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A note on the generalized inverse Nyquist stability criterion

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The purpose of this paper is to provide a self-contained proof of the generalized inverse Nyquist stability criterion which is based on the theory of algebraic functions and Riemann surfaces. It is shown how the fixed modes are related to the polynomial determining the inverse characteristic gains.

Notation

$\mathbb{R}$ The field of real numbers.

$\mathbb{R}(s)$ The field of rational functions over $\mathbb{R}$.

$\mathbb{R}^{m\times m}(s)$ The set of $m \times m$ matrices with entries in $\mathbb{R}(s)$.

$\mathbb{C}$ The field of complex numbers.

$\mathbb{C}_+ := \{s \in \mathbb{C} : \text{Re}(s) \geqslant 0\}$.

$\mathbb{C}[s, g]$ The ring of polynomials (over $\mathbb{C}$) in two indeterminates, s, g.

Let $U \subset V \subset \mathbb{C}$, where V is an open set and let $f: V \to \mathbb{C}$ be meromorphic; then

$Z(f(s), U)$ The number of zeros (counting multiplicities) of f in U.

$P(f(s), U)$ The number of poles (counting multiplicities) of f in U.

Let $G(s) \in \mathbb{R}^{m\times m}(s)$ and let the zero and pole polynomials of $G(s)$ denoted by $z(s)$ and $p(s)$; then

$Z(G(s), U) := Z(z(s), U)$.

$P(G(s), U) := P(p(s), U)$.

Let $a \in \mathbb{C}$ and let Γ be a cycle in $\mathbb{C}\backslash\{a\}$; then

$\nu(\Gamma, a)$ The winding number of the cycle Γ about the point a (clockwise convention).

1. Introduction

Postlethwaite (1977) and Postlethwaite and MacFarlane (1979) gave a proof of the generalized inverse Nyquist stability criterion using the tools of algebraic function theory and Riemann surfaces. Following Smith (1981) the main difference from Postlethwaite and MacFarlane is the use here of the matrix fraction decomposition of a transfer function and the absence of any reference to a state space description.

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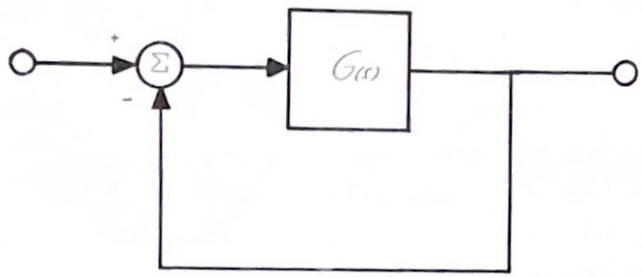
2. Some preliminaries

We will consider the feedback arrangement of the Figure, in which $G(s) \in \mathbb{R}^{m \times m}(s)$, where $\det G(s) \not\equiv 0$ and $\det (I_m + G(s)) \not\equiv 0$. For a feedback kI_m we will need to assume that $\det (I_m + kG(s)) \not\equiv 0$. Let $N(s)D^{-1}(s)$ be a right coprime factorization of $G(s)$; then the closed-loop transfer function $R(s)$ is given by

$$R(s) = N(s)(N(s) + D(s))^{-1} \tag{1}$$



Multivariable feedback arrangement.

It can be shown (see, for example, Smith 1981), that the pole polynomial $p(s)$ and the zero polynomial $z(s)$ of $G(s)$ and the pole polynomial $p'(s)$ of $R(s)$ are given by

$$p(s) = \det D(s) \tag{2}$$

$$z(s) = \det N(s) \tag{3}$$

$$p'(s) = \det (N(s) + D(s)) \tag{4}$$

In the proof of the generalized direct Nyquist stability criterion (MacFarlane and Postlethwaite 1977, Postlethwaite and MacFarlane 1979, Smith 1981, Logemann and Schwarting 1981) the return difference matrix $F(s) := I_m + G(s)$ is an important object of study. For the generalized inverse Nyquist criterion we have to consider the matrix $\hat{F}(s) := I_m + \hat{G}(s)$, where $\hat{G}(s) := G^{-1}(s)$. We can express $\hat{F}(s)$ in the following form

$$\hat{F}(s) = (N(s) + D(s))N^{-1}(s) \tag{5}$$

Like Smith (1981) we define the characteristic gains by the following expression

$$\Delta(s, g) := \det (gD(s) - N(s)) = 0 \tag{6}$$

The inverse characteristic gains are defined by the equation

$$\hat{\Delta}(s, \hat{g}) := \det (\hat{g}N(s) - D(s)) = 0 \tag{7}$$

The relationship between $\Delta(s, y)$ and $\hat{\Delta}(s, y)$ is given by

$$\hat{\Delta}(s, y) = (-1)^m y^m \Delta\left(s, \frac{1}{y}\right) \tag{8}$$

where y is a complex variable. We will express $\Delta(s, g)$ as follows

$$\Delta(s, g) = e(s)\Delta_1(s, g)\Delta_2(s, g) \dots \Delta_l(s, g)c(g) \tag{9}$$

where $\Delta_i(s, g)$ are irreducible polynomials in the ring $\mathbb{C}[s, g]$, $1 \leqslant i \leqslant l$. The term $e(s)$ is the product of all irreducible factors which are independent of g. The polynomial $c(g)$ is the product of all irreducible factors independent of s. Each $\Delta_i(s, g)$ determines an algebraic function g_i defined on a corresponding Riemann surface R_i (see, for example, Jänich 1980, Farkas and Kra 1980 or Springer 1957).

