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Mathematics of Control, Signals, and Systems

Adaptive Exponential Stabilization for a Class of Nonlinear Retarded Processes*

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Abstract. This paper considers the problem of adaptive exponential stabilization for a class of single-input single-output nonlinear retarded processes. The class includes certain linear retarded systems which are subject to sector-bounded actuator and sensor nonlinearities. It is shown that there is a wide range of high-gain adaptive compensators which achieve exponential stability for the class of processes under consideration.

Key words. Global adaptive stabilization, Retarded systems, Time-delay systems, Sensor and actuator nonlinearities.

1. Introduction

Robust (nonadaptive) high-gain control of retarded systems has been studied in some detail by Logemann and Owens [LO1], [LO2]. In this paper a theory of high-gain adaptive exponential stabilization for a class of single-input single-output nonlinear retarded systems is developed. It is fairly obvious that any technical (or biological) system will almost certainly involve time delays. These arise because a certain amount of time is required to sense a signal and then respond to it. It is therefore important to find adaptive control laws which apply to retarded systems. The approach adopted here is not based on any parameter-identification algorithms and can be regarded as being in the spirit of several previous studies. In papers by Nussbaum [N], Willems and Byrnes [WB], Heymann *et al.* [HLM], and Mårtensson [M1] linear finite-dimensional systems are considered. Owens *et al.* [OPI] study linear finite-dimensional systems with certain nonlinear perturbations in the state. The paper by Dahleh and Hopkins [DH] extends the main result of Willems and Byrnes [WB] to a class of linear differential-delay systems, while Kobayashi [K2] and Byrnes [B] show that it carries over to certain linear distributed parameter systems. Logemann and Owens [LO3] develop an input–output theory of high-gain adaptive stabilization of infinite-dimensional systems with

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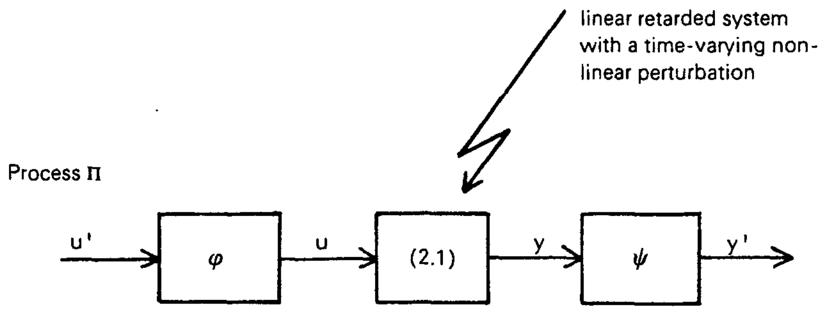

Fig. 1

actuator and sensor nonlinearities, which in particular includes the results of Dahleh and Hopkins [DH], Kobayashi [K2], and Byrnes [B]. Apart from Owens *et al.* [OPI], who consider the problem of exponential stabilization for the special case when the sign of the high-frequency gain is known, none of the above references investigates the possibility of stabilizing a system with an exponential rate of decay.

The process Π considered in this paper is shown in Fig. 1. We assume:

(1) The functions φ and ψ are memoryless unbiased time-varying nonlinearities lying either in a positive or negative sector.
(2) The nonlinear retarded system (2.1) satisfies conditions paralleling those imposed on the finite-dimensional system by Owens *et al.* [OPI] (see Section 2 for details).

We mention that the nonlinearities included in process Π are more general than the ones considered in [OPI] and by Logemann and Owens [LO3].

In Section 2 we consider the process Π shown in Fig. 1 without actuator and sensor nonlinearities and establish some preliminary results. Section 3 gives a general class of adaptive compensators achieving exponential stability for the process Π provided that (1) and (2) are satisfied. We emphasize that the adaptive control laws of Section 3 differ from the control laws introduced in the above references. For example, the approach adopted here includes gain adaptation rules of the form

derivative of the gain = modulus of the process output.

Moreover, exponential weighting factors have to be employed, since we want to achieve exponential stability. In order to deal with the actuator and sensor nonlinearities φ and ψ, the notion of *scaling invariant* switching functions (introduced in [LO3]) proves useful. As in [OPI] and [LO3] we allow switching as a function of both current and past gain and input data. This leads to a wide class of stabilizing adaptive high-gain compensators with the convergence of the switching mechanism being independent of the gain-adaptation rules. It is to be noted in particular that the adaptive control laws of Section 3 give a solution to the problem of stabilizing an unknown first-order system with a *prescribed* rate of exponential decay. This problem was solved for the linear finite-dimensional case by Polderman [P1], [P2] using adaptive pole-placement. The approach taken here is different in nature from the one pursued in [P1] and [P2] and it applies to a much larger class of systems.

Section 4 concludes the paper. The proofs of some technical lemmas are relegated to the appendices.

Nomenclature

$\mathbb{R}_+ :=$ set of nonnegative real numbers.

$\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \operatorname{Re}(s) > \alpha\} \qquad (\alpha \in \mathbb{R})$.

Let $J \subset \mathbb{R}$ be a finite or infinite interval, then

$C(J, \mathbb{R}^n) :=$ vector space of $\mathbb{R}^n$-valued continuous functions on J,

$L^p(J, \mathbb{R}^n) :=$ vector space of $\mathbb{R}^n$-valued p-integrable functions on J,

$LL^p(J, \mathbb{R}^n) :=$ vector space of $\mathbb{R}^n$-valued locally p-integrable functions on J,

$L^p_\alpha(J, \mathbb{R}^n) := \{f\colon J \to \mathbb{R}^n \mid f\exp(\alpha\cdot) \in L^p(J, \mathbb{R}^n)\} \qquad (\alpha \in \mathbb{R})$.

$BV(J, \mathbb{R}^{n\times n}) :=$ vector space of $\mathbb{R}^{n\times n}$-valued functions of bounded variation on J.

