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On the transfer matrix of a neutral system: Characterizations of exponential stability in input–output terms

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Abstract: This paper investigates the stability of linear autonomous multivariable neutral systems from an input–output viewpoint. Several frequency-domain and input–output characterizations for exponential stability of neutral systems are given. We provide two examples which illustrate that the behaviour of neutral systems may be quite different from that of retarded systems. Moreover we give necessary and sufficient conditions for the transfer functions of a neutral system to belong to certain algebras of meromorphic functions introduced in this paper.

Keywords: Neutral systems, Multivariable systems, Frequency-domain methods, Transfer matrices, Algebras of meromorphic functions, Exponential stability, Input–output stability.

Introduction

Most of the work on neutral systems is based on the state space approach to linear systems (cf. e.g. Salamon [22]). This paper studies neutral systems using frequency-domain and input–output ideas. In particular we shall study the relationship between exponential (or internal) stability, input–output (or external) stability and analyticity of the transfer matrix in the closed right-half plane.

It is well known that for exponential stability of a neutral system it is not sufficient to have all the roots of the characteristic equation in the open left half plane. We provide two examples which illustrate how this fact is reflected in the behaviour of the system in the frequency domain. We show that there exist neutral systems being internally and externally unstable, but having transfer functions which are holomorphic in the closed right half plane. Moreover we demonstrate that it is possible for an exponentially unstable canonical neutral system to have a transfer function which is bounded and holomorphic in the closed righ half plane. Neither phenomenon occurs in the theory of retarded sytems. Apart from these examples we provide a result on the transfer matrix of a general neutral system. More precisely: we give a necessary and sufficient condition for the entries of the transfer matrix of a neutral system to belong to one of the two algebras of transfer functions introduced in Section 1 of this paper.

Finally we prove several characterizations of exponential stability in input–output and frequency-domain terms. The latter result is based on Chapter 3 of Harris and Valenca [10].

This paper is organized as follows. Section 1 is devoted to notation and preliminaries. In Section 2 we introduce the class of neutral systems we shall deal with. In Section 3 we study the transfer function of a neutral system. Section 4 contains several necessary and sufficient input–output conditions for exponential stability.

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1. Notation and preliminaries

Let $\mathrm{BV}([a, b], \mathbb{R}^{k\times l})$ denote the space of functions of bounded variation on $[a, b]$ with values in $\mathbb{R}^{k\times l}$. For $1 \le p \le \infty$ set $L^p(\mathbb{R}^l) := L^p([0, \infty), \mathbb{R}^l)$ and for $\sigma \in \mathbb{R}$ define $L^p_\sigma(\mathbb{R}^l) := \{f: [0, \infty) \to \mathbb{R}^l \mid f(\cdot)\mathrm{e}^{\sigma\cdot} \in L^p(\mathbb{R}^l)\}$. We equip $L^p_\sigma(\mathbb{R})$ with the norm $\|f\|_{p,\sigma} := \|f(\cdot)\mathrm{e}^{\sigma\cdot}\|_p$. It is trivial that $L^p_0(\mathbb{R}^l) = L^p(\mathbb{R}^l)$ and that $L^p_{\sigma_1}(\mathbb{R}^l) \subset L^p_{\sigma_2}(\mathbb{R}^l)$ if $\sigma_1 \ge \sigma_2$. The space $X^p(\mathbb{R}^l) := \cup_{\sigma>0} L^p_\sigma(\mathbb{R}^l)$ can be made into a sequential convergence space (cf. [10], 3.6.1–3.6.3 and Dudley [8]) by the following rule of convergence: a sequence $f_i \in X^p(\mathbb{R}^l)$ converges to $f \in X^p(\mathbb{R}^l)$ as $i \to \infty$ if there exists $\sigma_0 > 0$ such that $f \in L^p_\sigma(\mathbb{R}^l)$, $f_i \in L^p_\sigma(\mathbb{R}^1)$ $\forall i \in \mathbb{N}$ and $\lim_{i\to\infty} \|f - f_i\|_{p,\sigma_0} = 0$. Furthermore let $\mathscr{L}(L^p_{\sigma_1}(\mathbb{R}^k), L^p_{\sigma_2}(\mathbb{R}^l))$ denote the space of linear bounded operators mapping $L^p_{\sigma_1}(\mathbb{R}^k)$ into $L^p_{\sigma_2}(\mathbb{R}^l)$ and let $\mathscr{L}(X^p(\mathbb{R}^k), X^p(\mathbb{R}^l))$ represent the space of linear sequantially continuous operators defined on $X^p(\mathbb{R}^k)$ with values in $X^p(\mathbb{R}^l)$ (a linear operator $T: X^p(\mathbb{R}^k) \to X^p(\mathbb{R}^l)$ is called sequentially continuous iff Tf_i converges to zero whenever f_i converges to zero). For $\sigma \in \mathbb{R}$ set $\mathbb{C}_\sigma := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > \sigma\}$. H^∞_σ denotes the algebra of all $\mathbb{C}$-valued functions which are holomorphic and bounded on $\mathbb{C}_\sigma$. Moreover we define the algebra $H^\infty_- := \cup_{\sigma<0} H^\infty_\sigma$.

