

STABILIZATION OF WELL-POSED INFINITE-DIMENSIONAL SYSTEMS BY DYNAMIC SAMPLED-DATA FEEDBACK*

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Abstract. It is shown that a set of six natural conditions is necessary and sufficient for the existence of a finite-dimensional stabilizing sampled-data controller for a well-posed infinite-dimensional system. The underlying stability concept for the sampled-data system is reminiscent of the notion of input-to-state stability from nonlinear control theory.

Key words. dynamic feedback, frequency-domain methods, input-to-state stability, sampled-data control, stabilization, state-space methods, well-posed infinite-dimensional systems

AMS subject classifications. 93B52, 93C05, 93C20, 93C25, 93C35, 93C57, 93D15

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1. Introduction. The analysis and synthesis of sampled-data control schemes is not only important for digital control applications, but is also interesting from a theoretical point of view, because of the interaction of continuous-time and discrete-time dynamics. Over the last 20 years, a variety of sampled-data control problems for infinite-dimensional systems have been studied; for example,

- tracking for stable systems by low-gain sampled-data control [6, 7, 8, 10, 13, 14],
- stabilization by discretization of continuous-time feedback, so-called indirect sampled-data control [12],
- stabilization by periodic sampled-data output feedback [11, 15, 21, 22],
- stabilization by piecewise polynomial control [16],
- robust sampled-data control of a class of semilinear parabolic systems via linear matrix inequalities [4].

In this paper, we consider the problem of existence of stabilizing (finite-dimensional) dynamic sampled-data controllers for well-posed linear continuous-time infinite-dimensional systems. Somewhat surprisingly, it seems that this problem has not been systematically studied in the literature. The contributions in the present paper fill this gap and complement the results in the above references.

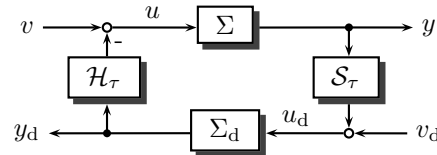
There exists a highly developed state-space and frequency-domain theory for the class of well-posed infinite-dimensional systems; see, for example, [17, 19, 20, 25, 26]. Systems in this class allow for considerable unboundedness of the control and observation operators B and C and they encompass many of the most commonly studied partial differential equations with boundary control and observation, and all functional differential equations of retarded and neutral type with delays in the inputs and outputs.

A well-posed system Σ has generating operators (A, B, C) , where A is the generator of a strongly continuous semigroup $\mathbf{T} = (\mathbf{T}_t)_{t \geq 0}$ governing the state evolution of the uncontrolled system, B is the control operator, and C the observation operator. Denote by u and y the input and output of Σ , respectively. For a given sampling

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FIG. 1.1. *Sampled-data system.*

period $\tau > 0$, we consider sampled-data feedback of the form

$$(1.1) \quad u = v - \mathcal{H}_\tau y_d, \quad u_d = v_d + \mathcal{S}_\tau y;$$

see Figure 1.1. Here the continuous-time signal v and the discrete-time signal v_d are external inputs to the closed-loop sampled-data system, u_d and y_d are the input and output, respectively, of a linear strictly causal discrete-time system Σ_d (the controller), \mathcal{H}_τ is the zero-order hold operator, and \mathcal{S}_τ is a generalized sampling operator, that is,

$$(\mathcal{S}_\tau y)(k) = \int_0^\tau w(t)y(k\tau + t) dt \quad \forall k \in \mathbb{Z}_+,$$

where the scalar weighting function w is in $L^2(0, \tau)$. This kind of generalized sampling is natural for well-posed systems, since their outputs are in L^2_{loc} , but can otherwise be quite irregular, making the ideal sampling operation $y \mapsto (y(k\tau))_{k \in \mathbb{Z}_+}$ meaningless. Obviously, if w is nonnegative, $\int_0^\tau w(t) dt = 1$, and $w(t) = 0$ for all $t \in [\varepsilon, \tau]$, where $\varepsilon > 0$ is small as compared to the sampling period τ , then the generalized sampling operation “mimics” ideal sampling (because, under these conditions, w can be considered as an “approximation” of the Dirac delta function).

In this paper, we give a set of conditions which are necessary and sufficient for the existence of a linear strictly causal discrete-time system with input u_d and output y_d such that the feedback interconnection given by (1.1) leads to a stable sampled-data system. By “stable” we mean exponentially L^q/l^q -input-to-state stable in the sense that there exist positive constants Γ and γ such that the state x of the continuous-time well-posed system and the state x_d of the discrete-time controller satisfy

$$\left\| \begin{pmatrix} x(k\tau + t) \\ x_d(k) \end{pmatrix} \right\| \leq \Gamma \left(e^{-\gamma(k\tau + t)} \left\| \begin{pmatrix} x(0) \\ x_d(0) \end{pmatrix} \right\| + \|v\|_{L^q} + \|v_d\|_{l^q} \right)$$

for all $t \in [0, \tau)$, all $k \in \mathbb{Z}_+$, all initial values $x(0)$ and $x_d(0)$, and all inputs $v \in L^q(0, \infty)$ and $v_d \in l^q(\mathbb{Z}_+)$, where $2 \leq q \leq \infty$.

Loosely speaking, the main result of this paper (Theorem 9) states that, for a given well-posed system Σ , there exists a stabilizing linear sampled-data controller if and only if the following six conditions are satisfied:

- (i) the unstable portion of the spectrum of A consists of at most finitely many eigenvalues with finite algebraic multiplicities;
- (ii) the semigroup generated by the stable part of A is exponentially stable;
- (iii) the unstable (finite-dimensional) part of the controlled discrete-time system (\mathbf{T}_τ, B) is controllable;
- (iv) the unstable (finite-dimensional) part of the observed discrete-time system (C, \mathbf{T}_τ) is observable;
- (v) The numbers $2k\pi i/\tau$ are in the resolvent set of A for all integers $k \neq 0$;
- (vi) $\int_0^\tau w(t)e^{\lambda t} dt \neq 0$ for all unstable eigenvalues λ of A .

As it turns out, if these conditions are satisfied, then there even exists a finite-dimensional stabilizing sampled-data controller (see Theorem 9).

The paper is organized as follows. Preliminaries on well-posed systems are dealt with in section 2. Discretization of well-posed systems by sample and hold is discussed in section 3. In section 4, the basic sample-data system is studied and a characterization of exponential L^q/l^q -input-to-state stability of the sampled-data system in terms of its behavior at the sampling instants is derived. In section 5, the above six conditions are studied in some detail and a number of consequences of these conditions are discussed. In particular, it is shown that if conditions (i) and (ii) hold and the sampled-data system is exponentially L^2/l^2 -input-to-state stable, then $\|y\|_{L^2} + \|y_d\|_{l^2}$ can be “nicely” estimated in terms of $\|x(0)\|$, $\|x_d(0)\|$, $\|v\|_{L^2}$, and $\|v_d\|_{l^2}$. The main result is stated and proved in section 6. Since spectral theory (in particular, spectral projections) plays an important role in this paper, it is convenient to work with complex vector spaces. Therefore the theory in sections 2–6 is developed in a complex setting. The question of the existence of stabilizing real sampled-data controllers for real systems is addressed in section 7. Finally, the proofs of two results have been relegated to an appendix (section 8).

Notation and terminology. Let \mathbb{Z}_+ denote the set of all nonnegative integers. For $\alpha \in \mathbb{R}$, set $\mathbb{C}_\alpha := \{s \in \mathbb{C} : \operatorname{Re} s > \alpha\}$ and, for $\eta > 0$, set $\mathbb{E}_\eta := \{z \in \mathbb{C} : |z| > \eta\}$. Let Y and Z be real or complex Banach spaces. The space of all linear bounded operators mapping Y to Z is denoted by $\mathcal{B}(Y, Z)$. We write $\mathcal{B}(Y)$ for $\mathcal{B}(Y, Y)$. An operator $T \in \mathcal{B}(Y)$ is said to be *power stable* if there exist constants $\Gamma \geq 1$ and $\theta \in (0, 1)$ such that

$$\|T^k\| \leq \Gamma \theta^k \quad \forall k \in \mathbb{Z}_+.$$

For a linear operator T defined in Y and mapping into Z , we write $\operatorname{dom}(T)$ for the domain of T . The resolvent set and spectrum of a linear operator $T : \operatorname{dom}(T) \subset Y \rightarrow Y$ are denoted by $\varrho(T)$ and $\sigma(T)$, respectively.

Let $\Omega = \mathbb{C}_\alpha$ or $\Omega = \mathbb{E}_\eta$. The space of all holomorphic and bounded functions $\Omega \rightarrow \mathbb{C}^{p \times m}$ is denoted by $H^\infty(\Omega, \mathbb{C}^{p \times m})$. We write $H^\infty(\Omega)$ for $H^\infty(\Omega, \mathbb{C})$. For $\alpha \in \mathbb{R}$, we define the exponentially weighted L^q -space $L^q_\alpha(\mathbb{R}_+, Y) := \{f \in L^q_{\text{loc}}(\mathbb{R}_+, Y) : f(\cdot) \exp(-\alpha \cdot) \in L^q(\mathbb{R}_+, Y)\}$ and we endow $L^q_\alpha(\mathbb{R}_+, Y)$ with the norm $\|f\|_{L^q_\alpha} := \|e^{-\alpha \cdot} f(\cdot)\|_{L^q}$. The space of all functions $\mathbb{Z}_+ \rightarrow Y$ (unilateral sequences in Y) is denoted by $F(\mathbb{Z}_+, Y)$. Finally, \mathcal{L} denotes the Laplace transform.

2. Preliminaries. We start by providing some background material on well-posed infinite-dimensional linear systems. There are a number of equivalent definitions of well-posed systems; see [17, 19, 20, 25, 26]. We will be brief in the following and refer the reader to the above references for more details. Throughout, we shall be considering a well-posed system Σ with state space X , input space \mathbb{C}^m , and output space \mathbb{C}^p , generating operators (A, B, C) , input-output operator G , and transfer function \mathbf{G} . Here X is a separable complex Hilbert space, A is the generator of a strongly continuous semigroup $\mathbf{T} = (\mathbf{T}_t)_{t \geq 0}$ on X , $B \in \mathcal{B}(\mathbb{C}^m, X_{-1})$, and $C \in \mathcal{B}(X_1, \mathbb{C}^p)$. Since we will be using spectral theory and spectral projections it is convenient to work with complex spaces. However, the important question of existence of *real* stabilizing sampled-data controllers for *real* systems needs to be addressed and this will be done in section 7. The spaces X_1 and X_{-1} , respectively, are interpolation and extrapolation spaces associated with X . The space X_1 is given by $X_1 := \operatorname{dom}(A)$, endowed with the graph norm of A , while X_{-1} denotes the completion of X with respect to the norm $\|\xi\|_{-1} := \|(\lambda I - A)^{-1} \xi\|$, where $\lambda \in \varrho(A)$ (different choices of λ

lead to equivalent norms), and $\|\cdot\|$ denotes the norm on X . Clearly, $X_1 \subset X \subset X_{-1}$ and the canonical injections are bounded and dense. The semigroup \mathbf{T} restricts to a strongly continuous semigroup on X_1 and extends to a strongly continuous semigroup on X_{-1} with the exponential growth constant being the same on all three spaces; the generator of the restriction (extension) of \mathbf{T} is a restriction (extension) of A ; we shall use the same symbol \mathbf{T} (respectively, A) for the original semigroup (respectively, generator) and the associated restrictions and extensions: with this convention, we may write $A \in \mathcal{B}(X, X_{-1})$ (considered as a generator on X_{-1} , the domain of A is X). The spectra of A and its extension coincide. If $\lambda \in \rho(A)$, then $\lambda I - A$, considered as an operator in $\mathcal{B}(X, X_{-1})$, provides an isometric isomorphism from X to X_{-1} (we refer the reader to [3] and [24] for more details on the extrapolation space X_{-1}).

Moreover, the operator B is an *admissible control operator* for \mathbf{T} , i.e., for each $t \geq 0$ there exists $b_t \geq 0$ such that

$$\left\| \int_0^t \mathbf{T}_{t-s} B u(s) ds \right\| \leq b_t \|u\|_{L^2(0,t)} \quad \forall u \in L^2([0, t], \mathbb{C}^m).$$

The operator C is an *admissible observation operator* for \mathbf{T} , i.e., for each $t \geq 0$ there exists $c_t \geq 0$ such that

$$\left(\int_0^t \|C \mathbf{T}_s \xi\|^2 ds \right)^{1/2} \leq c_t \|\xi\| \quad \forall \xi \in X_1.$$

The control operator B is said to be *bounded* if it is so as a map from the input space \mathbb{C}^m to the state space X ; otherwise, it is said to be *unbounded*; the observation operator C is said to be *bounded* if it can be extended continuously to X ; otherwise, C is said to be *unbounded*.