Now let t be the degree of $c(g)$ and let t_i be the degree of $\Delta_i(s, g)$ in g, $1 \leqslant i \leqslant l$. Certainly

$$t + \sum_{i=1}^{l} t_i = m \tag{10}$$

and we define

$$\hat{e}(s) := (-1)^m e(s) \tag{11}$$

$$\hat{c}(y) := y^t c\left(\frac{1}{y}\right) \tag{12}$$

$$\hat{\Delta}_i(s, y) := y^{t_i}\Delta_i\left(s, \frac{1}{y}\right), \quad 1 \leqslant i \leqslant l \tag{13}$$

By definition (eqn. (11)), the zeros of $\hat{e}(s)$ are the fixed modes in the sense of Smith (1981).

It follows from (8) and (9) that

$$\hat{\Delta}(s, \hat{g}) = \hat{e}(s)\hat{\Delta}_1(s, \hat{g})\hat{\Delta}_2(s, \hat{g}) \dots \hat{\Delta}_l(s, \hat{g})\hat{c}(\hat{g}) \tag{14}$$

It is easy to check that the polynomials $\hat{\Delta}_i(s, \hat{g})$ are irreducible in $\mathbb{C}[s, \hat{g}]$. Therefore, each $\hat{\Delta}_i(s, \hat{g})$ determines an algebraic function $\hat{g}_i$ and a corresponding Riemann surface $\hat{R}_i$. Note that R_i and $\hat{R}_i$ are the same Riemann surfaces (modulo a biholomorphic homeomorphism) and that, in a certain sense, $\hat{g}_i$ is the inverse of g_i (see Appendix).

3. Two statements of the inverse generalized Nyquist stability criterion

Now consider

$$\hat{\delta}(s, \hat{f}) := \det(\hat{f}N(s) - N(s) + D(s)) \tag{15}$$

$$= \hat{\Delta}(s, \hat{f} - 1) \tag{16}$$

$$= \hat{e}(s)\hat{\Delta}_1(s, \hat{f} - 1) \dots \hat{\Delta}_l(s, \hat{f} - 1)\hat{c}(\hat{f} - 1) \tag{17}$$

Since $\hat{\Delta}_i(s, \hat{g})$ is irreducible, the polynomial $\hat{\Delta}_i(s, \hat{f} - 1) \in \mathbb{C}[s, \hat{f}]$ is also irreducible. Therefore each $\hat{\delta}_i(s, \hat{f}) := \hat{\Delta}_i(s, \hat{f} - 1)$ defines an algebraic function $\hat{f}_i$. Let

$$\hat{\delta}_i(s, \hat{f}) = \hat{b}_{m_i}{}^{(i)}(s)\hat{f}^{m_i} + \dots + \hat{b}_0{}^{(i)}(s) \tag{18}$$

then

$$\left.\begin{aligned} k_1\hat{e}(s) \prod_{i=1}^{l} \hat{b}_{m_i}{}^{(i)}(s) &= \det N(s) \\ k_2\hat{e}(s) \prod_{i=1}^{l} \hat{b}_0{}^{(i)}(s) &= (-1)^m \det(N(s) + D(s)) \end{aligned}\right\} \tag{19}$$

for some $k_1, k_2 \in \mathbb{R}$. (19) follows from

$$\det(\hat{f} I_m - \hat{F}(s)) = \hat{f}^m + \dots + (-1)^m \det \hat{F}(s)$$

$$= \hat{f}^m + \dots + (-1)^m \frac{\det(N(s)+D(s))}{\det N(s)} \quad \text{(by (5))}$$

and

$$\hat{\delta}(s, \hat{f}) = \det(\hat{f} I_m - \hat{F}(s)) \det N(s) \quad \text{(by (5))}$$

Now we draw on each Riemann surface R_i a generalized Nyquist contour N_{Q_i} (see, for example, Smith 1981), such that all poles and zeros of $\hat{g}_i + 1$ lying over $\mathbb{C}_+$ are inside the contour N_{Q_i}, $1 \leqslant i \leqslant l$. (Note that the poles of $\hat{g}_i + 1$ are the poles of $\hat{g}_i$, which are the zeros of g_i). Under this assumption the following theorem holds.

