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Transfer-Function Conditions for the Stability of Neutral and Volterra Integrodifferential Systems

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In this paper we study the feedback interconnection (a) of two neutral systems and (b) of two Volterra integrodifferential systems. We show that under very mild assumptions the closed-loop system pair is internally stable iff every transfer function which occurs around the loop is holomorphic in a right half plane containing $\{s : \operatorname{Re} s \geqslant 0\}$.

1. Introduction

CONSIDER the feedback interconnection of the transfer matrices G_1 and G_2 shown in Fig. 1. Following Desoer & Chan (1975) we consider every transfer function $u_i \mapsto y_j$ which occurs around the loop. The transfer matrix G_c of the closed-loop system relating $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is given by

$$G_c = \begin{bmatrix} G_1(I+G_2G_1)^{-1} & -G_1G_2(I+G_1G_2)^{-1} \\ G_2G_1(I+G_2G_1)^{-1} & G_2(I+G_1G_2)^{-1} \end{bmatrix}.$$

Suppose that G_i has a finite-dimensional stabilizable and detectable state-space realization (A_i, B_i, C_i, D_i) $(i = 1, 2)$. Then it is known that the composite state-space realization of the closed-loop system is exponentially stable iff G_c has no poles in the closed right half plane. This theorem has been a 'folk result' in the control community for several years. The proof can be found in Bhattacharyya & Howze (1985). Jacobson (1984) has shown that the same is true if G_i admits a stabilizable and detectable Hilbert-space realization (A_i, B_i, C_i, D_i), where B_i, C_i, D_i are bounded operators and A_i is the infinitesimal generator of a strongly

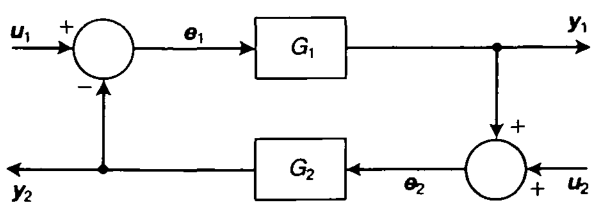

FIG 1.

continuous semigroup ($i = 1, 2$). Moreover A_i is required to satisfy certain conditions.

The purpose of the present paper is to prove similar results for two classes of infinite-dimensional systems which do not fit into the set-up of Jacobson (1984). More precisely, we show under very mild assumptions that:

(1) the feedback interconnection of two neutral systems is exponentially stable iff the transfer matrix G_c has no poles in $\{s : \operatorname{Re} s \geq \alpha\}$ for some $\alpha < 0$;

(2) the feedback interconnection of two Volterra integrodifferential systems is uniformly asymptotically stable iff G_c has no poles in $\operatorname{Re} s \geq 0$.

In Section 2 of this paper we introduce the notion of a holomorphic realization of a meromorphic matrix. Moreover, we study a certain holomorphic realization of the closed-loop matrix G_c. In Section 3 we use Theorem 2.5—the main result of Section 2—in order to prove the results which we have just mentioned. In Section 4 we give the proof of a technical lemma which we need in Section 3.

We use the following notation: $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{C}_\sigma := \{s \in \mathbb{C} : \operatorname{Re} s \geq \sigma\}$ for $\sigma \in \mathbb{R}$. Let $\Omega \subset \mathbb{C}$ be a region, then $\mathrm{H}(\Omega)$ is the ring of holomorphic functions on Ω and $\mathrm{M}(\Omega)$ is the field of meromorphic functions on Ω. Finally let R be a ring, then $R^{m \times n}$ is the set of $m \times n$-matrices over R. Other notation will be introduced as it is needed.