The so-called Λ -*extension* C_Λ of C is defined by

$$C_\Lambda \xi = \lim_{s \rightarrow \infty, s \in \mathbb{R}} C s (sI - A)^{-1} \xi,$$

with $\text{dom}(C_\Lambda)$ consisting of all $\xi \in X$ for which the above limit exists. For every $\xi \in X$, $\mathbf{T}_t \xi \in \text{dom}(C_\Lambda)$ for a.e. $t \geq 0$ and, if $\alpha > \omega(\mathbf{T})$, then $C_\Lambda \mathbf{T} \xi \in L_\alpha^2(\mathbb{R}_+, \mathbb{C}^p)$, where

$$\omega(\mathbf{T}) := \lim_{t \rightarrow \infty} \frac{1}{t} \ln \|\mathbf{T}_t\|$$

denotes the exponential growth constant of \mathbf{T} .

The transfer function \mathbf{G} satisfies

$$\frac{1}{s - \lambda} (\mathbf{G}(s) - \mathbf{G}(\lambda)) = -C(sI - A)^{-1}(\lambda I - A)^{-1}B \quad \forall s, \lambda \in \mathbb{C}_{\omega(\mathbf{T})}, \quad s \neq \lambda,$$

and $\mathbf{G} \in H^\infty(\mathbb{C}_\alpha, \mathbb{C}^{p \times m})$ for every $\alpha > \omega(\mathbf{T})$. Moreover, the input-output operator $G : L_{\text{loc}}^2(\mathbb{R}_+, \mathbb{C}^m) \rightarrow L_{\text{loc}}^2(\mathbb{R}_+, \mathbb{C}^p)$ is continuous and shift invariant; for every $\alpha > \omega(\mathbf{T})$, $G \in \mathcal{B}(L_\alpha^2(\mathbb{R}_+, \mathbb{C}^m), L_\alpha^2(\mathbb{R}_+, \mathbb{C}^p))$ and

$$(\mathfrak{L}(Gu))(s) = \mathbf{G}(s)(\mathfrak{L}(u))(s) \quad \forall s \in \mathbb{C}_\alpha, \quad \forall u \in L_\alpha^2(\mathbb{R}_+, \mathbb{C}^m).$$

While, a priori, \mathbf{G} is only defined on the half-plane $\mathbb{C}_{\omega(\mathbf{T})}$, we say that \mathbf{G} is holomorphic (meromorphic) on \mathbb{C}_α (where $\alpha < \omega(\mathbf{T})$) if there exists a holomorphic (meromorphic) function $\mathbb{C}_\alpha \rightarrow \mathbb{C}^{p \times m}$ extending \mathbf{G} . This function (if it exists) will also be denoted by \mathbf{G} .

In the following, let $\lambda \in \mathbb{C}_{\omega(\mathbf{T})}$ be fixed, but arbitrary. For $x^0 \in X$ and $u \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^m)$, let x and y denote the state and output functions of Σ , respectively, corresponding to the initial condition $x(0) = x^0 \in X$ and the input function u . Then $x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-s} B u(s) ds$ for all $t \geq 0$, $x(t) - (\lambda I - A)^{-1} B u(t) \in \text{dom}(C_\Lambda)$ for a.e. $t \geq 0$, and

$$(2.1) \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x^0 \in X \text{ a.e. } t \geq 0, \\ y(t) = C_\Lambda(x(t) - (\lambda I - A)^{-1} B u(t)) + \mathbf{G}(\lambda)u(t) & \text{a.e. } t \geq 0. \end{cases}$$

Of course, the differential equation in (2.1) has to be interpreted in X_{-1} . Note that the second equation in (2.1) yields the following formula for the input-output operator G :

$$(2.2) \quad (Gu)(t) = C_\Lambda \left[\int_0^t \mathbf{T}_{t-s} B u(s) ds - (\lambda I - A)^{-1} B u(t) \right] + \mathbf{G}(\lambda)u(t) \\ \forall u \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m), \text{ a.e. } t \geq 0.$$

In the following, we identify Σ and (2.1) and refer to (2.1) as a well-posed system.

The above formulas for the output, the input-output operator, and the transfer function reduce to a more recognizable form for the subclass of regular systems. Recall that the well-posed system (2.1) is called *regular* if the limit

$$\lim_{s \rightarrow \infty, s \in \mathbb{R}} \mathbf{G}(s) =: D$$

exists. In this case, $x(t) \in \text{dom}(C_\Lambda)$ for a.e. $t \geq 0$, the output equation in (2.1) and the formula (2.2) for the input-output operator simplify to

$$y(t) = C_\Lambda x(t) + Du(t) \quad \text{a.e. } t \geq 0$$

and

$$(Gu)(t) = C_\Lambda \int_0^t \mathbf{T}_{t-s} B u(s) ds + Du(t) \quad \forall u \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^m), \text{ a.e. } t \geq 0,$$

respectively; moreover, $(sI - A)^{-1} B C^m \subset \text{dom}(C_\Lambda)$ for all $s \in \rho(A)$ and we have that

$$\mathbf{G}(s) = C_\Lambda (sI - A)^{-1} B + D \quad \forall s \in \mathbb{C}_{\omega(\mathbf{T})}.$$

The matrix D is called the *feedthrough matrix* of (2.1). It can be shown that, if B is a bounded control operator or if C is a bounded observation operator, then (2.1) is regular.

The following result, the proof of which can be found in [11], relates to the asymptotic behavior of the output of an exponentially stable well-posed system under the assumption that a certain “smoothness” condition is satisfied.

PROPOSITION 1. *Assume that (2.1) is exponentially stable. Let $x^0 \in X$ and $u \in W^{1,2}(\mathbb{R}_+, \mathbb{C}^m)$. If there exists $t_0 \geq 0$ such that $\mathbf{T}_{t_0}(Ax^0 + Bu(0)) \in X$, then the output y of (2.1) is continuous¹ on $[t_0, \infty)$ and $\lim_{t \rightarrow \infty} y(t) = 0$.*

¹The output y of the well-posed system (2.1) is an element in $L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^p)$ and so, strictly speaking, y is not a function, but an equivalence class of functions coinciding a.e. in \mathbb{R}_+ . We say that y is continuous on $[t_0, \infty)$ if there exists a representative in the equivalence class which is continuous on $[t_0, \infty)$.

3. Discretization of (2.1) by sample and hold. Let $\tau > 0$ denote the sampling period. As usual, the zero-order hold operator $\mathcal{H}_\tau : F(\mathbb{Z}_+, \mathbb{C}^m) \rightarrow L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^m)$ is defined by

$$(\mathcal{H}_\tau f)(t) = f(k) \quad \forall t \in [k\tau, (k+1)\tau), \quad \forall k \in \mathbb{Z}_+.$$

Moreover, $\mathcal{S}_\tau : L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^p) \rightarrow F(\mathbb{Z}_+, \mathbb{C}^p)$ denotes the generalized sampling operator defined by

$$(\mathcal{S}_\tau g)(k) = \int_0^\tau w(t)g(k\tau + t) dt \quad \forall k \in \mathbb{Z}_+,$$

where the scalar weighting function w is in $L^2(0, \tau)$. Note that, for $g \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^p)$, the ideal sampling operation $g \mapsto (g(k\tau))_{k \in \mathbb{Z}_+}$ is meaningless since point evaluations do not make sense for L^2_{loc} functions. This becomes relevant in the context of well-posed infinite-dimensional systems because, in general, their outputs (which are functions in $L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^p)$) are too irregular for ideal sampling to be meaningful.

Define

$$A_\tau := \mathbf{T}_\tau \in \mathcal{B}(X).$$

Invoking the admissibility of B , we conclude that the operator

$$B_\tau : L^2([0, \tau], \mathbb{C}^m) \rightarrow X, \quad g \mapsto \int_0^\tau \mathbf{T}_t B g(\tau - t) dt$$

is in $\mathcal{B}(L^2(0, \tau), X)$. Setting

$$B_\tau \xi := B_\tau(\xi \mathbf{1}) \quad \forall \xi \in \mathbb{C}^m,$$

where $\mathbf{1} \in L^2(0, \tau)$ denotes the function identically equal to 1, it follows that the map $\mathbb{C}^m \rightarrow X$, $\xi \mapsto B_\tau \xi$ is in $\mathcal{B}(\mathbb{C}^m, X)$. Furthermore, by admissibility of C , there exists a constant $c_\tau \geq 0$ such that $(\int_0^\tau \|C_\Lambda \mathbf{T}_t \xi\|^2 dt)^{1/2} \leq c_\tau \|\xi\|$ for all $\xi \in X$. Consequently, the operator C_τ defined by

$$C_\tau \xi = \int_0^\tau w(t) C_\Lambda \mathbf{T}_t \xi dt \quad \forall \xi \in X,$$

is in $\mathcal{B}(X, \mathbb{C}^p)$. Finally, we define

$$D_\tau : L^2([0, \tau], \mathbb{C}^m) \rightarrow \mathbb{C}^p, \quad g \mapsto \int_0^\tau w(t)(Gg)(t) dt.$$

Setting

$$D_\tau \xi := D_\tau(\xi \mathbf{1}) \quad \forall \xi \in \mathbb{C}^m,$$

then, trivially, the map $\mathbb{C}^m \rightarrow \mathbb{C}^p$, $\xi \mapsto D_\tau \xi$ is in $\mathcal{B}(\mathbb{C}^m, \mathbb{C}^p)$.

LEMMA 2. Let $u = \mathcal{H}_\tau f + g$, where $f \in F(\mathbb{Z}_+, \mathbb{C}^m)$ and $g \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^m)$, and let $x^0 \in X$. Set

$$x(t) := \mathbf{T}(t)x^0 + \int_0^t \mathbf{T}_{t-s} B u(s) ds \quad \forall t \geq 0.$$

Then

$$\begin{aligned} x((k+1)\tau) &= A_\tau x(k\tau) + B_\tau f(k) + B_\tau \mathbf{L}_{k\tau} g, \\ (\mathcal{S}_\tau y)(k) &= C_\tau x(k\tau) + D_\tau f(k) + D_\tau \mathbf{L}_{k\tau} g, \end{aligned}$$

where $\mathbf{L}_{k\tau} g \in L^2([0, \tau], \mathbb{C}^m)$ is defined by $(\mathbf{L}_{k\tau} g)(t) = g(k\tau + t)$ for all $t \in [0, \tau]$.

Proof. To obtain the first identity note that, for all $k \in \mathbb{Z}_+$,

$$\begin{aligned} x((k+1)\tau) &= \mathbf{T}_\tau x(k\tau) + \int_{k\tau}^{(k+1)\tau} \mathbf{T}_{(k+1)\tau-s} B u(s) \, ds \\ &= A_\tau x(k\tau) + \int_0^\tau \mathbf{T}_s B f(k) \, ds + \int_0^\tau \mathbf{T}_s B g((k+1)\tau - s) \, ds \\ &= A_\tau x(k\tau) + B_\tau f(k) + B_\tau \mathbf{L}_{k\tau} g. \end{aligned}$$

To prove the second identity, let $k \in \mathbb{Z}_+$ and $t \in [0, \tau]$. Then

$$\begin{aligned} y(k\tau + t) &= C_\Lambda \mathbf{T}_t x(k\tau) + (G(\mathbf{L}_{k\tau} u))(t) \\ &= C_\Lambda \mathbf{T}_t x(k\tau) + (G(f(k)\mathbf{1}))(t) + (G(\mathbf{L}_{k\tau} g))(t). \end{aligned}$$

Hence,

$$\begin{aligned} (\mathcal{S}_\tau y)(k) &= \int_0^\tau w(t) C_\Lambda \mathbf{T}_t x(k\tau) \, dt + D_\tau f(k) + D_\tau \mathbf{L}_{k\tau} g \\ &= C_\tau x(k\tau) + D_\tau f(k) + D_\tau \mathbf{L}_{k\tau} g. \quad \square \end{aligned}$$

