Some spectral properties of operator-valued positive-real functions

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In memory of Ruth F Curtain

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Abstract. We consider operator-valued positive real functions $\mathbf{H}$ and show that the intersections of the point, continuous and residual spectra of $\mathbf{H}(s)$ with the imaginary axis do not depend on s. In particular, if $\mathbf{H}$ is positive real and $\mathbf{H}(z)$ is invertible for some z in the open right-half plane, then $\mathbf{H}(s)$ is invertible for all s in then open right-half plane. Furthermore, we prove that the eigenspace of $\mathbf{H}(s)$ corresponding to an imaginary eigenvalue does not depend on s. It is also shown that the intersection of the numerical range of $\mathbf{H}(s)$ with the imaginary axis is independent of s. Finally, we prove that, under suitable assumptions, application of a "sufficiently positive real" static output feedback to a positive real transfer function leads to a strictly positive real closed-loop system.

Keywords. Infinite-dimensional systems, numerical range, operator-valued transfer functions, positive real, spectrum, strictly positive real.

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1 Introduction

I met Ruth in November 1984 when I gave a seminar to the Systems & Control Group in the Department of Mathematics at the University of Groningen. At the time I was a PhD student (supervised by Diederich Hinrichsen) at the University of Bremen and Ruth had invited me to talk about my research.[†] Her interest in my work provided strong encouragement. In subsequent years, Ruth and I became collaborators and friends. I visited Groningen many times, and, with great fondness, I remember the relaxed and stimulating atmosphere in the Groningen Systems & Control Group and the many discussions I had with Ruth and other members of the group (including Hans Nieuwenhuis, Martin Weiss, Jan Willems and Hans Zwart) over coffee, lunch and dinner, not only on systems & control (or, more generally, on mathematics), but also on politics and on art, movies and music. Sadly, these inspiring conversations and discussions with Ruth are now a thing of the past.

Ruth had a longstanding interest in frequency-domain methods, positive realness and their role in absolute stability theory [7, 8, 9, 10, 11, 12, 20] and therefore I believe that the topic of the current paper is very appropriate for this memorial issue of *Systems & Control Letters*.

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† My seminar was well received and when I got on the train back to Bremen, I felt quite pleased with myself. But this did not last long: just after the train had crossed the border into to Germany, I was confronted and searched by two plain clothes German police officers who suspected me to be a drug smuggler. They did not hide their frustration when it became clear that I did not have any drugs on me. Being a gentleman, I apologized to the two officers for not being a drug smuggler, but they were not amused. Bizzarely, in 1998/99, Ruth suffered a similar fate: due to a clerical error by Australian customs and excise staff, Ruth came under suspicion of being a drug smuggler.

Positive realness and the corresponding time-domain notion of passivity are key concepts in the theory of circuits and networks and in systems & control (e.g. [1, 13, 14, 15, 18, 19, 21, 26, 32, 33]), with the positive-real lemma (or Kalman-Yakubovich-Popov lemma) being one of the best known results of the field. Here we will consider operator-valued positive real functions which, for example, arise as resolvent operators of dissipative operators or as transfer functions of impedance-passive well-posed infinite-dimensional systems (or, more generally, impedance-passive system nodes) [13, 25, 28, 29, 31]. Specific instances of the latter are distributed parameter systems with certain passivity properties and with control and observation applied on infinitely many points of the boundary (a submanifold of the boundary, for example).

The question which triggered the research presented in this paper is as follows: given an operator-valued positive-real function $\mathbf{H}$, and assuming that there exists a point z in the open right-half plane such that $\mathbf{H}(z)$ is invertible, does it follow that $\mathbf{H}(s)$ is invertible for every s with $\operatorname{Re} s > 0$ (in which case $\mathbf{H}^{-1}$ is positive real)? The answer to the above question is yes and it follows from the main result of this paper which says that the intersections of the imaginary axis with the point, continuous and residual spectra of $\mathbf{H}(s)$, where $\mathbf{H}$ is an operator-valued positive-real function, do not depend on s, with the intersection of the imaginary axis with the residual spectrum of $\mathbf{H}(s)$ being empty for all s with $\operatorname{Re} s > 0$. Moreover, the eigenspaces of $\mathbf{H}(s)$ corresponding to imaginary eigenvalues do not depend on s. We also prove that 0 is not in the residual spectrum of the real part of $\mathbf{H}(s)$ for any s with $\operatorname{Re} s > 0$, and that if 0 is in the point (continuous) spectrum of the real part of $\mathbf{H}(z)$ for some z in the open right-half plane, then 0 is in the point (continuous) spectrum of the real part of $\mathbf{H}(s)$ for all s with $\operatorname{Re} s > 0$. The intersection of the numerical range of $\mathbf{H}(s)$ with the imaginary axis is also shown to be independent of s.

Invertibility of positive-real functions plays a key role in Section 5 which addresses the problem of obtaining a strictly positive-real closed-loop transfer function by the application of suitable static output feedback to a positive-real transfer function. The results in Section 5 provide easily checkable sufficient conditions for the existence of such output feedback operators.

The paper is structured as follows. Section 2 is devoted to a number of preliminaries. In Section 3, we investigate certain spectral properties of matrix-valued positive-real functions, and, in Section 4, we consider the infinite-dimensional operator-valued case. Section 5 focuses on the problem of achieving stric positive realness by application of a "suffciently positive real" static output feedback to a positive-real transfer function.

2 Preliminaries

As usual, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the natural, real and complex numbers, respectively. For $\alpha \in \mathbb{R}$, we define $\mathbb{C}_\alpha := \{s \in \mathbb{C} : \text{Re } s > \alpha\}$. We denote the set of all imaginary numbers by $\mathbb{I}$, that is,

$$\mathbb{I} := i\mathbb{R} = \{ir : r \in \mathbb{R}\} \subset \mathbb{C}_0.$$

For a set $S \subset \mathbb{C}$, the closure of S is denoted by $\operatorname{cl} S$. For the rest of this paper, U denotes a complex Hilbert space with inner product $\langle \cdot\,, \cdot\rangle$ and we set $\mathbb{E}_U := \{u \in U : \|u\| = 1\}$, where $\|\cdot\|$ is the norm induced by the inner product. The Banach algebra of all bounded linear operators $U \to U$ is denoted by $\mathcal{L}(U)$. We say that $T \in \mathcal{L}(U)$ is *positive semi-definite*, and write $T \succeq 0$, if $\langle Tu, u\rangle \geq 0$ for all $u \in U$. Since U is a complex vector space, every positive semi-definite operator is necessarily self-adjoint. For $T_1, T_2 \in \mathcal{L}(U)$, we write $T_1 \succeq T_2$ if $T_1 - T_2 \succeq 0$.

Let $\sigma(T)$ denote the spectrum of $T \in \mathcal{L}(U)$. We recall that

$$\sigma(T) = \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{c}}(T) \cup \sigma_{\mathrm{r}}(T),$$

where $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{c}}(T)$ and $\sigma_{\mathrm{r}}(T)$ denote the point, continuous and residual spectra of T, respectively, that is, $\sigma_{\mathrm{p}}(T)$ is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective (the set of all eigenvalues of T), $\sigma_{\mathrm{c}}(T)$ consists of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is injective, not surjective and $\operatorname{im}(T - \lambda I)$ is dense, and

$$\sigma_{\mathrm{r}}(T) = \{\lambda \in \mathbb{C} \colon T - \lambda I \text{ is injective and } \operatorname{im}(T - \lambda I) \text{ is not dense}\}.$$

Clearly, if $\lambda \in \sigma_{\mathrm{c}}(T)$, then the densely defined operator $(T - \lambda I)^{-1}$ is unbounded. Moreover, it is obvious that the sets $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{c}}(T)$ and $\sigma_{\mathrm{r}}(T)$ are mutually disjoint. As usual, we say that $\lambda \in \mathbb{C}$ is an *approximate eigenvalue* of T if $\inf_{u \in \mathbb{E}_U} \|(T - \lambda I)u\| = 0$. Setting

$$\sigma_{\mathrm{ap}}(T) := \{\lambda \in \mathbb{C} : \lambda \text{ is an approximate eigenvalue of } T\},$$

we have that $\sigma_{\mathrm{c}}(T) \subset \sigma_{\mathrm{ap}}(T) \backslash \sigma_{\mathrm{p}}(T)$. If $\lambda \in \sigma_{\mathrm{ap}}(T)$, then every sequence $(u_j)_{j \in \mathbb{N}}$ in $\mathbb{E}_U$ such that $(T - \lambda I)u_j \to 0$ as $j \to \infty$ is said to be an *eigensequence* of T associated with λ.

The numerical range $W(T)$ of an operator $T \in \mathcal{L}(U)$ is defined by

$$W(T) := \{\langle Tu, u \rangle : u \in \mathbb{E}_U\}.$$

It is well-known that $W(T)$ is convex, $\sigma(T) \subset \operatorname{cl} W(T)$ and, if T is normal, then $\operatorname{cl} W(T)$ is the convex hull of $\sigma(T)$, see, for example, [16, 23].

For $T \in \mathcal{L}(U)$, we define the real part $\operatorname{Re} T$ of T by

$$\operatorname{Re} T := \frac{1}{2}(T + T^*),$$

where T^* is the Hilbert space adjoint of T.

In the following lemma, we present some simple properties of operators which have positive semi-definite real part.

Lemma 2.1. *Let $T \in \mathcal{L}(U)$ and assume that $\operatorname{Re} T \succeq 0$. Then the following statements hold.*

(1) *If $\lambda \in \sigma(T)$, then $\operatorname{Re} \lambda \geq 0$.*

(2) *If $0 \notin \sigma(T)$ (that is, T is invertible), then $\operatorname{Re} T^{-1} \succeq 0$.*

(3) *$\ker(T - \lambda I) = \ker(T^* + \lambda I)$ for all $\lambda \in \mathbb{I}$.*

(4) *$\mathbb{I} \cap \sigma_{\mathrm{r}}(T) = \emptyset$.*

Proof. (1) Let $z \in \mathbb{C}$ with $\operatorname{Re} z < 0$. Then

$$\|(T - zI)u\| \geq |\langle (T - zI)u, u \rangle| \geq |\operatorname{Re} \langle (T - zI)u, u \rangle| = \langle (\operatorname{Re} T + |\operatorname{Re} z| I)u, u \rangle \geq |\operatorname{Re} z| > 0 \quad \forall\, u \in \mathbb{E}_U,$$

that is, $T - zI$ is bounded away from 0. By an identical argument, it can be shown that the adjoint $(T - zI)^* = T^* - \bar{z}I$ of $T - zI$ is also bounded away from 0, showing that $T - zI$ is invertible [23, Proposition 3.2.6], that is, $z \notin \sigma(T)$. Consequently, if $\lambda \in \sigma(T)$, then $\operatorname{Re} \lambda \geq 0$.