2. A holomorphic realization of the closed-loop transfer matrix

For the convenience of the reader we recall some facts concerning the algebraic structure of $\mathrm{H}(\Omega)$. The units of the integral domain $\mathrm{H}(\Omega)$ are exactly those functions in $\mathrm{H}(\Omega)$ which have no zeros in Ω. It is well known (Helmer's theorem) that $\mathrm{H}(\Omega)$ is a Bezout domain, i.e. every finitely generated ideal in $\mathrm{H}(\Omega)$ is principal (Helmer, 1940, or Narasimhan, 1985, p. 136). Thus, any finite set of functions in $\mathrm{H}(\Omega)$ has a greatest common divisor. Moreover, it is known (Wedderburn's theorem) that $\mathrm{H}(\Omega)$ forms a so called elementary divisor ring, i.e. every matrix of holomorphic functions admits a Smith normal form (Wedderburn, 1915, or Narasimhan, 1985, p. 141).

The following lemma can be proved in the same way as for polynomial matrices (Rosenbrock, 1970, p. 71).

LEMMA 2.1 *For $P \in \mathrm{H}(\Omega)^{m \times l}$ and $Q \in \mathrm{H}(\Omega)^{n \times l}$ the following statements are equivalent:*

(i) *P and Q are right coprime. (This means that every common right divisor $D \in \mathrm{H}(\Omega)^{l \times l}$ of P and Q is unimodular, i.e.* $\det D(s) \neq 0$ *for all $s \in \Omega$.)*

(ii) $\operatorname{rk} \begin{bmatrix} P(s) \\ Q(s) \end{bmatrix} = l \qquad (s \in \Omega)$.

(iii) *There exist matrices $X \in \mathrm{H}(\Omega)^{l \times m}$ and $Y \in \mathrm{H}(\Omega)^{l \times n}$ such that*

$$XP + YQ = I \quad \text{on} \quad \Omega.$$

An immediate consequence of the above lemma is stated in

COROLLARY 2.2 *Let* $a \in \Omega$, $P \in \mathrm{H}(\Omega)^{m\times l}$, *and* $Q \in \mathrm{H}(\Omega)^{n\times l}$. *The following are equivalent:*

(i) $\mathrm{rk}\begin{bmatrix} P(a) \\ Q(a) \end{bmatrix} = l.$

(ii) *There exists a neighbourhood* $U \subset \Omega$ *of* a *and matrices* $X \in \mathrm{H}(U)^{l\times m}$ *and* $Y \in \mathrm{H}(U)^{l\times n}$ *such that*

$$XP + YQ = I \quad \text{on} \quad U.$$

Completely analogous results can be stated for left coprimeness.

Since a Smith form exists for holomorphic matrices, a Smith–McMillan form exists for meromorphic matrices. This implies that every meromorphic matrix admits left and right coprime factorizations. These factorizations are unique up to unimodular factors.

We have to introduce some notation. Let $a \in \Omega$, $f \in \mathrm{H}(\Omega)$, and $M \in \mathrm{M}(\Omega)^{p\times m}$. Assume that $M = N_r D_r^{-1}$ and $M = D_\ell^{-1} N_\ell$ are right coprime and left coprime factorizations, respectively. We define

$$\mathrm{ord}_a f := \min\left\{n \in \mathbb{N}_0 : \frac{\mathrm{d}^n f}{\mathrm{d}z^n}(a) \neq 0\right\} \qquad (\mathbb{N}_0 := \{0, 1, 2, \ldots\}),$$

$$\begin{aligned} \mathrm{P}_a(M) &:= \mathrm{ord}_a \det D_r \\ &= \mathrm{ord}_a \det D_l. \end{aligned}$$

If $\mathrm{ord}_a f > 0$ then a is a zero of f and $\mathrm{ord}_a f$ is the order of the zero of f at a. Of course, if $\mathrm{P}_a(M) > 0$ we call a a pole of M and $\mathrm{P}_a(M)$ is called the order of the pole of M at a.