4. Dynamic sampled-data feedback. Consider the (strictly causal) discrete-time controller Σ_d given by

$$(4.1) \quad \begin{cases} x_d^\nabla(k) = P x_d(k) + Q u_d(k), & x_d(0) = x_d^0 \in X_d, \\ y_d(k) = R x_d(k), \end{cases}$$

where X_d is a complex Hilbert space, $P \in \mathcal{B}(X_d)$, $Q \in \mathcal{B}(\mathbb{C}^p, X_d)$, $R \in \mathcal{B}(X_d, \mathbb{C}^m)$, and $x_d^\nabla(k) := x_d(k+1)$. As illustrated in Figure 1.1, we connect the continuous-time system (2.1) and the discrete-time controller (4.1) via the following sampled-data feedback law:

$$(4.2) \quad u = v - \mathcal{H}_\tau y_d, \quad u_d = v_d + \mathcal{S}_\tau y,$$

where $v \in L^q_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^m)$ and $v_d \in F(\mathbb{Z}_+, \mathbb{C}^p)$ are the inputs of the closed-loop system. The resulting sampled-data system is given by

$$(4.3) \quad \begin{cases} \dot{x} = Ax + B(v - \mathcal{H}_\tau y_d), & x(0) = x^0 \in X, \\ y = C_\Lambda(x - (\lambda I - A)^{-1} B(v - \mathcal{H}_\tau y_d)) + \mathbf{G}(\lambda)(v - \mathcal{H}_\tau y_d), \\ x_d^\nabla = P x_d + Q(v_d + \mathcal{S}_\tau y), & x_d(0) = x_d^0 \in X_d, \\ y_d = R x_d. \end{cases}$$

In the following, let $q \in [2, \infty]$. We say that the sampled-data system (4.3) is *exponentially L^q/l^q -input-to-state stable* if there exist $\Gamma \geq 1$ and $\gamma > 0$ such that the solution (x, x_d) satisfies

$$(4.4) \quad \begin{cases} \left\| \begin{pmatrix} x(k\tau + t) \\ x_d(k) \end{pmatrix} \right\| \leq \Gamma \left(e^{-\gamma(k\tau+t)} \left\| \begin{pmatrix} x^0 \\ x_d^0 \end{pmatrix} \right\| + \|\mathbf{P}_{k\tau+t} v\|_{L^q} + \|\mathbf{P}_k v_d\|_{l^q} \right) \\ \forall k \in \mathbb{Z}_+, \forall t \in [0, \tau), \forall x^0 \in X, \forall x_d^0 \in X_d, \\ \forall v \in L^q_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^m), \forall v_d \in F(\mathbb{Z}_+, \mathbb{C}^p), \end{cases}$$

where $\mathbf{P}_{k\tau+t} : L^q_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^m) \rightarrow L^q(\mathbb{R}_+, \mathbb{C}^m)$ and $\mathbf{P}_k : F(\mathbb{Z}_+, \mathbb{C}^p) \rightarrow l^q(\mathbb{Z}_+, \mathbb{C}^p)$ denote the usual projection (respectively, truncation) operators defined by

$$(\mathbf{P}_{k\tau+t}v)(s) := \begin{cases} v(s), & 0 \leq s \leq k\tau + t, \\ 0, & s > k\tau + t, \end{cases} \quad (\mathbf{P}_k v_d)(j) := \begin{cases} v_d(j), & 0 \leq j \leq k, \\ 0, & j > k. \end{cases}$$

PROPOSITION 3. *Let $q \in [2, \infty]$. The sampled-data system (4.3) is exponentially L^q/l^q -input-to-state stable if and only if (4.1) stabilizes $(A_\tau, B_\tau, C_\tau, D_\tau)$ (the discretization of (2.1) obtained by sampling and holding) in the sense that*

$$(4.5) \quad \Delta := \begin{pmatrix} A_\tau & -B_\tau R \\ QC_\tau & P - QD_\tau R \end{pmatrix}$$

is power stable.

Furthermore, if $q < \infty$ and the sampled-data system (4.3) is exponentially L^q/l^q -input-to-state stable, then, for all $v \in L^q(\mathbb{R}_+, \mathbb{C}^m)$ and all $v_d \in l^q(\mathbb{Z}_+, \mathbb{C}^p)$, $x(t) \rightarrow 0$ (in X) as $t \rightarrow \infty$ and $x_d(k) \rightarrow 0$ as $k \rightarrow \infty$.

Proof. Let $x^0 \in X$, $x_d^0 \in X_d$, $v \in L^q_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^m)$, $v_d \in F(\mathbb{Z}_+, \mathbb{C}^p)$, and let (x, x_d) be the solution of the sampled-data system (4.3). Invoking Lemma 2, we obtain

$$\begin{aligned} x((k+1)\tau) &= A_\tau x(k\tau) - B_\tau y_d(k) + B_\tau \mathbf{L}_{k\tau} v \\ &= A_\tau x(k\tau) - B_\tau R x_d(k) + B_\tau \mathbf{L}_{k\tau} v \end{aligned}$$

and, furthermore,

$$\begin{aligned} x_d(k+1) &= P x_d(k) + Q(v_d(k) + (\mathcal{S}_\tau y)(k)) \\ &= P x_d(k) + Q(v_d(k) + C_\tau x(k\tau) - D_\tau R x_d(k) + D_\tau \mathbf{L}_{k\tau} v) \\ &= QC_\tau x(k\tau) + (P - QD_\tau R)x_d(k) + QD_\tau \mathbf{L}_{k\tau} v + Qv_d(k). \end{aligned}$$

Consequently,

$$(4.6) \quad \begin{pmatrix} x((k+1)\tau) \\ x_d(k+1) \end{pmatrix} = \Delta \begin{pmatrix} x(k\tau) \\ x_d(k) \end{pmatrix} + f(k),$$

where $f : \mathbb{Z}_+ \rightarrow X \times X_d$ is defined by

$$(4.7) \quad f(k) := \begin{pmatrix} B_\tau \\ QD_\tau \end{pmatrix} \mathbf{L}_{k\tau} v + \begin{pmatrix} 0 \\ Q \end{pmatrix} v_d(k).$$

To prove necessity, assume that the sampled-data system (4.3) is exponentially L^q/l^q -input-to-state stable, that is, there exist constants $\Gamma \geq 1$ and $\gamma > 0$ such that (4.4) holds. Then, for the unforced sampled-data system (4.3) (that is, $v = 0$ and $v_d = 0$), (4.6) and (4.7) yield

$$\left\| \Delta^k \begin{pmatrix} x^0 \\ x_d^0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} x(k\tau) \\ x_d(k) \end{pmatrix} \right\| \leq \Gamma e^{-\gamma k\tau} \left\| \begin{pmatrix} x^0 \\ x_d^0 \end{pmatrix} \right\|,$$

showing that Δ is power stable.

To prove sufficiency, assume that Δ is power stable. Then there exist $\Gamma_1 \geq 1$ and $\theta \in (0, 1)$ such that

$$\|\Delta^k\| \leq \Gamma_1 \theta^k \quad \forall k \in \mathbb{Z}_+.$$

Here Γ_1 and θ depend only on Δ . Therefore, by (4.6), (4.7), and Hölder’s inequality,

$$(4.8) \quad \left\| \begin{pmatrix} x(k\tau) \\ x_d(k) \end{pmatrix} \right\| \leq \Gamma_1 \left(\theta^k \left\| \begin{pmatrix} x^0 \\ x_d^0 \end{pmatrix} \right\| + \sum_{j=0}^{k-1} \theta^{k-1-j} \|f(j)\| \right)$$

$$(4.9) \quad \leq \Gamma_2 \left(\theta^k \left\| \begin{pmatrix} x^0 \\ x_d^0 \end{pmatrix} \right\| + \|\mathbf{P}_{k-1}f\|_{l^q} \right) \quad \forall k \in \mathbb{Z}_+,$$

where $\Gamma_2 \geq \Gamma_1$ depends only on Γ_1 and θ . Set

$$f_1(k) := \begin{pmatrix} B_\tau \\ QD_\tau \end{pmatrix} \mathbf{L}_{k\tau}v, \quad f_2(k) := \begin{pmatrix} 0 \\ Q \end{pmatrix} v_d(k).$$

Then $f = f_1 + f_2$. Furthermore, defining

$$r := \frac{q}{q-2} \in [1, \infty],$$

we have that $1/r + 2/q = 1$. Denoting the norm of the bounded operator

$$\begin{pmatrix} B_\tau \\ QD_\tau \end{pmatrix}$$

by α , it follows from Hölder’s inequality that

$$\|f_1(k)\|^2 \leq \alpha^2 \|\mathbf{L}_{k\tau}v\|_{L^2(0,\tau)}^2 \leq \alpha^2 \tau^{1/r} \left(\int_{k\tau}^{(k+1)\tau} \|v(s)\|^q ds \right)^{2/q} \quad \forall k \in \mathbb{Z}_+.$$

Hence,

$$(4.10) \quad \|f_1(k)\| \leq \alpha \tau^{1/(2r)} \|v\|_{L^q(k\tau, (k+1)\tau)} \quad \forall k \in \mathbb{Z}_+,$$

showing that

$$\|\mathbf{P}_{k-1}f_1\|_{l^q} \leq \alpha \tau^{1/(2r)} \|v\|_{L^q(0, k\tau)} = \alpha \tau^{1/(2r)} \|\mathbf{P}_{k\tau}v\|_{L^q} \quad \forall k \in \mathbb{Z}_+.$$

Trivially, we have that

$$\|\mathbf{P}_{k-1}f_2\|_{l^q} \leq \beta \|\mathbf{P}_{k-1}v_d\|_{l^q} \quad \forall k \in \mathbb{Z}_+,$$

where β denotes the norm of the operator Q . Consequently, by (4.9),

$$(4.11) \quad \left\| \begin{pmatrix} x(k\tau) \\ x_d(k) \end{pmatrix} \right\| \leq \Gamma_3 \left(\theta^k \left\| \begin{pmatrix} x^0 \\ x_d^0 \end{pmatrix} \right\| + \|\mathbf{P}_{k\tau}v\|_{L^q} + \|\mathbf{P}_{k-1}v_d\|_{l^q} \right) \quad \forall k \in \mathbb{Z}_+,$$

where $\Gamma_3 \geq \Gamma_2$ depends only on Γ_2 , τ , r , α , and β . Moreover, for $k \in \mathbb{Z}_+$ and $t \in [0, \tau)$,

$$x(k\tau + t) = \mathbf{T}_t x(k\tau) + \int_{k\tau}^{k\tau+t} \mathbf{T}_{k\tau+t-s} Bu(s) ds,$$

and so

$$(4.12) \quad \begin{aligned} \|x(k\tau + t)\| &\leq \|\mathbf{T}_t x(k\tau)\| + \left\| \int_0^t \mathbf{T}_s Bu(k\tau + t - s) ds \right\| \\ &\leq \Gamma_4 (\|x(k\tau)\| + \|u\|_{L^2(k\tau, k\tau+t)}) \quad \forall k \in \mathbb{Z}_+, \forall t \in [0, \tau), \end{aligned}$$

where Γ_4 depends only on \mathbf{T} and the admissibility constant b_τ . Since

$$\begin{aligned} \|u\|_{L^2(k\tau, k\tau+t)} &\leq \|v\|_{L^2(k\tau, k\tau+t)} + \tau^{1/2} \|y_d(k)\| \\ (4.13) \quad &\leq t^{1/(2r)} \|v\|_{L^q(k\tau, k\tau+t)} + \tau^{1/2} \|R\| \|x_d(k)\| \\ &\leq \tau^{1/(2r)} \|\mathbf{P}_{k\tau+t} v\|_{L^q} + \tau^{1/2} \|R\| \|x_d(k)\| \quad \forall k \in \mathbb{Z}_+, \forall t \in [0, \tau), \end{aligned}$$

it follows from (4.12) that

$$(4.14) \quad \|x(k\tau + t)\| \leq \Gamma_5 (\|x(k\tau)\| + \|x_d(k)\| + \|\mathbf{P}_{k\tau+t} v\|_{L^q}) \quad \forall k \in \mathbb{Z}_+, \forall t \in [0, \tau),$$

where $\Gamma_5 \geq \Gamma_4$ depends only on Γ_4 , τ , r , and $\|R\|$. Setting

$$\tilde{x}^0 := \begin{pmatrix} x^0 \\ x_d^0 \end{pmatrix}$$

and combining (4.11) and (4.14), we obtain

$$\|x(k\tau + t)\| \leq \Gamma_6 (\theta^k \|\tilde{x}^0\| + \|\mathbf{P}_{k\tau+t} v\|_{L^q} + \|\mathbf{P}_{k-1} v_d\|_{l^q}) \quad \forall k \in \mathbb{Z}_+, \forall t \in [0, \tau).$$

Here Γ_6 depends only on Γ_3 and Γ_5 . Set $\gamma := -(\ln \theta)/\tau > 0$. Then

$$(4.15) \quad \theta^k = e^{-\gamma k\tau} \leq e^{\gamma\tau} e^{-\gamma(k\tau+t)} \quad \forall k \in \mathbb{Z}_+, \forall t \in [0, \tau).$$

Hence,

$$(4.16) \quad \|x(k\tau + t)\| \leq \Gamma_7 (e^{-\gamma(k\tau+t)} \|\tilde{x}^0\| + \|\mathbf{P}_{k\tau+t} v\|_{L^q} + \|\mathbf{P}_{k-1} v_d\|_{l^q}) \quad \forall k \in \mathbb{Z}_+, \forall t \in [0, \tau),$$

where $\Gamma_7 := e^{\gamma\tau} \Gamma_6$. Furthermore, by (4.11) and (4.15),

$$(4.17) \quad \|x_d(k)\| \leq \Gamma_8 (e^{-\gamma(k\tau+t)} \|\tilde{x}^0\| + \|\mathbf{P}_{k\tau+t} v\|_{L^q} + \|\mathbf{P}_{k-1} v_d\|_{l^q}) \quad \forall k \in \mathbb{Z}_+ \forall t \in [0, \tau),$$

where $\Gamma_8 := e^{\gamma\tau} \Gamma_3$. Exponential L^q/l^q -input-to-state stability now follows from (4.16) and (4.17).