Let $f \in L^p_\alpha(J, \mathbb{R}^n)$, then $\|f\|_{p,\alpha} := (\int_J |f(t)\exp(\alpha t)|_e^p\, dt)^{1/p}$, where $|\cdot|_e$ denotes the Euclidean norm on $\mathbb{R}^n$.

Let $f \in LL^p([a, \infty), \mathbb{R}^n)$, then for $t \geq a$

$$(\pi_t f)(\tau) = \begin{cases} f(\tau), & a \leq \tau \leq t, \\ 0, & \tau > t. \end{cases}$$

An operator $T\colon D_T \subset LL^p([a_1, \infty), \mathbb{R}^m) \to LL^q([a_2, \infty), \mathbb{R}^n)(a_2 \leq a_1)$ is called *causal* if $\pi_t T = \pi_t T \pi_t$ for all $t \geq a_1$.

$S(\delta, \Delta)\,(\Delta \geq \delta > 0)$ denotes the set of all Borel functions $f\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$ and f satisfies either $\Delta x^2 \geq xf(t, x) \geq \delta x^2$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ or $(-\delta)x^2 \geq xf(t, x) \geq (-\Delta)x^2$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

$$S := \bigcup_{\Delta \geq \delta > 0} S(\delta, \Delta).$$

Given $f \in S$ then either $\operatorname{sign}(x) = \operatorname{sign}(f(t, x))$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ or $\operatorname{sign}(x) = -\operatorname{sign}(f(t, x))$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. In the first case we write $\sigma(f) = +1$ and in the second $\sigma(f) = -1$.

2. The Process Without Sensor and Actuator Nonlinearities

In the following we extend any function $F \in BV([0, r], \mathbb{R}^{n\times n})$ to the whole real axis by setting $F(t) = F(0)$ for all $t < 0$ and $F(t) = F(r)$ for all $t > r$. Any measurable function $f\colon \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}$ will be extended to the whole real axis by defining $f(t) = 0$ for all $t \notin \Omega$. For $F = (F_{ij}) \in BV([0, r], \mathbb{R}^{n\times n})$ and $f = (f_1, \ldots, f_n)^T$, $f_i \in LL^1(\mathbb{R}, \mathbb{R})$ $(1 \leq i \leq n)$, we define

$$dF * f := \begin{bmatrix} \sum_{j=1}^{n} dF_{1j} * f_j \\ \sum_{j=1}^{n} dF_{nj} * f_j \end{bmatrix},$$

where dF_{ij} denotes the measure on $\mathbb{R}$ induced by F_{ij} and $dF_{ij} * f$ denotes the convolution of the measure dF_{ij} and the function f_j. If f is continuous on $[-r, \infty)$, then of course

$$(dF * f)(t) = \int_0^r dF(\tau)f(t-\tau) \qquad \text{for all} \quad t \geq 0.$$

In this section we consider the process Π in Fig. 1 stripped of its actuator and sensor nonlinearities. It is assumed to be given by

$$\begin{aligned} \dot{x} &= dA * x + P_1 x + b(P_2 x + u), \\ y &= c^T x, \\ x|_{[-r,0]} &= x_0 \in C([-r, 0], \mathbb{R}^n), \end{aligned} \tag{2.1}$$

where $A \in BV([0, r], \mathbb{R}^{n\times n})$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, and P_1 and P_2 are operators mapping $LL^1([-r, \infty), \mathbb{R}^n)$ into $LL^1([-r, \infty), \mathbb{R}^n)$ and $LL^1([-r, \infty), \mathbb{R})$, respectively. We assume that

$$c^T b \neq 0, \tag{2.2}$$

$$\chi(s) := \det\begin{bmatrix} sI - \hat{A}(s) & -b \\ c^T & 0 \end{bmatrix} \neq 0 \qquad \text{for all} \quad s \in \bar{\mathbb{C}}_{-\alpha} \quad (\text{for some } \alpha \geq 0), \tag{2.3}$$

$$P_i(0) = 0, \tag{2.4}$$

and

$$\|\pi_t(P_i f - P_i f')\|_{1,\alpha} \leq \gamma_i \|\pi_t(f - f')\|_{1,\alpha} \tag{2.5}$$

for all $f, f' \in LL^1([-r, \infty), \mathbb{R}^n)$, for all $t \geq 0$ (for some $\gamma_i \geq 0$), $i = 1, 2$.

In (2.3) the function $\hat{A}$ is given by $\hat{A}(s) = \int_0^r \exp(-s\tau)\, dA(\tau)$. In the case when $\gamma_1 = \gamma_2 = 0$, (2.1) is a linear retarded system with transfer function

$$g(s) = c^T(sI - \hat{A}(s))^{-1}b.$$

Remark 2.1.

(i) Let $\gamma_1 = \gamma_2 = 0$. If condition (2.3) is satisfied, then the system (2.1) is called $(-\alpha)$*-minimum-phase.*

(ii) It can be shown that (2.3) holds if and only if

$$\left.\begin{aligned} g(s) &\neq 0, \\ \text{rank}(sI - \hat{A}(s), b) &= n, \\ \text{and} \qquad \text{rank}\begin{bmatrix} sI - \hat{A}(s) \\ c^T \end{bmatrix} &= n, \end{aligned}\right\} \qquad \text{for all} \quad s \in \bar{\mathbb{C}}_{-\alpha}.$$

(iii) It is a trivial consequence of condition (2.5) that the operator P_i is causal, $i = 1, 2$.

