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On the existence of finite-dimensional compensators for retarded and neutral systems

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In this paper we study a large class of neutral systems containing the class of retarded systems. A necessary and sufficient condition is given for the existence of a finite-dimensional compensator achieving exponential stability.

1. Introduction

A fundamental problem in the control of time-delay systems is determining whether or not there exists a finite-dimensional system (compensator) (A, B, C) such that the closed-loop system shown in Fig. 1 is exponentially stable. In the present paper we discuss this problem for a fairly large class of neutral systems with general delays in the state variables and input/output variables. The term 'general delay' indicates that we allow for distributed delays and infinitely many point delays. In particular, the class of neutral systems under consideration contains the class of retarded systems.

There are several recent papers (Byrnes *et al.* 1984, Emre and Knowles 1984, Kamen *et al.* 1984, 1985) on the stabilization of time-delay systems based on a delay-operator approach in the context of the theory of linear systems over rings. In these papers retarded systems are considered as systems over $\mathbb{R}[z_1, \ldots, z_n]$ (ring of polynomials in n variables with real coefficients). In case of neutral systems the ring $\mathbb{R}[z_1, \ldots, z_n]$ has to be replaced by a certain localization of $\mathbb{R}[z_1, \ldots, z_n]$. It should be clear that systems with distributed delays and/or infinitely many point delays do not fit into the framework of the ring approach to neutral systems.

Byrnes *et al.* (1984) are concerned with stabilization by non-dynamic state feedback, while Emre and Knowles (1984) and Kamen *et al.* (1984) consider dynamic output feedback. However, the resulting stabilization schemes are infinite-dimensional. As far as I am aware the paper of Kamen *et al.* (1985) is the only one which uses the ring approach to time-delay systems in order to tackle the problem of finite-dimensional stabilization via dynamic output feedback. Using the systems-over-rings approach in Kamen *et al.* (1984) it is shown that for the finite-dimensional stabilization of a retarded system with finitely many point delays it is necessary and sufficient that the generalized Hautus conditions are satisfied in the closed right half-plane (See Definition 3.6).

The purpose of the present paper is to show that the same holds for a much more general class of a neutral systems containing infinitely many point delays and distributed delays. This cannot be done in the context of the ring approach to neutral systems. Our results depend on frequency-domain methods for distributed-parameter systems as developed by Callier and Desoer (1978, 1980 a, b), Nett (1984) and Pandolfi (1982, 1983).

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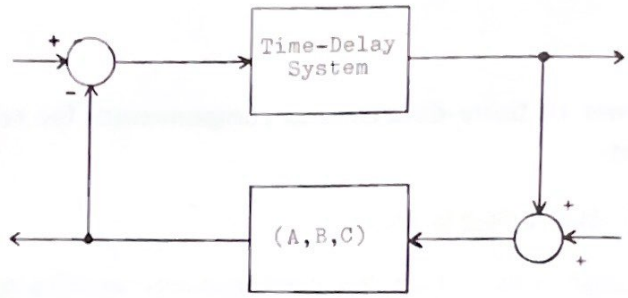

Figure 1.

So far we have not mentioned the functional analytic semigroup approach to the stabilization of infinite-dimensional systems (see Curtain (1983) for a survey). In this context there have been several articles in the literature dealing with the problem of finite-dimensional compensators for infinite-dimensional systems, see for example Curtain and Salamon (1984), Jacobson (1984) and Schumacher (1983) to name but a few. These papers are concerned with the following class of systems:

$$\left.\begin{aligned} \dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}x(t) \end{aligned}\right\} \qquad (1.1)$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are linear operators. Furthermore $\mathbf{A}$ is assumed to be the infinitesimal generator of a strongly continuous semigroup on the state space (a Hilbert space or a Banach space). A large class of neutral systems can be interpreted as an abstract evolution system of the form (1.1). However, it seems that in the context of the semigroup approach to distributed systems the problem of finite-dimensional stabilization of time delay systems with *simultaneous input and output delays* has not yet been solved (cf. Curtain and Salamon (1984), Remark 3.7)

As far as the present paper is concerned, the article of Jacobson (1984) is very interesting. He has shown for the case of bounded operators $\mathbf{B}$ and $\mathbf{C}$ that stabilizability and detectability of (1.1) are necessary and sufficient for the existence of a finite-dimensional compensator. It is notable that this is done using both semigroup methods and frequency-domain methods for distributed systems (Callier and Desoer (1978, 1980 a, b), Nett (1984)). His paper is close in spirit to the present one.

The paper is organized as follows: Section 2 is devoted to preliminaries concerning the system and the feedback configuration under consideration. In § 3, it is shown that under certain conditions the exponential stability and the input/output stability of the closed-loop system are equivalent concepts. A necessary and sufficient condition for the existence of a finite-dimensional compensator (achieving exponential stability) is presented in § 4. In § 5 we collect some facts concerning the algebra of transfer functions introduced by Callier and Desoer (1978, 1980 a). Furthermore, we show that the transfer function of the neutral system under consideration is an element of this algebra.

We use the following notation:

$\mathbb{R}$ denotes the field of real numbers, and $\mathbb{C}$ the field of complex numbers.