To prove the claim of Proposition 3 relating to the convergence of x and x_d to 0, assume that $q < \infty$ and that the sampled-data system (4.3) is exponentially L^q/l^q -input-to-state stable. By what we have already proved, we know that Δ is power stable. Let $v \in L^q(\mathbb{R}_+, \mathbb{C}^m)$ and $v_d \in l^q(\mathbb{Z}_+, \mathbb{C}^p)$. Then, by (4.7) and (4.10), $f(k) \rightarrow 0$ as $k \rightarrow \infty$, and therefore, invoking (4.6) and the power stability of Δ , we conclude that $x(k\tau) \rightarrow 0$ (in X) and $x_d(k)$ as $k \rightarrow \infty$. Combining this with (4.12) and (4.13) shows that $x(t) \rightarrow 0$ (in X) as $t \rightarrow \infty$. \square

Note that Proposition 3 implies that if the sampled-data system (4.3) is exponentially L^q/l^q -input-to-state stable for some $q \in [2, \infty]$, then (4.3) is exponentially L^q/l^q -input-to-state stable for every $q \in [2, \infty]$.

5. Assumptions on the well-posed system (2.1). In the following, we impose six assumptions on the well-posed system (2.1).

(A1) There exists $\varepsilon > 0$ such that $\sigma(A) \cap \overline{\mathbb{C}}_{-\varepsilon}$ consists of finitely many isolated eigenvalues of A with finite algebraic multiplicities.

If (A1) holds, then there exists a smooth, positively oriented, and simple closed curve Φ in \mathbb{C} not intersecting $\sigma(A)$, enclosing $\sigma(A) \cap \overline{\mathbb{C}}_0$ in its interior and having $\sigma(A) \cap (\mathbb{C} \setminus \overline{\mathbb{C}}_0)$ in its exterior. The operator

$$(5.1) \quad \Pi := \frac{1}{2\pi i} \int_{\Phi} (sI - A)^{-1} ds$$

is a projection on X and we have

$$(5.2) \quad X = X^+ \oplus X^-, \quad \text{where } X^+ := \Pi X, \quad X^- := (I - \Pi)X.$$

It follows from a standard result (see, for example, Lemma 2.5.7 in [1] or p.178 in [5]) that $\dim X^+ < \infty$, $X^+ \subset X_1$, X^+ and $X^- := (I - \Pi)X$ are \mathbf{T}_t -invariant for all $t \geq 0$, and

$$(5.3) \quad \sigma(A^+) = \sigma(A) \cap \overline{\mathbb{C}}_0, \quad \sigma(A^-) = \sigma(A) \cap (\mathbb{C} \setminus \overline{\mathbb{C}}_0),$$

where

$$(5.4) \quad A^+ := A|_{X^+}, \quad A^- := A|_{X_1 \cap X^-}.$$

Moreover, we define

$$(5.5) \quad \mathbf{T}_t^+ := \mathbf{T}_t|_{X^+}, \quad \mathbf{T}_t^- := \mathbf{T}_t|_{X^-}, \quad C^+ := C|_{X^+}, \quad C^- := C|_{X_1 \cap X^-}.$$

Note that $\mathbf{T}^+ = (\mathbf{T}_t^+)_{t \geq 0}$ and $\mathbf{T}^- = (\mathbf{T}_t^-)_{t \geq 0}$ are C_0 -semigroups with generators A^+ and A^- , respectively. Since the spectrum of A considered as an operator on X coincides with the spectrum of A considered as an operator on X_{-1} ,² the projection operator Π on X defined in (5.1) extends to a projection Π_{-1} on X_{-1} . Note that, if $\lambda \in \rho(A)$, then $\Pi_{-1} = (\lambda I - A)\Pi(\lambda I - A)^{-1}$, where $\lambda I - A$ is considered as an operator in $\mathcal{B}(X, X_{-1})$. Moreover, it is important to observe that $\Pi_{-1}X_{-1} = \Pi X = X^+$. We define

$$(5.6) \quad B^+ := \Pi_{-1}B, \quad B^- := (I - \Pi_{-1})B.$$

In the following, we will use the same symbol Π for the original projection and its associated extension Π_{-1} . Obviously, the operator A^- extends to an operator in $\mathcal{B}(X^-, (X^-)_{-1})$ and the same symbol A^- will be used to denote this extension. The following simple lemma, the proof of which can be found in [11], will be useful in the following.

LEMMA 4. Assume that (A1) holds. There exists a well-posed system Σ^- with generating operators³ (A^-, B^-, C^-) and input-output operator $G^- := G - G^+$, where G^+ denotes the input-output operator of the (finite-dimensional) system (A^+, B^+, C^+) , that is, $(G^+u)(t) = \int_0^t C^+ e^{A^+(t-s)} B^+ u(s) ds$ for all $t \in \mathbb{R}_+$ and all $u \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^m)$. For every $x^0 \in X$ and $u \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^m)$, the output y of the well-posed system (2.1) can be written in the form

$$y(t) = (C^-)_\Lambda \mathbf{T}_t^- (I - \Pi)x^0 + (G^-u)(t) + C^+ \Pi x(t) \quad \text{a.e. } t \geq 0,$$

where $x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-s} B u(s) ds$ for all $t \geq 0$.

² The point and approximate point spectra coincide as well.

³ For (A^-, B^-, C^-) to be the generating operators of a well-posed system it is of course necessary that B^- maps into $(X^-)_{-1} = ((I - \Pi)X)_{-1}$, the extrapolation space associated with A^- . Since, by definition, B^- maps into $(I - \Pi)X_{-1} =: (X_{-1})^-$, there seems to be a difficulty. However, it is clear that the spaces $(X^-)_{-1}$ and $(X_{-1})^-$ are both completions of X^- endowed with the norm $\|\cdot\|_{-1}$. Hence there exists an isometric isomorphism $(X^-)_{-1} \rightarrow (X_{-1})^-$, the restriction of which to X^- is the identity, and so we can safely identify $(X^-)_{-1}$ and $(X_{-1})^-$.

We are now in position to formulate the remaining assumptions.

(A2) The C_0 -semigroup $\mathbf{T}^- = (\mathbf{T}_t^-)_{t \geq 0}$ is exponentially stable.

(A3) (\mathbf{T}_τ^+, B^+) is discrete-time controllable.

(A4) (C^+, \mathbf{T}_τ^+) is discrete-time observable.

(A5) $2k\pi i/\tau \notin \sigma(A^+) \forall k \in \mathbb{Z} \setminus \{0\}$.

(A6) $\int_0^\tau w(t)e^{\lambda t} dt \neq 0 \forall \lambda \in \sigma(A^+)$.

Under suitable conditions, assumptions (A3)–(A5) can be replaced by the following assumptions.

(A3') (A^+, B^+) is continuous-time controllable.

(A4') (C^+, A^+) is continuous-time observable.

(A5') $\tau(\lambda - \mu) \neq 2k\pi i \forall \lambda, \mu \in \sigma(A^+) \forall k \in \mathbb{Z} \setminus \{0\}$.

Assumptions (A3') and (A4') seem slightly more natural than (A3) and (A4), because (A3') and (A4') are formulated entirely in terms of continuous-time data. Assumptions (A1)–(A4), (A3'), and (A4') are quite common in feedback control of infinite-dimensional systems; see [1] for the continuous-time case, [9] for the discrete-time case and [11, 16] for sampled-data systems. In particular, (A1), (A2), (A3'), and (A4') have been invoked in the past to establish the existence and facilitate the construction of stabilizing finite-dimensional dynamic continuous-time controllers (but, to the best of my knowledge, in settings more restrictive than the well-posed systems framework). Here, (A1)–(A4) will be used to guarantee the existence of stabilizing finite-dimensional dynamic sampled-data controllers for well-posed systems and for sampling periods τ satisfying the constraints imposed by (A5) and (A6). Spectral conditions such as (A5) and (A5') arise naturally in sampled-data control; see [18]. Finally, assumption (A6), which also appears in [11], implies discrete-time detectability of (C_τ, A_τ) (see proof of Theorem 9).

The following proposition shows that (A3)–(A5) and (A3')–(A5') are closely related.

PROPOSITION 5. *Assume that (A1) is satisfied. Then the following statements hold.*

(1) *If (A3') and (A5') hold, then (A3) holds. Conversely, assume that (A3) holds. Then (A3') holds and, if $m = 1$, (A5') is also satisfied. If $m > 1$, then (A3) does not imply (A5').*

(2) *If (A4') and (A5') hold, then (A4) holds. Conversely, assume that (A4) holds. Then (A4') holds and, if $p = 1$, (A5') is also satisfied. If $p > 1$, then (A4) does not imply (A5').*

(3) *(A5') implies (A5) (the converse is not true).*

(4) *If w is equal to a nonzero constant, then (A5) and (A6) are equivalent.*

Proof. To prove statement (1), note that the first implication mentioned in the statement is a consequence of [18, Theorem 4, p. 102]. Now assume that (A3) holds. Let φ be in the dual space of X^+ and $\lambda \in \sigma(A^+)$. Assume that $\varphi \circ A^+ = \lambda\varphi$ and $\varphi \circ B^+ = 0$. By the Hautus criterion, (A3') will follow if we can show that $\varphi = 0$. Noting that $\varphi \circ \mathbf{T}_\tau^+ = \varphi \circ e^{A^+\tau} = e^{\lambda\tau}\varphi$, the Hautus criterion, applied to the controllable discrete-time system (\mathbf{T}_τ^+, B^+) , yields that $\varphi = 0$. Furthermore, under the additional assumption that $m = 1$, controllability of (\mathbf{T}_τ^+, B^+) together with the Hautus criterion yields that $\dim \ker(zI - \mathbf{T}_\tau^+) \leq \text{rk } B^+ = 1$ for every $z \in \mathbb{C}$. If (A5') is not satisfied, then there exist $\lambda, \mu \in \sigma(A^+)$ such that $\lambda \neq \mu$ and $e^{\lambda\tau} = e^{\mu\tau} =: \zeta$. As in [18, Remark 3.4.5, p. 103], it then follows that $\dim \ker(\zeta I - \mathbf{T}_\tau^+) > 1$ which is impossible. Therefore, (A5') holds. It is a straightforward exercise to construct examples which show that, if $m > 1$, then (A3) does not imply (A5'). The details are left to the reader.

Statement (2) can be obtained from statement (1) by duality. Statements (3) and (4) are trivial. \square

The following proposition shows that if (A1) and (A2) hold and if the sampled-data system (4.3) is exponentially L^2/l^2 -input-to-state stable, then the output (y, y_d) behaves nicely in the sense that $\|y\|_{L^2} + \|y_d\|_{l^2}$ can be estimated in terms of $\|v\|_{L^2}$, $\|v_d\|_{l^2}$, and the norm of the initial condition.