(2) Assume that T is invertible. Let $u \in U$ and set $v := T^{-1}u$. Then

$$\langle \operatorname{Re} T^{-1}u, u \rangle = \operatorname{Re} \langle T^{-1}u, u \rangle = \operatorname{Re} \langle v, Tv \rangle = \operatorname{Re} \langle Tv, v \rangle \geq 0,$$

and since $u \in U$ was arbitrary, it follows that $\operatorname{Re} T^{-1} \succeq 0$.

(3) Let $\lambda \in \mathbb{I}$ and $u \in \ker(T - \lambda I)$. Then

$$\langle (T - \lambda I)u, u \rangle = 0 = \langle u, (T^* + \lambda I)u \rangle,$$

and so $\langle (T^* + \lambda I)u, u\rangle = 0$. Consequently,

$$\langle (T + T^*)u, u\rangle = \langle (T - \lambda I)u, u\rangle + \langle (T^* + \lambda I)u, u\rangle = 0,$$

showing that $\langle (\mathrm{Re}\, T)u, u\rangle = 0$. Since $\mathrm{Re}\, T \succeq 0$, we conclude that $(\mathrm{Re}\, T)u = 0$. Thus, $T^*u = -Tu$, implying that $(T^* + \lambda I)u = -(T - \lambda I)u = 0$, whence $u \in \ker(T^* + \lambda I)$. We have now shown that

$$\ker(T - \lambda I) \subset \ker(T^* + \lambda I) \tag{2.1}$$

Replacing in (2.1) T by T^* and λ by $-\lambda$ and using that $(T^*)^* = T$ and $\mathrm{Re}\, T^* = \mathrm{Re}\, T \succeq 0$, we see that $\ker(T^* + \lambda I) \subset \ker(T - \lambda I)$, completing the proof of statement (3).

(4) It is well-known that $\overline{\sigma_{\mathrm{r}}(T)} \subset \sigma_{\mathrm{p}}(T^*)$ (because $(\mathrm{im}(T - \lambda I))^\perp = \ker(T^* - \bar{\lambda} I)$ for all $\lambda \in \mathbb{C}$). As $\sigma_{\mathrm{p}}(T) \cap \sigma_{\mathrm{r}}(T) = \emptyset$, it now follows from statement (3) that $\mathbb{I} \cap \sigma_{\mathrm{r}}(T) = \emptyset$. $\square$

For a non-empty open set $\Omega \subset \mathbb{C}$ and a complex Banach space X, the vector space of all holomorphic functions $\Omega \to X$ is denoted by $\mathcal{H}(\Omega, X)$. Furthermore, $\mathcal{H}^*(\Omega, X)$ is the set of all X-valued functions which are holomorphic on Ω with exception of isolated points, namely poles and essential singularities (where it is understood that any removable singularities have been removed holomorphic extension).

For $\alpha \in \mathbb{R}$, we set

$$\mathcal{H}_\alpha(X) := \mathcal{H}(\mathbb{C}_\alpha, X) \quad \text{and} \quad \mathcal{H}^*_\alpha(X) := \mathcal{H}^*(\mathbb{C}_\alpha, X).$$

If $\dim X = 1$, that is, $X = \mathbb{C}$, we write $\mathcal{H}_\alpha := \mathcal{H}_\alpha(\mathbb{C})$ and $\mathcal{H}^*_\alpha := \mathcal{H}^*_\alpha(\mathbb{C})$.

Definition 2.2. A function $\mathbf{H} \in \mathcal{H}^*_0(\mathcal{L}(U))$ is said to be positive real if $\mathrm{Re}\, \mathbf{H}(s) \succeq 0$ for all $s \in \mathbb{C}_0$ which are not singularities of $\mathbf{H}$. $\lozenge$

Note that holomorphy of $\mathbf{H}$ on $\mathbb{C}_0$ is not assumed in the above definition. In fact, the holomorphy of $\mathbf{H}$ on the open right-half plane, or, equivalently, the absence of any singularities in $\mathbb{C}_0$, is a consequence of the positive-real property as was shown in [13].

Proposition 2.3. *If $\mathbf{H} \in \mathcal{H}^*_0(\mathcal{L}(U))$ is positive real, then $\mathbf{H}$ does not have any singularities in $\mathbb{C}_0$, that is, $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$.*

We close this section with two examples of operator-valued positive-real functions.

Example 2.4. (a) Consider the following heat equation on the square $(0, 1) \times (0, 1)$:

$$\begin{aligned}
&\frac{\partial w}{\partial t}(x_1, x_2, t) = \frac{\partial^2 w}{\partial x_1^2}(x_1, x_2, t) + \frac{\partial^2 w}{\partial x_2^2}(x_1, x_2, t),\\
&w(0, x_2, t) = 0, \quad w(1, x_2, t) = 0,\\
&\frac{\partial w}{\partial x_2}(x_1, 0, t) = 0, \quad \frac{\partial w}{\partial x_2}(x_1, 1, t) = u(x_1, t),\\
&y(x_1, t) = w(x_1, 1, t),
\end{aligned}$$

where u is the control function and y is the observation. It is shown in [13] that the transfer function of the above system is given by

$$\mathbf{H}(s)v = \sum_{n=1}^{\infty} \frac{\cosh(\sqrt{s + n^2\pi^2})\gamma_n(v)}{\sqrt{s + n^2\pi^2}\, \sinh(\sqrt{s + n^2\pi^2})} \sqrt{2} \sin(n\pi\, \cdot) \quad \forall\, v \in L^2(0, 1),$$

where the linear functional γ_n is defined by

$$\gamma_n(v) := \sqrt{2} \int_0^1 v(x_1) \sin(n\pi x_1) dx_1.$$

We note that $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$ with $U = L^2(0,1)$. For each $n \in \mathbb{N}$, the scalar-valued function

$$s \mapsto \frac{1}{\sqrt{s + n^2\pi^2}\ \tanh(\sqrt{s + n^2\pi^2})}$$

is positive real, implying the positve realness of $\mathbf{H}$.

(b) Let $A : D(A) \subset U \to U$ be densely defined and closed, and let $\mathbf{H}$ be the resolvent of A, that is, $\mathbf{H}(s) = (sI - A)^{-1}$. Then the following statements are equivalent.

(1) A is dissipative and there exists $z \in \mathbb{C}_0$ such that $z \notin \sigma(A)$.

(2) $\mathbf{H}$ is positive real.

(3) A is the generator of a strongly continuous contraction semigroup.

The equivalence of statements (1) and (3) is the Lumer-Phillips Theorem (see, for example, [28, Theorem 3.4.8]), and the equivalence of statements (1) and (2) is proved in [13]. Note that, under the assumption that (1) holds, $0 \in \sigma_c(\mathbf{H}(s))$ for every $s \in \mathbb{C}_0$ if A is unbounded, whilst there does not exist $s \in \mathbb{C}_0$ such that $0 \in \sigma(\mathbf{H}(s))$ if A is bounded. ◊

3 Spectral properties of positive-real functions: the case of finite-dimensional U

In this section, we assume that $\dim U = m < \infty$, that is, $U = \mathbb{C}^m$, for some positive integer m.

Proposition 3.1. *Assume that $\mathbf{H} \in \mathcal{H}_0(\mathbb{C}^{m\times m})$ is positive real. Then the set $\mathbb{I} \cap \sigma(\mathbf{H}(s))$ does not depend on s.*

Proof. Let $z \in \mathbb{C}_0$ and $\lambda \in \mathbb{I}$ be fixed and assume that $\lambda \notin \sigma(\mathbf{H}(z))$. It is sufficient to show that $\lambda \notin \sigma(\mathbf{H}(s))$ for all $s \in \mathbb{C}_0$. To this end, set $\mathbf{G}(s) := \mathbf{H}(s) - \lambda I$. Obviously, $\mathbf{G} \in \mathcal{H}_0(\mathbb{C}^{m\times m})$, $\mathbf{G}$ is positive real, $\mathbf{G}(z)$ is invertible and, for each $s \in \mathbb{C}_0$, the invertibility of $\mathbf{G}(s)$ is equivalent to $\lambda \notin \sigma(\mathbf{H}(s))$. We therefore have to show that $\mathbf{G}(s)$ is invertible for $s \in \mathbb{C}_0$. Since $\mathbf{G}(z)$ is invertible, the set of zeros of $\det \mathbf{G}(s)$ does not have accumulation points in $\mathbb{C}_0$ and therefore, $\mathbf{G}^{-1}$ is a meromorphic $\mathbb{C}^{m\times m}$-valued function on $\mathbb{C}_0$, and, *a fortiori*, $\mathbf{G}^{-1} \in \mathcal{H}_0^*(\mathbb{C}^{m\times m})$. By statement (2) of Lemma 2.1, $\operatorname{Re} \mathbf{G}(s) \succeq 0$ for $s \in \mathbb{C}_0$ such that $\det \mathbf{G}(s) \neq 0$, showing that $\mathbf{G}^{-1}$ is positive real. Proposition 2.3 now guarantees that $\mathbf{G}^{-1}$ does not have any singularities in $\mathbb{C}_0$, implying that $\mathbf{G}(s)$ is invertible for $s \in \mathbb{C}_0$. □

The following corollary is an immediate consequence of Proposition 3.1.

Corollary 3.2. *Assume that $\mathbf{H} \in \mathcal{H}_0(\mathbb{C}^{m\times m})$ is positive real. If there exists $z \in \mathbb{C}_0$ such that $\mathbf{H}(z)$ is invertible, then $\mathbf{H}(s)$ is invertible for all $s \in \mathbb{C}_0$. Equivalently, if $\det \mathbf{H}(z) = 0$ for some $z \in \mathbb{C}$, then $\det \mathbf{H}(s) = 0$ for all $s \in \mathbb{C}_0$.*

Another consequence of Proposition 3.1 is the following result in which the focus is on the scalar case ($m = 1$).