Let σ be a real number, then

$$\mathbb{C}_\sigma := \{s \in \mathbb{C}: \quad \mathrm{Re}(s) \geqslant \sigma\}.$$

Let R be a ring, then $R^{m \times n}$ denotes the set of (m, n) matrices over R.

2. System definition and stability

Consider the neutral system $(\mathcal{N})$:

$$\frac{d}{dt}\left(\int_{-r}^{0} dL(\tau)\, x(t+\tau)\right) = \int_{-r}^{0} dA(\tau)\, x(t+\tau) + \int_{-r}^{0} dB(\tau)\, u(t+\tau) \tag{2.1 a}$$

$$y(t) = \int_{-r}^{0} dC(\tau)\, x(t+\tau) + \int_{-r}^{0} dD(\tau)\, u(t+\tau), \quad t \geq 0 \tag{2.1 b}$$

where $r > 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^q$, $y(t) \in \mathbb{R}^p$. A, B, C, D and L are functions of bounded variation on the interval $[-r, 0]$ with values in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times q}$, $\mathbb{R}^{p\times n}$, $\mathbb{R}^{p\times q}$ and $\mathbb{R}^{n\times n}$, respectively. Without loss of generality we can assume that the functions A, B, C, D and L are normalized, i.e. they are left-continuous on $(-r, 0)$ and vanish at 0. We call the system $(\mathcal{N})$ retarded if $L = \theta I$, where θ is defined by

$$\theta(x) := \begin{cases} -1, & x < 0 \\ 0, & x \geq 0 \end{cases}$$

In order to get a reasonable theory we have to impose some additional conditions on L (see Hale 1977):

$$L = \theta I - \tilde{L}$$

where $\tilde{L}$ is a function of bounded variation on $[-r, 0]$ with values in $\mathbb{R}^{n\times n}$ which is continuous at 0.

It is well known (see Hale 1977) that eqn. (2.1 *a*) has a unique solution if the function u is continuous on $[-r, \infty)$, and if we specify that the function x is equal to a given continuous initial function φ on the interval $[-r, 0]$. (Of course the continuity assumptions could be relaxed, but we do not need more than this.)

Let $S(t)$ denote the strongly continuous solution semigroup of the corresponding homogeneous problem

$$\frac{d}{dt}\left(\int_{-r}^{0} dL(\tau)\, x(t+\tau)\right) = \int_{-r}^{0} dA(\tau)\, x(t+\tau), \quad \tau \geq 0$$

Definition 2.1

The system $(\mathcal{N})$ is called exponentially stable if there exist $\varepsilon > 0$ and $M > 0$ such that

$$\|S(t)\| \leq M \exp(-\varepsilon t) \quad \forall t \geq 0$$

We shall need the following assumption on the function L:

(A1) The function L of bounded variation contains no singular part (see for example Kolmogorov and Fomin (1975)).

As far as applications are concerned, the restriction of generality induced by (A1) is not very serious.

Theorem 2.2

Assume that (A1) holds. The neutral system $(\mathcal{N})$ is exponentially stable if and only if there exists $\alpha < 0$ such that

$$\det\left(s \int_{-r}^{0} \exp(s\tau)\, dL(\tau) - \int_{-r}^{0} \exp(s\tau)\, dA(\tau)\right) \neq 0 \quad \forall s \in \mathbb{C}_{\alpha}.$$

In the retarded case (i.e. $L = \theta I$) the negative constant α may be replaced by 0.

Proof

It has been proved by Henry (1974) (cf. also Hale 1977) that, under assumption (A1), the exponential growth of the semigroup $S(t)$ is determined by the spectrum of its infinitesimal generator H. More precisely,

$$\begin{aligned}\omega_0 &= \inf\{\omega: \exists\, M_\omega \text{ such that } \|S(t)\| \leqslant M_\omega \exp(\omega t)\} \\ &= \lim_{t\to\infty} t^{-1} \ln \|S(t)\| \\ &= \sup\{\operatorname{Re}(z): z \in \operatorname{spec}(H)\}.\end{aligned}$$

The conclusion follows from the well-known fact that spec(H) coincides with the zeros of

$$d(s) := \det\left(s \int_{-r}^{0} \exp(s\tau)\, dL(\tau) - \int_{-r}^{0} \exp(s\tau)\, dA(\tau)\right)$$

(see Hale 1977). In the retarded case the characteristic function $d(s)$ has at most finitely many zeros in every right halfplane. Therefore the condition

$$d(s) \neq 0 \quad \forall\, s \in \mathbb{C}_0$$

implies the existence of $\alpha < 0$ such that $d(s) \neq 0\ \forall\, s \in \mathbb{C}_\alpha$. □

In order to introduce the transfer matrix of the system $(\mathcal{N})$ we define

$$A(s) := \int_{-r}^{0} \exp(s\tau)\, dA(\tau), \quad B(s) := \int_{-r}^{0} \exp(s\tau)\, dB(\tau), \quad C((s) := \int_{-r}^{0} \exp(s\tau)\, dC(\tau)$$

$$D(s) := \int_{-r}^{0} \exp(s\tau)\, dD(\tau), \quad L(s) := \int_{-r}^{0} \exp(s\tau)\, dL(\tau), \quad \Delta(s) := sL(s) - A(s)$$

Observe that, for example, $A(\tau)$ and $A(s)$ have different meanings. $A(s)$ is a complex matrix, defined in terms of the real matrix $A(\tau)$. We shall consistently use s to denote complex variables, so that no confusion will arise. The entries of $A(s)$, $B(s)$, $C(s)$, $D(s)$ and $L(s)$ are entire functions which are bounded on every right half-plane.