PROPOSITION 6. *Assume that (A1) and (A2) hold. Furthermore assume that the sampled-data system (4.3) is exponentially L^2/l^2 -input-to-state stable. Then there exists $\Gamma_{\text{out}} > 0$ such that the output (y, y_d) of the sampled-data system satisfies the estimate*

$$(5.7) \quad \left\{ \begin{aligned} \|y\|_{L^2} + \|y_d\|_{l^2} &\leq \Gamma_{\text{out}} \left(\left\| \begin{pmatrix} x^0 \\ x_d^0 \end{pmatrix} \right\| + \|v\|_{L^2} + \|v_d\|_{l^2} \right) \\ \forall x^0 \in X, \forall x_d^0 \in X_d, \forall v \in L^2(\mathbb{R}_+, \mathbb{C}^m), \forall v_d \in l^2(\mathbb{Z}_+, \mathbb{C}^p). \end{aligned} \right.$$

Proof. Let $x^0 \in X$, $x_d^0 \in X_d$, $v \in L^2(\mathbb{R}_+, \mathbb{C}^m)$, and $v_d \in l^2(\mathbb{Z}_+, \mathbb{C}^p)$ and let (x, x_d) and (y, y_d) be the corresponding state trajectory and output, respectively, of the sampled-data system (4.3). Write

$$\mathbf{G}^+(s) := C^+(sI - A^+)^{-1}B^+, \quad \mathbf{G}^-(s) := \mathbf{G}(s) - \mathbf{G}^+(s).$$

It follows from Lemma 4 that \mathbf{G}^- is the transfer function of an exponentially stable well-posed system with generating operators (A^-, B^-, C^-) and input-output operator G^- and the output y of (2.1) can be written in the form

$$(5.8) \quad y = y^+ + y^-,$$

with

$$y^+ := C^+\Pi x; \quad y^-(t) := (C^-)_\Lambda \mathbf{T}_t^-(I - \Pi)x^0 + (G^-u)(t) \quad \text{a.e. } t \geq 0,$$

where $u = v - \mathcal{H}_\tau y_d$. By Proposition 3, Δ is power stable and thus (4.8) holds with $\theta \in (0, 1)$. Therefore,

$$\left(\sum_{j=0}^{\infty} \left\| \begin{pmatrix} x(j\tau) \\ x_d(j) \end{pmatrix} \right\|^2 \right)^{1/2} \leq \Gamma_2 \left(\left\| \begin{pmatrix} x^0 \\ x_d^0 \end{pmatrix} \right\| + \|f\|_{l^2} \right)$$

for a suitable constant $\Gamma_2 \geq 0$ depending only on Γ_1 and θ . By the definition of f (see (4.7)), there exist constants $\alpha, \beta \geq 0$ such that

$$\|f(j)\|^2 \leq \alpha \int_{j\tau}^{(j+1)\tau} \|v(s)\|^2 ds + \beta \|v_d(j)\|^2 \quad \forall j \in \mathbb{Z}_+.$$

Thus there exists $\Gamma_3 \geq 0$ (depending only on Γ_2 , α , and β) such that

$$(5.9) \quad \left(\sum_{j=0}^{\infty} \left\| \begin{pmatrix} x(j\tau) \\ x_d(j) \end{pmatrix} \right\|^2 \right)^{1/2} \leq \Gamma_3 \left(\left\| \begin{pmatrix} x^0 \\ x_d^0 \end{pmatrix} \right\| + \|v\|_{L^2} + \|v_d\|_{l^2} \right).$$

Since $u = v - \mathcal{H}_\tau y_d$, we have

$$(5.10) \quad \|u\|_{L^2(k\tau, (k+1)\tau)} \leq \|v\|_{L^2(k\tau, (k+1)\tau)} + \|R\|\sqrt{\tau}\|x_d(k)\| \quad \forall k \in \mathbb{Z}_+,$$

and so, by (4.12),

$$\|x(k\tau + t)\| \leq \Gamma_4 (\|x(k\tau)\| + \|v\|_{L^2(k\tau, (k+1)\tau)} + \|R\| \sqrt{\tau} \|x_d(k)\|) \\ \forall k \in \mathbb{Z}_+, \forall t \in [0, \tau).$$

Therefore, there exists $\Gamma_5 \geq 0$ (depending only on Γ_4 , $\|R\|$, and τ) such that

$$\int_{k\tau}^{(k+1)\tau} \|x(s)\|^2 ds \leq \Gamma_5^2 (\|x(k\tau)\|^2 + \|v\|_{L^2(k\tau, (k+1)\tau)}^2 + \|x_d(k)\|^2) \quad \forall k \in \mathbb{Z}_+.$$

Hence,

$$\|x\|_{L^2}^2 \leq \Gamma_5^2 \left(\sum_{k=0}^{\infty} \|x(k\tau)\|^2 + \|v\|_{L^2}^2 + \|x_d\|_{l^2}^2 \right),$$

and thus

$$\|x\|_{L^2} \leq \Gamma_5 \left(\left(\sum_{k=0}^{\infty} \|x(k\tau)\|^2 \right)^{1/2} + \|v\|_{L^2} + \|x_d\|_{l^2} \right),$$

which in turn implies via (5.9) that

$$\|\Pi x\|_{L^2} \leq \|x\|_{L^2} \leq \Gamma_6 \left(\left\| \begin{pmatrix} x^0 \\ x_d^0 \end{pmatrix} \right\| + \|v\|_{L^2} + \|v_d\|_{l^2} \right),$$

where $\Gamma_6 \geq 0$ is a suitable constant which depends only on Γ_3 and Γ_5 . As a consequence,

$$(5.11) \quad \|y^+\|_{L^2} \leq \Gamma_7 \left(\left\| \begin{pmatrix} x^0 \\ x_d^0 \end{pmatrix} \right\| + \|v\|_{L^2} + \|v_d\|_{l^2} \right),$$

where $\Gamma_7 := \|C^+\| \Gamma_6$. By (5.10),

$$\|u\|_{L^2}^2 \leq 2 \left(\|v\|_{L^2}^2 + \|R\|^2 \tau \sum_{k=0}^{\infty} \|x_d(k)\|^2 \right),$$

leading to

$$\|u\|_{L^2} \leq \sqrt{2} (\|v\|_{L^2} + \|R\| \sqrt{\tau} \sum_{k=0}^{\infty} \|x_d(k)\|).$$

Consequently, by (5.9),

$$(5.12) \quad \|u\|_{L^2} \leq \Gamma_8 \left(\left\| \begin{pmatrix} x^0 \\ x_d^0 \end{pmatrix} \right\| + \|v\|_{L^2} + \|v_d\|_{l^2} \right)$$

for some suitable $\Gamma_8 > 0$ depending only on Γ_3 , $\|R\|$, and τ . By exponential stability of \mathbf{T}^- , there exists $\Gamma_9 \geq 0$ depending only on \mathbf{T}^- and G^- such that

$$\|y^-\|_{L^2} \leq \Gamma_9 (\|x^0\| + \|u\|_{L^2}).$$

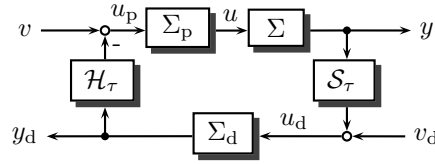


FIG. 5.1. *Sampled-data system with precompensator.*

Invoking (5.12), we conclude

$$(5.13) \quad \|y^-\|_{L^2} \leq \Gamma_{10} \left(\left\| \begin{pmatrix} x^0 \\ x_d^0 \end{pmatrix} \right\| + \|v\|_{L^2} + \|v_d\|_{l^2} \right),$$

where $\Gamma_{10} > 0$ depends only on Γ_8 and Γ_9 . Finally, as a trivial consequence of (5.9),

$$(5.14) \quad \|y_d\|_{l^2} \leq \Gamma_{11} \left(\left\| \begin{pmatrix} x^0 \\ x_d^0 \end{pmatrix} \right\| + \|v\|_{L^2} + \|v_d\|_{l^2} \right),$$

where the constant Γ_{11} depends only on Γ_3 and $\|R\|$. The claim now follows from (5.8), (5.11), (5.13), and (5.14). \square

Under the assumptions of Proposition 6, while $y_d(k) \rightarrow 0$ as $k \rightarrow \infty$ (since $y_d \in l^2(\mathbb{Z}_+, \mathbb{C}^m)$), it is of course not guaranteed that $y(t) \rightarrow 0$ as $t \rightarrow \infty$ (even under zero initial conditions). This issue can be addressed by using a smoothing stable precompensator Σ_p of the form

$$(5.15) \quad \dot{x}_p = -ax_p + u_p, \quad x_p(0) = x_p^0 \in \mathbb{C}^m,$$

where $a > 0$. Consider the sampled-data system shown in Figure 5.1. Formally, this system is given by (2.1), (5.15), (4.1), and the feedback law

$$(5.16) \quad u = x_p, \quad u_p = v - \mathcal{H}_\tau y_d, \quad u_d = v_d + \mathcal{S}_\tau y,$$

that is

$$(5.17) \quad \begin{cases} \dot{x} = Ax + Bx_p, & x(0) = x^0 \in X, \\ y = C_\Lambda(x - (\lambda I - A)^{-1}Bx_p) + \mathbf{G}(\lambda)x_p, \\ \dot{x}_p = -ax_p + v - \mathcal{H}_\tau y_d, & x_p(0) = x_p^0 \in \mathbb{C}^m, \\ x_d^\nabla = Px_d + Q(v_d + \mathcal{S}_\tau y), & x_d(0) = x_d^0 \in X_d, \\ y_d = Rx_d. \end{cases}$$

PROPOSITION 7. *Assume that (A1) and (A2) hold for (2.1) and that the sampled-data system (5.17) is exponentially L^2/l^2 -input-to-state stable. Then there exists $\Gamma_{\text{out}} > 0$ such that the output (y, y_d) of the sampled-data system satisfies the estimate*

$$(5.18) \quad \begin{cases} \|y\|_{L^2} + \|y_d\|_{l^2} \leq \Gamma_{\text{out}} \left(\left\| \begin{pmatrix} x^0 \\ x_p^0 \\ x_d^0 \end{pmatrix} \right\| + \|v\|_{L^2} + \|v_d\|_{l^2} \right) & \forall x^0 \in X, \\ \forall x_p^0 \in \mathbb{C}^m, \forall x_d^0 \in X_d, \forall v \in L^2(\mathbb{R}_+, \mathbb{C}^m), \forall v_d \in l^2(\mathbb{Z}_+, \mathbb{C}^p). \end{cases}$$

Furthermore, if $v \in L^2(\mathbb{R}_+, \mathbb{C}^m)$, $v_d \in l^2(\mathbb{Z}_+, \mathbb{C}^p)$, and $\mathbf{T}_{t_0}(Ax^0 + Bx_p^0) \in X$ for some $t_0 \geq 0$, then y is continuous on $[t_0, \infty)$ and $y(t) \rightarrow 0$ as $t \rightarrow \infty$.

The proof of Proposition 7 can be found in the Appendix. For later purposes we record another consequence of assumptions (A1) and (A2).

LEMMA 8. *Assume that (A1) and (A2) hold and set*

$$\mathbf{G}_\tau(z) := C_\tau(zI - A_\tau)^{-1}B_\tau + D_\tau.$$

Then $\mathbf{G}_\tau = \mathbf{G}_\tau^- + \mathbf{G}_\tau^+$, where $\mathbf{G}_\tau^- \in H^\infty(\mathbb{E}_\eta, \mathbb{C}^{p \times m})$ for some $\eta \in (0, 1)$ and \mathbf{G}_τ^+ is rational and strictly proper.

Proof. Define

$$(5.19) \quad A_\tau^+ := \mathbf{T}_\tau^+, A_\tau^- := \mathbf{T}_\tau^-, B_\tau^+ := \Pi B_\tau, B_\tau^- := (I - \Pi)B_\tau, C_\tau^+ := C_\tau|_{X^+}, C_\tau^- := C_\tau|_{X^-}.$$

Then

$$C_\tau(zI - A_\tau)^{-1}B_\tau + D_\tau = C_\tau^-(zI - A_\tau^-)^{-1}B_\tau^- + D_\tau + C_\tau^+(zI - A_\tau^+)^{-1}B_\tau^+.$$

Setting $\mathbf{G}_\tau^-(z) := C_\tau^-(zI - A_\tau^-)^{-1}B_\tau^- + D_\tau$ and $\mathbf{G}_\tau^+(z) := C_\tau^+(zI - A_\tau^+)^{-1}B_\tau^+$, it follows that $\mathbf{G}_\tau = \mathbf{G}_\tau^- + \mathbf{G}_\tau^+$. It is clear that \mathbf{G}_τ^+ is rational (since, as a consequence of (A1), $\dim X^+ < \infty$) and strictly proper. Furthermore, by (A2), $A_\tau^- = \mathbf{T}_\tau^-$ is power stable, implying that \mathbf{G}_τ^- is holomorphic and bounded on \mathbb{E}_η for some $\eta \in (0, 1)$. \square

6. Stabilization by dynamic sampled-data feedback. We are now in the position to state and prove the main result of this paper.

