mathcal{K}(y(\cdot))(\tau)\,d\tau$$

is a solution of (3.19) and hence, by uniqueness, $w(t) = \tilde{w}(t)$ for all $t \geq 0$. Finally realize that $\lim_{t\to 0} \tilde{w}(t) = 0$ which contradicts (3.18).

Remark 14. All the results in this section have been formulated for square plants. However, an inspection of the proofs shows that they carry over to the nonsquare case.

4. THE CIRCLE CRITERION

For the proof of the circle criterion we need the following loop shifting result.

Lemma 6. Suppose that the assumptions of Lemma 4 are satisfied. For $K \in \mathbb{R}^{p\times p}$ let $S_K(t)$ denote the strongly continuous semigroup on W and V which is the unique solution of (2.16) and let φ_K denote the function given by $\varphi_K(t, y) := \varphi(t, y) - Ky$. If $x(\cdot)$ solves

$$x(t) = S_K(t)x_0 + \int_0^t S_K(t-\tau)B\varphi_K(\tau, Cx(\tau))\,d\tau \tag{4.1}$$

then $x(\cdot)$ solves

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)B\varphi(\tau, Cx(\tau))\,d\tau \tag{4.2}$$

as well. In (4.1) and (4.2) it is assumed that $t \geq 0$ and $x_0 \in W$.

Remark 15. Notice that Lemma 6 makes sense of the following (purely formal) equation

$$\begin{aligned}\dot{x}(t) &= (A + BKC)x(t) + B\varphi_K(t, Cx(t)) \\ &= Ax(t) + B\varphi(t, Cx(t)).\end{aligned}$$

Proof of Lemma 6. From (4.1), (2.16) and Remark 7(i) it follows

$$\begin{aligned}x(t) &= S(t)x_0 + \int_0^t S(t-\tau)BKCS_K(\tau)x_0\,d\tau \\ &\quad + \int_0^t S(t-\tau)B\varphi_K(\tau, Cx(\tau))\,d\tau \\ &\quad + \int_0^t\int_0^{t-\tau} S(t-\tau-s)BKCS_K(s) \\ &\quad \times B\varphi_K(\tau, Cx(\tau))\,ds\,d\tau \\ &= S(t)x_0 + \int_0^t S(t-\tau)B\varphi(\tau, Cx(\tau))\,d\tau \\ &\quad + X(t) + Y(t) + Z(t),\end{aligned}$$

where

$$X(t) := -\int_0^t S(t-\tau)BKCx(\tau)\,d\tau,$$

$$Y(t) := \int_0^t S(t-\tau)BKCS_K(\tau)x_0\,d\tau$$

and

$$Z(t) := \int_0^t\int_0^{t-\tau} S(t-\tau-s)BKCS_K(s) \times B\varphi_K(\tau, Cx(\tau))\,ds\,d\tau.$$

It remains to show that $X(t) + Y(t) + Z(t) = 0$. Using (4.1), Lemma 2(ii), Remark 7(i) and Proposition 2 and defining all integrants to be zero for negative arguments we obtain

$$\begin{aligned}X(t) + Y(t) &= -\int_0^t S(t-\tau)BKCx(\tau)\,d\tau \\ &\quad + \int_0^t S(t-\tau)BKCx(\tau)\,d\tau \\ &\quad - \int_0^t S(t-\tau)BKC\int_0^\tau S_K(\tau-s) \\ &\quad \times B\varphi_K(s, Cx(s))\,ds\,d\tau \\ &= -\int_0^t\int_0^t S(t-\tau)BKCS_K(\tau-s) \\ &\quad \times B\varphi_K(s, Cx(s))\,ds\,d\tau \\ &= -\int_0^t\int_s^{s+t} S(t-\tau)BKCS_K(\tau-s) \\ &\quad \times B\varphi_K(s, Cx(s))\,d\tau\,ds \\ &= -\int_0^t\int_0^t S(t-s-\lambda)BKCS_K(\lambda) \\ &\quad \times B\varphi_K(s, Cx(s))\,d\lambda\,ds \\ &= -\int_0^t\int_0^{t-s} S(t-s-\lambda)BKCS_K(\lambda) \\ &\quad \times B\varphi_K(s, Cx(s))\,d\lambda\,ds \\ &= -Z(t).\end{aligned}$$

In order to formulate the circle criterion we have to make precise what we mean by multivariable Nyquist-diagrams.