If $x(t) = 0$, $u(t) = 0$ for all $t \leqslant 0$, then by Laplace transformation we obtain the following expression for the transfer matrix G of the system $(\mathcal{N})$:

$$G(s) = C(s)\Delta^{-1}(s)B(s) + D(s)$$

G is a meromorphic matrix on the whole complex plane, i.e. the entries of G are meromorphic functions on $\mathbb{C}$.

Consider the finite-dimensional system

$$(\mathcal{F}) \qquad \begin{aligned} \dot z(t) &= A_F z(t) + B_F v(t) \\ w(t) &= C_F z(t) \end{aligned}$$

where $z(t) \in \mathbb{R}^m$, $v(t) \in \mathbb{R}^p$, $w(t) \in \mathbb{R}^q$ and A_F, B_F and C_F are real matrices of appropriate dimensions.

The transfer matrix of the system $(\mathcal{F})$ is given by

$$F(s) := C_F \Delta_F^{-1}(s) B_F$$

where $\Delta_F(s) := sI - A_F$.

In what follows we shall be concerned with the feedback configuration shown in Fig. 2, where e and f denote external inputs. The closed-loop equations are given by

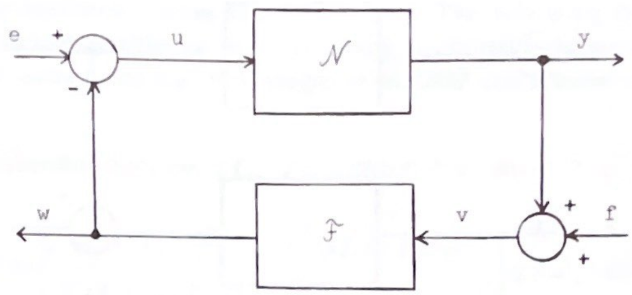

Figure 2.

$$\frac{d}{dt}\left(\int_{-r}^{0} dL_c(\tau)\,x_c(t+\tau)\right) = \int_{-r}^{0} dA_c(\tau)\,x_c(t+\tau) + \int_{-r}^{0} dB_c(\tau)\,u_c(t+\tau)$$

$$y_c(t) = \int_{-r}^{0} dC_c(\tau)\,x_c(t+\tau) + \int_{-r}^{0} dD_c(\tau)\,u_c(t+\tau)$$

where

$$x_c(t) := \begin{bmatrix} x(t) \\ z(t) \end{bmatrix},\quad u_c(t) := \begin{bmatrix} e(t) \\ f(t) \end{bmatrix},\quad y_c(t) := \begin{bmatrix} y(t) \\ w(t) \end{bmatrix}$$

$$A_c(\tau) := \begin{bmatrix} A(\tau) & -B(\tau)C_F \\ B_F C(\tau) & A_F\theta(\tau) - B_F D(\tau) C_F \end{bmatrix}$$

$$B_c(\tau) := \begin{bmatrix} B(\tau) & 0 \\ B_F D(\tau) & B_F\theta(\tau) \end{bmatrix}$$

$$C_c(\tau) := \begin{bmatrix} C(\tau) & -D(\tau)C_F \\ 0 & C_F\theta(\tau) \end{bmatrix}$$

$$D_c(\tau) := \begin{bmatrix} D(\tau) & 0 \\ 0 & 0 \end{bmatrix}$$

$$L_c(\tau) := \begin{bmatrix} L(\tau) & 0 \\ 0 & I\theta(\tau) \end{bmatrix}$$

Moreover we define

$$\Delta_c(s) := s\int_{-r}^{0} \exp(s\tau)\, dL_c(\tau) - \int_{-r}^{0} \exp(s\tau)\, dA_c(\tau)$$

The closed-loop system (denoted by $(\Sigma) = (\mathcal{N}, \mathcal{F})$) is a neutral system which satisfies the assumption (A1).

3. Input/output description

We shall need two further assumptions on $(\mathcal{N})$:

(A2) The singular parts (see Kolmogorov and Fomin 1975) of the functions A, B, C and D vanish identically.

(A3) $\inf_{s\in\mathbb{C}_0} |\det(L(s))| > 0$.

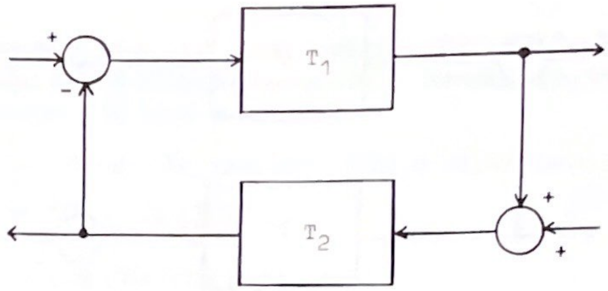

Figure 3.

Remark 3.1

(*a*) As far as applications are concerned the restriction of generality induced by (A2) is not very serious.