A (*holomorphic*) *realization of* $M \in \mathrm{M}(\Omega)^{p\times m}$ is a representation of the form

$$M = VT^{-1}U + W \tag{2.1}$$

where T, U, V, and W are holomorphic matrices on Ω of size $n \times n$, $n \times m$, $p \times n$, and $p \times m$, respectively, and $\det T \not\equiv 0$. The realization (2.1) is called *canonical in* $a \in \Omega$ if

$$\mathrm{rk}\,[T(a) \quad U(a)] = n, \qquad \mathrm{rk}\begin{bmatrix} T(a) \\ V(a) \end{bmatrix} = n. \tag{2.2}$$

The realization (2.1) is called *canonical in* $\Omega' \subset \Omega$ if the conditions (2.2) are satisfied for all $a \in \Omega'$. In particular, every right or left coprime factorization of M is a realization of M which is canonical in Ω.

Remark 2.3. For the polynomial/rational case (i.e. M in (2.1) is a rational matrix and T, U, V, and W in (2.1) are polynomial matrices) Coppel (1974) has developed a theory of realizations, which is not based on state-space techniques like the theory presented in Rosenbrock (1970). Because of the nice algebraic properties of $\mathrm{H}(\Omega)$ mentioned at the beginning of this section the theory of Coppel (1974) extends to the holomorphic/meromorphic case.

The next lemma follows from Pandolfi (1983) (cf. also Pandolfi (1982)).

LEMMA 2.4 *Assume that* $M \in \mathrm{M}(\Omega)^{p\times m}$ *admits the realization* $M = VT^{-1}U + W$. *Then for arbitrary* $a \in \Omega$

$$\mathrm{P}_a(M) \leqslant \mathrm{ord}_a \det T.$$

Moreover equality holds iff the realization $M = VT^{-1}U + W$ *is canonical in* a.

Consider the feedback configuration shown in Fig. 1. For the sake of simplicity we shall deal with realizations of the form $VT^{-1}U$ only, i.e. we assume that $W \equiv 0$ in (2.1).

THEOREM 2.5 *Let* $G_1 \in \mathrm{M}(\Omega)^{p\times m}$ *and* $G_2 \in \mathrm{M}(\Omega)^{m\times p}$ *and assume that the condition*

$$\det(I + G_2G_1) \not\equiv 0$$

is satisfied. Moreover suppose that G_1 *and* G_2 *admit the realizations*

$$G_i = V_iT_i^{-1}U_i \qquad (i = 1, 2). \tag{2.3}$$

Define:

$$T_c := \begin{bmatrix} T_1 & U_1V_2 \\ -U_2V_1 & T_2 \end{bmatrix}, \qquad V_c := \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}, \qquad U_c := \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}.$$

Under these conditions we have:

(a) $\det T_c = \det T_1 \det T_2 \det(I + G_2G_1)$ *and*

$$G_c = V_cT_c^{-1}U_c. \tag{2.4}$$

(b) *The realization* (2.4) *is canonical in* $a \in \Omega$ *iff the realizations* (2.3) *are canonical in* a.

(c) *For* $a \in \Omega$ *the following statements are equivalent:*
   (i) *The realizations* (2.4) *are canonical in* a.
   (ii) $\mathrm{P}_a(G_c) = \mathrm{ord}_a \det T_c$.

Proof. (a) The first equation is a generalization of a well-known formula of Hsu & Chen (1968) and can be proved in a completely analogous way.

In order to prove the second equation note that

$$T_c^{-1} = \begin{bmatrix} \Lambda_1 & -T_1^{-1}U_1V_2\Lambda_2 \\ T_2^{-1}U_2V_1\Lambda_1 & \Lambda_2 \end{bmatrix}$$

where

$$\Lambda_1 := (T_1 + U_1V_2T_2^{-1}U_2V_1)^{-1}, \qquad \Lambda_2 := (T_2 + U_2V_1T_1^{-1}U_1V_2)^{-1}$$

(Zurmühl & Falk (1984), p. 303).