Definition 5. Let G be in $\hat{\mathscr{B}}^{p\times p}$ and suppose that G has no poles on the $j\omega$-axis. Furthermore let $\iota : [0, 1] \to \mathbb{C}$ denote a parametrization of the $j\omega$-axis which has the property that $\iota(t)$ moves downwards from $j\infty$ to $-j\infty$. The Nyquist diagram $N(G)$ of G is formed by the path of the eigenvalues of $G(\iota(t))$ as t traverses the interval $[0, 1]$.

Remark 16. If $\lim_{\substack{|s|\to\infty \\ s\in\bar{\mathbb{C}}_0}} G(s)$ exists (in $\mathbb{C}^{p\times p}$) then it can be shown that $N(G)$ is a closed chain and $\text{ind}\ (N(G), -1/k) = \text{ind}\ (\det(I + kG)\circ\iota, 0)$ for all $k \in \mathbb{R}$, $k \neq 0$ such that $-(1/k) \notin \bigcup_{\omega\in\mathbb{R}} \sigma(G(j\omega))$ (see Desoer and Wang, 1980; Logemann, 1986).

Definition 6. A function $f : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^p$ is said to be in $\sec(a, b)$, where $a, b \in \mathbb{R}$, $a \geq b$, if

$$(f(t, x) - ax)^T(f(t, x) - bx) \leq 0$$

for all $t \geq 0$ and $x \in \mathbb{R}^p$.

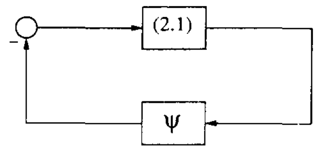

FIG. 2. Feedback system refered to in Theorem 4 (circle criterion).

If $p = 1$ then Definition 6 says that the graph of f lies in a sector between lines of slopes a and b.

For the formulation of the circle criterion it is convenient to define $D[a, b]$ $(a, b \in \mathbb{R})$ to be the closed disk in $\mathbb{C}$ whose diameter is the line segment joining the points $a + j0$ and $b + j0$.

Theorem 4. (Circle criterion). Suppose that

(a) Assumptions 1–4 are satisfied
(b) The system (2.1) is unbounded exponentially stabilizable on W and exponentially detectable on W
(c) The transfer matrix G of (2.1) has no poles on the $j\omega$-axis
(d) The matrix $G(j\omega)$ is normal for all $\omega \in \mathbb{R}$
(e) The function $\psi : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^p$ is in $\sec(a, b)$, $\psi(t, y)$ is continuous in t and locally Lipschitzian in y, uniformly in t on bounded intervals.

Let n denote the number of poles of G in $\mathbb{C}_0$ and assume either

(i) if $0 < a \le b$, $\sigma(G(j\omega)) \cap D[-(1/a), -(1/b)] = \varnothing\ \forall \omega \in \mathbb{R}$ and $\operatorname{ind}(N(G), s) = -n\ \forall s \in D[-(1/a), -(1/b)]$
(ii) if $0 = a < b$, $\sigma(G(j\omega)) \subset \mathbb{C}_{-(1/b)}\ \forall \omega \in \mathbb{R}$ and $n = 0$
(iii) if $a < 0 < b$, $\sigma(G(j\omega)) \subset \mathring{D}[-(1/a), -(1/b)]\ \forall \omega \in \mathbb{R}$ and $n = 0$†
(iv) if $a < 0 = b$, $\sigma(G(j\omega)) \subset \mathbb{C} \setminus \bar{\mathbb{C}}_{-(1/a)}\ \forall \omega \in \mathbb{R}$ and $n = 0$†
(v) if $a < b < 0$, $\sigma(G(j\omega)) \cap D[-(1/a), -(1/b)] = \varnothing\ \forall \omega \in \mathbb{R}$ and $\operatorname{ind}(N(G), s) = -n\ \forall s \in D[-(1/a), -(1/b)]$.

Then the solution $x(t; x_0)$ of

$$x(t) = S(t)x_0 - \int_0^t S(t - \tau)B\psi(\tau, Cx(\tau))\, d\tau \quad (4.3)$$

(i.e. the state trajectory of the feedback system shown in Fig. 2) is globally defined and there exists constants $M > 0$ and $\mu > 0$ such that

$$\|x(t; x_0)\|_W \le M e^{-\mu t} \|x_0\|_W$$

for all $t \ge 0$ and $x_0 \in W$.