It follows from (2.2) that there exists a nonsingular real matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$Q^{-1}b = \begin{bmatrix} c^T b \\ 0 \end{bmatrix}, \qquad c^T Q = (1, 0).$$

Setting $\bar{\eta}(t) = Q^{-1}x(t)$ it follows from (2.1) that

$$\begin{aligned} \dot{\bar{\eta}} &= d(Q^{-1}AQ)*\bar{\eta} + (Q^{-1}(P_1 + bP_2)Q)(\bar{\eta}) + (Q^{-1}b)u, \\ y &\triangleq (c^TQ)\bar{\eta}, \\ \bar{\eta}|_{[-r,0]} &= Q^{-1}x_0. \end{aligned} \tag{2.6}$$

Partition the matrix $Q^{-1}A(\cdot)Q$ as follows

$$Q^{-1}A(\cdot)Q = \begin{bmatrix} A_{11}(\cdot) & A_{12}(\cdot) \\ A_{21}(\cdot) & A_{22}(\cdot) \end{bmatrix},$$

where $A_{11}(\cdot)$, $A_{12}(\cdot)$, $A_{21}(\cdot)$, and $A_{22}(\cdot)$ are matrices with entries in $BV([0, r], \mathbb{R})$ of size 1×1, $1 \times (n-1)$, $(n-1) \times 1$, and $(n-1) \times (n-1)$, respectively. Furthermore, write $Q^{-1}P_1Q$ as $(\tilde{P}_1, \tilde{P}_2)^T$, where the operators $\tilde{P}_1$ and $\tilde{P}_2$ are mapping $LL^1([-r, \infty), \mathbb{R}^n)$ into $LL^1([-r, \infty), \mathbb{R})$ and $LL^1([-r, \infty), \mathbb{R}^{n-1})$, respectively. If we realize that $\bar{\eta}$ can be written in the form $\bar{\eta} = (y, \eta^T)^T$, then it follows that (2.6) can be expressed as

$$\dot{y} = (c^Tb)u_1, \tag{2.7}$$

$$\dot{\eta} = dA_{22} * \eta + dA_{21} * u_2 + \tilde{P}_2\begin{pmatrix} u_2 \\ \eta \end{pmatrix}, \tag{2.8a}$$

$$z = -\frac{1}{c^Tb}\left[dA_{12} * \eta + dA_{11} * u_2 + \tilde{P}_1\begin{pmatrix} u_2 \\ \eta \end{pmatrix}\right] - (P_2Q)\begin{pmatrix} u_2 \\ \eta \end{pmatrix}, \tag{2.8b}$$

$$u_1 = u - z, \qquad u_2 = y, \tag{2.9}$$

$$y|_{[-r,0]} = \eta_1, \qquad \eta|_{[-r,0]} = \eta_2, \tag{2.10}$$

where $(\eta_1, \eta_2)^T = \bar{\eta}|_{[-r,0]}$ and in particular $\eta_1 = c^Tx_0$, i.e., (2.6) is the feedback interconnection of the integrator (2.7) and the retarded system (2.8) (see Fig. 2).

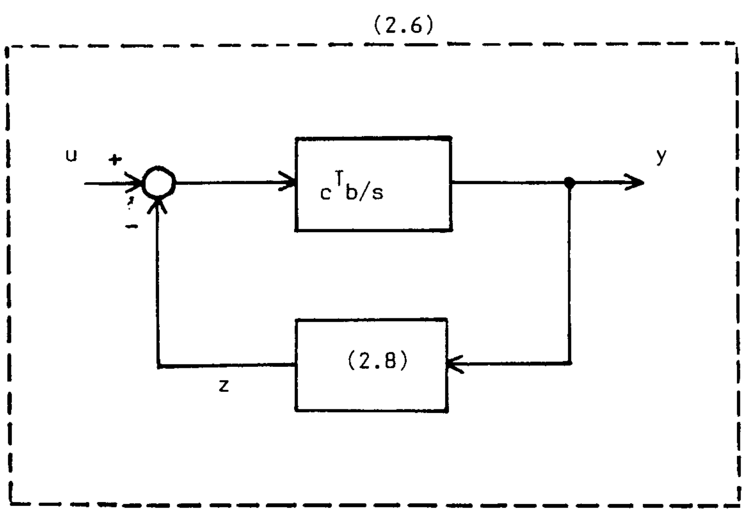

Fig. 2

Lemma 2.2. *Assume that* (2.2)–(2.5) *are satisfied and consider the initial value problem*

$$\dot{\eta} = dA_{22} * \eta + dA_{21} * u_2 + \tilde{P}_2\begin{pmatrix} u_2 \\ \eta \end{pmatrix},$$
$$u_2|_{[-r,0]} = \eta_1, \qquad \eta|_{[-r,0]} = \eta_2, \tag{IVP}$$

when an input signal $u_2(t) = v(t)$, $t \geq 0$, *is applied. Under these conditions:*

(i) *For* $v \in LL^1(\mathbb{R}_+, \mathbb{R})$ (IVP) *has a unique absolutely continuous solution* $S(v)$ *on* $[-r, \infty)$ *provided that* γ_1 *is sufficiently small. The map* $v \mapsto S(v)$ *is causal.*

(ii) *If* $v \in L^1_\alpha(\mathbb{R}_+, \mathbb{R})$ *the solution* $S(v)$ *of* (IVP) *is in* $L^1_\alpha([-r, \infty), \mathbb{R}^{n-1})$ *for sufficiently small* γ_1. *The operator* S *satisfies*

$$\|S(v)\|_{1,\alpha} \leq K(\eta_1, \eta_2) + L\|v\|_{1,\alpha}, \tag{2.11}$$

where $K(\eta_1, \eta_2)$ *and* L *are positive constants.*

Proof. We prove (ii) first.