(b) Assume that the realizations (2.3) are canonical in a. By Corollary 2.2 there exists a neighbourhood $A \subset \Omega$ of a and holomorphic matrices P_i, Q_i, R_i, S_i $(i = 1, 2)$ on A such that

$$P_iT_i + Q_iV_i \equiv I \quad \text{on } A, \qquad T_iR_i + U_iS_i \equiv I \quad \text{on } A.$$

Define

$$P := \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \quad Q := \begin{bmatrix} Q_1 & -P_1 U_1 \\ P_2 U_2 & Q_2 \end{bmatrix}, \quad R := \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}, \quad S := \begin{bmatrix} S_1 & -V_2 R_2 \\ V_1 R_1 & S_2 \end{bmatrix}.$$

We obtain

$$PT_c + QV_c \equiv I \quad \text{on } A, \qquad T_c R + U_c S \equiv I \quad \text{on } A. \tag{2.5}$$

Equation (2.5) implies that the realization (2.4) is canonical in a.

Conversely, assume that the realization (2.4) is canonical in a. Then there exists a neighbourhood $A \subset \Omega$ of a and holomorphic matrices P, Q, R, S on A such that (2.5) holds. With

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \qquad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

it follows that

$$P_{11} T_1 + (Q_{11} - P_{12} U_2) V_1 \equiv I \quad \text{on } A,$$
$$P_{22} T_2 + (P_{21} U_1 + Q_{22}) V_2 \equiv I \quad \text{on } A.$$

Corollary 2.2 yields that $\operatorname{rk} \begin{bmatrix} T_i(a) \\ V_i(a) \end{bmatrix}$ is full ($i = 1, 2$). By a similar argument it can be shown that $\operatorname{rk} [T_i(a) \quad U_i(a)]$ is full ($i = 1, 2$).

(c) The assertion follows from Lemma 2.4 and (b). □

3. Application of Theorem 2.5 to neutral and Volterra integrodifferential systems

In this section we study feedback interconnections of neutral and Volterra integrodifferential systems. We shall use Theorem 2.5 in order to show that (under very mild assumptions) the transfer matrix G_c of the closed loop gives the correct stability information. The reader is referred to Hale (1977) and Burton (1983) for the basic facts concerning neutral functional differential equations and Volterra integrodifferential equations.

3.1 *Neutral Systems*

Consider the neutral systems $\mathcal{N}_i$ for $i = 1, 2$:

$$\begin{aligned} \frac{d}{dt} \int_0^r [dD_i(\tau)\, x_i(t-\tau)] &= \int_0^r [dA_i(\tau)\, x_i(t-\tau)] + \int_0^r [dB_i(\tau)\, e_i(t-\tau)], \\ y_i(t) &= \int_0^r [dC_i(\tau)\, x_i(t-\tau)] \qquad (t \geqslant 0) \end{aligned} \tag{3.1}$$

where $r > 0$, $x_i(t) \in \mathbb{R}^{n_i}$, $e_i(t) \in \mathbb{R}^{q_i}$, and $y_i(t) \in \mathbb{R}^{p_i}$. The functions A_i, B_i, C_i, and D_i are of bounded variation on the interval $[0, r]$, with values in $\mathbb{R}^{n_i \times n_i}$, $\mathbb{R}^{n_i \times q_i}$, $\mathbb{R}^{p_i \times n_i}$, and $\mathbb{R}^{n_i \times n_i}$, respectively. Moreover, in order to get a reasonable theory we we have to impose an additional hypothesis on D_i (Henry, 1974):

$$D_i = \theta I - \tilde{D}_i$$

where $\theta(x) := \begin{cases} 0 & (x \leq 0) \\ 1 & (x > 0) \end{cases}$ and $\tilde{D}_i$ is a function of bounded variation on $[0, r]$ which is continuous at 0 $(i = 1, 2)$.