Proof. (1) Assume that condition (i) is satisfied and set $k := \frac{1}{2}(a + b)$. By assumption (a) and Lemma 2 there exists a strongly continuous semigroup $S_k(t)$ which is the unique solution of

$$S_k(t)x = S(t)x - k\int_0^t S(t - \tau)BCS_k(\tau)x\, d\tau$$

for all $x \in W$ and $t \ge 0$. Furthermore $S_k(t)$, B and C satisfy assumptions 2–4. Now realize that $\mathscr{F}[G, kI]$ is input–output stable by the multivariable Nyquist criterion (see Desoer and Wang, 1980; Logemann, 1986) and hence using assumption (b) and a result by Curtain (1988) it follows that $S_k(t)$ is exponentially stable on W. Define

$$\psi_k(t, x) := \psi(t, x) - kx \quad \text{and} \quad \gamma := \tfrac{1}{2}(b - a).$$

Since $\psi \in \sec(a, b)$ by assumption (e), it is easy to show that

$$|\psi_k(t, y)| \le \gamma\, |y|\ \forall t \ge 0\ \forall y \in \mathbb{R}^p.$$

Let $x_k(t; x_0)(x_0 \in W)$ denote the solution of

$$x(t) = S_k(t)x_0 - \int_0^t S_k(t - \tau)B\psi_k(\tau, Cx(\tau))\, d\tau.$$

We obtain using Lemma 6 that $x_k(\cdot; x_0)$ is the unique solution of (4.3) as well. Therefore, by Theorem 1, it is sufficient to show

$$\gamma\, \|\mathbb{L}(CS_k(\cdot)B)\|_\infty < 1. \quad (4.4)$$

Denote the eigenvalues of $G(j\omega)$ by $\lambda_i(\omega)$, $i = 1, \ldots, p$. Then it follows from normality [assumption (d)] that the eigenvalues $\eta_i(\omega)$ $(i = 1, \ldots, p)$ of $G(j\omega)(I + kG(j\omega))^{-1}$ are given by

$$\eta_i(\omega) = \lambda_i(\omega)(1 + k\lambda_i(\omega))^{-1}. \quad (4.5)$$

Since $G(j\omega)(I + kG(j\omega))^{-1}$ is normal for all $\omega \in \mathbb{R}$ there exists $i_\omega \in \{1, \ldots, p\}$ such that

$$\bar{\sigma}(G(j\omega)(I + kG(j\omega))^{-1}) = \eta_{i_\omega}(\omega). \quad (4.6)$$

Using $0 < a \le b$ it is easy to show that for all $s \in \mathbb{C}$ we have

$$\gamma \frac{|s|}{|1 + ks|} < 1 \Leftrightarrow s \notin D\left[-\frac{1}{a}, -\frac{1}{b}\right]. \quad (4.7)$$

Now $\sigma(G(j\omega)) \cap D[-(1/a), -(1/b)] = \varnothing$ and hence by (4.5)–(4.7)

$$\gamma\, \|G(I + kG)^{-1}\|_\infty < 1. \quad (4.8)$$

Finally it is obvious that $\mathbb{L}(CS_k(\cdot)B) = G(I + kG)^{-1}$ and therefore (4.4) is implied by (4.8).

(2) The claim is proved in a similar way if we assume that conditions (ii) or (iii) are satisfied. Moreover if (iv) or (v) hold replace G by $-G$, ψ by $-\psi$, a by $-b$, b by $-a$ and apply (ii) or (i) as appropriate.

Remark 17. If G has $j\omega$-axis poles then in Definition 5 and Theorem 4 the $j\omega$-axis has to be replaced by an intended $j\omega$-axis. This modification is standard.

The normality assumption in the above theorem is trivially satisfied for single-input single-output systems. It is clear that in the multivariable case the usefulness of Theorem 4 is

† For $M \subset \mathbb{C}$ denote the interior and the closure of M by $\mathring{M}$ and $\bar{M}$, respectively.

severely limited by the straight jacket of normality. The following remark gives sufficient conditions which ensure normality for transfer matrices of size 2×2.