We shall need the following lemma on bounded holomorphic functions which can be found in Corduneanu [4], pp. 72.

1.1. Lemma. *Let $a < b < c < d$ be given and suppose that f is a bounded holomorphic function on $a < \mathrm{Re}(s) < d$. Then f is uniformly continuous on the closed strip $b \le \mathrm{Re}(s) \le c$.*

In order to deal with unstable systems it is useful to introduce the algebra $\mathscr{T}$ of meromorphic functions which is defined by

$$\mathscr{T} := \left\{ \frac{n}{d} \mid n,\ d \in H^\infty_- \wedge \exists R > 0\colon \inf_{\substack{s\in\mathbb{C}_0 \\ |s|\ge R}} |d(s)| > 0 \right\}.$$

1.2. Remark. (i) $\mathscr{T}$ contains the algebra $\hat{\mathscr{B}}(0)$ of transfer functions introduced by Callier and Desoer [2,3].

(ii) Suppose that $f \in \mathscr{T}$. It follows from Lemma 1.1 that there exists $\sigma < 0$ such that f has at most finitely many poles in $\mathbb{C}_\sigma$. Consequently f can be represented as $f = h + r$, where $h \in H^\infty_-$ and r is a strictly proper rational function.

(iii) A function $f \in \mathscr{T}$ is a unit in $\mathscr{T}$ iff there exists $R > 0$ such that $\inf_{s\in\mathbb{C}_0: |s|\ge R} |f(s)| > 0$. Callier and Desoer proved an analogous result for the algebra $\hat{\mathscr{B}}(0)$ in [2,3]. Their proof extends to the more general case (cf. Logemann [17]).

2. System description

Consider the neutral system

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^r \mathrm{d}D(\tau)x(t-\tau) = \int_0^r \mathrm{d}A(\tau)x(t-\tau) + \int_0^r \mathrm{d}B(\tau)u(t-\tau), \tag{2.1a}$$

$$x|_{[-r,0]} = x_0 \in C([-r,], \mathbb{R}^n), \tag{2.1b}$$

$$y(t) = \int_0^r \mathrm{d}C(\tau)x(t-\tau), \tag{2.1c}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^q$ and $y(t) \in \mathbb{R}^m$.

The functions A, B, C and D are elements in the spaces $\mathrm{BV}([0, r], \mathbb{R}^{n\times m})$, $\mathrm{BV}([0, r], \mathbb{R}^{n\times q})$, $\mathrm{BV}([0, r], \mathbb{R}^{m\times n})$ and $\mathrm{BV}([0, r], \mathbb{R}^{n\times n})$, respectively. Moreover we suppose that

$$D = \theta I - E, \tag{2.1d}$$

where θ is the unit step and $E \in \mathrm{BV}([0, r], \mathbb{R}^{n\times n})$ is assumed to be continuous at 0 (cf. Hale and Meyer [9], Henry [12] and Kappel [13]).