In order to introduce the transfer matrix G_i of the system $\mathcal{N}_i$ we define

$$\hat{A}_i(s) := \int_0^r e^{-s\tau}\, dA_i(\tau), \qquad \hat{B}_i(s) := \int_0^r e^{-s\tau}\, dB_i(\tau), \qquad \hat{C}_i(s) := \int_0^r e^{-s\tau}\, dC_i(\tau),$$

$$\hat{D}_i(s) := \int_0^r e^{-s\tau}\, dD_i(\tau), \qquad \Delta_i(s) := s\hat{D}_i(s) - \hat{A}_i(s).$$

The entries of $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$, and Δ_i are entire functions. If we extend A_i, B_i, C_i, and D_i to the complete positive real axis by defining $A_i(t) = A_i(r)$, $B_i(t) = B_i(r)$, $C_i(t) = C_i(r)$, and $D_i(t) = D_i(r)$ for $t > r$, then the functions $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$, and $\hat{D}_i$ are the Laplace–Stieltjes transforms (Widder (1972), p. 37) of A_i, B_i, C_i, and D_i, respectively.

We obtain the following expression for G_i:

$$G_i = \hat{C}_i \Delta_i^{-1} \hat{B}_i. \tag{3.2}$$

We shall now consider the feedback interconnection of $\mathcal{N}_1$ and $\mathcal{N}_2$ ($p_1 = q_2$, $p_2 = q_1$, $\boldsymbol{e}_1 = \boldsymbol{u}_1 - \boldsymbol{y}_2$, $\boldsymbol{e}_2 = \boldsymbol{u}_2 + \boldsymbol{y}_1$). The state-space equations of the closed-loop system $\mathcal{N}_c$ relating $\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix}$ and $\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix}$ are given by

$$\frac{d}{dt}\int_0^{2r} [dD_c(\tau)\,\boldsymbol{x}(t-\tau)] = \int_0^{2r} [dA_c(\tau)\,\boldsymbol{x}(t-\tau)] + \int_0^{2r} [dB_c(\tau)\,\boldsymbol{u}(t-\tau)], \tag{3.3a}$$

$$\boldsymbol{y}(t) = \int_0^{2r} [dC_c(\tau)\,\boldsymbol{x}(t-\tau)], \tag{3.3b}$$

where

$$A_c := \begin{bmatrix} A_1 & -B_1 \times C_2 \\ B_2 \times C_1 & A_2 \end{bmatrix}, \tag{3.3c}$$

($\times$ denotes the Stieltjes convolution, i.e. $(B_1 \times C_2)(t) := \int_0^t B_1(t-s)\, dC_2(s)$)

$$\left.\begin{aligned} B_c &:= \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, & C_c &:= \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, & D_c &:= \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \\ \boldsymbol{x}(t) &:= \begin{bmatrix} \boldsymbol{x}_1(t) \\ \boldsymbol{x}_2(t) \end{bmatrix}, & \boldsymbol{y}(t) &:= \begin{bmatrix} \boldsymbol{y}_1(t) \\ \boldsymbol{y}_2(t) \end{bmatrix}, & \boldsymbol{u}(t) &:= \begin{bmatrix} \boldsymbol{u}_1(t) \\ \boldsymbol{u}_2(t) \end{bmatrix}. \end{aligned}\right\} \tag{3.3d}$$

See Section 4 for the verification of (3.3).

Note that the closed-loop system is again a neutral system. The transfer matrix G_c of (3.3) is given by

$$G_c = \hat{C}_c \Delta_c^{-1} \hat{B}_c$$

where

$$\hat{C}_c := \begin{bmatrix} \hat{C}_1 & 0 \\ 0 & \hat{C}_2 \end{bmatrix}, \quad \hat{B}_c := \begin{bmatrix} \hat{B}_1 & 0 \\ 0 & \hat{B}_2 \end{bmatrix}, \quad \Delta_c := \begin{bmatrix} \Delta_1 & \hat{B}_1\hat{C}_2 \\ -\hat{B}_2\hat{C}_1 & \Delta_2 \end{bmatrix}.$$

Here we have used the convolution theorem for the Laplace–Stieltjes transform (see Widder, 1972, p. 88).

We shall need a further assumption on the system $\mathcal{N}_i$ $(i = 1, 2)$:

(NS) The function D_i of bounded variation contains no singular part (see e.g. Kolmogorov & Fomin (1975), p. 341).