Remark 18. Let G be a transfer matrix of size 2×2 which has no poles on the imaginary axis. If G is of the form

$$G(s) = g(s)\begin{pmatrix} s+\alpha & -\alpha \\ -\alpha & s+\beta \end{pmatrix}$$

or

$$G(s) = \begin{pmatrix} g_1(s) & g_2(s) \\ g_2(s) & g_1(s) \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{R}$, g, g_1 and g_2 are scalar transfer functions, then $G(j\omega)$ is normal for all $\omega \in \mathbb{R}$.

Example 1. (A finite-dimensional plant with output-delay)

Let $F \in \mathbb{R}(s)^{p \times p}$ be strictly proper and set $G(s) := e^{-hs}F(s)$, where $h > 0$. It can be shown (see e.g. Curtain and Salamon, 1986; Pritchard and Salamon, 1987) that G admits a state-space realization of the form (2.1) satisfying assumptions 1–4. The spaces V and W are given by

$$V = M^2(-h, 0; \mathbb{R}^m) := \mathbb{R}^m \times L^2(-h, 0; \mathbb{R}^m)$$

and

$$W = \{(f(0), f) : f \in W^{1,2}(-h, 0; \mathbb{R}^m)\},$$

where m is the McMillan degree of F and $W^{1,2}$ denotes the usual Sobolev space. The realization can be chosen to be exponentially stable on W and V if and only if F has no poles in $\bar{\mathbb{C}}_0$. Let us consider the specific example given by $p = 2$, $h = 0.75$ and

$$F(s) = \begin{pmatrix} \dfrac{1}{s+1} & \dfrac{1}{s+2} \\ \dfrac{1}{s+2} & \dfrac{1}{s+1} \end{pmatrix}.$$

Notice that $G(j\omega) = e^{-0.75j\omega}F(j\omega)$ is normal for all $\omega \in \mathbb{R}$. The eigenvalues $\lambda_1(\omega)$ and $\lambda_2(\omega)$ of $G(j\omega)$ are given by

$$\lambda_1(\omega) = \frac{e^{-0.75j\omega}}{(j\omega+1)(j\omega+2)}$$

and

$$\lambda_2(\omega) = \frac{(2j\omega+3)e^{-0.75j\omega}}{(j\omega+1)(j\omega+2)}.$$

We have $\bar{\sigma}(G(j\omega)) = |\lambda_2(j\omega)|$ for all $\omega \in \mathbb{R}$ and hence $\|G\|_\infty = \lambda_2(0) = 1.5$. It follows from Theorem 1 (small-gain) that the feedback system consisting of G in the forward-loop and ψ in the feedback-loop is globally exponentially stable on W for all $\psi \in \sec(-\gamma, \gamma)$,† where γ is any number in $[0, \frac{2}{3})$. Of course, this follows from the circle criterion part (iii) as well. We claim that the circle criterion tells us something more. Indeed a computation shows that

$$\min\left\{\mathrm{Re}(\lambda) \mid \lambda \in \bigcup_{\omega \in \mathbb{R}} \sigma(G(j\omega))\right\} = -0.66548$$

and hence, by part (ii) of Theorem 4, the closed-loop system is globally exponentially stable on W for all $\psi \in \sec(0, 1.5)$. We see that "positive" nonlinearities with gains up to $1.5 > \frac{2}{3} = 1/\|G\|_\infty$ do not cause destabilization. Compared with the undelayed case the prediction is less "optimistic": an application of the circle criterion [part (ii)] to F shows that we have stability for all nonlinearities $\psi \in \sec(0, 17.4)$.

† We assume that ψ satisfies the regularity assumptions of Lemma 4.

5. CONCLUSIONS

In this paper we have presented a rigorous treatment of the small-gain condition and the circle criterion for a large class of infinite-dimensional systems which allows for unbounded control action. In particular, the following has been shown:

- A feedback system consisting of an exponentially stable infinite dimensional plant in the forward-loop and a sector-bounded memoryless nonlinearity (locally Lipschitz continuous operator of finite-gain) in the feedback-loop is globally exponentially stable (globally asymptotically stable) if the product of the L^2-gains is smaller than one.
- There exist unstable feedback systems of loop-gain equal to one, i.e. for a given exponentially stable infinite-dimensional real plant there exists a finite-dimensional real compensator making the L^2-loop gain equal to one and causing destabilization.
- The circle criterion ensures global exponential stability for the class of plants under consideration, provided that the transfer matrix is normal on the imaginary axis.

Using ideas presented in Mees (1981) it might be possible to develop multivariable circle criteria for internal stability which do not require the normality assumption. This is a topic for future research.

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