2.1 Remark. (i) In the following we extend any function $F \in \mathrm{BV}([a, b], \mathbb{R}^{k\times l})$ to the whole real axis by setting $F(t) = F(a)\ \forall t < a$ and $F(t) = F(b)\ \forall t > b$. Any measurable function $f: \Omega \to \mathbb{R}^k$, $\Omega \subset \mathbb{R}$, will be extended to the whole real axis by defining $f(t) = 0\ \forall t \notin \Omega$.

(ii) At the first glance an expression of the form $\int_0^r \mathrm{d}F(\tau)h(t-\tau)$, where $F = (f_{ij}) \in \mathrm{BV}([0, r], \mathbb{R}^{k\times l})$, is only well defined for $t \geq 0$ if $h(t) \in \mathbb{R}^l$ is continuous for $t \geq -r$. However, the following equation holds:

$$\int_0^r \mathrm{d}F(\tau)h(t-\tau) = \begin{bmatrix} \sum_{j=1}^{l} (\mathrm{d}f_{ij} * h_j)(t) \\ \vdots \\ \sum_{j=1}^{l} (\mathrm{d}f_{kj} * h_j)(t) \end{bmatrix} =: \mathrm{d}F * h, \tag{2.2}$$

where $\mathrm{d}f_{ij}$ denotes the measure on $\mathbb{R}$ induced by f_{ij} and $\mathrm{d}f_{ij} * h_j$ denotes the convolution of the measure $\mathrm{d}f_{ij}$ and the function h_j. The expression $\mathrm{d}f_{ij} * h_j$ makes sense if h_j is a locally integrable function on $\mathbb{R}$. Moreover the mapping $h \mapsto \mathrm{d}F * h$ is a linear bounded operator from $L^p_{\sigma_1}(\mathbb{R}^l)$ into $L^p_{\sigma_2}(\mathbb{R}^k)$ for all $\sigma_1 \geq \sigma_2 \geq 0$, $1 \leq p \leq \infty$ (cf. Dieudonné [6], pp. 282 and 285).

(iii) It follows via (ii) from [9] that for every $x_0 \in C([-r, 0], \mathbb{R}^n)$ and every $u \in L^1_{\mathrm{loc}}([0, \infty), \mathbb{R}^q)$ $(1 \leq p \leq \infty)$ there is a unique solution of (2.1a) and (2.1b) which is continuous on $[-r, \infty)$.

2.2. Definition. (i) The system (2.1) is called *exponentially stable* if the strongly continuous solution semigroup on $C([-r, 0], \mathbb{R}^n)$ of the homogenous part of (2.1a) is exponentially stable.

(ii) The system (2.1) is called *L^p-stable* (X^p-stable) if under zero initial conditions (i.e. $x_0 \equiv 0$ in (2.1b)) the i/o-operator $u \mapsto y$ associated with (2.1) is an element of $\mathscr{L}(L^p(\mathbb{R}^q), L^p(\mathbb{R}^m))$ ($\mathscr{L}(X^p(\mathbb{R}^q), X^p(\mathbb{R}^m))$), $1 \leq p \leq \infty$.

In order to introduce the transfer matrix of the system (2.1) we define $\hat{A}(s) := \int_0^r \mathrm{e}^{-s\tau}\mathrm{d}A(\tau)$ and it is obvious what is meant by $\hat{B}(s)$, $\hat{C}(s)$ and $\hat{D}(s)$. The entries of $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are entire functions. Moreover they are bounded on every right half plane. We obtain the following expression for the transfer matrix G of the system (2.1)

$$G = \hat{C}\Delta^{-1}\hat{B} \tag{2.3}$$

where Δ is given by

$$\Delta(x) := s\hat{D}(s) - \hat{A}(s). \tag{2.4}$$

The system (2.1) is called *canonical* in $\Omega \subset \mathbb{C}$ if

$$\mathrm{rk}(\Delta(s), \hat{B}(s)) = n \quad \forall\, s \in \Omega \tag{2.5a}$$

and

$$\mathrm{rk}\begin{bmatrix} \Delta(s) \\ \hat{C}(s) \end{bmatrix} = n \quad \forall\, s \in \Omega. \tag{2.5b}$$

We shall need the following assumption on the system (2.1)

(NS) The function D contains no singular part (see e.g. Natanson [20]).