(*b*) Since $\det(L(s))$ is holomorphic and bounded on every right half-plane $\mathbb{C}_\sigma$, it follows by Corduneanu (1968, p. 72) that $\det(L(s))$ is uniformly continuous in every vertical strip $\{s \in \mathbb{C} : \sigma_1 \leqslant \operatorname{Re}(s) \leqslant \sigma_2\}$. Hence if assumption (A3) is satisfied there even exists $\alpha < 0$ such that

$$\inf_{s \in \mathbb{C}_\alpha} |\det(L(s))| > 0$$

Therefore the assumption (A3) is the same as the corresponding one in Emre and Knowles (1984). Furthermore it is easy to see that (A3) is weaker than the stability hypothesis on $L(s)$ used in Byrnes *et al.* (1984) and Kamen *et al.* (1984) (formal stability).

(*c*) The closed-loop system $(\mathscr{N}, \mathscr{F})$ is again a neutral system satisfying (A1)–(A3).

Under the additional assumption (A3) it is possible to simplify the characterization of exponential stability given in Theorem 2.2.

Corollary 3.2

Assume that (A1) and (A3) hold. The system $(\mathscr{N})$ is exponentially stable if and only if

$$\det(sL(s) - A(s)) \neq 0 \quad \forall s \in \mathbb{C}_0$$

Proof

By Theorem 2.2 it is sufficient so show the existence of a constant $\alpha < 0$ such that $\det(\Delta(s))$ has at most finitely many zeros in $\mathbb{C}_\alpha$. $\det(\Delta)$ can be written in the form

$$\det(\Delta(s)) = \det(L(s)) \det(sI - L^{-1}(s)A(s))$$

By assumption (A1) and Remark 3.1 (*b*) there exists $\alpha < 0$ such that $\inf_{s \in \mathbb{C}_\alpha} |\det(L(s))| > 0$. Therefore the entries of $L^{-1}(s)A(s)$ are holomorphic and bounded on $\mathbb{C}_\alpha$. We can use the same reasoning as in Driver (1977, p. 322) in order to show that $\det(sI - L^{-1}(s)A(s))$ has at most finitely many zeros in $\mathbb{C}_\alpha$. □

In what follows we shall deal with the sets of transfer functions $\hat{\mathscr{A}}(\sigma)$, $\hat{\mathscr{A}}_-(\sigma)$, $\hat{\mathscr{A}}_-^\infty(\sigma)$ and $\hat{\mathscr{B}}(\sigma)$ which were introduced by Callier and Desoer (1978, 1980 a, b) (see § 5 of the present paper). Under the assumptions (A1)–(A3), it is not difficult to show that there exists $\alpha < 0$ such that for all $\sigma \geqslant \alpha$ the transfer matrix G of $(\mathscr{N})$ is an element of $(\hat{\mathscr{B}}(\sigma))^{p \times q}$ (see § 5).

Consider the feedback system shown in Fig. 3. The following notion of input/output stability is well established in the so-called fractional-representation theory of feedback system design (see e.g. Vidyasagar *et al.* 1982 and Desoer *et al.* 1980).

Definition 3.3

A pair of transfer matrices $(T_1, T_2) \in \hat{\mathscr{B}}(\sigma)^{m\times n} \times \hat{\mathscr{B}}(\sigma)^{n\times m}$ is called α-stable $(\alpha \geqslant \sigma)$ if

$$H(T_1, T_2) := \begin{bmatrix} (I + T_2T_1)^{-1} & -T_2(I + T_1T_2)^{-1} \\ T_1(I + T_2T_1)^{-1} & (I + T_1T_2)^{-1} \end{bmatrix} \in (\hat{\mathscr{A}}_-(\alpha))^{(m+n)\times(m+n)}$$

In the case $\alpha = 0$ we shall simply say that (T_1, T_2) is stable.

Lemma 3.4

Let $(T_1, T_2) \in \hat{\mathscr{B}}(\sigma)^{m\times n} \times \hat{\mathscr{B}}(\sigma)^{n\times m}$, $T_1 = N_1D_1^{-1}$ be a right-coprime factorization and $T_2 = D_2^{-1}N_2$ be a left-coprime factorization (see § 5, Theorem 5.4). The pair (T_1, T_2) is α-stable if and only if $D_2D_1 + N_2N_1$ is unimodular in $\hat{\mathscr{A}}_-(\alpha)^{n\times n}$, i.e. $\inf_{s\in\mathbb{C}_\alpha} |\det((D_2D_1 + N_2N_1)(s))| > 0$.

The proof follows immediately from Vidyasagar *et al.* (1982).

The main purpose of this section is to relate the exponential stability of $(\Sigma) = (\mathscr{N}, \mathscr{F})$ and the stability of (G, F). An important fact which we need is the following

Lemma 3.5

Suppose that the assumptions (A1)–(A3) are satisfied. For appropriate $\sigma \in \mathbb{R}$ consider G and F as elements of $\hat{\mathscr{B}}(\sigma)^{p\times q}$ and $\hat{\mathscr{B}}(\sigma)^{q\times p}$ respectively. Let $G = N_GD_G^{-1}$ be a right-coprime factorization and let $F = D_F^{-1}N_F$ be a left-coprime factorization. Then the following equation holds:

$$\det(\Delta_c) = \frac{\det(\Delta)\ \det(\Delta_F)}{\det(D_G)\ \det(D_F)} \det(D_FD_G + N_FN_G) \tag{3.1}$$

Proof

Using the same calculations as in the finite-dimensional case (see Hsu and Chen 1968) we obtain

$$\det(\Delta_c) = \det(\Delta)\det(\Delta_F)\det(I + FG) \tag{3.2}$$

Now by substituting the coprime factorizations of G and F into eqn. (3.2) we obtain eqn. (3.1). □

In order to show that, under certain conditions, the two stability concepts are equivalent, the following notion is useful (cf. Pandolfi 1982, 1983).