Corollary 3.3. *Assume that $\mathbf{H} \in \mathcal{H}_0$ is positive real. If there exists $z \in \mathbb{C}_0$ such that $\operatorname{Re} \mathbf{H}(z) = 0$, then $\mathbf{H}(s) = i \operatorname{Im} \mathbf{H}(z)$ for all $s \in \mathbb{C}_0$, and, in particular, $\operatorname{Re} \mathbf{H}(s) = 0$ for all $s \in \mathbb{C}_0$.*

Of course, Corollary 3.3 is well-known, but the way we have derived it here seems to be new. The usual proof involves an application of the maximum principle for holomorphic (harmonic) functions to the function $s \mapsto e^{-\mathbf{H}(s)}$ ($s \mapsto -\operatorname{Re} \mathbf{H}(s)$) to establish that $\operatorname{Re} \mathbf{H}(s) \equiv 0$, followed by an application of the Cauchy-Riemann equations (or, alternatively, of the open mapping theorem for holomorphic functions) to show that $\mathbf{H}(s) \equiv i \operatorname{Im} \mathbf{H}(z)$.

4 Spectral properties of positive-real functions: the case of infinite-dimensional U

We start by stating the generalization of Proposition 3.1 to the operator-valued case.

Proposition 4.1. *Assume that $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$ is positive real. Then the set $\mathbb{I} \cap \sigma(\mathbf{H}(s))$ does not depend on s.*

The proof of Proposition 3.1 relies on the fact that, for $\mathbf{H} \in \mathcal{H}_0(\mathbb{C}^{m\times m})$ and $\lambda \in \mathbb{C}$, the set $\{s \in \mathbb{C}_0 : \mathbf{H}(s) - \lambda I \text{ is not invertible}\}$ does not have any accumulation points in $\mathbb{C}_0$, provided that $\mathbf{H}(z) - \lambda I$ is invertible for some z. The following simple example shows that this does not necessarily hold when $\dim U = \infty$.

Example 4.2. Let $U = \ell^2(\mathbb{N}, \mathbb{C})$, let $(\xi_j)_{j\in\mathbb{N}}$ be a bounded sequence in $\mathbb{C}_0$ and set $\Xi := \{\xi_j : j \in \mathbb{N}\}$. Then the function $\mathbf{H} : \mathbb{C}_0 \to \mathcal{L}(U)$ given by

$$\mathbf{H}(s) := \operatorname{diag}_{j\in\mathbb{N}}(s - \xi_j) \quad \forall\, s \in \mathbb{C}_0 \tag{4.1}$$

is in $\mathcal{H}_0(\mathcal{L}(U))$ and $0 \in \sigma_{\mathrm{p}}(\mathbf{H}(\xi))$ for every $\xi \in \Xi$

(a) Assume that $\xi_j \to \xi_\infty \in \mathbb{C}_0$ as $j \to \infty$ and $\xi_j \neq \xi_\infty$ for all $j \in \mathbb{N}$. Obviously, $\operatorname{cl}\Xi = \Xi \cup \{\xi_\infty\} \subset \mathbb{C}_0$, $\mathbf{H}(z)$ is invertible for every $z \in \mathbb{C}_0\backslash\operatorname{cl}\Xi$, $0 \in \sigma_{\mathrm{c}}(\mathbf{H}(\xi_\infty))$ and $\xi_\infty \in \mathbb{C}_0$ is an accumulation point of the set $\{s \in \mathbb{C}_0 : \mathbf{H}(s) \text{ is not invertible}\} = \operatorname{cl}\Xi$.

(b) It is easy to construct more striking examples. Let $\Gamma \subset \mathbb{C}_0$ be a compact set (a closed disc, for example) and let $(\xi_j)_{j\in\mathbb{N}}$ be an enumeration of the countable set $\Gamma \cap \{p + iq : p, q \in \mathbb{Q}\}$. Then $\Gamma = \operatorname{cl}\Xi$, $\mathbf{H}(z)$ is invertible for every $z \in \mathbb{C}_0\backslash\Gamma$, $0 \in \sigma_{\mathrm{c}}(\mathbf{H}(\zeta))$ for every $\zeta \in \Gamma\backslash\Xi$ and $\{s \in \mathbb{C}_0 : \mathbf{H}(s) \text{ is not invertible}\} = \Gamma$. ◊

Whilst the above example shows that the above proof of Proposition 3.1 does not extend to the operator-valued case, there is an alternative argument which is based on certain subharmonicity properties of the spectrum [3, 30]. For the basic theory of subharmonic functions we the reader to [22, 24].

Proof of Proposition 4.1. By [30, Proposition 2.5] (or [3, Corollary 3.4.9]) the function

$$\theta : \mathbb{C}_0 \to \mathbb{R},\ s \mapsto \max\{\operatorname{Re}\lambda : \lambda \in \sigma(-\mathbf{H}(s))\}$$

is subharmonic. Invoking statement (1) of Lemma 2.1, we have that $\theta(s) \le 0$ for all $s \in \mathbb{C}_0$. Let $z \in \mathbb{C}_0$ and assume that $\mathbb{I} \cap \sigma(\mathbf{H}(z)) \neq \emptyset$. Then $\theta(z) = 0$, and thus, the maximum principle for subharmonic functions implies that $\theta(s) \equiv 0$. It follows now from [30, Proposition 2.10] that the set $\mathbb{I} \cap \sigma(\mathbf{H}(s))$ does not depend on s. □

The following corollary is a straightforward consequence of statement (2) of Lemma 2.1 and Proposition 4.1.

Corollary 4.3. *Assume that $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$ is positive real and there exists $z \in \mathbb{C}_0$ such that $\mathbf{H}(z)$ is invertible (that is, $0 \notin \sigma(\mathbf{H}(z))$. Then $\mathbf{H}(s)$ is invertible for every $s \in \mathbb{C}_0$ and $\mathbf{H}^{-1}$ is positive real.*

The main result of this paper (see Theorem 4.8 below) is a refinement of Proposition 4.1: the point, continuous and residual spectra will be considered separately and properties of eigenvectors and eigensequences will be proved. To this end, it is convenient to establish first some spectral properties of the real part of a positive real function.

Theorem 4.4. *Assume that $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$ is positive real and let $z \in \mathbb{C}_0$. The following statements hold.*

(1) *For every* $s \in \mathbb{C}_0$, $0 \notin \sigma_{\mathrm{r}}(\operatorname{Re}\mathbf{H}(s))$.

(2) *If* $0 \in \sigma_{\mathrm{p}}(\operatorname{Re}\mathbf{H}(z))$, *then* $0 \in \sigma_{\mathrm{p}}(\operatorname{Re}\mathbf{H}(s))$ *for all* $s \in \mathbb{C}_0$ *and the subspace* $\ker(\operatorname{Re}\mathbf{H}(s))$ *does not depend on* s.

(3) *If* $0 \in \sigma_{\mathrm{ap}}(\operatorname{Re}\mathbf{H}(z))$, *then* $0 \in \sigma_{\mathrm{ap}}(\operatorname{Re}\mathbf{H}(s))$ *for all* $s \in \mathbb{C}_0$, *and, furthermore, if* $(u_j)_{j\in\mathbb{N}}$ *is an eigensequence of* $\operatorname{Re}\mathbf{H}(z)$ *associated with the approximate eigenvalue* 0, *then there exists a subsequence* $(u_{j_k})_{k\in\mathbb{N}}$ *of* $(u_j)_{j\in\mathbb{N}}$ *which, for every* $s \in \mathbb{C}_0$, *is an eigensequence of* $\operatorname{Re}\mathbf{H}(s)$ *associated with* 0.

(4) *If* $0 \in \sigma_{\mathrm{c}}(\operatorname{Re}\mathbf{H}(z))$, *then* $0 \in \sigma_{\mathrm{c}}(\operatorname{Re}\mathbf{H}(s))$ *for all* $s \in \mathbb{C}_0$.

Proof. Applying statement (4) of Lemma 2.1 with $T = \operatorname{Re}\mathbf{H}(s)$, we obtain that $0 \notin \sigma_{\mathrm{r}}(\operatorname{Re}\mathbf{H}(s))$ for all $s \in \mathbb{C}_0$, establishing statement (1).

To prove statement (2), assume that $0 \in \sigma_{\mathrm{p}}(\operatorname{Re}\mathbf{H}(z))$ and let u be an arbitrary element in $\ker(\operatorname{Re}\mathbf{H}(z))$. Defining a scalar-valued function $h \in \mathcal{H}_0$ by $h(s) = \langle \mathbf{H}(s)u, u\rangle$, it is clear that h is positive real and $\operatorname{Re} h(z) = 0$. Hence, by Corollary 3.3, $\operatorname{Re} h(s) = 0$ for all $s \in \mathbb{C}_0$. By positive realness of $\mathbf{H}$, the operator $\operatorname{Re}\mathbf{H}(s)$ has a self-adjoint positive semi-definite square root $(\operatorname{Re}\mathbf{H}(s))^{1/2}$ for each $s \in \mathbb{C}_0$. Consequently,

$$\|(\operatorname{Re}\mathbf{H}(s))^{1/2}u\|^2 = \langle (\operatorname{Re}\mathbf{H}(s))^{1/2}u, (\operatorname{Re}\mathbf{H}(s))^{1/2}u\rangle = \langle (\operatorname{Re}\mathbf{H}(s))u, u\rangle = \operatorname{Re} h(s) = 0 \quad \forall\, s \in \mathbb{C}_0.$$

Hence, $(\operatorname{Re}\mathbf{H}(s))u = 0$ for all $s \in \mathbb{C}_0$ and thus $0 \in \sigma_{\mathrm{p}}(\operatorname{Re}\mathbf{H}(s))$ and $u \in \ker(\operatorname{Re}\mathbf{H}(s))$ for all $s \in \mathbb{C}_0$, establishing statement (2).