2.3. Remark. (i) Condition (2.5) is a generalization of the so-called Hautus conditions in the finite-dimensional case (cf. Hautus [11]) and has been used by many researchers working on the control of functional differential equations (cf. e.g. [22] or Pandolfi [21]).

(ii) If (NS) is satisfied then the exponential growth of the strongly continuous semigroup on $C([-r, 0], \mathbb{R}^n)$ corresponding to the homogenous part of (2.1a) is determined by the spectrum of its generator (cf. [12]). In particular it follows that the system (2.1) is exponentially stable iff there exists $\sigma < 0$ such that $\det(\Delta(s)) \neq 0 \ \forall s \in \mathbb{C}_\sigma$.

(iii) Suppose that (NS) is satisfied. Then it is clear that E (cf. eq. (2.1d)) contains no singular part and it follows that we can express $D(s)$ in the following way:

$$\hat{D}(s) = I - \sum_{j=0}^{\infty} E_j \, \mathrm{e}^{-r_j s} - \int_0^r E_\infty(\tau) \, \mathrm{e}^{-s\tau} \, \mathrm{d}\tau, \tag{2.6}$$

where $0 < r_j \leq r \ \forall j \geq 0$ and

$$\sum_{j=0}^{\infty} \|E_j\| + \int_0^r \|E_\infty(\tau)\| \, \mathrm{d}\tau < \infty.$$

We define

$$\Delta_0(s) := I - \sum_{j=0}^{\infty} E_j \, \mathrm{e}^{-r_j s}. \tag{2.7}$$

3. On the transfer matrix of the neutral system (2.1)

Consider the neutral system

$$\dot{x}(t) - \dot{x}(t-r) = -ax(t) + bx(t-r) + u(t), \tag{3.1a}$$

$$y(t) = x(t), \tag{3.1b}$$

where $x(t)$, $u(t)$ and $y(t) \in \mathbb{R}$, $r > 0$ and a, $b \in \mathbb{R}$. The transfer function g of (3.1) is given by

$$g(s) = \frac{1}{s(1 - \mathrm{e}^{-rs}) + a - b\mathrm{e}^{-rs}}. \tag{3.2}$$

The following result holds true:

3.1. Proposition. *For all values of the parameters a and b satisfying $a > |b|$ the transfer function g is holomorphic in $\overline{\mathbb{C}}_0$. However the system* (3.1) *is not exponentially stable, nor is it L^p-stable or X^p-stable for $p = 1, 2, \infty$.*

For the proof the following general lemma is required.

3.2. Lemma. *Suppose that* (NS) *holds and let $\alpha < \beta$ be given. Then the following statements are equivalent:*

(i) *There exists some $s_0 \in \mathbb{C}$ such that $\alpha < \mathrm{Re}(s_0) < \beta$ and $\det(\Delta_0(x_0)) = 0$.*

(ii) *There exists some $\epsilon > 0$ and a sequence $s_k \in \mathbb{C}$ such that $|\mathrm{Im}(s_k)| \to \infty$ as $k \to \infty$, $\alpha + \epsilon < \mathrm{Re}(s_k) < \beta - \epsilon$, and $\det(\Delta(s_k)) = 0 \ \forall k \in \mathbb{N}$.*

Proof. Cf. [22], p. 160.

Proof of Proposition 3.1. Define $\delta_0(x) := 1 - \mathrm{e}^{-rs}$ and $\delta(s) := 1/g(s)$. It is obvious that g is holomorphic in $\overline{\mathbb{C}}_0$ since

$$|s + a| > |(s + b) \, \mathrm{e}^{-rs}| \quad \forall s \in \overline{\mathbb{C}}_0, \qquad \text{i.e.} \quad \delta(s) \neq 0 \quad \forall s \in \overline{\mathbb{C}}_0.$$

If we realize that $\delta_0(0)=0$, then by Lemma 3.2 there exists a sequence $s_k \in \mathbb{C}$ such that $\mathrm{Re}(s_k)<0$, $\mathrm{Re}(s_k)\to 0$ and $|\mathrm{Im}(s_k)|\to\infty$ as $k\to\infty$ and $\delta(s_k)=0$ $\forall k\in\mathbb{N}$. It follows from Remark 2.3 (ii) that system (3.1) is not exponentially stable.