Step 1. Consider the linear homogeneous initial value problem

$$\dot{\eta} = dA_{22} * \eta, \tag{2.12a}$$

$$\eta|_{[-r,0]} = \eta_2. \tag{2.12b}$$

Let Y denote the fundamental solution of (2.12a), i.e., $\dot{Y} = dA_{22} * Y$ on $\mathbb{R}_+$, $Y(0) = I$, and $Y(t) = 0$ for $t \in [-r, 0)$ (see [H] or [K1]). The Laplace transform of Y is given by $\Delta^{-1}(s)$, where $\Delta(s) := sI - \hat{A}_{22}(s)$ and $\hat{A}_{22}$ is defined by $\hat{A}_{22}(s) := \int_0^r \exp(-s\tau)\, dA_{22}(\tau)$. It follows from (2.2) that the zeros of χ (cf. (2.3)) and $\det(\Delta)$ coincide (see Appendix 1 for a proof) and hence (by (2.3)) $\det(\Delta)$ has no zeros in $\bar{\mathbb{C}}_{-\alpha}$. Since $\det(\Delta)$ has at most finitely many zeros in every right half-plane there exists $\beta > \alpha$ such that $\det(\Delta)$ has no zeros in $\bar{\mathbb{C}}_{-\beta}$. As a consequence Δ^{-1} is holomorphic in $\mathbb{C}_{-\beta}$ and if we realize that $\Delta^{-1}(s) = O(s^{-1})$ as $|s| \to \infty$ in $\mathbb{C}_{-\beta}$ we obtain, using a result of Mossaheb [M2] (see also [L]), $Y \in L^1_\alpha(\mathbb{R}_+, \mathbb{R}^{(n-1)\times(n-1)})$.

Step 2. Denote the solution of (2.12) by η^*. It is not difficult to show that (IVP) is equivalent to

$$\eta(t) = \eta^*(t) + \int_0^t Y(t-\tau)(dA_{21} * u_2)(\tau)\, d\tau$$
$$+ \int_0^t Y(t-\tau)\left(\tilde{P}_2\begin{pmatrix} u_2 \\ \eta \end{pmatrix}\right)(\tau)\, d\tau, \qquad t \geq 0, \tag{2.13}$$
$$u_2|_{[-r,0]} = \eta_1, \qquad \eta|_{[-r,0]} = \eta_2.$$

Using standard arguments based on Banach's contraction mapping theorem it follows that, for $u_2|_{[0,\infty)} = v \in L^1_\alpha(\mathbb{R}_+, \mathbb{R})$, (2.13) has a unique solution $S(v) \in L^1_\alpha([-r, \infty), \mathbb{R}^{n-1})$ if

$$\gamma_1 \|Y\|_{1,\alpha}(\bar{\sigma}/\underline{\sigma}) < 1, \tag{2.14}$$

where $\bar{\sigma}$ and $\underline{\sigma}$ denote the largest and smallest singular value of the matrix Q, respectively.

Step 3. Assume that (2.14) is satisfied. It is clear that the solution operator $S: L_\alpha^1(\mathbb{R}_+, \mathbb{R}) \to L_\alpha^1([-r, \infty), \mathbb{R}^{n-1})$ is causal. Moreover, it is straightforward to show that (2.11) holds true for

$$K(\eta_1, \eta_2) = \frac{1}{1 - \|Y\|_{1,\alpha}\gamma^*}\left[\|\eta^*\|_{1,\alpha} + r\|Y\|_{1,\alpha}\left(\exp(\alpha r)\overset{r}{\underset{0}{\mathrm{V}}}(A_{21}) + \gamma^*\right)\|\eta_1\|_{\infty,\alpha}\right]$$

and

$$L = \frac{1}{1 - \|Y\|_{1,\alpha}\gamma^*}\left(\|Y\|_{1,\alpha}\left(\exp(\alpha r)\overset{r}{\underset{0}{\mathrm{V}}}(A_{21}) + \gamma^*\right)\right),$$

where $\gamma^* := (\bar{\sigma}/\underline{\sigma})\gamma_1$ and $\mathrm{V}_0^r(A_{21})$ denotes the total variation of A_{21} on $[0, r]$.

In order to prove (i) we assume that (2.14) is satisfied. Define the operator T by

$$(Tf)(t) = (S\pi_\tau f)(t) \qquad \text{for all} \quad -r \le t \le \tau.$$

It is clear that T maps $LL^1(\mathbb{R}_+, \mathbb{R})$ into $LL^1([-r, \infty), \mathbb{R}^{n-1})$. Realize that $\pi_\tau T = \pi_\tau S\pi_\tau$ for all $\tau \ge 0$ and hence $\pi_\tau T = (\pi_\tau S\pi_\tau)\pi_\tau = \pi_\tau T\pi_\tau$ for all $\tau \ge 0$, i.e., T is causal. Moreover, it follows from the causality of S that

$$\pi_\tau Tf = \pi_\tau S\pi_\tau f = \pi_\tau Sf \qquad \text{for all} \quad \tau \ge 0 \quad \text{and for all} \quad f \in L_\alpha^1(\mathbb{R}_+, \mathbb{R})$$

which means that T extends S to $LL^1(\mathbb{R}_+, \mathbb{R})$. It is easy to verify that T is the unique causal extension of S to $LL^1(\mathbb{R}_+, \mathbb{R})$. Finally, we claim that for $u_2|_{[0,\infty)} = v \in LL^1(\mathbb{R}_+, \mathbb{R})$ the function Tv is the unique solution of (IVP). In order to see that Tv solves (IVP), pick $\tau > 0$ and notice that $S\pi_\tau v$ is a solution of (IVP) with $u_2|_{[0,\infty)}$ given by $\pi_\tau v$. Since $Tv = S\pi_\tau v$ on $[-r, \tau]$ and $v = \pi_\tau v$ on $[0, \tau]$ and by the causality of the operations occuring on the right-hand side of (IVP) we conclude that Tv is a solution of (IVP) (with $u_2|_{[0,\infty)}$ given by v) on the interval $[-r, \tau)$. The number $\tau > 0$ was arbitrary and hence Tv solves (IVP) on $[-r, \infty)$. For the proof of uniqueness let f be a solution of (IVP) with $u_2|_{[0,\infty)} = v$. We want to show that $f = Tv$. Since f and Tv are solutions of (2.13) an easy calculation yields

$$(1 - \gamma_1\|Y\|_{1,\alpha}(\bar{\sigma}/\underline{\sigma}))\|\pi_\tau(Tv - f)\|_{1,\alpha} \le 0 \qquad \text{for all} \quad \tau \ge 0.$$