Remark 3.1. As far as applications are concerned the restriction of generality induced by the assumption (NS) is not very serious.

COROLLARY 3.2 *Assume that the condition* (NS) *is satisfied. Then the closed-loop system* (3.3) *is exponentially stable* (*i.e. the strongly continuous solution semigroup of the homogeneous part of* (3.3a) *is exponentially stable*) *iff there exists* $\alpha < 0$ *such that*

(a) *the realizations* (3.2) *are canonical in* $\mathbb{C}_\alpha$,

(b) *the transfer matrix* G_c *has no poles in* $\mathbb{C}_\alpha$.

Proof.
Step 1. Of course, condition (NS) implies that D_c contains no singular part and therefore it follows from Henry (1974) that $\mathcal{N}_c$ is exponentially stable iff there exists $\sigma > 0$ such that

$$\det \Delta_c(s) = \det [s\hat{D}_c(s) - \hat{A}_c(s)] \neq 0 \quad (s \in \mathbb{C}_\sigma).$$

Step 2. Assume that (a) and (b) hold. Then it follows from Theorem 2.5(c) that $\det \Delta_c(s) \neq 0$ $(s \in \mathbb{C}_\alpha)$. Step 1 yields the claim. Conversely assume that $\mathcal{N}_c$ is exponentially stable. Therefore (by Step 1) there exists $\sigma < 0$ such that

$$\text{ord}_s \det \Delta_c = 0 \qquad (s \in \mathbb{C}_\sigma).$$

Hence by Lemma 2.4

$$\text{P}_s(G)_c = 0 \qquad (s \in \mathbb{C}_\sigma).$$

which means that G_c has no poles in $\mathbb{C}_\sigma$. Moreover we conclude via Theorem 2.5(c) that the realizations (3.2) are canonical in $\mathbb{C}_\sigma$. □

Remark 3.3. For the interpretation of condition (a) in terms of controllability and observability see Salamon (1984).

If the function D_i is required to satisfy an additional assumption $(i = 1, 2)$, the constant α in Corollary 3.2 may be replaced by 0. More precisely we have:

COROLLARY 3.4 *Assume that*

(i) *Condition* (NS) *is satisfied.*

(ii) $\inf_{s \in \mathbb{C}_0} |\det \hat{D}_i(s)| > 0 \qquad (i = 1, 2).$

Under these conditions the closed-loop system $\mathcal{N}_c$ *is exponentially stable iff*

(a) *the realizations* (3.2) *are canonical in* $\mathbb{C}_0$.

(b) *the transfer matrix* G_c *has no poles in* $\mathbb{C}_0$.

Proof. Trivially the assumptions (i) and (ii) hold also for D_c. Therefore $\mathcal{N}_c$ is exponentially stable iff $\det \Delta_c(s) \neq 0$ $(s \in \mathbb{C}_0)$ (Logemann, 1985). Now use the same arguments as in Step 2 of the proof of Corollary 3.2.

Remark 3.5. The assumptions (i) and (ii) are satisfied by every retarded system ('retarded' means that there are no delays in the derivative). For every finite-dimensional system and for certain classes of retarded systems condition (a) admits an interpretation in terms of stabilizability and detectability (Hautus, 1970; Pandolfi, 1975; Olbrot, 1978). In particular Corollary 3.4 contains the result of Battacharyya & Howze (1985).

3.2 *Volterra Integrodifferential Systems*

Consider the Volterra integrodifferential system $\mathcal{V}_i$ for $i = 1, 2$:

$$\dot{x}_i(t) = \int_0^t A_i(t-s)x_i(s)\,ds + \int_0^t B_i(t-s)e_i(s)\,ds$$

$$y_i(t) = \int_0^t C_i(t-s)x_i(s)\,ds, \qquad (t \geq 0)$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $e_i(t) \in \mathbb{R}^{q_i}$, and $y_i(t) \in \mathbb{R}^{p_i}$. The entries of the matrices A_i, B_i, and C_i are all functions in

$$L^1_- := \{f \in L^1(\mathbb{R}^+, \mathbb{R}) : \exists\ \alpha > 0 \text{ s.t. } f(t)e^{\alpha t} \in L^1(\mathbb{R}^+, \mathbb{R})\}.$$

The sizes of A_i, B_i, and C_i are given by $n_i \times n_i$, $n_i \times q_i$, and $p_i \times n_i$, respectively $(i = 1, 2)$.