In order to prove that the system (3.1) is not L^p-stable or X^p-stable for $p=1, 2, \infty$, it is sufficient of show that $g\notin H_0^\infty$ (this follows from [10], Ch. 3). Define

$$\phi_k := \sqrt{\pi^2k^2+r(a-b)}-\pi k \quad \text{and} \quad \omega_k := \frac{1}{r}(2\pi k+\phi_k), \qquad k\in\mathbb{N}.$$

We claim

$$\lim_{k\to\infty}\delta(\mathrm{j}\omega_k)=\lim_{k\to\infty}\frac{1}{g(\mathrm{j}\omega_k)}=0. \tag{3.3}$$

Note that

$$\delta(\mathrm{j}\omega_k)=a-b\cos(r\omega_k)+\omega_k\sin(r\omega_k)+\mathrm{j}(\omega_k(1-\cos(r\omega_k))+b\sin(r\omega_k)) \tag{3.4}$$

and

$$\phi_k=\mathrm{O}\left(\frac{1}{k}\right) \quad (\text{as } k\to\infty). \tag{3.5}$$

We have to show

$$\lim_{k\to\infty}(a-b\cos(r\omega_k)-\omega_k\sin(r\omega_k))=0 \tag{3.6}$$

and

$$\lim_{k\to\infty}(\omega_k(1-\cos(r\omega_k))+b\sin(r\omega_k))=0. \tag{3.7}$$

We shall prove that (3.6) holds true. The verification of (3.7) is left to the reader. We have

$$\begin{aligned} a-b\cos(r\omega_k)-\omega_k\sin(r\omega_k) &= a-b-\frac{1}{r}(2\pi k+\phi_k)\phi_k-b\sum_{i=1}^{\infty}\frac{(-1)^i}{(2i)!}(\phi_k)^{2i} \\ &\quad -\frac{1}{r}(2\pi k+\phi_k)\sum_{i=1}^{\infty}\frac{(-1)^i}{(2i+1)!}(\phi_k)^{2i+1}. \end{aligned} \tag{3.8}$$

It follows from the definition of the ϕ_k that $a-b-r^{-1}(2\pi k+\phi_k)\phi_k=0$ $\forall k\in\mathbb{N}$. Moreover we obtain from (3.5) that the two remaining terms on the right-hand side of (3.8) tend to zero as $k\to\infty$.

3.3. Remark. Proposition 3.1 shows that analyticity of the transfer function of a neutral system in $\overline{\mathbb{C}}_0$ does not imply internal or external stability. In particular, Proposition 3.1 provides an example for a transfer function that is holomorphic in $\overline{\mathbb{C}}_0$ but whose associated i/o-operator is not L^∞-stable. See Desoer [5] and Baker and Vakharia [1] for another example.

Let us consider the following example of a neutral system. It will provide another interesting phenomenon arising in the theory of neutral systems which does not occur in the theory of retarded systems.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t)-\dot{x}_2(t-r) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -a \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t), \tag{3.9a}$$

$$y(t)=(0 \quad 1)\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{3.9b}$$

where $x_1(t)$, $x_2(t)$, $u(t)$ and $y(t) \in \mathbb{R}$, $r>0$ and $a \in \mathbb{R}$. The transfer function of the system (3.9) is given by

$$h(s)=\frac{1}{s+1}\,\frac{1}{s(1-e^{-rs})+a}. \tag{3.10}$$

Moreover it should by noticed that the system (3.9) is canonical in $\mathbb{C}$. Nevertheless we have the following result:

3.4. Proposition. *The transfer function h given by* (3.10) *is in* H_0^∞ (*i.e. the system* (3.9) *is* L^2*-stable* (*cf.* [10], *p.* 83)) *for all* $a>0$. *However, the system* (3.9) *is not exponentially stable.*

Proof. It can be shown in exactly the same way as in the proof of Proposition 3.1 that the system is not exponentially stable. It remains to show that $h \in H_0^\infty$ $\forall a>0$. It is easy to verify that h is holomorphic in $\overline{\mathbb{C}}_0$. The boundedness of h is a consequence of the following lemma, which is proved in Logemann [18].