Definition 3.6

Let Ω be a subset of $\mathbb{C}$. The system $(\mathscr{N})$ is said to be canonical in Ω if

(i) $\operatorname{rank}(sL(s) - A(s), B(s)) = n \quad \forall s \in \Omega$

(ii) $\operatorname{rank}\begin{bmatrix} sL(s) - A(s) \\ C(s) \end{bmatrix} = n \quad \forall s \in \Omega$

The conditions (i) and (ii) are sometimes called generalized Hautus conditions (cf. Hautus 1969, 1970).

Remark 3.7

(*a*) In the finite-dimensional case (i.e. $L(s) \equiv I$, $A(s)$, $B(s)$, $C(s)$ and $D(s)$ are constant matrices) the following equivalences are valid:
Conditions (i) holds for all $s \in \mathbb{C}$ ($\mathbb{C}_0$) if and only if the system is controllable (stabilizable). Condition (ii) holds for all $s \in \mathbb{C}$ ($\mathbb{C}_0$) if and only if the system is observable (detectable) (see Hautus 1969, 1970).

(*b*) As far as the general (infinite-dimensional) situation is concerned, see Salamon (1984) for the interpretation of the conditions (i) and (ii) in terms of controllability and observability.

Let $\Omega \subset \mathbb{C}$ be open. For a holomorphic function f on Ω and $z \in \Omega$ we define

$$Z(f, z) := \begin{cases} 0, & \text{if } f(z) \neq 0 \\ \text{multiplicity of } z & \text{if } f(z) = 0 \end{cases}$$

Lemma 3.8

Assume that (A1)–(A3) hold. Consider the transfer matrix G of the neutral system $(\mathcal{N})$ as an element of $\hat{\mathscr{B}}(\sigma)$. Let $G = ND^{-1}$ be a right coprime factorization. Then the following are equivalent:

(i) $(\mathcal{N})$ is canonical in $\mathbb{C}_\sigma$

(ii) $Z(\det(D), s) = Z(\det(\Delta), s) \quad \forall s \in \mathbb{C}_\sigma$

A similar equivalence is valid if we use a left-coprime factorization instead of a right-coprime factorization.

Proof

Pandolfi (1982, 1983) defines the poles of a meromorphic matrix via a local Smith–MacMillan form (which is essentially due to Kappel and Wimmer (1976)). It is not difficult to show that the zeros of $\det(D)$ in $\mathbb{C}_\sigma$ coincides with the set of Smith–MacMillan poles of G in $\mathbb{C}_\sigma$. It now follows from Pandolfi (1983) (for the retarded case cf. also Pandolfi (1982)) that

$$Z(\det(D), s) \leqslant Z(\det(\Delta), s) \quad \forall s \in \mathbb{C}_\sigma$$

Moreover equality holds for all $s \in \mathbb{C}_\sigma$ if and only if $(\mathcal{N})$ is canonical in $\mathbb{C}_\sigma$. □

The following theorem shows the interrelation between exponential stability of $(\mathcal{N}, \mathcal{F})$ and stability of (G, F).

Theorem 3.9

Under the assumptions (A1)–(A3) the following are true.
The system $(\mathcal{N}, \mathcal{F})$ is exponential stable if and only if

(i) (G, F) is stable.

(ii) The systems $(\mathcal{N})$ and $(\mathcal{F})$ are canonical in $\mathbb{C}_0$.

Proof

if: it follows from eqn. (3.1), Lemma 3.4 and Lemma 3.8 that $\det(\Delta_c)$ has no zeros in $\mathbb{C}_0$. Therefore the closed-loop system is exponential stable by Remark 3.1 *c* and Corollary 3.2.

only if: by Theorem 2.2 we have

$$\sup\{\text{Re}(s) : \det(\Delta_c(s)) = 0\} < 0$$

Since all three factors on the right-hand side of equation (3.1) are holomorphic on $\mathbb{C}_0$ (cf. the proof of Lemma 3.8) we obtain

$$(*)\quad \frac{\det(\Delta)}{\det(D_G)},\ \frac{\det(\Delta_F)}{\det(D_F)}\quad \text{have no zeros inside } \mathbb{C}_0.$$

$$(**)\ \det(D_F D_G + N_F N_G)\quad \text{has no zeros inside } \mathbb{C}_0.$$

With $(*)$ it follows from Lemma 3.8 that $(\mathcal{N})$ and $(\mathcal{F})$ are canonical in $\mathbb{C}_0$.

It remains to show that the pair (G, F) is stable. Realize that $\det(D_F D_G) \in \hat{\mathcal{A}}_-^\infty(0)$ and $\lim\limits_{\substack{|s|\to\infty \\ s\to\in\mathbb{C}_0}} (N_F N_G)(s) = 0$.