We proceed to prove statement (3). Assume that $0 \in \sigma_{\mathrm{ap}}(\operatorname{Re}\mathbf{H}(z))$ and let $(u_j)_{j\in\mathbb{N}}$ be an eigensequence of $\operatorname{Re}\mathbf{H}(z)$ corresponding to the approximate eigenvalue 0, that is, $u_j \in \mathbb{E}_U$ for all $j \in \mathbb{N}$ and $(\operatorname{Re}\mathbf{H}(z))u_j \to 0$ as $j \to \infty$. We define a sequence $(h_j)_{j\in\mathbb{N}}$ of functions in $\mathcal{H}_0$ by setting

$$h_j(s) := \langle \mathbf{H}(s)u_j, u_j\rangle \quad \forall\, s \in \mathbb{C}_0,\ \forall\, j \in \mathbb{N}.$$

Then, obviously, h_j is positive real for all $j \in \mathbb{N}$ and $\operatorname{Re} h_j(z) \to 0$ as $j \to \infty$. Moreover,

$$|h_j(s)| \le \|\mathbf{H}(s)\| \quad \forall\, s \in \mathbb{C}_0,\ \forall\, j \in \mathbb{N},$$

showing that the sequence $(h_j)_{j\in\mathbb{N}}$ is locally bounded. An application of Montel's theorem [22, Theorem 2, p. 34] yields the existence of a subsequence $(h_{j_k})_{k\in\mathbb{N}}$ which converges locally uniformly to a function $g \in \mathcal{H}_0$ as $k \to \infty$. It is clear that g is positive real and $\operatorname{Re} g(z) = 0$, and thus, by Corollary 3.3, $\operatorname{Re} g(s) = 0$ for all $s \in \mathbb{C}_0$. As a consequence, for every $s \in \mathbb{C}_0$,

$$\langle \operatorname{Re}\mathbf{H}(s)u_{j_k}, u_{j_k}\rangle = \operatorname{Re}\langle \mathbf{H}(s)u_{j_k}, u_{j_k}\rangle = \operatorname{Re} h_{j_k}(s) \to \operatorname{Re} g(s) = 0 \quad \text{as } k \to \infty.$$

Thus, for every $s \in \mathbb{C}_0$, $(\operatorname{Re}\mathbf{H}(s))^{1/2}u_{j_k} \to 0$ as $k \to \infty$, which in turn implies that, for every $s \in \mathbb{C}_0$, $(\operatorname{Re}\mathbf{H}(s))u_{j_k} \to 0$ as $k \to \infty$. We conclude that, for all $s \in \mathbb{C}_0$, $0 \in \sigma_{\mathrm{ap}}(\operatorname{Re}\mathbf{H}(s))$ and the subsequence $(u_{j_k})_{k\in\mathbb{N}}$ of $(u_j)_{j\in\mathbb{N}}$ is an eigensequence of $\operatorname{Re}\mathbf{H}(s)$ associated with 0.

Finally, to prove statement (4), assume that $0 \in \sigma_{\mathrm{c}}(\operatorname{Re}\mathbf{H}(z))$. As $\sigma_{\mathrm{c}}(\operatorname{Re}\mathbf{H}(z)) \subset \sigma_{\mathrm{ap}}(\operatorname{Re}\mathbf{H}(z))$, it follows from statement (3), that $0 \in \sigma_{\mathrm{ap}}(\operatorname{Re}\mathbf{H}(s))$ for all $s \in \mathbb{C}_0$. Since $0 \in \sigma_{\mathrm{c}}(\operatorname{Re}\mathbf{H}(z))$, statement (2) yields that $0 \notin \sigma_{\mathrm{p}}(\operatorname{Re}\mathbf{H}(s))$ for all $s \in \mathbb{C}_0$. Furthermore, appealing to statement (1), we see that $0 \notin \sigma_{\mathrm{r}}(\operatorname{Re}\mathbf{H}(s))$ for all $s \in \mathbb{C}_0$. It now follows that $0 \in \sigma_{\mathrm{c}}(\operatorname{Re}\mathbf{H}(s))$ for every $s \in \mathbb{C}_0$, completing the proof. □

An interesting consequence of Theorem 4.4 is provided by the following corollary.

Corollary 4.5. *Assume that* $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$ *is positive real and let* $z \in \mathbb{C}_0$. *If* $\operatorname{Re}\mathbf{H}(z)$ *is invertible, then, for every compact set* $\Gamma \subset \mathbb{C}_0$, *there exists* $\gamma > 0$ *such that* $\operatorname{Re}\mathbf{H}(s) \succeq \gamma I$ *for all* $s \in \Gamma$.

Proof. Define the function $h : \mathbb{C}_0 \to [0, \infty)$ by

$$h(s) := \inf_{u \in \mathbb{E}_U} \operatorname{Re} \langle \mathbf{H}(s)u, u\rangle = \inf_{u \in \mathbb{E}_U} \langle (\operatorname{Re} \mathbf{H}(s))u, u\rangle \quad \forall\, s \in \mathbb{C}_0,$$

and note that h is continuous. We proof the claim by contraposition. To this end, assume that there exists compact $\Gamma \subset \mathbb{C}_0$ such that there does not exist $\gamma > 0$ with $\operatorname{Re} \mathbf{H}(s) \succeq \gamma I$ for all $s \in \Gamma$, in which case $\min_{s\in\Gamma} h(s) = 0$. Consequently, $h(s_0) = 0$ for some $s_0 \in \Gamma$, and so

$$\inf_{u \in \mathbb{E}_U} \|(\operatorname{Re} \mathbf{H}(s_0))^{1/2} u\|^2 = \inf_{u \in \mathbb{E}_U} \langle (\operatorname{Re} \mathbf{H}(s_0))u, u\rangle = 0,$$

showing that $(\operatorname{Re} \mathbf{H}(s_0))^{1/2}$ is not invertible. Consequently, $\operatorname{Re} \mathbf{H}(s_0)$ is not invertible, and so, by Theorem 4.4, for every $s \in \mathbb{C}_0$, the operator $\operatorname{Re} \mathbf{H}(s)$ is not invertible, completing the contraposition argument. □

Recall that $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$, where $\alpha < 0$, is said to be *strictly positive real* if there exists $\beta \in [\alpha, 0)$ such that the function $s \mapsto \mathbf{H}(s + \beta)$ is positive real (in which case $\mathbf{H}$ is holomorphic on $\mathbb{C}_\beta$).

Corollary 4.6. *Let $\alpha < 0$, $\mathbf{H} \in \mathcal{H}_\alpha(\mathcal{L}(U))$ and $D \in \mathcal{L}(U)$. Assume that $\mathbf{H}$ is strictly positive real and $\mathbf{H}(s)$ converges to D in the uniform topology of $\mathcal{L}(U)$ as $|s| \to \infty$ in $\mathbb{C}_0$. If $\operatorname{Re} D$ is invertible, then there exists $\gamma > 0$ such that $\operatorname{Re} \mathbf{H}(s) \succeq \gamma I$ for all $s \in \mathbb{C}_0$.*

A function $\mathbf{H} \in \mathcal{H}^*_0(\mathcal{L}(U))$ for which there exists $\gamma > 0$ such that $\operatorname{Re} \mathbf{H}(s) \succeq \gamma I$ for all $s \in \mathbb{C}_0$ which are are not singularities of $\mathbf{H}$ is somestimes called strongly positive real, see, for example, [13]. Adopting this terminology, Corollary 4.6 says that strict positive realness together with the existence of a uniform limit with invertible real part guarantees strong positive realness.

Proof of Corollary 4.6. By the positive realness of $\mathbf{H}$ and the convergence assumption, it is clear that $\operatorname{Re} D \succeq 0$, and, so by invertibilty of $\operatorname{Re} D$, there exists $\gamma_1 > 0$ such that $\operatorname{Re} D \succeq 2\gamma_1 I$. Consequently, there exists $\rho > 0$, such that $\operatorname{Re} \mathbf{H}(s) \succeq \gamma_1 I$ for all $s \in \operatorname{cl} \mathbb{C}_0$ with $|s| > \rho$. By strict positive realness of $\mathbf{H}$, there exists $\beta \in [\alpha, 0)$ such that $\tilde{\mathbf{H}}$, defined by $\tilde{\mathbf{H}}(s) := \mathbf{H}(s + \beta)$ for all $s \in \mathbb{C}_0$, is positive real. Choose $z \in \mathbb{C}_0$ such that $z + \beta \in \mathbb{C}_0$ and $|z + \beta| > \rho$. Then $\operatorname{Re} \tilde{\mathbf{H}}(z) = \operatorname{Re} \mathbf{H}(z + \beta) \succeq \gamma_1 I$, implying that $\operatorname{Re} \tilde{\mathbf{H}}(z)$ is invertible. Setting $\Gamma := \{s \in \operatorname{cl} \mathbb{C}_0 : |s| \le \rho\}$, it is clear that the set $\tilde{\Gamma} := \Gamma - \beta = \Gamma + |\beta|$ is compact and contained in $\mathbb{C}_0$, and an application of Corollary 4.6 to $\tilde{\mathbf{H}}$ yields the existence of a number $\gamma_2 > 0$ such that $\operatorname{Re} \tilde{\mathbf{H}}(s) \ge \gamma_2 I$ for all $s \in \tilde{\Gamma}$. Consequently, $\operatorname{Re} \mathbf{H}(s) \ge \gamma_2 I$ for all $s \in \Gamma$. Setting $\gamma := \min(\gamma_1, \gamma_2) > 0$, it now follows that $\operatorname{Re} \mathbf{H}(s) \ge \gamma I$ for all $s \in \mathbb{C}_0$, completing the proof. □

The following result on the numerical range of a positive-real function can be proved by arguments similar to those used in the proof of Theorem 4.4, and we therefore omit the details.

Proposition 4.7. *Assume that $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$ is positive real. The following statements hold.*

(1) *The set $\mathbb{I} \cap W(\mathbf{H}(s))$ does not depend on s. If there exists $z \in \mathbb{C}_0$ such that $\mathbb{I} \cap W(\mathbf{H}(z)) \ne \emptyset$, then, for every $u \in \mathbb{E}_U$ such that $\langle \mathbf{H}(z)u, u\rangle \in \mathbb{I}$, it follows that $\langle \mathbf{H}(s)u, u\rangle = \langle \mathbf{H}(z)u, u\rangle$ for all $s \in \mathbb{C}_0$.*

(2) *The set $\mathbb{I} \cap \operatorname{cl} W(\mathbf{H}(s))$ does not depend on s. If there exists $z \in \mathbb{C}_0$ and $\lambda \in \mathbb{I}$ such that $\lambda \in \operatorname{cl} W(\mathbf{H}(z)) \backslash W(\mathbf{H}(z))$, then there exists a sequence $(u_j)_{j\in\mathbb{N}}$ in $\mathbb{E}_U$, such that $\langle \mathbf{H}(s)u_j, u_j\rangle \to \lambda$ as $j \to \infty$ for all $s \in \mathbb{C}_0$.*

We are now in the position to state and prove the main result of this paper.

Theorem 4.8. *Assume that $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$ is positive real. The following statements hold.*

(1) *For all $s \in \mathbb{C}_0$, $\mathbb{I} \cap \sigma_{\mathrm{r}}(\mathbf{H}(s)) = \emptyset$.*

(2) *The set $\mathbb{I} \cap \sigma_{\mathrm{p}}(\mathbf{H}(s))$ does not depend on s and, for every $\lambda \in \mathbb{I}$, the subspace $\ker(\mathbf{H}(s) - \lambda I)$ is independent of s.*

(3) *The set* $\mathbb{I} \cap \sigma_{\mathrm{c}}(\mathbf{H}(s))$ *does not depend on* s*. Moreover, if there exists* $z \in \mathbb{C}_0$ *such that* $\mathbb{I} \cap \sigma_{\mathrm{c}}(\mathbf{H}(z)) \neq \emptyset$ *and if* $(u_j)_{j\in\mathbb{N}}$ *is an eigensequence of* $\mathbf{H}(z)$ *associated with the approximate eigenvalue* $\lambda \in \mathbb{I} \cap \sigma_{\mathrm{c}}(\mathbf{H}(z))$*, then there exists a subsequence* $(u_{j_k})_{k\in\mathbb{N}}$ *of* $(u_j)_{j\in\mathbb{N}}$ *which, for every* $s \in \mathbb{C}_0$*, is an eigensequence of* $\mathbf{H}(s)$ *associated with* λ*.*

It is obvious that Proposition 4.1 can be obtained as an immediate corollary of Theorem 4.8.