3.5. Lemma. *Let* $\epsilon>0$ *be given. If* $a>0$ *then* $\inf_{s\in\overline{\mathbb{C}}_0:|s|\geq\epsilon} |s(s(1-e^{-rs})+a)|>0$.

The following result holds for the transfer matrix G of the general neutral system (2.1).

3.6. Theorem. (i) *The entries of the transfer matrix given by* (2.3) *are of the form* n/d, *where* n, $d \in H_\sigma^\infty$ *and* $\sigma \in \mathbb{R}$ *is arbitrary.*

(ii) *Suppose that* (NS) *holds. If* $\inf_{s\in\mathbb{C}_0} |\det(\Delta_0(s))|>0$ *then* $G \in \mathcal{F}^{m\times q}$.

(iii) *Assume that* (NS) *is satisfied and that A, B and C contain no singular part. If* $\inf_{s\in\mathbb{C}_0} |\det(\Delta_0(s))| > 0$ *then* $G \in \hat{\mathcal{B}}(0)^{m\times q}$.

Proof. (i) Notice that

$$G(s)=\hat{C}(s)\frac{1}{(s-\sigma+1)^n}\,\mathrm{ad}(\Delta(s))\left(\frac{1}{(s-\sigma+1)^n}\,\det(\Delta(s))\right)^{-1}\hat{B}(s)$$

and that $\hat{C}\in H_\sigma^{\infty m\times n}$, $\hat{B}\in H_\sigma^{\infty n\times q}$, $(s-\sigma+1)^{-n}$ $\mathrm{ad}(\Delta(s))\in H_\sigma^{\infty n\times n}$ and $(s-\sigma+1)^{-n}\det(\Delta(s))\in H_\sigma^\infty$.

(ii) It is possible to write G in the following form:

$$G(s)=\hat{C}(s)\frac{1}{s^n}\,\mathrm{ad}(F(s))\left(\frac{1}{s^n}\,\det(F(s))\right)^{-1}\hat{D}^{-1}(s)\hat{B}(s), \tag{3.11}$$

where $F(s):=sI-\hat{D}^{-1}(s)\hat{A}(s)$. It is sufficient to show that there exists $R>0$ such that

$$\inf_{\substack{s\in\mathbb{C}_0\\|s|\geq R}} |\det(\hat{D}(s))|>0. \tag{3.12}$$

Indeed it follows then via Remark 1.2(iii) that $\hat{C}(s)s^{-n}$ $\mathrm{ad}(F(s))\in\mathcal{F}^{m\times n}$ and $\hat{D}^{-1}\hat{B}\in\mathcal{F}^{n\times q}$. Moreover we have that

$$\det(F(s))=s^n+\gamma_1(s)s^{n-1}+\gamma_2(s)s^{n-2}+\cdots+\gamma_n(s),$$

where the γ_i belong to the subalgebra of $\mathcal{F}$ generated by the entries of $\hat{D}^{-1}\hat{A}$. Hence we obtain from Remark 1.2(iii) that $(s^{-n}\det(F(s)))^{-1}\in\mathcal{F}^{n\times n}$.

It remains to show that (3.12) is true. Because (NS) is satisfied we can write (cf. (2.6) and (2.7))

$$\hat{D}(s)=\Delta_0(s)-\int_0^r E_\infty(\tau)\,e^{-s\tau}\,d\tau. \tag{3.13}$$

Since $\inf_{s\in\mathbb{C}_0} |\det(\Delta_0(s))|>0$ (by assumption) and $\lim_{|s|\to\infty,s\in\mathbb{C}_0}\int_0^r E_\infty(\tau)\,e^{-s\tau}\,d\tau=0$ (cf. Doetsch [7], p. 159) it follows from (3.13) that there exists $R>0$ such that $\inf_{s\in\mathbb{C}_0:|s|\geq R} |\det(\hat{D}(s))|>0$.