Generalized inverse Nyquist criterion (statement 1)

The feedback arrangement of the Figure will be closed-loop stable if and only if

$$\sum_{i=1}^{l} (\hat{g}_i \bigcirc N_{Q_i}, -1) = -Z(G(s), \mathbb{C}_+)$$

Proof

We have chosen N_{Q_i} so that we can apply the generalized argument principle to $\hat{g}_i + 1$ and N_{Q_i} for each i. Then it follows from Smith (1981) or Logemann and Schwarting (1981) that

$$\nu(\hat{g}_i \bigcirc N_{Q_i}, -1) = \nu((\hat{g}_i + 1) \bigcirc N_{Q_i}, 0)$$

$$= Z(\hat{b}_0^{(i)}(s), \mathbb{C}_+) - Z(\hat{b}_{m_i}^{(i)}(s), \mathbb{C}_+) \quad (20)$$

Hence

$$\sum_{i=1}^{l} \nu(\hat{g}_i \bigcirc N_{Q_i}, -1) = \sum_{i=1}^{l} Z(\hat{b}_0^{(i)}(s), \mathbb{C}_+) + Z(\hat{e}(s), \mathbb{C}_+)$$

$$- \sum_{i=1}^{l} Z(\hat{b}_{m_i}^{(i)}(s), \mathbb{C}_+) - Z(\hat{e}(s), \mathbb{C}_+) \quad (21)$$

and from (19) we obtain

$$\sum_{i=1}^{l} \nu(\hat{g}_i \bigcirc N_{Q_i}, -1) = Z(\det(N(s)+D(s)), \mathbb{C}_+) - Z(\det N(s), \mathbb{C}_+) \quad (22)$$

$$= Z(p'(s), \mathbb{C}_+) - Z(z(s), \mathbb{C}_+) \quad \text{(by (3) and (4))}$$

$$= P(R(s), \mathbb{C}_+) - Z(G(s), \mathbb{C}_+) \quad (23)$$

□

For a feedback of the form kI_m, the critical point is $-k$. The criterion for this case is obtained using the above arguments exactly with the alternative definition $\hat{F}(s) := kI_m + \hat{G}(s)$.

Now let us make the additional assumption, that all zeros of $\hat{g}_i$ lying over $\mathbb{C}_+$ are inside the contour N_{Q_i}, $1 \leqslant i \leqslant l$. Then we can apply the (generalized) argument principle to $\hat{g}_i$ and N_{Q_i} for each i and by analogous arguments leading us to (23) we obtain

$$\sum_{i=1}^{l} \nu(\hat{g}_i \bigcirc N_{Q_i}, 0) = P(G(s), \mathbb{C}_+) - Z(G(s), \mathbb{C}_+) \tag{24}$$

It follows from (23) and (24) that

$$\sum_{i=1}^{l} \nu(\hat{g}_i \bigcirc N_{Q_i}, -1) - \sum_{i=1}^{l} \nu(\hat{g}_i \bigcirc N_{Q_i}, 0) = P(R(s), \mathbb{C}_+) - P(G(s), \mathbb{C}_+) \tag{25}$$

Hence we have proved the following theorem

Generalized inverse Nyquist criterion (*statement* 2)

Under the above assumptions the feedback arrangement of the Figure will be closed-loop stable if and only if

$$\sum_{i=1}^{l} \nu(\hat{g}_i \bigcirc N_{Q_i}, -1) - \sum_{i=1}^{l} \nu(\hat{g}_i \bigcirc N_{Q_i}, 0) = -P(G(s), \mathbb{C}_+)$$

Appendix

In the following we shall use some terms and facts of complex analysis. For definitions and details the reader is referred to Jänich (1980), Farkas and Kra (1980) and Springer (1957).

Let (h, p) be a germ, which satisfies the equation

$$\Delta_i(s, h(s)) \equiv 0$$

in a neighbourhood of p, where $p \in \mathbb{C}$. The Riemann surface R_i is the union of the set of all analytic continuations of (h, p) and the set of the corresponding branch points. R_i is assorted with a certain topology and a certain atlas, such that the conditions in the definition of an abstract Riemann surface are satisfied. Now let $(l, q) \in R_i$, i.e. $\Delta_i(s, l(s)) \equiv 0$ in a neighbourhood of q. The algebraic function $g_i: \ R_i \rightarrow \mathbb{C} \cup \{\infty\}$ is defined by $g_i((l, q)) = l(q)$.

$\hat{R}_i$ is the abstract Riemann surface determined by

$$\hat{\Delta}_i(s, y) = y^{l_i} \Delta_i\left(s, \frac{1}{y}\right) = 0$$

Since $\hat{\Delta}_i(s, l^{-1}(s)) \equiv 0$ in a neighbourhood of q, we see that $(l^{-1}, q) \in \hat{R}_i$ and we realize that in general (in the sense of set theory) $\hat{R}_i$ is not the same Riemann surface as R_i. Nevertheless, from an abstract standpoint of view, R_i and $\hat{R}_i$ are the same surfaces. To see this, consider the map

$$\Phi: \ R_i \rightarrow \hat{R}_i$$

$$\Phi((l, q)) = (l^{-1}, q)$$

It is not difficult to show, that Φ is a biholomorphic homeomorphism and that the following equation holds

$$\hat{g}_i \bigcirc \Phi = g_i^{-1}$$

Therefore, in the sense of topology and complex function theory, R_i and $\hat{R}_i$ are the same Riemann surfaces and $\hat{g}_i$ is the inverse of g_i.

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