For the transfer matrix G_i of $\mathcal{V}_i$ we obtain the following expression:

$$G_i(s) = \hat{C}_i(s)[sI - \hat{A}_i(s)]^{-1}\hat{B}_i(s), \tag{3.4}$$

where $\hat{A}_i$, $\hat{B}_i$, and $\hat{C}_i$ denote the Laplace transforms of A_i, B_i, and C_i. The matrix G_i is meromorphic on an open right half plane containing $\mathbb{C}_0$ $(i = 1, 2)$.

We shall consider the feedback interconnection of $\mathcal{V}_1$ and $\mathcal{V}_2$ $(p_1 = q_2,\ p_2 = q_1,\ e_1 = u_1 - y_2,\ e_2 = u_2 + y_1)$. The state equations of the closed-loop system $\mathcal{V}_c$ relating $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ are given by

$$\dot{x}(t) = \int_0^t A_c(t-s)x(s)\,ds + \int_0^t B_c(t-s)u(s)\,ds, \tag{3.5a}$$

$$y(t) = \int_0^t C_c(t-s)x(s)\,ds, \tag{3.5b}$$

where

$$A_c := \begin{bmatrix} A_1 & -B_1\star C_2 \\ B_2\star C_1 & A_2 \end{bmatrix}, \qquad B_c := \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \qquad C_c := \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$$

($\star$ denotes the convolution in $L^1(\mathbb{R}^+, \mathbb{R})$),

$$x(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad y(t) := \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \qquad u(t) := \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

We note that the closed-loop system is again a Volterra integrodifferential system.

The transfer matrix G_c of (3.5) is given by

$$G_c(s) = \hat{C}_c(s)[sI - \hat{A}_c(s)]^{-1}\hat{B}_c(s),$$

where ^ denotes the Laplace transformation.

COROLLARY 3.6 *The closed-loop system* (3.5) *is uniformly asymptotically stable* (*i.e. the trivial solution of the homogeneous part of* (3.5a) *is uniformly asymptotically stable in the sense of Miller* (1971, 1972)) *iff*

(a) *the realizations* (3.4) *are canonical in* $\mathbb{C}_0$,

(b) *the transfer matrix* G_c *has no poles in* $\mathbb{C}_0$.

Proof. It is known that $\mathcal{V}_c$ is uniformly asymptotically stable iff

$$\det\,[sI - \hat{A}_c(s)] \neq 0 \qquad (s \in \mathbb{C}_0). \tag{3.6}$$

The necessity of condition (3.5) is proved in Miller (1971). For the sufficiency of (3.6) see Miller (1972). Now the assertion of the corollary can be proved using the same arguments as in Step 2 of the proof of Corollary 3.2. □

4. Derivation of equations (3.3)

We now derive the closed-loop equations (3.3) for the system $\mathcal{N}_c$. By setting $e_1 = u_1 - y_2$ and $e_2 = u_2 + y_1$ we obtain from (3.1) the state-space equations describing the closed-loop system $\mathcal{N}_c$:

