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Finitely generated ideals in certain algebras of transfer functions for infinite-dimensional systems†

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In this note we show that in certain algebras of stable transfer functions for infinite-dimensional systems there exist finitely generated ideals that are not principal. As a consequence there exist unstable transfer functions that have no coprime factorizations.

1. Introduction

The recently developed fractional representation theory of feedback-system design (see e.g. Desoer *et al.* 1980, Vidyasagar *et al.* 1982) is based on the notion of coprime factorization of an unstable plant. Coprime factorizations have played a major role in designing feedback controllers for unstable plants since the work of Youla *et al.* (1976).

For time-invariant finite-dimensional systems the existence of such factorizations is easily shown. In the (time-invariant) infinite-dimensional case there are plants that do not admit a coprime factorization. This is probably a well-known fact to many researchers working in the field of infinite-dimensional systems theory. The purpose of the present note is to provide a unified proof of this fact, which includes all commonly used algebras of transfer functions for infinite-dimensional systems. Our main result shows that in every algebra under consideration there exists a finitely generated ideal that is not principal. As a consequence (cf. Vidyasagar *et al.* 1982, Corollary 2.2) there are unstable transfer functions (i.e. elements of the quotient field of the particular algebra) that do not have coprime factorizations.

2. Notation and preliminaries

For $\alpha \in R$ define $\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \operatorname{Re}(s) > \alpha\}$. The $\mathbb{C}$-algebra of all holomorphic and bounded functions on $\mathbb{C}_\alpha$ is denoted by $H^\infty(\alpha)$. Let $A(\alpha)$ denote the $\mathbb{C}$-algebra of all functions f that are holomorphic on $\mathbb{C}_\alpha$, continuous on $\overline{\mathbb{C}_\alpha}$ and for which $\lim_{\substack{|s| \to \infty \\ s \in \mathbb{C}_\alpha}} f(s)$ exists. Furthermore let $\hat{\mathscr{A}}(\alpha)$ denote the well known $\mathbb{C}$-algebra of transfer functions studied by Callier and Desoer (1978, 1980 a, b).

The algebras $H^\infty(\alpha)$, $A(\alpha)$ and $\hat{\mathscr{A}}(\alpha)$ have been used by many authors working in the field of frequency-domain methods for the control of distributed-parameter systems. For the algebra $H^\infty(\alpha)$ see Zames (1981), Harris and Valenca (1983), Logemann (1984), Pandolfi and Olbrot (1986), Feintuch and Tannenbaum (1986), for the algebra $A(\alpha)$ see Logemann (1984) and Kamen *et al.* (1985 a, b), and for the algebra

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$\hat{\mathscr{A}}(\alpha)$ see Callier and Desoer (1980 b), Chen and Desoer (1982), Postlethwaite and Foo (1985) and Logemann (1986).

Since a 'real-world system' has a transfer function with real coefficients (i.e. any power-series expansion about a real point has real coefficients) it is useful to consider the $\mathbb{R}$-algebras $H_r^\infty(\alpha)$, $A_r(\alpha)$ and $\hat{\mathscr{A}}_r(\alpha)$, where $H_r^\infty(\alpha) := \{f \in H^\infty(\alpha) \mid \overline{f(s)} = f(\bar{s})\ \forall s \in \mathbb{C}_\alpha\}$ and $A_r(\alpha)$, $\hat{\mathscr{A}}(\alpha)$ are defined analogously. Note that the $\mathbb{R}$-algebras $H_r^\infty(\alpha)$, $A_r(\alpha)$ and $\hat{\mathscr{A}}_r(\alpha)$ are subrings, but not subalgebras, of the $\mathbb{C}$-algebras $H^\infty(\alpha)$, $A(\alpha)$ and $\hat{\mathscr{A}}(\alpha)$ respectively. Moreover, define $H_-^\infty(\alpha) := \{f \in H^\infty(\alpha) \mid \exists\, \sigma < \alpha : f \in H^\infty(\sigma)\}$.