Therefore there exists $R > 0$ such that

$$\inf_{\substack{|s|>R \\ s\in\mathbb{C}_0}} |\det((D_F D_G + N_F N_G)(s))| > 0$$

Since $\det(D_F D_G + N_F N_G)$ has no finite zeros in $\mathbb{C}_0$ (see $(**)$), we conclude that

$$\inf_{s\in\mathbb{C}_0} |\det(D_F D_G + N_F N_G)(s)| > 0$$

Now Lemma 3.4 yields that (G, F) is stable. □

4. Existence of finite-dimensional compensators

Nett (1984) proved that for every transfer matrix $T \in \hat{\mathcal{B}}(\sigma)^{m\times n}$ there exists a finite-dimensional compensator. More precisely, we can state the following result.

Theorem 4.1

Let $T \in \hat{\mathcal{B}}(\sigma)^{m\times n}$, then there exists a strictly proper rational matrix R such that the pair (T, R) is σ-stable.

For the proof see Nett (1984) or Logemann (1984).

It should be mentioned that the state-space results presented by Curtain and Salamon (1984) and Schumacher (1983) for example are more constructive than the frequency-domain result Theorem 4.1. Although the proof of Theorem 4.1 is constructive in nature, there still remains much to be done in order to develop implementable algorithms.

Now we can state our main result.

Theorem 4.2

Under the assumptions (A1)–(A3) the following statements are equivalent:

(i) There exists a finite-dimensional compensator for $(\mathcal{N})$, i.e. there exists a finite-dimensional system $(\mathcal{F}) = (A_F, B_F, C_F)$ such that the closed-loop system $(\mathcal{N}, \mathcal{F})$ is exponential stable.

(ii) The system $(\mathcal{N})$ is canonical in $\mathbb{C}_0$.

Proof

(i) ⇒ (ii) follows from eqn. (3.1).

(ii) ⇒ (i) It is shown in Lemma 5.5 that $G \in \hat{\mathcal{B}}(0)^{p\times q}$. By Theorem 4.1 there exists a strictly proper matrix R such that (G, R) is stable. Now choose a minimal realization

$(\mathscr{F}) = (A_F, B_F, C_F)$ of R. (This means, in particular, that $(\mathscr{F})$ is canonical in $\mathbb{C}_0$.) It follows from Theorem 3.9 that the closed-loop system $(\mathscr{N}, \mathscr{F})$ is exponential stable. □

Remark 4.3

It can be shown that condition (ii) in Theorem 4.2 is also a necessary condition for the stabilization of $(\mathscr{N})$ by an infinite-dimensional neutral compensator. Therefore we conclude from Theorem 4.2 that we cannot achieve stability by using neutral compensators if it is impossible to stabilize the system $(\mathscr{N})$ by a finite-dimensional compensator.

There is a graphical criterion checking condition (ii) of Theorem 4.2.

Lemma 4.4

Suppose that (A1)–(A3) are satisfied. Choose a right-coprime factorization $G = ND^{-1}$ with D rational (see Theorem 5.4). Then condition (ii) of Theorem 4.2 is satisfied if and only if

$$\text{(a)} \quad \frac{\det(i\omega I - L^{-1}(i\omega)A(i\omega))}{(i\omega + 1)^n \det(D(i\omega))} \neq 0 \quad \forall \omega \in \mathbb{R}$$

$$\text{(b)} \quad \frac{\det(i\omega I - L^{-1}(i\omega)A(i\omega))}{(i\omega + 1)^n \det(D(i\omega))}$$

does not encircle the origin if ω tends from $-\infty$ to $+\infty$.

Proof It follows from Lemma 3.8 that the neutral system $(\mathscr{N})$ is canonical in $\mathbb{C}_0$ if and only if

$$(*) \quad \psi(s) := \frac{\det(sI - L^{-1}(s)A(s))}{(s + 1)^n \det(D(s))} \text{ has no zeros in } \mathbb{C}_0$$

Note that

$$\lim_{\substack{|s| \to \infty \\ s \in \mathbb{C}_0}} \psi(s) = \frac{1}{\det(D(\infty))} \neq 0$$

Therefore it follows by the argument principle that $(*)$ holds if and only if the conditions (a) and (b) are satisfied. □

5. Appendix

In this section we collect some definitions and facts concerning the frequency-domain description of distributed-parameter systems developed by Callier and Desoer (1978, 1980 a, b). Furthermore we show that the transfer matrix of the neutral system $(\mathscr{N})$ fits into this framework.