Proof of Theorem 4.8. Statement (1) follows immediately from statement (4) of Lemma 2.1. To establish statements (2) and (3), we will use a representation theorem for positive-real functions which guarantees that there exist $H_0, H_1 \in \mathcal{L}(U)$ and a $\mathcal{L}(U)$-valued measure H [33, Definition 2.2-1] defined on the σ-algbra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$ such that $H_0^* = H_0$, $H_1 \succeq 0$, $H(B) \succeq 0$ for all $B \in \mathcal{B}$ and

$$\mathbf{H}(s) = iH_0 + sH_1 + \int_{-\infty}^{\infty} \frac{1 - i\tau s}{s - i\tau} \mathrm{d}H(\tau) \quad \forall\, s \in \mathbb{C}_0, \tag{4.2}$$

see [27, Satz 9.2] or [33, Theorem 8.11-2].[†] Note that (4.2) implies in particular

$$\operatorname{Re} \mathbf{H}(1) = H_1 + H(\mathbb{R}). \tag{4.3}$$

It is useful to note that, as the map $B \mapsto \langle H(B)u, u\rangle$ defines a finite positive measure on $\mathcal{B}$ [33, Theorem 2.2-1], we have

$$H(\mathbb{R}) \succeq H(B) \quad \forall\, B \in \mathcal{B}. \tag{4.4}$$

Furthermore, for every $s \in \mathbb{C}_0$, let $(f_{s,n})_{n\in\mathbb{N}}$ be a sequence of simple complex-valued functions defined on $\mathbb{R}$ (that is, $f_{n,s}$ is Borel measurable and attains only a finite number of values) such that

$$\sup_{\tau\in\mathbb{R}} \left| \frac{1 - i\tau s}{s - i\tau} - f_{s,n}(\tau) \right| \to 0 \ \text{ as } n \to \infty \quad \forall\, s \in \mathbb{C}_0.$$

We then have that

$$\int_{-\infty}^{\infty} f_{s,n}(\tau) \mathrm{d}H(\tau) \to \int_{-\infty}^{\infty} \frac{1 - i\tau s}{s - i\tau} \mathrm{d}H(\tau) \ \text{ in } \mathcal{L}(U) \text{ as } n \to \infty, \quad \forall\, s \in \mathbb{C}_0. \tag{4.5}$$

We proceed to prove statement (2). To this end, let $z \in \mathbb{C}_0$ and assume that $\mathbb{I} \cap \sigma_{\mathrm{p}}(\mathbf{H}(z)) \neq \emptyset$. Let $\lambda \in \mathbb{I} \cap \sigma_{\mathrm{p}}(\mathbf{H}(z))$ and $u \in \ker(\mathbf{H}(z) - \lambda I)$. Consequently, as $2\operatorname{Re} \mathbf{H}(z) = (\mathbf{H}(z) - \lambda I) + (\mathbf{H}^*(z) + \lambda I)$, it follows from statement (3) of Lemma 2.1 that $\operatorname{Re} \mathbf{H}(z)u = 0$. Invoking statement (2) of Theorem 4.4, we obtain that $u \in \ker(\operatorname{Re} \mathbf{H}(1))$. Hence, as $H_1 \succeq 0$ and $H(\mathbb{R}) \succeq 0$, it follows from (4.3) that

$$H_1 u = 0, \tag{4.6}$$

and, furthermore, $H(\mathbb{R})u = 0$. Combining the latter with (4.4), we obtain that $H(B)u = 0$ for all $B \in \mathcal{B}$, and so,

$$\left(\int_{-\infty}^{\infty} f_{s,n}(\tau) \mathrm{d}H(\tau) \right) u = 0 \quad \forall\, n \in \mathbb{N},\ \forall\, s \in \mathbb{C}_0. \tag{4.7}$$

Together with (4.5) this yields

$$\left(\int_{-\infty}^{\infty} \frac{1 - i\tau s}{s - i\tau} \mathrm{d}H(\tau) \right) u = 0 \quad \forall\, s \in \mathbb{C}_0.$$

Invoking (4.2) and (4.6), we conclude that $(\mathbf{H}(s) - \lambda I)u = (iH_0 - \lambda I)u$ for all $s \in \mathbb{C}_0$. But $(\mathbf{H}(z) - \lambda I)u = 0$, showing that $(iH_0 - \lambda I)u = 0$. Therefore, $(\mathbf{H}(s) - \lambda I)u = 0$ for all $s \in \mathbb{C}_0$, establishing statement (2).

[†] The representation formula (4.2) is the operator-valued version of a result by Cauer [6] which in turn is the right-half plane version of a theorem due to Herglotz, see, for example, [5, Theorem 31].

To prove statement (3), let $z \in \mathbb{C}_0$, assume that $\mathbb{I} \cap \sigma_c(\mathbf{H}(z)) \neq \emptyset$, let $\lambda \in \mathbb{I} \cap \sigma_c(\mathbf{H}(z))$ and let $(u_j)_{j\in\mathbb{N}}$ be an eigensequence of $\mathbf{H}(z)$ corresponding to λ, that is,

$$\lim_{j\to\infty} (\mathbf{H}(z) - \lambda I)u_j = 0, \qquad \|u_j\| = 1 \quad \forall\, j \in \mathbb{N}.$$

As an obvious consequence we have that

$$\langle \operatorname{Re} \mathbf{H}(z)u_j, u_j\rangle = \operatorname{Re} \langle \mathbf{H}(z)u_j, u_j\rangle = \operatorname{Re} \langle (\mathbf{H}(z) - \lambda I)u_j, u_j\rangle \to 0 \quad \text{as } j \to \infty.$$

As $\operatorname{Re} \mathbf{H}(z) \succeq 0$, we conclude that $\operatorname{Re} \mathbf{H}(z)u_j \to 0$ as $j \to \infty$, and so, in particular, $0 \in \sigma_{\mathrm{ap}}(\operatorname{Re} \mathbf{H}(z))$. Statement (3) of Theorem 4.4 now guarantees the existence of a subsequence $(u_{j_k})_{k\in\mathbb{N}}$ of $(u_j)_{j\in\mathbb{N}}$ such that $\operatorname{Re} \mathbf{H}(1)u_{j_k} \to 0$ as $k \to \infty$. Hence, as $H_1 \succeq 0$ and $H(\mathbb{R}) \succeq 0$, it follows from (4.3) that

$$H_1 u_{j_k} \to 0 \quad \text{as } k \to \infty \tag{4.8}$$

and $H(\mathbb{R})u_{j_k} \to 0$ as $k \to \infty$. The latter, together with (4.4) and the fact that $H(B) \succeq 0$ for all $B \in \mathcal{B}$ implies that $H(B)u_{j_k} \to 0$ as $k \to \infty$ for every $B \in \mathcal{B}$. Therefore,

$$\left(\int_{-\infty}^{\infty} f_{s,n}(\tau)\mathrm{d}H(\tau)\right) u_{j_k} \to 0 \ \text{ as } k \to \infty, \quad \forall\, n \in \mathbb{N},\ \forall\, s \in \mathbb{C}_0. \tag{4.9}$$

For each $s \in \mathbb{C}_0$ and $n \in \mathbb{N}$, we define operators $J(s)$ and $J_n(s)$ in $\mathcal{L}(U)$ by setting

$$J(s) := \int_{-\infty}^{\infty} \frac{1 - i\tau s}{s - i\tau}\mathrm{d}H(\tau) \quad \text{and} \quad J_n(s) := \int_{-\infty}^{\infty} f_{s,n}(\tau)\mathrm{d}H(\tau)$$

Let $\varepsilon > 0$. By (4.5), for every $s \in \mathbb{C}_0$, there exists $m = m(s) \in \mathbb{N}$ such that

$$\|J_m(s) - J(s)\| \le \frac{\varepsilon}{2}. \tag{4.10}$$

Moreover, (4.9) shows that, for every $s \in \mathbb{C}_0$, there exists $k^* = k^*(s) \in \mathbb{N}$ such that

$$\|J_m(s)u_{j_k}\| \le \frac{\varepsilon}{2} \quad \forall\, k \ge k^*. \tag{4.11}$$

Now, for every $s \in \mathbb{C}_0$ and every $k \in \mathbb{N}$,

$$\|J(s)u_{j_k}\| \le \|J(s)u_{j_k} - J_m(s)u_{j_k}\| + \|J_m(s)u_{j_k}\| \le \|J(s) - J_m(s)\| + \|J_m(s)u_{j_k}\|.$$

Thus, by (4.10) and (4.11), we have that, for every $s \in \mathbb{C}_0$,

$$\|J(s)u_{j_k}\| \le \varepsilon \quad \forall\, k \ge k^*,$$

whence

$$\left(\int_{-\infty}^{\infty} \frac{1 - i\tau s}{s - i\tau}\mathrm{d}H(\tau)\right) u_{j_k} = J(s)u_{j_k} \to 0 \quad \text{as } k \to \infty, \quad \forall\, s \in \mathbb{C}_0. \tag{4.12}$$

As $(\mathbf{H}(z) - \lambda I)u_{j_k} \to 0$ as $k \to \infty$ by hypothesis, it now follows from (4.8) and the representation formula (4.2) with $s = z$ that $(iH_0 - \lambda I)u_{j_k} \to 0$ as $k \to \infty$. Another application of (4.2) together with (4.8) and (4.12) now shows that, for every $s \in \mathbb{C}_0$, $(\mathbf{H}(s) - \lambda I)u_{j_k} \to 0$ as $k \to \infty$. Hence, for every $s \in \mathbb{C}_0$, (u_{j_k}) is an eigensequence of $\mathbf{H}(s)$ associated with λ. As $\lambda \in \sigma_c(\mathbf{H}(z))$, it is now a consequence of statements (1) and (2) that $\lambda \in \sigma_c(\mathbf{H}(s))$ for all $s \in \mathbb{C}_0$, completing the proof. ☐

An alternative proof of statement (2) of Theorem 4.8 which does not rely on the representation formula (4.2) can be found in the Appendix.