It is now obvious what is meant by $A_-(\alpha)$, $\hat{\mathscr{A}}_-(\alpha)$, $H_{r,-}^\infty(\alpha)$, $A_{r,-}(\alpha)$ and $\hat{\mathscr{A}}_{r,-}(\alpha)$. Finally, define the convolution algebras

$$L_1(\alpha) := \{f : \mathbb{R}_+ \to \mathbb{C} \mid f(t)\exp(-\alpha t) \text{ is integrable}\}$$

$$L_{1,r}(\alpha) := \{f : \mathbb{R}_+ \to \mathbb{R} \mid f(t)\exp(-\alpha t) \text{ is integrable}\}$$

Let $\hat{L}_1(\alpha)$ $(\hat{L}_{1,r}(\alpha))$ denote the algebra whose elements are the Laplace transforms of the functions in $L_1(\alpha)$ $(L_{1,r}(\alpha))$.

The following inclusions hold:

$$\hat{L}_1(\alpha) \subset A(\alpha) \subset H^\infty(\alpha)$$

$$\hat{L}_1(\alpha) \subset \hat{\mathscr{A}}(\alpha) \subset H^\infty(\alpha)$$

$$\bigcup_{\sigma<\alpha} \hat{L}_1(\sigma) \subset A_-(\alpha) \subset H_-^\infty(\alpha)$$

$$\bigcup_{\sigma<\alpha} \hat{L}_1(\sigma) \subset \hat{\mathscr{A}}_-(\alpha) \subset H_-^\infty(\alpha)$$

The inclusions $\hat{L}_1(\alpha) \subset A(\alpha)$ and $\bigcup_{\sigma<\alpha} \hat{L}_1(\sigma) \subset A(\alpha)$ follow from Doetsch (1976, p. 159). The other inclusions are trivial.

In order to prove our main results we need the following.

Lemma (Mossaheb 1980)

Let g be a holomorphic function on $\mathbb{C}_\alpha$ and suppose that $sg(s)$ is bounded as $|s| \to \infty$, $s \in \mathbb{C}_\alpha$. Then there exists $\forall\, \sigma > \alpha$ a function $f \in L_1(\sigma)$ such that

$$g(s) = \int_0^\infty f(t)\exp(-st)\,dt\ \forall\, s \in \mathbb{C}_\sigma$$

(Of course, if g has real coefficients then $f \in L_{1,r}(\sigma)$.)

3. Main result

Theorem

Let S be a subring of $H^\infty(\alpha)$. In S there exists a finitely generated ideal that is not principal if there is a real number $\sigma \leqslant \alpha$ such that $S \supseteq \hat{L}_{1,r}(\sigma)$.

Proof

It is sufficient to prove the assertion for $\alpha = 0$, because the map

$$H^\infty(\alpha) \to H^\infty(0)$$

$$h \to h(\cdot + \alpha)$$

and its restriction to S are isomorphisms of rings. Define

$$\left.\begin{aligned} \alpha_n &:= (1-\sigma) + in, \\ \beta_n &:= (1-\sigma) + i\left(n + \frac{1}{n}\right), \end{aligned}\right\} \quad n \geq 1$$

Then the Blaschke products in the half-plane $\mathbb{C}_0$,

$$B_1(s) := \prod_{n\geq 1} \frac{|1-\alpha_n^2|}{1-\alpha_n^2}\frac{s-\alpha_n}{s+\overline{\alpha_n}}$$

$$B_1^*(s) := \prod_{n\geq 1} \frac{|1-\bar{\alpha}_n^2|}{1-\bar{\alpha}_n^2}\frac{s-\overline{\alpha_n}}{s+\alpha_n}$$

$$B_2(s) := \prod_{n\geq 1} \frac{|1-\beta_n^2|}{1-\beta_n^2}\frac{s-\beta_n}{s+\overline{\beta_n}}$$

$$B_2^*(s) := \prod_{n\geq 1} \frac{|1-\overline{\beta}_n^2|}{1-\overline{\beta}_n^2}\frac{s-\overline{\beta_n}}{s+\beta_n}$$

are well defined (cf. Hoffman 1962, p. 132). Choose $\gamma \in (\sigma - 1, \sigma)$ and $\delta \in (|\gamma|, |\sigma - 1|)$ and define