It now follows from (2.14) that $\pi_\tau(Tv - f) = 0$ for all $\tau \ge 0$ and hence $Tv = f$. ■

An immediate consequence of the previous lemma is

Corollary 2.3. *Consider the system (2.8) with $\eta|_{[-r,0]} = \eta_2$, $u_2|_{[-r,0]} = \eta_1$ and $u_2|_{[0,\infty)} = v$ and assume that γ_1 is sufficiently small (i.e., γ_1 satisfies (2.14)). Then the corresponding input–output operator $H: v \mapsto z$ is causal and maps $LL^1(\mathbb{R}_+, \mathbb{R})$ into itself. Moreover, $L_\alpha^1(\mathbb{R}_+, \mathbb{R})$ is an invariant subspace of H and*

$$\|H(v)\|_{1,\alpha} \le h_1(\eta_1, \eta_2) + h_2\|v\|_{1,\alpha}, \tag{2.15}$$

where $h_1(\eta_1, \eta_2)$ and h_2 are positive constants.

3. A General Class of Adaptive Stabilizing Compensators

We study the behavior of the process Π if the following control law is applied:

$$u'(t) = N(\xi(t))k(t)y'(t), \tag{3.1a}$$

$$\dot{\xi}(t) = \exp(\alpha t)k(t)|y'(t)|, \qquad \xi(0) = \xi_0, \tag{3.1b}$$

i.e.,

$$u(t) = \varphi[t, N(\xi(t))k(t)\psi(t, y(t))], \tag{3.2a}$$

$$\dot{\xi}(t) = \exp(\alpha t)k(t)|\psi(t, y(t))|, \qquad \xi(0) = \xi_0, \tag{3.2b}$$

where $k: \mathbb{R}_+ \to \mathbb{R}$ is a strictly positive function (i.e., $k(t) \geq \varepsilon > 0$ for all $t \in \mathbb{R}_+$) and N is a so-called scaling invariant switching function.

Definition 3.1.

(i) A function $N \in LL^\infty(\mathbb{R}, \mathbb{R})$ is called a *switching function* if for some $a \in \mathbb{R}$

$$\sup_{x>a} \frac{1}{x-a} \int_a^x N(\lambda)\, d\lambda = +\infty \tag{3.3a}$$

and

$$\inf_{x>a} \frac{1}{x-a} \int_a^x N(\lambda)\, d\lambda = -\infty. \tag{3.3b}$$

(ii) A switching function N is called a *scaling invariant* if for arbitrary positive θ_1 and θ_2 the function

$$\lambda \mapsto \begin{cases} \theta_1 N(\lambda) & \text{if } N(\lambda) > 0, \\ 0 & \text{if } N(\lambda) = 0, \\ \theta_2 N(\lambda) & \text{if } N(\lambda) < 0 \end{cases}$$

is a switching function.

Remark 3.2.

(i) It is easily seen that if conditions (3.3) are satisfied for some $a \in \mathbb{R}$, they are satisfied for all $a \in \mathbb{R}$.

(ii) An example of a scaling invariant switching function is given by $N(\lambda) = \cos(\frac{1}{2}\pi\lambda)\exp(\lambda^2)$. For a proof, see [LO3].

(iii) It is not difficult to find switching functions which are not scaling invariant, e.g., $N(\lambda) = N_0(\lambda)\lambda$, with

$$N_0(\lambda) = \begin{cases} 1 & n^2 \leq |\lambda| < (n+1)^2, \quad n \text{ even}, \\ -1 & n^2 \leq |\lambda| < (n+1)^2, \quad n \text{ odd}. \end{cases}$$

(iv) The concept of a switching function has its origin in the papers by Nussbaum [N] and Willems and Byrnes [WB]. ◆

The following result shows that the control law (3.1) stabilizes the process shown in Fig. 1.

Theorem 3.3. *Let φ and ψ be in S and assume that* (2.2)–(2.5) *are satisfied. Moreover, assume that the feedback system given by* (2.1) *and* (3.2) *has a unique absolutely continuous solution which can be continued uniquely to the right as long as it remains bounded. Then for all sufficiently small γ_1 (i.e., all γ_1 satisfying* (2.14)) *the following is true:*

(i) $y, y' \in \bigcap_{p=1}^{\infty} L_\alpha^p(\mathbb{R}_+, \mathbb{R})$ *and* $x \in \bigcap_{p=1}^{\infty} L_\alpha^p(\mathbb{R}_+, \mathbb{R}^n)$.
(ii) $\lim_{t\to\infty} \xi(t)$ *exists and is finite.*

In the case when $\alpha > 0$ (i) *implies that x, y, and y' are tending exponentially to zero as t tends to ∞. If $\alpha = 0$ we still have* $\lim_{t\to\infty} x(t) = 0$ *and* $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0$, *provided that k is bounded.*

Remark 3.4. Suppose that k is continuous, N is continuously differentiable, and φ and ψ are continuous with continuous partial derivatives $D_2\varphi$ and $D_2\psi$. Then it is straightforward to show that the feedback system given by (2.1) and (3.2) has a unique absolutely continuous solution which can be continued uniquely to the right as long as it remains bounded (see [LO3]).

For the proof of Theorem 3.3 we need the following two lemmas.