THEOREM 9. *The following statements are equivalent.*

- (1) (A1)–(A6) hold.
- (2) *There exists a discrete-time controller (4.1) such that the sampled-data system (4.3) is exponentially L^q/l^q -input-to-state stable for every $q \in [2, \infty]$.*
- (3) *There exists a finite-dimensional discrete-time controller (4.1) such that the sampled-data system (4.3) is exponentially L^q/l^q -input-to-state stable for every $q \in [2, \infty]$.*

Proposition 5 shows that statement (1) remains sufficient for statements (2) and (3) to hold if assumptions (A3)–(A5) are replaced by (A3')–(A5'). Furthermore, if $m = p = 1$, then Theorem 9 remains true if (A3)–(A5) are replaced by (A3')–(A5').

Proof of Theorem 9. Obviously, it suffices to show (2) \Rightarrow (1) \Rightarrow (3).

(2) \Rightarrow (1). To prove this implication, we start by noting that (A1) and (A2) hold by a general result on necessary conditions for stabilization of (2.1) by step-function controls (see [16, 23]). Moreover, by Proposition 3, the operator Δ is power stable. To show that (A3) and (A5) hold, note that, by (A1), $\dim X^+ < \infty$. We show first that $(\mathbf{T}_\tau^+, B_\tau^+)$ is discrete-time controllable. Seeking a contradiction, assume that this is not the case. Then, by the Hautus criterion, there exists a linear functional $\varphi \neq 0$ in the dual space of X^+ and $\lambda \in \sigma(\mathbf{T}_\tau^+)$ such that

$$\varphi \circ \mathbf{T}_\tau^+ = \lambda\varphi, \quad \varphi \circ B_\tau^+ = 0.$$

By the spectral mapping theorem, $\sigma(\mathbf{T}_\tau^+) = e^{\tau\sigma(A^+)}$ and therefore, by (5.3), $|\lambda| \geq 1$. Define the linear functional ψ in the dual space of $X \times X_d$ by $\psi(\xi, \xi_d) = \varphi(\Pi\xi)$ for all $(\xi, \xi_d) \in X \times X_d$. Then

$$(\psi \circ \Delta)(\xi, \xi_d) = \varphi(\Pi\mathbf{T}_\tau\xi - \Pi B_\tau R\xi_d) = \varphi(\mathbf{T}_\tau^+\Pi\xi) - \varphi(B_\tau^+ R\xi_d) = \lambda\varphi(\Pi\xi) = \lambda\psi(\xi, \xi_d).$$

Consequently,

$$\psi \circ \Delta^k = \lambda^k \psi \quad \forall k \in \mathbb{Z}_+,$$

which, combined with the fact that $|\lambda| \geq 1$, yields a contradiction to the power stability of Δ . Hence, $(\mathbf{T}_\tau^+, B_\tau^+)$ is discrete-time controllable. Since

$$\mathbf{T}_\tau^+ = e^{A^+\tau} \quad \text{and} \quad B_\tau^+ \xi = \Pi B_\tau \xi = \int_0^\tau e^{A^+t} B^+ \xi \, dt \quad \forall \xi \in \mathbb{C}^m,$$

it follows from finite-dimensional sampled-data control theory ([18, p. 100]) that (\mathbf{T}_τ^+, B^+) is discrete-time controllable and $2k\pi i/\tau \notin \sigma(A^+)$ for all $k \in \mathbb{Z} \setminus \{0\}$, showing that (A3) and (A5) hold.

To show that (A4) and (A6) hold, we note that, by an argument similar to that establishing discrete-time controllability of $(\mathbf{T}_\tau^+, B_\tau^+)$, it can be proved that $(C_\tau^+, \mathbf{T}_\tau^+)$ is discrete-time observable. Let \mathcal{O} and \mathcal{O}_τ denote the observability matrices of the pairs (C^+, \mathbf{T}_τ^+) and $(C_\tau^+, \mathbf{T}_\tau^+)$, respectively. Defining $W^+ \in \mathcal{B}(X^+)$ by

$$(6.1) \quad W^+ \xi = \int_0^\tau w(t) \mathbf{T}_t^+ \xi \, dt = \int_0^\tau w(t) e^{A^+t} \xi \, dt \quad \forall \xi \in X^+,$$

it follows that $C_\tau^+ = C^+ W^+$. Since W^+ and \mathbf{T}_τ^+ commute, we have that

$$\mathcal{O}_\tau = \mathcal{O} W^+.$$

Since \mathcal{O}_τ has full rank, we conclude that \mathcal{O} has full rank and W^+ is invertible. Consequently, (C^+, \mathbf{T}_τ^+) is discrete-time observable, that is, (A4) holds. Furthermore, note that

$$(6.2) \quad f : \mathbb{C} \rightarrow \mathbb{C}, \quad s \mapsto \int_0^\tau w(t) e^{st} \, dt$$

is an entire function and

$$(6.3) \quad W^+ = f(A^+).$$

By the spectral mapping theorem,

$$(6.4) \quad \sigma(W^+) = f(\sigma(A^+)).$$

Since W^+ is invertible, $0 \notin \sigma(W^+)$ and so $f(\lambda) \neq 0$ for every $\lambda \in \sigma(A^+)$, showing that (A6) holds.

(1) \Rightarrow (3). Using Lemma 8, it can be shown that there exists a strictly proper rational transfer function \mathbf{K} stabilizing \mathbf{G}_τ in the sense that

$$(6.5) \quad \begin{pmatrix} I & \mathbf{G}_\tau \\ -\mathbf{K} & I \end{pmatrix}^{-1} = (I + \mathbf{H}J)^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{l \times l}),$$

where $l := m + p$, $\mathbf{H} := \text{diag}(\mathbf{G}_\tau, \mathbf{K})$ and

$$J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

To prove the existence of such a strictly proper rational \mathbf{K} will require us to go into a number of technical details and therefore we relegate this argument to the end of the proof.

Let (P, Q, R) be a minimal realization of \mathbf{K} and define Δ by (4.5). By Proposition 3, it is sufficient to show that Δ is power stable. Note that the transfer function

of the discrete-time system $(\Delta, \text{diag}(B_\tau, Q), \text{diag}(C_\tau, R))$ is given by $(I + \mathbf{H}J)^{-1}\mathbf{H}$. Invertibility of J together with (6.5) implies that $(I + \mathbf{H}J)^{-1}\mathbf{H} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{l \times l})$. By a result in [9] on the equivalence of power stability and input-output stability, power stability of Δ will follow if we can show that $(\Delta, \text{diag}(B_\tau, Q))$ is discrete-time stabilizable and $(\text{diag}(C_\tau, R), \Delta)$ is discrete-time detectable. For this it suffices to show that (A_τ, B_τ) and (C_τ, A_τ) are discrete-time stabilizable and detectable, respectively.

With $A_\tau^+, A_\tau^-, B_\tau^+, B_\tau^-, C_\tau^+$, and C_τ^- as defined in (5.19), it follows that

$$A_\tau = \begin{pmatrix} A_\tau^+ & 0 \\ 0 & A_\tau^- \end{pmatrix}, \quad B_\tau = \begin{pmatrix} B_\tau^+ \\ B_\tau^- \end{pmatrix}, \quad C_\tau = (C_\tau^+, C_\tau^-).$$

Since $A_\tau^+ = \mathbf{T}_\tau^+ = e^{A^+\tau}$ and $B_\tau^+ = \int_0^\tau e^{A^+t} dt B^+$, it follows from (A3) and (A5) via a well-known result in finite-dimensional sampled-data control (see Lemma 3.4.1 in [18]) that the pair (A_τ^+, B_τ^+) is discrete-time controllable. Hence there exists a linear operator $F_\tau^+ : X^+ \rightarrow \mathbb{C}^m$ such that $A_\tau^+ + B_\tau^+ F_\tau^+$ is power stable. Consequently, invoking (A2), we conclude that

$$A_\tau + B_\tau(F_\tau^+, 0) = \begin{pmatrix} A_\tau^+ + B_\tau^+ F_\tau^+ & 0 \\ B_\tau^- F_\tau^+ & A_\tau^- \end{pmatrix}$$

is power stable, showing that (A_τ, B_τ) is discrete-time stabilizable.

To show that (C_τ, A_τ) is discrete-time detectable, note that, by (6.2)–(6.4) and (A6), the operator W^+ , defined in (6.1), is invertible. Hence,

$$(C_\tau^+, A_\tau^+) = (CW^+, e^{A^+\tau}) = (C^+W^+, (W^+)^{-1}e^{A^+\tau}W^+),$$

where we have made use of the fact that W^+ and $e^{A^+\tau}$ commute. Consequently, the observed discrete-time systems (C_τ^+, A_τ^+) and $(C^+, e^{A^+\tau}) = (C^+, \mathbf{T}_\tau^+)$ are similar. It follows from (A4) that (C_τ^+, A_τ^+) is discrete-time observable, and thus, there exists a linear operator $H_\tau^+ : \mathbb{C}^p \rightarrow X^+$ such that $A_\tau^+ + H_\tau^+ C_\tau^+$ is power stable. Combining this with (A2) then shows that

$$A_\tau + \begin{pmatrix} H_\tau^+ \\ 0 \end{pmatrix} C_\tau = \begin{pmatrix} A_\tau^+ + H_\tau^+ C_\tau^+ & H_\tau^+ C_\tau^- \\ 0 & A_\tau^- \end{pmatrix}$$

is power stable. Hence, the pair (C_τ, A_τ) is discrete-time detectable.

It remains to prove that there exists a strictly proper rational \mathbf{K} such that (6.5) holds. By Lemma 8, $\mathbf{G}_\tau = \mathbf{G}_\tau^- + \mathbf{G}_\tau^+$, where $\mathbf{G}_\tau^- \in H^\infty(\mathbb{E}_\eta, \mathbb{C}^{p \times m})$ for some $\eta \in (0, 1)$ and \mathbf{G}_τ^+ is rational and strictly proper. It is well known that there exist matrices \mathbf{N}_+ , \mathbf{D}_+ , \mathbf{Y}_+ , and \mathbf{Z}_+ with rational entries in $H^\infty(\mathbb{E}_1)$, with \mathbf{N}_+ and \mathbf{Y}_+ strictly proper and such that

$$\mathbf{G}_\tau^+(z) = \mathbf{D}_+^{-1}\mathbf{N}_+, \quad \mathbf{N}_+\mathbf{Y}_+ + \mathbf{D}_+\mathbf{Z}_+ = I;$$

see, for example, Theorem 7.3.8 in [1]. In particular, $\mathbf{D}_+^{-1}\mathbf{N}_+$ is a left-coprime factorization of \mathbf{G}_τ^+ over $H^\infty(\mathbb{E}_1) \cap \mathbb{C}(z)$. Setting

$$(6.6) \quad \mathbf{N} := \mathbf{D}_+\mathbf{G}_\tau^- + \mathbf{N}_+, \quad \mathbf{Z} := \mathbf{Z}_+ - \mathbf{G}_\tau^-\mathbf{Y}_+,$$

we have that the entries of the matrices \mathbf{N} and \mathbf{Z} are in $H^\infty(\mathbb{E}_1)$, $\mathbf{G}_\tau = \mathbf{D}_+^{-1}\mathbf{N}$, and

$$(6.7) \quad \mathbf{N}\mathbf{Y}_+ + \mathbf{D}_+\mathbf{Z} = I.$$

Obviously, the Taylor expansion of \mathbf{G}_τ^- at ∞ , $\mathbf{G}_\tau^-(z) = \sum_{k=0}^\infty G_k z^{-k}$, converges absolutely in \mathbb{E}_η . Fixing $\theta \in (\eta, 1)$, it follows that $\sum_{k=0}^\infty \|G_k\| \theta^{-k} < \infty$. Thus, there exists a constant $\mu > 0$ such that $\|G_k\| \leq \mu \theta^k$ for $k \in \mathbb{Z}_+$. Hence,

$$\sup_{z \in \mathbb{E}_1} \left\| \mathbf{G}_\tau^-(z) - \sum_{k=0}^n G_k z^{-k} \right\| \leq \sum_{k=n+1}^\infty \|G_k\| \leq \mu \frac{\theta^{n+1}}{1-\theta} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Consequently, there exists a rational matrix $\mathbf{R} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p \times m})$ such that

$$\|\mathbf{D}_+(\mathbf{G}_\tau^- - \mathbf{R})\mathbf{Y}_+\|_{H^\infty(\mathbb{E}_1)} = \sup_{z \in \mathbb{E}_1} \|\mathbf{D}_+(z)(\mathbf{G}_\tau^-(z) - \mathbf{R}(z))\mathbf{Y}_+(z)\| < 1.$$