The following corollary shows that if $\mathbf{H} \in \mathcal{H}_0^*(\Omega, \mathcal{L}(U))$ where $\Omega \supset \mathbb{C}_0$ and if the restriction of $\mathbf{H}$ to $\mathbb{C}_0$ is positive real, then some of the spectral properties of the restriction are inherited by $\mathbf{H}(s)$ for all s in $\Omega\backslash\mathbb{C}_0$ which are not singularities of $\mathbf{H}$.

Corollary 4.9. *Let $\Omega \subset \mathbb{C}$ be open, connected and such that $\mathbb{C}_0 \subset \Omega$. Let $\mathbf{H} \in \mathcal{H}^*(\Omega, \mathcal{L}(U))$ and set $\Omega^* := \{s \in \Omega : s$ not a singularity of $\mathbf{H}\}$. Assume that $\mathbf{H}|_{\mathbb{C}_0}$ is positive real. Then $\mathbb{C}_0 \subset \Omega^*$ and the following statements hold.*

(1) *Let $z \in \mathbb{C}_0$ be such that $\mathbb{I} \cap \sigma_{\mathrm{p}}(\mathbf{H}(z)) \neq \emptyset$ and let $\lambda \in \mathbb{I} \cap \sigma_{\mathrm{p}}(\mathbf{H}(z))$. Then $\lambda \in \sigma_{\mathrm{p}}(\mathbf{H}(s))$ and $\ker(\mathbf{H}(z) - \lambda I) \subset \ker(\mathbf{H}(s) - \lambda)$ for all $s \in \Omega^*$.*

(2) *Let $z \in \mathbb{C}_0$ be such that $\mathbb{I} \cap \sigma_{\mathrm{c}}(\mathbf{H}(z)) \neq \emptyset$ and let $\lambda \in \mathbb{I} \cap \sigma_{\mathrm{c}}(\mathbf{H}(z))$. Then $\lambda \in \sigma_{\mathrm{ap}}(\mathbf{H}(s))$ for all $s \in \Omega^*$; furthermore, if $(u_j)_{j\in\mathbb{N}}$ is an eigensequence of $\mathbf{H}(z)$ associated with λ, then there exists a subsequence $(u_{j_k})_{k\in\mathbb{N}}$ of $(u_j)_{j\in\mathbb{N}}$ which, for all $s \in \Omega^*$, is an eigensequence of $\mathbf{H}(s)$ associated with λ.*

Proof. By hypothesis, $\mathbf{H}|_{\mathbb{C}_0}$ is positive real, and so Proposition 2.3 guarantees that $\mathbf{H}$ does not have any singularities in $\mathbb{C}_0$. Since $\mathbb{C}_0 \subset \Omega$, we conclude that $\mathbb{C}_0 \subset \Omega^*$.

To prove statement (1), let $\lambda \in \mathbb{I} \cap \sigma_{\mathrm{p}}(\mathbf{H}(z))$ and $u \in \ker(\mathbf{H}(z) - \lambda I)$. By statement (2) of Theorem 4.8, $u \in \ker(\mathbf{H}(s) - \lambda I)$ for all $s \in \mathbb{C}_0$, that is, $(\mathbf{H}(s) - \lambda I)u = 0$ for all $s \in \mathbb{C}_0$. As $\mathbb{C}_0 \subset \Omega^*$ and Ω^* is connected, the identity theorem yields that $(\mathbf{H}(s) - \lambda I)u = 0$ for all $s \in \Omega^*$. Consequently, $\ker(\mathbf{H}(z) - \lambda I) \subset \ker(\mathbf{H}(s) - \lambda I)$ and $\lambda \in \mathbb{I} \cap \sigma_{\mathrm{p}}(\mathbf{H}(s))$ for all $s \in \Omega^*$, establishing statement (1).

We proceed to prove statement (2). To this end, let $z \in \mathbb{C}_0$ be such that $\mathbb{I} \cap \sigma_{\mathrm{c}}(\mathbf{H}(z)) \neq \emptyset$, let $\lambda \in \mathbb{I} \cap \sigma_{\mathrm{c}}(\mathbf{H}(z))$ and let $(u_j)_{j\in\mathbb{N}}$ be an eigensequence of $\mathbf{H}(z)$ associated with λ. Invoking statement (3) of Theorem 4.8, there exists a subsequence $(v_l)_{l\in\mathbb{N}} := (u_{n_l})_{l\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ which, for all $s \in \mathbb{C}_0$, is an eigensequence of $\mathbf{H}(s)$ associated with λ, whence

$$\lim_{l\to\infty} (\mathbf{H}(s) - \lambda I)v_l = 0 \quad \forall\, s \in \mathbb{C}_0.$$

Setting $h_l(s) := (\mathbf{H}(s) - \lambda)v_l$ for all $s \in \Omega^*$ and all $l \in \mathbb{N}$, it is clear that $(h_l)_{l\in\mathbb{N}}$ is a locally bounded sequence in $\mathcal{H}(\Omega^*, U)$ and $h_l(s) \to 0$ for all $s \in \mathbb{C}_0$ as $l \to \infty$. By the vector-valued version of Vitali's theorem (see [2, Theorem 2.1] or [17, Theorem 3.14.1]), there exists a subsequence $(h_{l_k})_{k\in\mathbb{N}}$ of $(h_l)_{l\in\mathbb{N}}$ such that $(h_{l_k})_{k\in\mathbb{N}}$ converges locally uniformly to a function $h \in \mathcal{H}(\Omega^*, U)$ as $k \to \infty$. As $h_l(s) \to 0$ for all $s \in \mathbb{C}_0$ as $l \to \infty$, we see that $h(s) = 0$ for all $s \in \mathbb{C}_0$ and an application of the identity theorem shows that $h(s) = 0$ for all $s \in \Omega^*$. Hence, setting $j_k := n_{l_k}$, we obtain

$$\lim_{k\to\infty} (\mathbf{H}(s) - \lambda I)u_{j_k} = 0 \quad \forall\, s \in \Omega^*.$$

We conclude that, for every $s \in \Omega^*$, $\lambda \in \sigma_{\mathrm{ap}}(\mathbf{H}(s))$ and $(u_{j_k})_{k\in\mathbb{N}}$ is an eigensequence of $\mathbf{H}(s)$ associated with λ. □

Inspecting the proof of statement (2) and given that $\lambda \in \sigma(\mathbf{H}(s))$ for all $s \in \mathbb{C}_0$, it may be tempting to expect that there is a version of the identity theorem for the spectrum which ensures that $\lambda \in \sigma(\mathbf{H}(s))$ for all $s \in \Omega^*$ (and which could replace the use of the vector-valued Vitali theorem). However, in the absence of any compactness assumptions on $\mathbf{H}(s)$, such an identity theorem does not hold as Example 4.2 shows.

5 Strict positive realness through feedback

Proposition 4.1 (or, alternatively, Theorem 4.8) shows that if $\mathbf{H}$ is an operator-valued positive-real function and $\mathbf{H}(z)$ is invertible at some point $z \in \mathbb{C}_0$, then $\mathbf{H}(s)$ is invertible for all $s \in \mathbb{C}_0$. This invertibility property of positive-real functions will play a key role in this section.

Let $\mathbf{H} \in \mathcal{H}_0^*(\mathcal{L}(U))$ be positive real and let $K \in \mathcal{L}(U)$ be such that $\operatorname{Re} K \succeq \kappa I$ for some $\kappa > 0$. Then, by [13, Lemma 2], the operator $I + K\mathbf{H}(s)$ is invertible for every $s \in \mathbb{C}_0$, and so,

$$\mathbf{H}^K := \mathbf{H}(I + K\mathbf{H})^{-1} \in \mathcal{H}_0(\mathcal{L}(U)).$$

In fact, by [13, Theorem 6.4], $\mathbf{H}^K(s)$ is bounded on $\mathbb{C}_0$ as a function of s, that is, K is a stabilizing feedback for $\mathbf{H}$. Furthermore, letting $u \in U$, $s \in \mathbb{C}_0$, and setting $v := (I + K\mathbf{H}(s))^{-1}u$, we have that

$$\operatorname{Re}\langle \mathbf{H}^K(s)u, u\rangle = \operatorname{Re}\langle \mathbf{H}(s)v, (I + K\mathbf{H}(s))v\rangle = \operatorname{Re}\langle \mathbf{H}(s)v, v\rangle + \operatorname{Re}\langle \mathbf{H}(s)v, K\mathbf{H}(s)v\rangle \geq 0,$$

and we see that $\mathbf{H}^K$ is positive real. We will show that, under suitable additional assumptions, $\mathbf{H}^K$ is strictly positive real. This can be useful in the context of absolute stability theory where frequently strict positve realness assumptions are made.

Theorem 5.1. *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$ with $\alpha < 0$ and let $\beta \in (\alpha, 0)$. Assume that $\mathbf{H}|_{\mathbb{C}_0}$ is positive real and that there exists $z \in \mathbb{C}_0$ such that $\mathbf{H}(z)$ is invertible. Then $\mathbf{H}(s)$ is invertible for every $s \in \mathbb{C}_0$ and $\mathbf{H}^{-1}$ is positive real. Under the additional assumptions*

(A1) *$\mathbf{H}^{-1}$ extends holomorphically to $\operatorname{cl}\mathbb{C}_\beta$,*

(A2) *there exists $\kappa > 0$ such that*

$$\operatorname{Re}\langle \mathbf{H}^{-1}(\beta + i\omega)u, u\rangle \geq -\kappa \quad \omega \in \mathbb{R},\ \forall u \in \mathbb{E}_U,$$

(A3) $\liminf_{|s|\to\infty} \left(\operatorname{Re}\langle \mathbf{H}^{-1}(s)u, u\rangle / \log|s|\right) \geq 0$ *for all $u \in \mathbb{E}_U$, where the limes inferior is taken in the vertical strip $\beta \leq \operatorname{Re} s \leq 0$,*

the function $\mathbf{H}^K$ is strictly positive real for every $K \in \mathcal{L}(U)$ such that $\operatorname{Re} K \succeq \kappa I$.

Theorem 5.1 combined with loop-shifting can be useful in the context of absolute stability theory where frequently strict positve realness assumptions are made: a Lur'e system with positive-real linear part $\mathbf{H}$ and nonlinearity N can be replaced, via an application of Theorem 5.1 together with loop-shifting, by an equivalent Lur'e system with strictly positive-real linear part $\mathbf{H}^K$ and nonlinearity $N + K$.