Lemma 3.5. *Suppose that $f: J \to \mathbb{R}$($J \subset \mathbb{R}$ is a closed interal) is absolutely continuous. Then*

$$\frac{d}{dt}|f(t)| = \operatorname{sign}(f(t))\dot{f}(t) \quad \textit{a.e. on } J.$$

Lemma 3.6. *Let $\varphi \in S(\Delta_\varphi, \delta_\varphi)$ and $\psi \in S(\Delta_\psi, \delta_\psi)$ such that $\sigma(\varphi) = \sigma(\psi)$. Then the inequalities*

$$\operatorname{sign}(x)\varphi(t, \lambda\rho\psi(t, x)) \le \Gamma_+(\lambda)\lambda\rho|\psi(t, x)|$$

and

$$\operatorname{sign}(x)\varphi(t, \lambda\rho\psi(t, x)) \ge \Gamma_-(\lambda)\lambda\rho|\psi(t, x)|$$

hold for all $(t, \rho, \lambda, x) \in \mathbb{R}_+^2 \times \mathbb{R}^2$, where

$$\Gamma_+(\lambda) := \begin{cases} \Delta_\varphi, & \lambda > 0, \\ 0, & \lambda = 0, \\ \delta_\varphi, & \lambda < 0, \end{cases} \tag{3.4a}$$

and

$$\Gamma_-(\lambda) := \Gamma_+(-\lambda). \tag{3.4b}$$

The proofs of Lemmas 3.5 and 3.6 are given in Appendices 2 and 3, respectively.

Proof of Theorem 3.3. Instead of the system given by (2.1) and (3.2) we consider the system given by (2.6) and (3.2). They are equivalent in the sense that $(\bar{\eta}, \xi)$ is a solution of (2.6) and (3.2) if and only if $(Q\bar{\eta}, \xi)$ is a solution of (2.1) and (3.2).

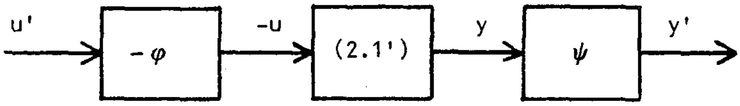

Fig. 3

Moreover, we can assume without loss of generality that $\sigma(\varphi) = \sigma(\psi)$ (this follows from the equivalence of the diagrams in Figs. 1 and 3). Consider the equation

$$\dot{y} = c^T b(\varphi[\cdot, (N \circ \xi)k\psi(\cdot, y)] - Hy) \tag{3.5}$$

which follows from (2.7)–(2.10) and (3.2) (H is defined as in Corollary 2.3). We obtain, from (3.5),

$$\left(\frac{d}{dt} y \exp(\alpha \cdot)\right) \operatorname{sign}(y) = \alpha \exp(\alpha \cdot)|y| - c^T b \exp(\alpha \cdot) \operatorname{sign}(y) Hy + c^T b \varphi[\cdot, (N \circ \xi)k\psi(\cdot, y)] \exp(\alpha \cdot) \operatorname{sign}(y). \tag{3.6}$$

Furthermore, we have (by Corollary 2.3)

$$\int_0^t \exp(\alpha\tau)|(Hf)(\tau)|\, d\tau \le h_1 + h_2 \int_0^t \exp(\alpha\tau)|f(\tau)|\, d\tau \quad \text{for all} \quad t \ge 0 \quad \text{and for all} \quad f \in LL^1(\mathbb{R}_+, \mathbb{R}). \tag{3.7}$$

Now let δ_φ, Δ_φ, δ_ψ, and Δ_ψ be positive numbers such that $\varphi \in S(\delta_\varphi, \Delta_\varphi)$ and $\psi \in S(\delta_\psi, \Delta_\psi)$. Integrating (3.6) from 0 to t using (3.7) and (3.4) and applying Lemmas 3.5 and 3.6 yields

$$|y(t)| \exp(\alpha t) - |y(0)| \le |c^T b| h_1 + (|c^T b| h_2 + \alpha) \int_0^t |y(\tau)| \exp(\alpha\tau)\, d\tau \pm |c^T b| \int_0^t (\Gamma_\pm \circ N)(\xi(\tau)) N(\xi(\tau)) k(\tau) |\psi(\tau, y(\tau))| \exp(\alpha\tau)\, d\tau.$$

Using (3.2b) and the change of variables formula for Lebesgue integrals it follows that

$$|y(t)| \exp(\alpha t) - |y(0)| \le |c^T b| h_1 + (|c^T b| h_2 + \alpha) \int_0^t |y(\tau)| \exp(\alpha\tau)\, d\tau \pm |c^T b| \int_{\xi_0}^{\xi(t)} (\Gamma_\pm \circ N)(\lambda) N(\lambda)\, d\lambda. \tag{3.8}$$

By the properties of k and (3.2b)

$$\int_0^t |y(\tau)| \exp(\alpha\tau)\, d\tau \le \frac{1}{\varepsilon} \int_0^t \exp(\alpha\tau) k(\tau) |y(\tau)|\, d\tau \le \frac{1}{\varepsilon\delta_\psi} \int_0^t \exp(\alpha\tau) k(\tau) |\psi(\tau, y(\tau))|\, d\tau = \frac{1}{\varepsilon\delta_\psi}(\xi(t) - \xi_0). \tag{3.9}$$

Setting $K_1 := |c^T b| h_1 + |y(0)|$ and $K_2 := (1/\varepsilon\delta_\psi)(|c^T b| h_2 + \alpha)$ and using (3.8) and (3.9) we obtain

$$|y(t)| \exp(\alpha t) \leq K_1 + (\xi(t) - \xi_0)\left(K_2 \pm \frac{|c^T b|}{\xi(t) - \xi_0} \int_{\xi_0}^{\xi(t)} (\Gamma_\pm \circ N)(\lambda) N(\lambda)\, d\lambda\right). \tag{3.10}$$