(iii) The proof of part (iii) is omitted for brevity but follows in a similar way as that of part (ii).

3.7. Remark. (i) Theorem 3.6(ii) shows in particular that the entries of the transfer matrix of a retarded system are functions in $\mathscr{T}$.

(ii) It follows in Theorem 3.6(ii) that every neutral system satisfying (NS) and $\inf_{s \in \mathbb{C}_0} |\det(\Delta_0(s))| > 0$ can be stabilized by a finite-dimensional compensator. This can be proved in the same way as in Logemann [15] and [17] where the stronger condition $\inf_{s \in \mathbb{C}_0} |\det(\hat{D}(s))| > 0$ was supposed to be satisfied.

Under certain conditions the inverse of Theorem 3.6(ii) and (iii) is valid. More precisely, we have:

3.8. Proposition. *Suppose that the condition* (NS) *is satisfied and that the system* (2.1) *is canonical in* $\mathbb{C}_\sigma$ *for some* $\sigma < 0$. *If* $G \in \mathscr{T}^{m \times q}$ *then* $\inf_{s \in \mathbb{C}_0} |\det(\Delta_0(s))| > 0$.

Proof. Since the system is canonical in $\mathbb{C}_\sigma$ it follows from [21], Sec. 5 and 6, or Logemann [16] that the zeros of $\det(\Delta)$ in $\mathbb{C}_\sigma$ and the poles of G in $\mathbb{C}_\sigma$ coincide. Hence, by Remark 1.2(ii), $\det(\Delta)$ has at most finitely many zeros in $\mathbb{C}_\alpha$ for some $\sigma \leq \alpha < 0$ and we obtain from Lemma 3.2 that

$$\det(\Delta_0(s)) \neq 0 \quad \forall\, s \in \mathbb{C}_\alpha. \tag{3.14}$$

Moreover it can be seen from (2.7) that

$$\inf_{s \in \mathbb{C}_\beta} |\det(\Delta_0(s))| > 0 \quad \forall\, s \in \mathbb{C}_\beta \tag{3.15}$$

for large enough $\beta > 0$.

Now realize that $\det(\Delta_0)$ is an almost periodic function (cf. [4]) and therefore by (3.14) and a result of Levin [14], p. 268,

$$\inf_{0 \leq \mathrm{Re}(s) \leq \beta} |\det(\Delta_0(s))| > 0. \tag{3.16}$$

The claim follows from (3.15) and (3.16).

4. Characterization of exponential stability in i/o-terms

The following theorem provides several necessary and sufficient i/o-conditions for exponential stability of the neutral system (2.1)

4.1. Theorem. *Suppose that* (NS) *is statisfied. Then the following statements are equivalent*:

(i) *The system* (2.1) *is exponentially stable.*
(ii) *The transfer matrix* G *is in* $H_{-}^{\infty m \times q}$ *and* (2.1) *is canonical in* $\mathbb{C}_\sigma$ *for some* $\sigma < 0$.
(iii) *The system* (2.1) *is* X^1*-stable and it is canonical in* $\mathbb{C}_\sigma$ *for some* $\sigma < 0$.
(iv) *The system* (2.1) *is* X^2*-stable and it is canonical in* $\mathbb{C}_\sigma$ *for some* $\sigma < 0$.
(v) *The system* (2.1) *is* X^∞*-stable and it is canonical in* $\mathbb{C}_\sigma$ *for some* $\sigma < 0$.
(vi) *The system* (2.1) *is* X^p*-stable* $\forall\, 1 \leq p \leq \infty$ *and it is canonical in* C_σ *for some* $\sigma < 0$.