Proof of Theorem 5.1. Since $\mathbf{H}|_{\mathbb{C}_0}$ is positive real and there exists $z \in \mathbb{C}_0$ such that $\mathbf{H}(z)$ is invertible, it follows from Corollary 4.3 that $\mathbf{H}(s)$ is invertible for all $s \in \mathbb{C}_0$ and $\mathbf{H}^{-1}$ is positive real. Let $K \in \mathcal{L}(U)$ be such that $\operatorname{Re} K \succeq \kappa I$. Noting that

$$\mathbf{H}^K = \mathbf{H}(I + K\mathbf{H})^{-1} = (\mathbf{H}^{-1} + K)^{-1},$$

it is clear that it is sufficient to show that $\mathbf{H}^{-1} + K$ is strictly positive real. Obviously, by assumption (A1), the function $s \mapsto \mathbf{H}^{-1}(\beta + s)$ is holomorphic on $\operatorname{cl}\mathbb{C}_0$ and, in particular, does not have any poles in $\operatorname{cl}\mathbb{C}_0$. Invoking assumption (A2), we obtain

$$\operatorname{Re}\langle (\mathbf{H}^{-1}(\beta + i\omega) + K)u, u\rangle \geq 0 \quad \forall \omega \in \mathbb{R},\ \forall u \in \mathbb{E}_U. \tag{5.1}$$

Furthermore, the positive realness of $\mathbf{H}^{-1}$ together with assumption (A3) yields

$$\liminf_{|s|\to\infty,\, s \in \operatorname{cl}\mathbb{C}_0} \left(\operatorname{Re}\langle (\mathbf{H}^{-1}(s + \beta) + K)u, u\rangle / \log|s + \beta|\right) \geq 0 \quad \forall u \in \mathbb{E}_U. \tag{5.2}$$

Inequalities (5.1) and (5.2) together with the Phragmén-Lindelöf principle (see [24, Corollary 2.3.3]) applied to the harmonic function $\mathbb{C}_0 \to \mathbb{R}$, $s \mapsto -\operatorname{Re}\langle (\mathbf{H}^{-1}(s + \beta) + K)u, u\rangle$ lead to

$$\operatorname{Re}\langle (\mathbf{H}^{-1}(s + \beta) + K)u, u\rangle \geq 0 \quad \forall s \in \mathbb{C}_0,\ \forall u \in \mathbb{E}_U,$$

establishing the strict positive realness of the function $s \mapsto \mathbf{H}^{-1}(s) + K$. $\square$

Corollary 5.2. *Let $\mathbf{H} \in \mathcal{H}^*_\alpha(\mathcal{L}(U))$ with $\alpha < 0$ and let $\beta \in (\alpha, 0)$. Assume that $\mathbf{H}|_{\mathbb{C}_0}$ is positive real and that there exists $z \in \mathbb{C}_0$ such that $\mathbf{H}(z)$ is invertible. Then $\mathbf{H}(s)$ is invertible for every $s \in \mathbb{C}_0$ and $\mathbf{H}^{-1}$ is positive real. If assumption* (A1) *of Theorem 5.1 holds and there exists $\kappa > 0$ such that*

$$\liminf_{|s|\to\infty,\ \beta \le \operatorname{Re} s \le 0} \Big(\inf_{u \in \mathbb{E}_U} \operatorname{Re} \langle \mathbf{H}^{-1}(s)u, u\rangle \Big) > -\kappa, \tag{5.3}$$

then the function $\mathbf{H}^K$ is strictly positive real for every $K \in \mathcal{L}(U)$ such that $\operatorname{Re} K \succeq \kappa I$.

Proof. By hypothesis there exists $\rho > 0$ such that

$$\inf_{u \in \mathbb{E}_U} \operatorname{Re} \langle \mathbf{H}^{-1}(s)u, u\rangle \ge -\kappa \quad \text{for all } s \in \mathbb{C} \text{ such that } \beta \le \operatorname{Re} s \le 0 \text{ and } |s| > \rho.$$

The function $\mathbb{C}_\beta \to \mathbb{R}$, $s \mapsto \inf_{u\in\mathbb{E}_U} \operatorname{Re} \langle \mathbf{H}^{-1}(s)u, u\rangle$ is continuous and since $\inf_{u\in\mathbb{E}_u} \operatorname{Re} \langle \mathbf{H}^{-1}(i\omega)u, u\rangle \ge 0$ for all $\omega \in \mathbb{R}$, it follows that there exists $\gamma \in (\beta, 0)$ such that

$$\inf_{u \in \mathbb{E}_U} \operatorname{Re} \langle \mathbf{H}^{-1}(s)u, u\rangle \ge -\kappa \quad \text{for all } s \in \mathbb{C} \text{ such that } \gamma \le \operatorname{Re} s \le 0 \text{ and } |s| \le \rho.$$

Consequently,

$$\operatorname{Re} \langle \mathbf{H}^{-1}(s)u, u\rangle \ge -\kappa \quad \text{for all } u \in \mathbb{E}_U \text{ and all } s \in \mathbb{C} \text{ such that } \gamma \le \operatorname{Re} s \le 0,$$

implying that assumptions (A2) and (A3) of Theorem 5.1 hold (with β in Theorem 5.1 replaced by γ). Hence, $\mathbf{H}^K$ is strictly positive real for every $K \in \mathcal{L}(U)$ such that $\operatorname{Re} K \succeq \kappa I$. $\square$

Example 5.3. Consider the matrix-valued function

$$\mathbf{H}(s) := \begin{pmatrix} \dfrac{1}{\sqrt{s}\tanh(\sqrt{s})} & \dfrac{1}{s} \\ \dfrac{1}{s} & \dfrac{2(s^2+s+1)}{s(s+1)} \end{pmatrix}.$$

Noting that the function $1/(\sqrt{s}\tanh(\sqrt{s}))$ is meromorphic on $\mathbb{C}$, it is clear that $\mathbf{H}$ is also meromorphic on $\mathbb{C}$. The poles of $\mathbf{H}$ are located at 0, -1 and $-n^2\pi^2$, where $n \in \mathbb{N}$. It is straightforward to check that $2(s^2+s+1)/(s(s+1))$ is positive real. Furthermore, it is well-known that $1/(\sqrt{s}\tanh(\sqrt{s}))$ is positive real, see [10, Section 1.1] or [13, Example 7.11]. Combining this with [13, Theorem 3.7], it is a routine exercise to show that $\mathbf{H}$ is positive real.

We claim that the conditions of Corollary 5.2 hold. This means we need to show that assumption (A1) of Theorem 5.1 and (5.3) are satisfied.

Assumption (A1) of Theorem 5.1. Setting

$$\mathbf{N}(s) := \begin{pmatrix} \dfrac{\sqrt{s}}{\tanh(\sqrt{s})} & 1 \\ 1 & \dfrac{2(s^2+s+1)}{s+1} \end{pmatrix} \qquad \text{and} \qquad \mathbf{D}(s) := \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix},$$

we see that $\mathbf{H} = \mathbf{N}\mathbf{D}^{-1}$. Obviously, for all sufficiently large $x > 0$, $\mathbf{N}(x)$ is invertible, and, hence, so is $\mathbf{H}(x)$. Therefore, $\mathbf{N}^{-1}$ and $\mathbf{H}^{-1}$ are meromorphic on $\mathbb{C}$ and

$$\mathbf{H}^{-1}(s) = \mathbf{D}(s)\mathbf{N}^{-1}(s) = \frac{s}{\det \mathbf{N}(s)} \begin{pmatrix} \dfrac{2(s^2+s+1)}{s+1} & -1 \\ -1 & \dfrac{\sqrt{s}}{\tanh(\sqrt{s})} \end{pmatrix}. \tag{5.4}$$

As $\mathbf{H}$ is positive real, it follows from Proposition 3.1 that $\mathbf{H}(s)$ is invertible for all $s \in \mathbb{C}_0$. Consequently, $\mathbf{N}(s)$ is invertible for all $s \in \mathbb{C}_0$. Furthermore, it can be shown that

$$\left|\frac{\sqrt{s}}{\tanh(\sqrt{s})}\right| \geq 1 \quad \text{and} \quad \left|\frac{2(s^2+s+1)}{s+1}\right| > \frac{4}{3} \quad \forall\, s \in \mathbb{I},$$

and thus, $\mathbf{N}(s)$ is invertible for all $s \in \mathbb{I}$. It now follows that,

$$\mathbf{H}(s) \text{ is invertible for all } s \in \operatorname{cl}\mathbb{C}_0. \tag{5.5}$$

Let $\alpha \in (-1, 0)$. Since

$$\lim_{|s|\to\infty,\, s\in\mathbb{C}_\alpha} \left|\frac{\sqrt{s}}{\tanh(\sqrt{s})}\right| = \lim_{|s|\to\infty,\, s\in\mathbb{C}_\alpha} \left|\frac{2(s^2+s+1)}{s+1}\right| = \infty,$$

we see that there exists $\rho > 0$ such that $\mathbf{N}(s)$, and hence $\mathbf{H}(s)$, is invertible for all $s \in \mathbb{C}_\alpha$ with $|s| \geq \rho$. Together with (5.5) this implies that there exists $\beta \in (\alpha, 0)$ such that $\mathbf{H}(s)$ is invertible for all $s \in \operatorname{cl}\mathbb{C}_\beta$. Thus, $\mathbf{H}^{-1}$ is holomorphic on $\operatorname{cl}\mathbb{C}_\beta$, showing that the assumption (A1) of Theorem 5.1 holds.