Combining this with (6.7), it follows that

$$\|\mathbf{N}\mathbf{Y}_+ + \mathbf{D}_+(\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+) - I\|_{H^\infty(\mathbb{E}_1)} = \|\mathbf{D}_+(\mathbf{G}_\tau^- - \mathbf{R})\mathbf{Y}_+\|_{H^\infty(\mathbb{E}_1)} < 1,$$

implying that $\mathbf{U} := \mathbf{N}\mathbf{Y}_+ + \mathbf{D}_+(\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+)$ is unimodular over $H^\infty(\mathbb{E}_1)$, that is, the entries of \mathbf{U}^{-1} are in $H^\infty(\mathbb{E}_1)$. Now set $\mathbf{K} := \mathbf{Y}_+(\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+)^{-1}$. Then, obviously, \mathbf{K} is rational. Moreover, since $\mathbf{Y}_+(\infty) = 0$ and $\mathbf{Z}_+(\infty)$ is invertible, it follows that \mathbf{K} is strictly proper. Finally, noting that

$$\begin{aligned} \begin{pmatrix} I & \mathbf{G}_\tau \\ -\mathbf{K} & I \end{pmatrix}^{-1} &= \begin{pmatrix} (I + \mathbf{G}_\tau \mathbf{K})^{-1} & -\mathbf{G}_\tau (I + \mathbf{K} \mathbf{G}_\tau)^{-1} \\ \mathbf{K} (I + \mathbf{G}_\tau \mathbf{K})^{-1} & (I + \mathbf{K} \mathbf{G}_\tau)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} (I + \mathbf{G}_\tau \mathbf{K})^{-1} & -(I + \mathbf{G}_\tau \mathbf{K})^{-1} \mathbf{G}_\tau \\ \mathbf{K} (I + \mathbf{G}_\tau \mathbf{K})^{-1} & I - \mathbf{K} (I + \mathbf{G}_\tau \mathbf{K})^{-1} \mathbf{G}_\tau \end{pmatrix} \end{aligned}$$

and $(I + \mathbf{G}_\tau \mathbf{K})^{-1} = (\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+)\mathbf{U}^{-1}\mathbf{D}_+$, we obtain

$$\begin{pmatrix} I & \mathbf{G}_\tau \\ -\mathbf{K} & I \end{pmatrix}^{-1} = \begin{pmatrix} (\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+)\mathbf{U}^{-1}\mathbf{D}_+ & -(\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+)\mathbf{U}^{-1}\mathbf{N}_+ \\ \mathbf{Y}_+\mathbf{U}^{-1}\mathbf{D}_+ & I - \mathbf{Y}_+\mathbf{U}^{-1}\mathbf{N}_+ \end{pmatrix}.$$

Since the matrices \mathbf{D}_+ , \mathbf{N} , \mathbf{R} , \mathbf{Y}_+ , \mathbf{Z}_+ , and \mathbf{U}^{-1} have entries in $H^\infty(\mathbb{E}_1)$, it follows that (6.5) holds. \square

We discuss a simple example which illustrates Theorem 9 and the construction of \mathbf{K} in its proof.

Example. Consider the heating of a metal rod of length 1. Let $\theta(\xi, t)$ denote the temperature at position $\xi \in [0, 1]$ and at time $t \geq 0$. We assume that the rod is insulated at either end. The temperature is controlled by a heating element at $\xi_0 \in (0, 1)$ and it is measured at the point $\xi_1 \in (0, 1)$ (point control and point observation). The system is described by the following (formal) partial differential equation:

$$(6.8) \quad \begin{cases} \theta_t(\xi, t) = \theta_{\xi\xi}(\xi, t) + \delta_{\xi_0}(\xi)u(t), & \theta_\xi(0, t) = \theta_\xi(1, t) = 0, & \theta(\xi, 0) = \theta^0(\xi), \\ y(t) = \theta(\xi_1, t); & \xi \in [0, 1], t \geq 0. \end{cases}$$

Here δ_{ξ_0} denotes the delta function supported at the point ξ_0 .

It is well known (see, for example, [19]) that (6.8) defines a regular well-posed system (with feedthrough equal to zero) on the state space $X = L^2(0, 1)$. The generating operators (A, B, C) of this well-posed system are given by

$$Af = f'' \quad \forall f \in X_1 = \text{dom}(A) = \{f \in W^{2,2}(0, 1) : f'(0) = f'(1) = 0\},$$

$Bs = s\delta_{\xi_0}$ for all $s \in \mathbb{C}$, and $Cf = f(\xi_1)$ for all $f \in X_1$. Setting $s_n := -n^2\pi^2$ for $n \in \mathbb{Z}_+$ and defining $f_n \in X_1$ by

$$f_0(\xi) = 1, \quad f_n(\xi) = \sqrt{2} \cos(n\pi\xi) \quad (n = 1, 2, \dots) \quad \forall \xi \in [0, 1],$$

we have that $\sigma(A) = \{s_n : n \in \mathbb{Z}_+\}$, each s_n is an eigenvalue of A , the functions f_n form an orthonormal basis of $L^2(0, 1)$,

$$Af = \sum_{n=0}^{\infty} s_n \langle f, f_n \rangle f_n \quad \forall f \in X_1, \quad \text{and} \quad (sI - A)^{-1}f = \sum_{n=0}^{\infty} \frac{\langle f, f_n \rangle}{s - s_n} f_n \quad \forall f \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $X = L^2(0, 1)$ and $s \neq s_n$ for $n \in \mathbb{Z}_+$. The strongly continuous semigroup $(\mathbf{T}_t)_{t \geq 0}$ generated by A is given by

$$\mathbf{T}_t f = \sum_{n=0}^{\infty} e^{s_n t} \langle f, f_n \rangle f_n \quad \forall f \in X, \quad \forall t \geq 0.$$

The transfer function \mathbf{G} can be expressed as

$$\mathbf{G}(s) = \frac{1}{s} + \sum_{n=1}^{\infty} \frac{f_n(\xi_0) f_n(\xi_1)}{s - s_n} = \frac{1}{s} + 2 \sum_{n=1}^{\infty} \frac{\cos(n\pi\xi_0) \cos(n\pi\xi_1)}{s - s_n}.$$

The derivation of the above expressions for A , $(\mathbf{T}_t)_{t \geq 0}$, and \mathbf{G} can be found, for example, in [1].

Let $\varphi \in (0, \pi^2)$ and set $\Phi(t) = \varphi e^{2\pi i t}$ for $t \in [0, 1]$. Obviously, assumption (A1) holds. For the spectral projection Π we have

$$\Pi f = \frac{1}{2\pi i} \int_{\Phi} (sI - A)^{-1} f ds = \langle f, f_0 \rangle f_0 \quad \forall f \in X.$$

Hence, $X^+ = \Pi X = \{s f_0 : s \in \mathbb{C}\}$, $A^+ = 0$, $B^+ s = s f_0$, and $C^+ s f_0 = s$ for all $s \in \mathbb{C}$. Furthermore, the expansions of $\mathbf{T}_t f$ and $\mathbf{T}_t \Pi f$ have the same first term (namely, $\langle f, f_0 \rangle f_0$). Consequently, $(\mathbf{T}_t^-)_{t \geq 0}$ is exponentially stable, showing that (A2) holds. The assumptions (A3)–(A5) are trivially satisfied and (A6) holds, provided that the weighting function w is such that $\int_0^\tau w(t) dt \neq 0$.

In the following, we assume that w satisfies $\int_0^\tau w(t) dt = 1$. Since (A1)–(A6) hold, Theorem 9 applies. In particular, there exists a finite-dimensional discrete-time controller which achieves exponential L^q/l^q -input-to-state stability (for every $q \in [2, \infty]$). To compute such a stabilizing controller, we will follow the construction used in the proof of Theorem 9. To this end, note that $\mathbf{G}_\tau^+(z) = \tau/(z - 1)$. Define $\mathbf{H} \in H^\infty(\mathbb{C}_+)$ by

$$\mathbf{H}(s) := \mathbf{G}(s) - \frac{1}{s} = 2 \sum_{n=1}^{\infty} \frac{\cos(n\pi\xi_0) \cos(n\pi\xi_1)}{s - s_n},$$

and denote the corresponding input-output operator by H . Denoting the continuous-time integrator by J , the input-output operator G of (6.8) can be written as $G = J + H$. A routine calculation shows that $\mathcal{S}_\tau J \mathcal{H}_\tau = G_\tau^+ + (\tau/2)I$, where G_τ^+ is the discrete-time operator with transfer function \mathbf{G}_τ^+ . Hence, the transfer function \mathbf{G}_τ^- can be written in the form $\mathbf{G}_\tau^-(z) = \tau/2 + \mathbf{H}_\tau(z)$, where \mathbf{H}_τ is the transfer function of the operator $\mathcal{S}_\tau H \mathcal{H}_\tau$. The hold \mathcal{H}_τ , as an operator from $l^2(\mathbb{Z}_+)$ to $L^2(\mathbb{R}_+)$, has

norm $\sqrt{\tau}$, while \mathcal{S}_τ , as an operator from $L^2(\mathbb{R}_+)$ to $l^2(\mathbb{Z}_+)$, has norm $\nu := \|w\|_{L^2}$. Hence $\|\mathcal{S}_\tau H \mathcal{H}_\tau\| \leq \nu\sqrt{\tau}\|H\|$ and consequently,

$$\|\mathbf{H}_\tau\|_{H^\infty(\mathbb{E}_1)} \leq \nu\sqrt{\tau}\|\mathbf{H}\|_{H^\infty(\mathbb{C}_+)} \leq \frac{2\nu\sqrt{\tau}}{\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} = \frac{2\nu\sqrt{\tau}}{\pi^2} \frac{\pi^2}{6} = \frac{\nu\sqrt{\tau}}{3}.$$

Let $\alpha \in [0, 1)$ and define

$$\mathbf{N}_+(z) := \frac{\tau}{z - \alpha}, \quad \mathbf{D}_+(z) := \frac{z - 1}{z - \alpha}, \quad \mathbf{Y}_+(z) := \frac{(1 - \alpha)^2}{\tau} \frac{1}{z - \alpha}, \quad \mathbf{Z}_+(z) := \frac{z + 1 - 2\alpha}{z - \alpha}.$$

Then, $\mathbf{G}_\tau^+ = \mathbf{D}_+^{-1}\mathbf{N}_+$ and $\mathbf{N}_+\mathbf{Y}_+ + \mathbf{D}_+\mathbf{Z}_+ = 1$, that is, $\mathbf{D}_+^{-1}\mathbf{N}_+$ is a left-coprime factorization of \mathbf{G}_τ^+ over $H^\infty(\mathbb{E}_1) \cap \mathbb{C}(z)$. Routine calculations show that

$$\|\mathbf{D}_+\|_{H^\infty(\mathbb{E}_1)} = \mathbf{D}_+(-1) = \frac{2}{1 + \alpha}, \quad \|\mathbf{Y}_+\|_{H^\infty(\mathbb{E}_1)} = \mathbf{Y}_+(1) = \frac{1 - \alpha}{\tau}.$$

Choosing $\mathbf{R}(z) \equiv \tau/2$, it follows that

$$(6.9) \quad \|\mathbf{D}_+(\mathbf{G}_\tau^- - \mathbf{R})\mathbf{Y}_+\|_{H^\infty(\mathbb{E}_1)} = \|\mathbf{D}_+\mathbf{H}_\tau\mathbf{Y}_+\|_{H^\infty(\mathbb{E}_1)} \leq \frac{2\nu}{3\sqrt{\tau}} \frac{1 - \alpha}{1 + \alpha}.$$

By the construction used in the proof of Theorem 9, the controller

$$\mathbf{K}(z) := \mathbf{Y}_+(z)(\mathbf{Z}_+(z) - \mathbf{R}(z)\mathbf{Y}_+(z))^{-1} = \frac{(1 - \alpha)^2}{\tau} \frac{1}{z - (\alpha^2/2 + \alpha - 1/2)}$$

will be stabilizing (in the sense that the sampled-data system (4.3), with (P, Q, R) given by a minimal realization of \mathbf{K} , is exponentially L^q/l^q -input-to-state stable for every $q \in [2, \infty]$), provided the term on the right-hand side of (6.9) is smaller than 1. For given $\tau > 0$ and given w , this can be achieved by choosing α sufficiently close to 1.