Proof. (i) $\Rightarrow$ (ii): By Remark 2.3(ii) there exists $\sigma < 0$ such that $\det(\Delta(s)) \neq 0$ $\forall s \in \mathbb{C}_\sigma$. It follows in particular that the system is canonical in $\mathbb{C}_\sigma$. Moreover we obtain from Lemma 3.2 that $\det(\Delta_0(s)) \neq 0$ $\forall\, s \in \mathbb{C}_\sigma$. Therefore we can conclude in exactly the same way as in the proof of Proposition 3.8 that $\inf_{s \in \mathbb{C}_0} |\det(\Delta_0(s))| > 0$. Hence there exists a constant $R > 0$ such that (cf. Proof of Theorem 3.6(ii))

$$\inf_{\substack{s \in \mathbb{C}_0 \\ |s| \geq R}} |\det(\hat{D}(s))| > 0$$

and Remark 1.2(iii) yields that $\hat{D}^{-1} \in \mathscr{T}^{n\times n}$. It is clear that Δ^{-1} is holomorphic in $\mathbb{C}_\sigma$ and it follows from the identity

$$\Delta^{-1}(s) = \left(s\hat{D}(s) - \hat{A}(s)\right)^{-1} = \frac{\mathrm{ad}\left(sI - \hat{D}^{-1}(s)\hat{A}(s)\right)}{\det\left(sI - \hat{D}^{-1}(s)\hat{A}(s)\right)}\hat{D}^{-1}(s) \tag{4.1}$$

that the function Δ^{-1} is bounded in $\mathbb{C}_\alpha$ for some $\sigma \le \alpha \le 0$. This implies that $G = \hat{C}\Delta^{-1}\hat{B} \in H_-^{\infty m\times q}$.

(i) $\Rightarrow$ (vi): Inspection of (4.1) yields that $\Delta^{-1}(s) = \mathrm{O}(1/s)$ as $|s| \to \infty$ in $\mathbb{C}_\alpha$. If we choose $\alpha < \beta < 0$ then we obtain from a result of Mossaheb [19] (cf. also [17], p. 16) that there exists a matrix-valued function $K \in L^1_{-\beta}(\mathbb{R}^{n\times m})$ such that $\Delta^{-1}(s) = \int_0^\infty K(t)e^{-st}\,dt$ $\forall s \in \mathbb{C}_\beta$. Consider the system (2.1) and suppose that $x_0 \equiv 0$ and $u \in L^p(\mathbb{R}^q)$. The solution of (2.1a) is then given by $x = K * (\mathrm{d}B * u)$ (cf. [13], p. 22). It follows from Remark 2.1(ii) that the i/o-operator $T: u \mapsto \mathrm{d}C * (K * (\mathrm{d}B * u))$ of (2.1) is in $\mathscr{L}(L^p_{\sigma_1}(\mathbb{R}^q), L^p_{\sigma_2}(\mathbb{R}^m))$ for all numbers $\sigma_1 \ge 0$, $0 \le \sigma_2 \le -\beta$ satisfying $\sigma_1 \ge \sigma_2$, $1 \le p \le \infty$. Hence $T \in \mathscr{L}(X^p, X^p)$ $(1 \le p \le \infty)$ by [10], 3.6.2.

(ii) $\Rightarrow$ (i): Since the system (2.1) is canonical on $\mathbb{C}_\sigma$ for some $\sigma < 0$ the poles of G in $\mathbb{C}_\sigma$ and the zeros of $\det(\Delta)$ in $\mathbb{C}_\sigma$ coincide (cf. [21], Sec. 5 and 6, or [16]). Hence there exists $\sigma \le \alpha \le 0$ such that $\det(\Delta(s)) \ne 0$ $\forall s \in \mathbb{C}_\alpha$. The exponential stability of the system (2.1) follows now from Remark 2.3(ii).

The implications (iii) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (ii) hold by [10], 3.6.4 and 3.6.5.

4.2. Theorem. *Suppose that* (NS) *is satisfied and that* $\inf_{s\in\mathbb{C}_0} |\det(\Delta_0(s))| > 0$. *Then Theorem* 4.1 *remains true if we replace* H_-^∞ *by* H_0^∞, $\mathbb{C}_\sigma$ *by* $\overline{\mathbb{C}}_0$ *and* X^p *by* L^p $(p = 1,2,\ldots,\infty)$.

The proof is similar to that of Theorem 4.1 and is omitted for brevity.

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