$$F_i(s) := \frac{1}{s+\delta}(B_i B_i^*)(s+\delta), \quad i = 1, 2$$

Obviously, $F_i \in H_r^\infty(\gamma)$ and $sF_i(s)$ is bounded as $|s| \to \infty$, $s \in \mathbb{C}_\gamma$, $i = 1, 2$. Moreover, $F_i \in S$, because $F_i \in L_{1,r}(\tau)\ \forall\, \tau > \gamma$ (by the Lemma in § 2) and therefore in particular $F_i \in \hat{L}_{1,r}(\sigma)$, $i = 1, 2$.

We claim that the ideal $F_1 S + F_2 S$ is not principal. Assume the contrary, i.e. that there exist functions $d, f_1, f_2, g_1, g_2 \in S$ such that

$$d = f_1 F_1 + f_2 F_2 \tag{1}$$

$$F_1 = dg_1, \quad F_2 = dg_2 \tag{2}$$

The zeros of g_i are exactly the zeros of F_i, $i = 1, 2$ (because F_1 and F_2 have no common zeros). Therefore $g_1(\varepsilon_n) = 0$ and $g_2(\eta_n) = 0$, where $\varepsilon_n := \alpha_n - \delta$ and $\eta_n := \beta_n - \delta$, $n \geq 1$. Note that $\mathrm{Re}\,(\varepsilon_n)$, $\mathrm{Re}\,(\eta_n) = 1 - \sigma - \delta > 0\ \forall\, n \geq 1$. It now follows from (1) and (2) that

$$f_1 g_1(\eta_n) = 1 \quad \forall\, n \geq 1 \tag{3}$$

On the other hand, we have

$$f_1 g_1(\varepsilon_n) = 0 \quad \forall\, n \geq 1 \tag{4}$$

Since $\lim_{n\to\infty} (\eta_n - \varepsilon_n) = 0$, the equations (3) and (4) contradict the fact that a function in $H^\infty(0)$ (and hence $f_1 g_1$) is uniformly continuous on every vertical strip $a \leq \mathrm{Re}\,(s) \leq b$, $0 < a < b$ (see Corduneanu 1968, p. 72). □

An immediate consequence of the Theorem is the following.

Corollary

In each of the algebras $H^\infty(\alpha)$, $H_r^\infty(\alpha)$, $H_-^\infty(\alpha)$, $H_{r,-}^\infty(\alpha)$, $A(\alpha)$, $A_r(\alpha)$, $A_-(\alpha)$, $A_{r,-}(\alpha)$,

$\hat{\mathscr{A}}(\alpha)$, $\hat{\mathscr{A}}_r(\alpha)$, $\hat{\mathscr{A}}_-(\alpha)$, $\hat{\mathscr{A}}_{r,-}(\alpha)$, $\hat{L}_1(\alpha)$ and $\hat{L}_{1,r}(\alpha)$ there exists a finitely generated ideal that is not principal.

Remarks

(i) The idea behind the proof is due to Whittaker (1935). It is also used in the paper of Vidyasagar *et al.* (1982).

(ii) The fact that in the algebra $\hat{\mathscr{A}}(0)$ there exists a finitely generated ideal that is not principal was noted by Vidyasagar *et al.* (1982). However, they did not provide a proof.

(iii) As far as the finitely generated ideals are concerned, the algebra $\mathscr{H}(\Omega)$ of all holomorphic functions in the region $\Omega \subset \mathbb{C}$ is quite different from the algebras in the above Corollary. Indeed it is well known that every finitely generated ideal in $\mathscr{H}(\Omega)$ is principal (cf. e.g. Narasimhan 1985, p. 136).

(iv) It is easy to show that the above Corollary is also true for the algebra of α-exponential stable transfer functions, which has recently been introduced by Callier and Winkin (1984).

Note. After sending the final proofs of this paper to the publisher, it was pointed out to the author that it has been proved by M. von Renteln in 1977 (*Acta Sci. Math.*, **39**, 139) that there exist finitely generated ideals in $H^\infty(0)$ and $A(0)$ that are not principal. The author wishes to acknowledge priority.

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