Definition 5.1

(i) For $\sigma \in \mathbb{R}$ we define $\mathscr{A}(\sigma)$ to be the set of distributions of the form

$$f = \sum_{i=0}^{\infty} f_i \delta_{t_i} + f_a$$

where $t_0 := 0$, $t_i > 0 \ \forall i \geq 1$, δ_{t_i} is the Dirac distribution at the point t_i, $(f_i)_{i \geq 0}$ is a real sequence and $f_a: [0, \infty) \to \mathbb{R}$ is a measurable function such that

$$\sum_{i=0}^{\infty} |f_i| \exp(-\sigma t_i) + \int_0^{\infty} |f_a(t)| \exp(-\sigma t) \, dt < \infty$$

(ii) $\mathscr{A}_-(\sigma) := \{f \in \mathscr{A}(\sigma) : \exists \sigma' < \sigma \text{ such that } f \in \mathscr{A}(\sigma')\}$

$\hat{\mathscr{A}}_-(\sigma) := \{\hat{f} : f \in \mathscr{A}_-(\sigma)\}$, where $\hat{f}$ denotes the Laplace transform of f.

$$\hat{\mathscr{A}}_-^{\infty}(\sigma) := \{\hat{f} \in \hat{\mathscr{A}}_-(\sigma) : \exists R > 0 \text{ such that } \inf_{\substack{s \in \mathbb{C}_\sigma \\ |s| \geq R}} |f(s)| > 0\}$$

Remark 5.2

$\mathscr{A}(\sigma)$ and $\mathscr{A}_-(\sigma)$ are convolution algebras of distributions. $\hat{\mathscr{A}}_-(\sigma)$ is an algebra of holomorphic functions which are defined on some right half-plane $\mathbb{C}_{\sigma'}$, $\sigma' < \sigma$. $\hat{\mathscr{A}}_-^{\infty}(\sigma)$ is a multiplicative subset of $\hat{\mathscr{A}}_-(\sigma)$.

Lemma 5.3

$g \in \hat{\mathscr{A}}_-(\sigma)$ is a unit of $\hat{\mathscr{A}}_-(\sigma)$ if and only if $\inf_{s \in \mathbb{C}_\sigma} |g(s)| > 0$.

$\hat{\mathscr{B}}(\sigma) := (\hat{\mathscr{A}}_-(\sigma))(\hat{\mathscr{A}}_-^{\infty}(\sigma))^{-1}$ denotes the commutative algebra of fractions $g = nd^{-1}$, where $n \in \hat{\mathscr{A}}_-(\sigma)$ and $d \in \hat{\mathscr{A}}_-^{\infty}(\sigma)$. In particular, $\hat{\mathscr{B}}(\sigma)$ contains the set of proper rational functions.

It is an important property of $\hat{\mathscr{B}}(\sigma)$ that every matrix in $\hat{\mathscr{B}}(\sigma)^{m \times n}$ admits right- and left-coprime factorizations. This can be stated more precisely as follows.

Theorem 5.4

For $T \in \hat{\mathscr{B}}(\sigma)^{m \times n}$ there exists a right-coprime factorization, i.e. there exist $N \in \hat{\mathscr{A}}_-(\sigma)^{m \times n}$ and $D \in \hat{\mathscr{A}}_-(\sigma)^{n \times n}$ such that

(i) $\det(D) \in \hat{\mathscr{A}}_-^{\infty}(\sigma)$
(ii) $T = ND^{-1}$
(iii) N and D are right-coprime, i.e. there exist matrices $U \in \hat{\mathscr{A}}_-(\sigma)^{n \times m}$ and $V \in \hat{\mathscr{A}}_-(\sigma)^{n \times n}$ such that

$$UN + VD = I$$

Moreover, it is possible to choose D rational. A similar statement is valid for left-coprime factorizations.

Lemma 5.5

Under the assumptions (A1)–(A3), the transfer matrix G of the system $(\mathscr{N})$ is an element of $\hat{\mathscr{B}}(\sigma)^{p \times q}$ for all $\sigma \geq \alpha$, for some $\alpha < 0$.

Proof

Step 1

We show that the entries of $A(s)$, $B(s)$, $C(s)$, $D(s)$ and $L(s)$ are elements of $\hat{\mathscr{A}}_-(\sigma)$ for all $\sigma \in \mathbb{R}$.

Let $f: [-r, 0] \to \mathbb{R}$ be a normalized function of bounded variation containing no singular part. This means that $f(\tau)$ can be written in the form

$$f(\tau) = -\sum_{i=0}^{\infty} f_i \chi_{(-\infty,-r_i]}(\tau) - \int_{\tau}^{0} f_\infty(t)\,dt, \quad -r \leqslant \tau \leqslant 0 \tag{5.1 a}$$

where $0 < r_i \leqslant r$, $\chi_{(-\infty,-r_i]}$ denotes the indicator function of the interval $(-\infty, -r_i]$, $(f_i)_{i\geqslant 0}$ and f_∞ satisfy

$$\sum_{i=0}^{\infty} |f_i| + \int_{-r}^{0} |f_\infty(t)|\,dt < \infty \tag{5.1 b}$$

By assumptions (A1) and (A2) every entry of $A(\tau)$, $B(\tau)$, $C(\tau)$, $D(\tau)$ and $L(\tau)$ can be written in the form (5.1).

An easy calculation yields for all $s \in \mathbb{C}$

$$\int_{-r}^{0} \exp(s\tau)\,df(\tau) = \sum_{i=0}^{\infty} f_i \exp(-sr_i) + \int_{0}^{\infty} f_a(t)\exp(-st)\,dt$$

where the function f_a is defined by

$$f_a(t) := \begin{cases} f_\infty(-t), & 0 \leqslant t \leqslant r \\ 0, & t > r \end{cases}$$

Therefore we have

$$\int_{-r}^{0} \exp(s\tau)\,df(\tau) \in \hat{\mathscr{A}}_-(\sigma)$$

for all $\sigma \in \mathbb{R}$.