$$\frac{d}{dt}\int_0^{2r}\left(d\begin{bmatrix} D_1(\tau) & 0 \\ 0 & D_2(\tau) \end{bmatrix}\begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix}\right)$$
$$= \begin{bmatrix} \int_0^r [dA_1(\tau)\,x_1(t-\tau)] - \int_0^r \left(dB_1(\tau)\int_0^r [dC_2(\zeta)\,x_2(t-\tau-\zeta)]\right) \\ \int_0^r \left(dB_2(\tau)\int_0^r [dC_1(\zeta)\,x_1(t-\tau-\zeta)]\right) + \int_0^r [dA_2(\tau)\,x_2(t-\tau)] \end{bmatrix} +$$
$$\int_0^{2r}\left(d\begin{bmatrix} B_1(\tau) & 0 \\ 0 & B_2(\tau) \end{bmatrix}\begin{bmatrix} u_1(t-\tau) \\ u_2(t-\tau) \end{bmatrix}\right), \tag{4.1}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \int_0^{2r}\left(d\begin{bmatrix} C_1(\tau) & 0 \\ 0 & C_2(\tau) \end{bmatrix}\begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix}\right).$$

All that remains to show is that the homogeneous part of (4.1) can be written in the form

$$\frac{d}{dt}\int_0^{2r}\left(d\begin{bmatrix} D_1(\tau) & 0 \\ 0 & D_2(\tau)\end{bmatrix}\begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau)\end{bmatrix}\right)$$
$$=\int_0^{2r}\left(d\begin{bmatrix} A_1(\tau) & -(B_1\times C_2)(\tau) \\ (B_2\times C_1)(\tau) & A_2(\tau)\end{bmatrix}\begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau)\end{bmatrix}\right).$$

This follows from the next lemma.

LEMMA 4.1 *For $r>0$ let $f:[0,2r]\to\mathbb{R}$ be continuous and let $\alpha,\beta:[0,r]\to\mathbb{R}$ be of bounded variation; without loss of generality we may assume that α and β are normalized in the sense of Widder* (1972), *p.* 13. *Moreover we extend α and β to $\mathbb{R}$ by defining $\alpha(t)=\beta(t)=0$ $(t<0)$, and $\alpha(t)=\alpha(r)$ and $\beta(t)=\beta(r)$ $(t>r)$. The following formula holds:*

$$\int_0^r\int_0^r f(\tau+\zeta)\,d\alpha(\tau)\,d\beta(\zeta)=\int_0^{2r} f(\tau)\,d(\alpha\times\beta)(\tau).$$

Proof. Define:

$$\Phi:[0,r]\to\mathbb{R}:x\mapsto\int_0^{2r} f(s)\,d_s\alpha(s-x),$$

$$\psi:\mathbb{R}\to\mathbb{R}:s\mapsto\int_0^r\alpha(s-x)\,d\beta(x).$$

It is routine to show that Φ is continuous, and that ψ is of bounded variation on $[0,2r]$ and normalized in the sense of Widder (1972), p. 13. Furthermore we have $\psi(t)=0$ $(t<0)$ and $\psi(t)=\psi(2r)$ $(t>2r)$. Therefore we can use the same arguments as in the proof of Theorem 5 in Bray (1919) in order to establish the equation

$$\int_0^r\int_0^{2r} f(\tau)\,d_\tau\alpha(\tau-\zeta)\,d\beta(\zeta)=\int_0^{2r} f(\tau)\,d_\tau\int_0^r\alpha(\tau-\zeta)\,d\beta(\zeta). \tag{4.2}$$

The assertion follows from a straightforward calculation:

$$\int_0^{2r} f(\tau)\,d(\alpha\times\beta)(\tau)$$
$$=\int_0^{2r} f(\tau)\,d_\tau\int_0^r\alpha(\tau-\zeta)\,d\beta(\zeta)\quad\text{(by definition of the Stieltjes convolution)}$$
$$=\int_0^r\int_0^{2r} f(\tau)\,d_\tau\alpha(\tau-\zeta)\,d\beta(\zeta)\quad\text{(by (4.2))}$$
$$=\int_0^r\int_0^{r+\zeta} f(\tau)\,d_\tau\alpha(\tau-\zeta)\,d\beta(\zeta)\quad\text{(by definition of }\alpha)$$
$$=\int_0^r\int_0^r f(\tau+\zeta)\,d\alpha(\tau)\,d\beta(\zeta)\quad\text{(change of variables)}\ \square.$$

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