Condition (5.3). Note that

$$\det \mathbf{N}(s) = 2s^{3/2}h(s) \quad \forall\, s \in \mathbb{C}_\beta\backslash(\beta, 0]$$

where h is holomorphic on $\mathbb{C}_\beta\backslash(\beta, 0]$ and

$$\lim_{|s|\to\infty, s\in\mathbb{C}_\beta} h(s) = 1. \tag{5.6}$$

Setting

$$h_1(s) := \frac{1}{\sqrt{s}(s+1)} \quad \text{and} \quad h_2(s) := \frac{1}{2\tanh(\sqrt{s})} \quad \forall\, s \in \mathbb{C}_\beta\backslash(\beta, 0]$$

it follows from (5.7) that

$$\mathbf{H}^{-1}(s) = \frac{1}{h(s)} \begin{pmatrix} \sqrt{s} + h_1(s) & -\dfrac{1}{2\sqrt{s}} \\ -\dfrac{1}{2\sqrt{s}} & h_2(s) \end{pmatrix}.$$

Now $h_1(s) \to 0$ and $h_2(s) \to 1/2$ as $|s| \to \infty$ in $\mathbb{C}_\beta$, and so, invoking (5.6), we see that there exists $r > 0$ such that

$$\operatorname{Re}\big((\sqrt{s} + h_1(s))/h(s)\big) > 0 \quad \text{and} \quad \operatorname{Re}(h_2(s)/h(s)) > 0 \quad \forall\, s \in \mathbb{C}_\beta \text{ such that } |s| \geq r.$$

As a consequence, it is now straightforward to show that

$$\langle(\operatorname{Re}\mathbf{H}^{-1}(s))u, u\rangle \geq -2|1/(2h(s)\sqrt{s})| \quad \forall\, u \in \mathbb{E}_{\mathbb{C}^2}, \forall\, s \in \mathbb{C}_\beta \text{ such that } |s| \geq r,$$

which in turn implies that

$$\liminf_{|s|\to\infty,\, s\in\mathbb{C}_\beta} \big(\inf_{u\in\mathbb{E}_{\mathbb{C}^2}} \operatorname{Re}\langle\mathbf{H}^{-1}(s)u, u\rangle\big) \geq 0.$$

This means that (5.3) holds for every $\kappa > 0$ and it follows from Corollary 5.2 that $\mathbf{H}^K$ is strictly positive real for every $K \in \mathbb{C}^{2\times 2}$ such that $\operatorname{Re} K$ is positive definite. $\Diamond$

If $\mathbf{H}$ is a square rational matrix, then Theorem 5.1 takes a particularly simple form. We recall that $z \in \mathbb{C}$ is said be a zero of a square rational matrix $\mathbf{H} = \mathbf{N}\mathbf{D}^{-1}$ if $\det \mathbf{N}(z) = 0$, where $\mathbf{D}$ and $\mathbf{N}$ are right-coprime polynomial matrices. The poles of $\mathbf{H}$ coincide with the zeros of $\det \mathbf{D}$.

Corollary 5.4. *Let* $\mathbf{H}$ *be a rational* $\mathbb{C}^{m\times m}$*-valued positive-real function and assume that* $\mathbf{H}$ *does not have any zeros in* $\mathbb{I}$*. Then* $\mathbf{H}^K$ *is strictly positive real for every* $K \in \mathbb{C}^{m\times m}$ *such that* $\operatorname{Re} K$ *is positive definite.*

Proof. Write $\mathbf{H} = \mathbf{N}\mathbf{D}^{-1}$, where $\mathbf{D}$ and $\mathbf{N}$ are right-coprime polynomial matrices. It follows from the assumption on the zeros of $\mathbf{H}$ that $\det \mathbf{N}(s) \not\equiv 0$, and so there exists $z \in \mathbb{C}_0$ such that $\det \mathbf{N}(z) \neq 0$. By positive realness of $\mathbf{H}$, z is not a pole of $\mathbf{H}$, and thus, $\det \mathbf{D}(z) \neq 0$. Consequently, $\mathbf{H}(z)$ is invertible with inverse $\mathbf{D}(z)\mathbf{N}^{-1}(z)$. Therefore, by Corollary 4.3, $\mathbf{H}(s)$, and hence $\mathbf{N}(s)$, are invertible for all $s \in \mathbb{C}_0$ and $\mathbf{H}^{-1}$ is positive real. Furthermore,

$$\mathbf{H}^{-1}(s) = \mathbf{D}(s)\mathbf{N}^{-1}(s) \quad \forall\, s \in \mathbb{C}_0. \tag{5.7}$$

By hypothesis, $\det \mathbf{N}(s) \neq 0$ for all $s \in \mathbb{I}$ which, together with (5.7), implies that there exists $\alpha < 0$ such that $\mathbf{H}^{-1}$ does not have any poles in $\mathbb{C}_\alpha$, showing that assumption (A1) of Theorem 5.1 holds for any $\beta \in (\alpha, 0)$.

Since $\mathbf{H}^{-1}$ is rational and positive real, it follows from [1, Theorem 2.7.2] or [13, Corollary 3.10] that

$$\mathbf{H}^{-1}(s) = J_0 + J_1 s + \mathbf{J}(s) \quad \forall\, s \in \mathbb{C}, \tag{5.8}$$

where $J_0, J_1 \in \mathbb{C}^{m\times m}$ with $J_1 = J_1^* \succeq 0$ and $\mathbf{J}$ is a strictly proper rational matrix. As an immediate consequence of (5.8) we have that

$$\operatorname{Re} \mathbf{H}^{-1}(i\omega) = \operatorname{Re} J_0 + \operatorname{Re} \mathbf{J}(i\omega) \quad \forall\, \omega \in \mathbb{R}.$$

As $\operatorname{Re} \mathbf{H}^{-1}(i\omega) \succeq 0$ for all $\omega \in \mathbb{R}$ and $\operatorname{Re} \mathbf{J}(i\omega) \to 0$ as $|\omega| \to \infty$, we see that

$$\operatorname{Re} J_0 \succeq 0. \tag{5.9}$$

Let $K \in \mathbb{C}^{m\times m}$ be such that $\operatorname{Re} K$ is positive definite. Then there exists $\kappa > 0$ such that $\operatorname{Re} K \succeq \kappa I$. Invoking (5.8) and (5.9) and using that $\mathbf{J}$ is strictly proper, we conclude that there exist $\gamma \in (\alpha, 0)$ and $\rho > 0$ such that

$$\operatorname{Re} \mathbf{H}^{-1}(s) + \kappa I \succeq 0 \quad \forall\, s \in \Sigma(\gamma, 0) \text{ such that } |s| > \rho, \tag{5.10}$$

where $\Sigma(\gamma, 0)$ denotes he closed vertical strip given by $\gamma \le \operatorname{Re} s \le 0$. On the compact set $\Sigma(\gamma, 0) \cap \{\xi \in \mathbb{C} : |\xi| \le \rho\}$, the function $\operatorname{Re} \mathbf{H}^{-1}$ is uniformly continuous and, since $\operatorname{Re} \mathbf{H}^{-1}(i\omega) \succeq 0$ for all $\omega \in \mathbb{R}$, it is clear that there exists $\beta \in [\gamma, 0)$ such that

$$\operatorname{Re} \mathbf{H}^{-1}(s) + \kappa I \succeq 0 \quad \forall\, s \in \Sigma(\beta, 0) \text{ such that } |s| \le \rho, \tag{5.11}$$

It now follows from (5.10) and (5.11) that

$$\operatorname{Re} \mathbf{H}^{-1}(s) + \kappa I \succeq 0 \quad \forall\, s \in \Sigma(\beta, 0),$$

showing that assumptions (A2) and (A3) of Theorem 5.1 hold. Consequently, Theorem 5.1 guarantees that $\mathbf{H}^K$ is strictly positive real. $\square$

6 Appendix

Here we give an alternative proof of statement (2) of Theorem 4.8 which does not rely on the representation formula (4.2).

Alternative proof of statement (2) **of Theorem 4.8.** Let $\mathbf{H} \in \mathcal{H}_0(\mathcal{L}(U))$ be a positive-real function, let $z \in \mathbb{C}_0$, assume that $\mathbb{I} \cap \sigma_{\mathrm{p}}(\mathbf{H}(z)) \neq \emptyset$, let $\lambda \in \mathbb{I} \cap \sigma_{\mathrm{p}}(\mathbf{H}(z))$ and $u \in \ker(\mathbf{H}(z) - \lambda I)$. We need to prove that $u \in \ker(\mathbf{H}(s) - \lambda I)$ for all $s \in \mathbb{C}_0$. We claim that it is sufficient to show that

$$\mathbf{H}(s)v = \mathbf{H}(z)v \quad \forall\, s \in \mathbb{C}_0,\ \forall\, v \in \ker(\operatorname{Re}\mathbf{H}(z)) \tag{6.1}$$

Indeed, as $u \in \ker(\mathbf{H}(z) - \lambda I)$, it follows from statement (3) of Lemma 2.1 that $u \in \ker(\mathbf{H}^*(z) + \lambda I)$, and so $u \in \ker(\operatorname{Re}\mathbf{H}(z))$, which, together with (6.1), yields

$$(\mathbf{H}(s) - \lambda I)u = (\mathbf{H}(z) - \lambda I)u = 0 \quad \forall\, s \in \mathbb{C}_0.$$

To establish (6.1), let $v \in \ker(\operatorname{Re}\mathbf{H}(z))$. By statement (2) of Theorem 4.4, $v \in \ker(\operatorname{Re}\mathbf{H}(s))$ for all $s \in \mathbb{C}_0$, and so

$$\mathbf{H}(s)v = -\mathbf{H}^*(s)v \quad \forall\, s \in \mathbb{C}_0.$$

Consequently,

$$\langle \mathbf{H}(s)v, w\rangle = -\langle v, \mathbf{H}(s)w\rangle \quad \forall\, s \in \mathbb{C}_0,\ \forall\, w \in U.$$

Hence, for $s \in \mathbb{C}_0$ and $\zeta \in \mathbb{C}$, $\zeta \neq 0$, such that $s + \zeta \in \mathbb{C}_0$, the following identity holds

$$\left\langle \frac{1}{\zeta}(\mathbf{H}(s+\zeta) - \mathbf{H}(s))v, w \right\rangle = -\left\langle v, \frac{1}{\bar{\zeta}}(\mathbf{H}(s+\zeta) - \mathbf{H}(s))w \right\rangle \quad \forall\, w \in U.$$

Now consider the special cases $\zeta = r$ and $\zeta = ir$, where $r \in \mathbb{R}$, $r \neq 0$, and let $r \to 0$ to obtain

$$\langle \mathbf{H}'(s)v, w\rangle = \mp\langle v, \mathbf{H}'(s)w\rangle \quad \forall\, s \in \mathbb{C}_0,\ \forall\, w \in U.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle \mathbf{H}(s)v, w\rangle = 0 \quad \forall\, s \in \mathbb{C}_0,\ \forall\, w \in U,$$

whence

$$\langle \mathbf{H}(s)v, w\rangle = \langle \mathbf{H}(z)v, w\rangle \quad \forall\, s \in \mathbb{C}_0,\ \forall\, w \in U.$$

Therefore, $\mathbf{H}(s)v = \mathbf{H}(z)v$ for all $s \in \mathbb{C}_0$, establishing (6.1) and completing the proof. □

The above proof is similar to that of [4, Proposition 2.2] which shows that for a rational $\mathbb{C}^{m\times m}$-valued positive real function $\mathbf{H}$, the function $s \mapsto \mathbf{H}(s)u$ is constant for every $u \in \mathbb{C}^m$ for which there exists $z \in \mathbb{C}_0$ such that $\langle \mathbf{H}(z)u, u\rangle = 0$.

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