Equations (3.10) and (3.9) hold on each interval of the form $[0, a)$ where the solution $(\bar{\eta}, \xi) = (y, \eta, \xi)$ of the feedback system given by (2.6) and (3.2) exists. Since the right-hand side of (3.10) has to be nonnegative, we can conclude, using the properties of the function N, that $\xi(t)$ and hence (by (3.10)) $y(t)$ remain bounded. Moreover, it follows from the decomposition (2.7)–(2.10) of the system (2.6) and Lemma 2.2 that $\eta(t)$ remains bounded as well. As a consequence the solution $(\bar{\eta}, \xi)$ exists on $[0, \infty)$ and we obtain that $\lim_{t\to\infty} \xi(t)$ exists and is finite, which is (ii). Furthermore, by (3.10) and (3.9), $y \in L^1_\alpha(\mathbb{R}_+, \mathbb{R}) \cap L^\infty_\alpha(\mathbb{R}_+, \mathbb{R})$ and since $y' = \psi(\cdot, y(\cdot))$, where $\psi \in S$, it follows that $y' \in L^1_\alpha(\mathbb{R}_+, \mathbb{R}) \cap L^\infty_\alpha(\mathbb{R}_+, \mathbb{R})$. The decomposition (2.7)–(2.10) of the system (2.6) and Lemma 2.2 show that $\eta \in L^1_\alpha(\mathbb{R}_+, \mathbb{R}^{n-1})$ and hence (by (2.8a) and (2.9)) $\dot{\eta} \in L^1_\alpha(\mathbb{R}_+, \mathbb{R}^{n-1})$. Therefore $\eta \in L^\infty_\alpha(\mathbb{R}_+, \mathbb{R}^{n-1})$ which completes the proof of (i). Finally, it is easy to show that in the case when $\alpha = 0$ we still have $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0$ and $\lim_{t\to\infty} x(t) = 0$, provided that k is bounded. The proof is omitted for the sake of brevity. ■

It is natural to regard the gain k as the image of causal map Z operating on ξ and $\psi(\cdot, y(\cdot))$. Let us consider the following gain-adaptation rule:

$$k = Z(\xi(\cdot), \psi(\cdot, y(\cdot))), \tag{3.11}$$

where $Z\colon D_Z \to LL^1(\mathbb{R}_+, \mathbb{R})$ is a causal map whose domain D_Z contains the set $C(\mathbb{R}_+, \mathbb{R}) \times \bigcup_{\psi \in S} \{\psi(\cdot, f(\cdot)) | f \in LL^\infty(\mathbb{R}_+, \mathbb{R})\}$. Moreover, we assume that Z satisfies:

- (A1) $Z[D_Z \cap (L^\infty(\mathbb{R}_+, \mathbb{R}) \times (L^1(\mathbb{R}_+, \mathbb{R}) \cap L^\infty(\mathbb{R}_+, \mathbb{R})))] \subset L^\infty(\mathbb{R}_+, \mathbb{R})$.
- (A2) There exists $\varepsilon > 0$ such that $\inf_{t\geq 0} \{(Z(f))(t)\} \geq \varepsilon$ for all $f \in D_Z$.
- (A3) $Z(f)$ is nondecreasing for all $f \in D_Z$.

Consider the following example

$$\begin{aligned} &Z\colon C(\mathbb{R}_+) \times LL^\infty(\mathbb{R}_+, \mathbb{R}) \to LL^1(\mathbb{R}_+, \mathbb{R}), \\ &(f_1, f_2) \mapsto k_0 + \int_0^\cdot |f_2(\tau)| X(|f_1(\tau)|, |f_2(\tau)|)\, d\tau, \end{aligned} \tag{3.12}$$

where $k_0 > 0$ and X is a polynomial in two variables with positive coefficients. It is trivial that (A1)–(A3) are satisfied. The gain adaptation induced by (3.12) can be written in the form of an ordinary differential equation

$$\begin{aligned} \dot{k}(t) &= |\psi(t, y(t))| X(|\xi(t)|, |\psi(t, y(t))|), \\ k(0) &= k_0. \end{aligned} \tag{3.13}$$

The following theorem is a simple consequence of Theorem 3.3.

Theorem 3.7. *Let φ and ψ be in S and assume that* (2.2)–(2.5) *and* (A1)–(A3) *are satisfied. Moreover, assume that the feedback system given by* (2.1), (3.2), *and* (3.11) *has a unique absolutely continuous solution* (x, ξ) *which can be continued uniquely to the right as long as it remains bounded. Then for all sufficiently small* γ_1 *(i.e., all* γ_1 *satisfying* (2.14)) *the following hold*:

(i) $y, y' \in \bigcap_{p=1}^{\infty} L_\alpha^p(\mathbb{R}_+, \mathbb{R})$ *and* $x \in \bigcap_{p=1}^{\infty} L_\alpha^p(\mathbb{R}_+, \mathbb{R}^n)$.
(ii) $\lim_{t\to\infty} \xi(t)$ *exists and is finite.*
(iii) $\lim_{t\to\infty} k(t)$ *exists and is finite.*

In the case when $\alpha > 0$ (i) *implies that* x, y, *and* y' *are going exponentially to zero as* $t \to \infty$. *If* $\alpha = 0$ *we still have* $\lim_{t\to\infty} x(t) = 0$ *and* $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0$.

Remark 3.8. In the "first-order" case (i.e., $n = 1$) the above theorem holds *for all* $\gamma_1 > 0$, because P_1 can be written trivially as $P_1 = b(b^{-1}P_1)$ (i.e., P_1 can be absorbed into P_2). Moreover, it should be mentioned that every "first-order" system which fulfills (2.2) satisfies (2.3) for all $\alpha \geq 0$.

Remark 3.9. Assume that a given system satisfies (2.2) and (2.3) for $\alpha = 0$. Now it follows from Step 1 of the proof of Lemma 2.2 that (2.3) is satisfied for $\alpha > 0$, provided that α is sufficiently small. As a consequence the conclusions of Theorem 3.7 are true for sufficiently small positive α.