Step 2

It remains to show that there exists $\alpha < 0$ such that $(sL(s) - A(s))^{-1}$ is an element of $\hat{\mathscr{B}}(\sigma)^{n\times n}$ for all $\sigma \geqslant \alpha$. By Remark 3.1 *b* there exists $\alpha < 0$ such that $\inf_{s\in\mathbb{C}_\alpha} |\det(L(s))| > 0$. By Lemma 5.3 we have $L^{-1}(s) \in \hat{\mathscr{A}}_-(\alpha)$. Hence the claim follows if we show that

$$(sI - L^{-1}(s)A(s))^{-1} = (sL(s) - A(s))^{-1}L(s)$$

is an element of $\hat{\mathscr{B}}(\alpha)^{n\times n}$.

Let $\operatorname{Adj}(X(s))$ denote the adjoint of $X(s) := sI - L^{-1}(s)A(s)$, then by Cramer's rule

$$X^{-1}(s) = \frac{\operatorname{Adj}(X(s))}{(s+\beta)^n}\left(\frac{\det(X(s))}{(s+\beta)^n}\right)^{-1} \tag{5.2}$$

where we choose $\beta > |\alpha|$.

$\det(X(s))$ can be written in the form

$$\det(X(s)) = s^n + c_{n-1}(s)s^{n-1} + \cdots + c_0(s)$$

where the c_i belong to the subalgebra of $\hat{\mathscr{A}}_-(\alpha)$ generated by the entries of $L^{-1}(s)A(s)$. Therefore we have

$$\frac{\det(X(s))}{(s+\beta)^n} \in \hat{\mathscr{A}}_-^\infty(\alpha)$$

Finally we note that

$$\frac{\text{Adj}\,(X(s))}{(s+\beta)^n} \in \hat{\mathscr{A}}_{-}(\alpha)^{n\times n}$$

It follows from eqn. (5.2) that $X^{-1} \in \hat{\mathscr{B}}(\alpha)^{n\times n}$.

References

Byrnes, C. I., Spong, M. W., and Tarn, T. J., 1984, *Math. Syst. Theory*, **17**, 97.

Callier, F. M., and Desoer, C. A., 1978, *I.E.E.E. Trans. Circuits Syst.*, **25**, 651 (correction **26**, 360); 1980 a, *Ibid.*, **27**, 320; 1980 b, *Annls Soc. scient. Brux.*, **94**, 7.

Corduneanu, C., 1968, *Almost Periodic Functions* (New York: Wiley).

Curtain, R. F., 1983, *J. Franklin Inst.*, **315**, 331.

Curtain, R. F., and Salamon, D., 1984, Finite dimensional compensators for infinite dimensional systems with unbounded control action. Technical report 2684, Math. Research Center, University of Wisconsin-Madison.

Desoer, C. A., Liu, R. W., Murray, J., and Saeks, R., 1980, *I.E.E.E. Trans. autom. Control*, **25**, 399.

Driver, R. D., 1977, *Ordinary and Delay Differential Equations* (New York: Springer-Verlag).

Emre, E. and Knowles, G. J., 1984, *I.E.E.E. Trans. autom. Control*, **29**, 1083.

Hale, J. K., 1977, *Theory of Functional Differential Equations* (New York: Springer-Verlag).

Hautus, M. L. J., 1969, *Proc. Ned. Akad. Wetenschappen, ser.* A, **72**, 443; 1970, *Ibid.*, **73**, 448.

Henry, D., 1974, *J. diff. Eqns*, **15**, 106.

Hsu, C. -H., and Chen, C. -T., 1968, *Proc. Inst. Elect. electron. Engrs*, **56**, 2061.

Jacobson, C. A., 1984, Necessary and sufficient conditions for output feedback stabilization of a class of linear distributed systems. Preprint, ECSE Dept. Rensselaer Poly. Inst., Troy, NY.

Kamen, E. W., Khargonekar, P. P., and Tannenbaum, A., 1984, *Acta Appl. Math.*, **2**, 159; 1985, *I.E.E.E. Trans. autom. Control*, **30**, 75.

Kappel F., and Wimmer, H. K., 1976, *J. diff. Eqns*, **21**, 134.

Kolmogorov, A. N., and Fomin, S. V., 1975, *Introductory Real Analysis* (New York: Dover).

Logemann, H., 1984, Finite dimensional stabilization of infinite dimensional systems: a frequency domain approach. Report Nr. 124, Forschungsschwerpunkt Dynamische Systeme, Universität Bremen.

Nett, C. N., 1984, The fractional representation approach to robust linear feedback design: a self-contained exposition. M. Sc. Thesis, ESCE Dept., Rensselaer Poly. Inst., Troy, NY.

Pandolfi, L., 1982, *Int. J. Control*, **36**, 959; 1983, in *Control Theory for Distributed Parameter Systems and Applications*, ed. F. Kappel, K. Kunisch and W. Schappacher p. 163, (Berlin: Springer-Verlag).

Salamon, D., 1984, *Control and Observation of Neutral Systems* (London: Pitman).

Schumacher, J. M., 1983, *SIAM J. Control Optim.*, **21**, 823.

Vidyasagar, M., Schneider, H., and Francis, B. A., 1982, *I.E.E.E. Trans. autom. Control.*, **27**, 880.