4. Conclusions

A theory of adaptive exponential stabilization for certain nonlinear retarded processes has been developed. The resulting adaptive control laws are of high-gain type. In particular, the processes under consideration include a fairly large class of linear retarded systems which are subject to sector-bounded actuator and sensor nonlinearities. It can be shown that the results of Sections 2 and 3 also hold true for Volterra integrodifferential systems and for the classes of distributed parameter systems studied by Byrnes [B] and Kobayashi [K2]. This paper considers single-input single-output processes only. However, work is under way to extend the theory to the multivariable case. It should be mentioned that in contrast to most of the references neither the control law of Section 3 nor the algorithm proposed by Polderman [P1], [P2] is of the "standard form"

$$\dot{z} = f(z, y'),$$
$$u' = g(z, y')$$

(where u' and y' denote the input and the output of the process, see Fig. 1) investigated by Byrnes *et al.* [BHM]. This might lead to implementation problems. However, we do not claim that the control law of Section 3 is a practical one, and we emphasize that the controller proposed here is a more existential contribution. It is an interesting open problem to determine if there exist adaptive control laws of the above form achieving *exponential* stability under conditions which are similar to those in Section 2.

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Appendix 1. Proof that the Zeros of χ and $\det(\Delta)$ Coincide

It follows from the properties of Q that

$$\chi(s) = \det \begin{bmatrix} sI - \hat{A}_{11}(s) & -\hat{A}_{12}(s) & -c^T b \\ -\hat{A}_{21}(s) & sI - \hat{A}_{22}(s) & 0 \\ I & 0 & 0 \end{bmatrix}. \tag{A.1}$$

Define

$$T_1(s) := \begin{bmatrix} I & 0 & -(sI - \hat{A}_{11}(s)) \\ 0 & I & \hat{A}_{21}(s) \\ 0 & 0 & I \end{bmatrix}$$

and

$$T_2(s) := \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -(c^T b)^{-1}\hat{A}_{12}(s) & I \end{bmatrix}.$$

We obtain, from (A.1),

$$\begin{aligned} \chi(s) &= \det \left\{ T_1(s) \begin{bmatrix} sI - \hat{A}_{11}(s) & -\hat{A}_{12}(s) & -c^T b \\ -\hat{A}_{21}(s) & sI - \hat{A}_{22}(s) & 0 \\ I & 0 & 0 \end{bmatrix} T_2(s) \right\} \\ &= -(c^T b) \det(sI - \hat{A}_{22}(s)) \\ &= -(c^T b) \det(\Delta(s)). \end{aligned}$$

■

Appendix 2. Proof of Lemma 3.5

Let $M \subset J$ denote the set of measure zero where $\dot{f}$ does not exist. Moreover, define

$$M^* := \{t \in J \backslash M \mid f(t) = 0, \dot{f}(t) \neq 0\},$$

it is a matter of routine to show

$$\frac{d}{dt}|f(t)| = \operatorname{sign}(f(t))\dot{f}(t) \qquad \text{for all} \quad t \in J \backslash (M \cup M^*).$$

It remains to prove that M^* is of measure zero. But this follows easily from the fact that $|f|$ is not differentiable in any point of M^* and that $|f|$ as an absolutely continuous function is differentiable almost everywhere. ■

Appendix 3. Proof of Lemma 3.6

Without loss of generality we may assume that $\sigma(\varphi) = \sigma(\psi) = +1$. Indeed, if the claim is true in this case, then it is easy to show that the claim is true in the case when $\sigma(\varphi) = \sigma(\psi) = -1$. Moreover, we restrict ourselves to the proof of the inequality

$$\operatorname{sign}(x)\varphi(t, \lambda\rho\psi(t, x)) \leq \Gamma_+(\lambda)\lambda\rho|\psi(t, x)| \qquad ((t, \rho, \lambda, x) \in \mathbb{R}_+^2 \times \mathbb{R}^2). \quad (A.2)$$

The proof of the second inequality in Lemma 3.6 is very similar and is therefore omitted. In order to see why (A.2) holds realize that

$$\varphi(t, \lambda\rho\psi(t, x)) \leq \Delta_\varphi\lambda\rho\psi(t, x) \qquad ((t, \rho, \lambda, x) \in \mathbb{R}_+^2 \times \mathbb{R}_+ \times \mathbb{R}_+), \quad (A.3)$$

$$\varphi(t, \lambda\rho\psi(t, x)) \geq \Delta_\varphi\lambda\rho\psi(t, x) \qquad ((t, \rho, \lambda, x) \in \mathbb{R}_+^2 \times \mathbb{R}_+ \times (-\infty, 0]), \quad (A.4)$$

$$\varphi(t, \lambda\rho\psi(t, x)) \leq \delta_\varphi\lambda\rho\psi(t, x) \qquad ((t, \rho, \lambda, x) \in \mathbb{R}_2^2 \times (-\infty, 0] \times \mathbb{R}_+), \quad (A.5)$$

and

$$\varphi(t, \lambda\rho\psi(t, x)) \geq \delta_\varphi\lambda\rho\psi(t, x) \qquad ((t, \rho, \lambda, x) \in \mathbb{R}_+^2 \times (-\infty, 0] \times (-\infty, 0]). \quad (A.6)$$

We obtain, from (A.3) and (A.4),

$$\operatorname{sign}(x)\varphi(t, \lambda\rho\psi(t, x)) \leq \Delta_\varphi\lambda\rho|\psi(t, x)| \qquad ((t, \rho, \lambda, x) \in \mathbb{R}_+^2 \times [0, \infty) \times \mathbb{R}), \quad (A.7)$$

while (A.5) and (A.6) imply

$$\operatorname{sign}(x)\varphi(t, \lambda\rho\psi(t, x)) \leq \delta_\varphi\lambda\rho|\psi(t, x)| \qquad ((t, \rho, \lambda, x) \in \mathbb{R}_+^2 \times (-\infty, 0] \times \mathbb{R}). \quad (A.8)$$

Inequality (A.2) now follows from (A.7), (A.8), and (3.4a). ■

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