Specifically, for $\tau = 1$ and $w(t) \equiv 1$ (in which case $\nu = 1$), the choice $\alpha = \sqrt{2} - 1$, leads to the controller

$$(6.10) \quad \mathbf{K}(z) = \frac{2(3 - 2\sqrt{2})}{z}$$

which is stabilizing because

$$\frac{2\nu}{3\sqrt{\tau}} \frac{1 - \alpha}{1 + \alpha} = \frac{2}{3}(\sqrt{2} - 1) < 1.$$

Furthermore, if, with the aim to “mimic” ideal sampling, we choose $\tau = 1$ and

$$w(t) = \begin{cases} 12, & 0 \leq t \leq 1/12, \\ 0, & 1/12 < t \leq 1, \end{cases}$$

(in which case $\nu = \sqrt{12}$), then the controller (6.10) remains stabilizing because

$$\frac{2\nu}{3\sqrt{\tau}} \frac{1 - \alpha}{1 + \alpha} = \frac{2\sqrt{12}}{3}(\sqrt{2} - 1) < 0.957 < 1.$$

The next result shows that exponential L^∞/l^∞ -input-to-state stability guarantees the converging-input converging-state property. The proof makes essential use of

the fact that (A1), (A2), and (A5) are necessary conditions for exponential L^∞/l^∞ -input-to-state stability and therefore provides a nice illustration of the usefulness of Theorem 9.

THEOREM 10. *Assume that $v \in L^\infty(\mathbb{R}_+, \mathbb{C}^m)$ and $v_d \in l^\infty(\mathbb{Z}_+, \mathbb{C}^p)$ are convergent, that is, the limits*

$$\lim_{t \rightarrow \infty} v(t) =: v^\infty, \quad \lim_{k \rightarrow \infty} v_d(k) =: v_d^\infty$$

exist.⁴ If the sampled-data system (4.3) is exponentially L^∞/l^∞ -input-to-state stable, then, for all $x^0 \in X$ and all $x_d^0 \in X_d$, the corresponding state trajectory (x, x_d) of (4.3) satisfies

$$\lim_{t \rightarrow \infty} x(t) = x^\infty, \quad \lim_{k \rightarrow \infty} x_d(k) = x_d^\infty,$$

where

$$(6.11) \quad \begin{pmatrix} x^\infty \\ x_d^\infty \end{pmatrix} = (I - \Delta)^{-1} \begin{pmatrix} B_\tau v^\infty \\ QD_\tau v^\infty + Qv_d^\infty \end{pmatrix}.$$

Proof. Let $x^0 \in X$ and $x_d^0 \in X_d$ be arbitrary and let (x, x_d) denote the corresponding state trajectory of (4.3). Assume that (4.3) is exponentially L^∞/l^∞ -input-to-state stable. Then, by Proposition 3, the operator Δ given by (4.5) is power stable. Combining this with (4.6) and (4.7) shows that

$$\lim_{k \rightarrow \infty} x(k\tau) = x^\infty, \quad \lim_{k \rightarrow \infty} x_d(k) = x_d^\infty.$$

It remains to prove that $\lim_{t \rightarrow \infty} x(t) = x^\infty$. To this end, write $u := v - \mathcal{H}_\tau R x_d$ and note that

$$x(k\tau + t) = \mathbf{T}_t x(k\tau) + \int_0^t \mathbf{T}_s B u(k\tau + t - s) ds \quad \forall t \in [0, \tau].$$

For every $k \in \mathbb{Z}_+$, define $x_k \in C([0, \tau], X)$ and $u_k \in L^\infty([0, \tau], \mathbb{C}^m)$ by

$$x_k(t) = x(k\tau + t), \quad u_k(t) = u(k\tau + t) \quad \forall t \in [0, \tau].$$

Then

$$\lim_{k \rightarrow \infty} \|u_k - u^\infty \mathbf{1}\|_{L^\infty} = 0,$$

where $u^\infty := v^\infty - R x_d^\infty$ and $\mathbf{1}$ denotes the function identically equal to 1 on $[0, \tau]$. Defining $\zeta \in C([0, \tau], X)$ by

$$\zeta(t) = \mathbf{T}_t x^\infty + \int_0^t \mathbf{T}_s B u^\infty ds \quad \forall t \in [0, \tau],$$

it follows that

$$\sup_{t \in [0, \tau]} \|x_k(t) - \zeta(t)\| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

⁴The input v is in $L^\infty(\mathbb{R}_+, \mathbb{C}^m)$ and hence is an equivalence class of functions coinciding almost everywhere in \mathbb{R}_+ . We say that $\lim_{t \rightarrow \infty} v(t) = v^\infty$ if there exists a representative of v with limit equal to v^∞ as $t \rightarrow \infty$ or, equivalently, if $\text{ess sup}\{\|v(t) - v^\infty\| : t \geq T\} \rightarrow 0$ as $T \rightarrow \infty$.

To show that $x(t) \rightarrow x^\infty$ as $t \rightarrow \infty$, it is sufficient to prove that

$$(6.12) \quad \zeta(t) = x^\infty \quad \forall t \in [0, \tau].$$

To this end, note that, in X_{-1} ,

$$\dot{\zeta}(t) = A\zeta(t) + Bu^\infty \quad \forall t \in [0, \tau].$$

Since $\zeta(0) = x^\infty$, (6.12) will follow, provided that

$$(6.13) \quad Ax^\infty + Bu^\infty = 0.$$

By (4.5) and (6.11), $(I - \mathbf{T}_\tau)x^\infty + B_\tau R x_d^\infty = B_\tau v^\infty$ and thus

$$(6.14) \quad (\mathbf{T}_\tau - I)x^\infty + B_\tau u^\infty = 0.$$

Defining $J_\tau \in \mathcal{B}(X_{-1}, X)$ by $J_\tau \xi = \int_0^\tau \mathbf{T}_t \xi dt$ for all $\xi \in X_{-1}$, we have that $B_\tau = J_\tau B$ and $J_\tau A = \mathbf{T}_\tau - I$. Therefore, by (6.14), $J_\tau(Ax^\infty + Bu^\infty) = 0$ and (6.13) follows if J_τ has an inverse. Invoking Theorem 9, we see that (A1), (A2), and (A5) hold. In particular, $J_\tau = \text{diag}(J_\tau^+, J_\tau^-)$, where

$$J_\tau^+ \xi = \int_0^\tau e^{A^+ t} \xi dt \quad \forall \xi \in X^+ \quad J_\tau^- \xi = \int_0^\tau \mathbf{T}_t^- \xi dt \quad \forall \xi \in X^-.$$

It remains to show that J_τ^+ and J_τ^- are invertible. Note that $J_\tau^+ = f(A^+)$, where f is the entire function defined by $f(s) = \int_0^\tau e^{st} dt$. Obviously, $f(s) = 0$ if and only if $s = 2k\pi i/\tau$ for some $k \in \mathbb{Z} \setminus \{0\}$. Consequently, by (A5) and the spectral mapping theorem, $0 \notin f(\sigma(A^+)) = \sigma(J_\tau^+)$, showing that J_τ^+ is invertible. By (A2), A^- is invertible and thus $J_\tau^- = (A^-)^{-1}(\mathbf{T}_\tau^- - I)$. Furthermore, again invoking (A2), \mathbf{T}_τ^- is power stable (on X as well as on X_{-1}) and so $\mathbf{T}_\tau^- - I$ is invertible. Hence, J_τ^- has an inverse, completing the proof. \square

The next result relates to the sampled-data scheme (5.17) which includes the precompensator (5.15). Recall that, by Proposition 7, exponential L^2/l^2 -input-to-state stability of this scheme ensures that $y(t) \rightarrow 0$ as $t \rightarrow \infty$, provided that $v \in L^2(\mathbb{R}_+, \mathbb{C}^m)$, $v_d \in l^2(\mathbb{Z}_+, \mathbb{C}^p)$, and $\mathbf{T}_{t_0}(Ax^0 + Bx_p^0) \in X$ for some $t_0 \geq 0$.

THEOREM 11. *The following statements are equivalent.*

- (1) (A1)–(A6) hold for (2.1).
- (2) There exists a discrete-time controller (4.1) such that the sampled-data system (5.17) is exponentially L^q/l^q -input-to-state stable for every $q \in [2, \infty]$.
- (3) There exists a finite-dimensional discrete-time controller (4.1) such that the sampled-data system (5.17) is exponentially L^q/l^q -input-to-state stable for every $q \in [2, \infty]$.

The proof of Theorem 11 can be found in the Appendix.

If the sampled-data scheme (5.17) is exponentially L^∞/l^∞ -input-to-state stable, then, using Theorem 11, it can be shown that (5.17) has the convergent-input convergent-state property. If in addition, $\mathbf{T}_{t_0}(Ax^0 + Bx_p^0) \in X$ for some $t_0 \geq 0$, then (5.17) has also the convergent-input convergent-output property (that is, the outputs converge whenever the inputs converge). We omit the details for the sake of brevity.

7. Real sampled-data controllers for real systems. We now assume that the underlying well-posed system (2.1) is real in the sense that its state space X is a real Hilbert space and the input and output spaces are given by \mathbb{R}^m and \mathbb{R}^p ,

respectively. In order to use spectral theory and, in particular, the spectral projection Π defined in (5.1), we need to work with the complexification X^c of X . This is the complex Hilbert space $X \times X$, endowed with the scalar multiplication $\mathbb{C} \times (X \times X) \rightarrow X \times X$ given by

$$(\alpha_1 + i\alpha_2)(\xi_1, \xi_2) = (\alpha_1\xi_1 - \alpha_2\xi_2, \alpha_1\xi_2 + \alpha_2\xi_1).$$

Let $\xi = (\xi_1, \xi_2) \in X^c$. It is convenient to write $\xi = \xi_1 + i\xi_2$, $\operatorname{Re} \xi = \xi_1$, and $\operatorname{Im} \xi = \xi_2$ (real and imaginary parts of ξ). We define the complex conjugation operation by

$$\bar{\xi} := \operatorname{Re} \xi - i \operatorname{Im} \xi \quad \forall \xi \in X^c.$$

The inner product $\langle \cdot, \cdot \rangle$ on X extends to an inner product on X^c in a natural way:

$$\langle \xi, \zeta \rangle := \langle \operatorname{Re} \xi, \operatorname{Re} \zeta \rangle + \langle \operatorname{Im} \xi, \operatorname{Im} \zeta \rangle + i(\langle \operatorname{Im} \xi, \operatorname{Re} \zeta \rangle - \langle \operatorname{Re} \xi, \operatorname{Im} \zeta \rangle) \quad \forall \xi, \zeta \in X^c.$$

Consequently, $\|\xi\| = \sqrt{\|\operatorname{Re} \xi\|^2 + \|\operatorname{Im} \xi\|^2}$ for all $\xi \in X^c$.

Linear operators defined on X extend in an obvious way to linear operators defined on X^c . In particular, we define the complexification A^c of A by setting $\operatorname{dom}(A^c) := \{\xi \in X^c : \operatorname{Re} \xi, \operatorname{Im} \xi \in \operatorname{dom}(A)\}$ and

$$A^c \xi := A(\operatorname{Re} \xi) + iA(\operatorname{Im} \xi) \quad \forall \xi \in \operatorname{dom}(A^c).$$

Complexifications B^c and C^c of B and C , respectively, can be defined similarly. Obviously, A^c generates a C_0 -semigroup $\mathbf{T}^c = (\mathbf{T}^c)_{t \geq 0}$ which extends \mathbf{T} to X^c .

An operator $S : \operatorname{dom}(S) \subset X^c \rightarrow X^c$ is said to be *real* if, for all $\xi \in \operatorname{dom}(S)$, $\bar{\xi} \in \operatorname{dom}(S)$ and $\overline{S\xi} = S\bar{\xi}$. If S is real, then $S\xi \in X$ for all $\xi \in \operatorname{dom}(S) \cap X$. Trivially, the operator A^c defined above is real. Moreover, a function $\mathbf{H} \in H^\infty(\mathbb{E}_\eta, \mathbb{C}^{p \times m})$ is said to be *real* if $\overline{\mathbf{H}(z)} = \mathbf{H}(\bar{z})$ for all $z \in \mathbb{E}_\eta$ or, equivalently, if the coefficients of the Taylor expansion of \mathbf{H} at ∞ are in $\mathbb{R}^{p \times m}$.

Let $\lambda \in \varrho(A^c) \cap \mathbb{R}$. Then $(\lambda I - A^c)^{-1}$ is real, the operator $\lambda I - A : \operatorname{dom}(A) \rightarrow X$ is bijective, and $(\lambda I - A^c)^{-1} X = (\lambda I - A)^{-1} X$. With $\|\xi\|_{-1} = \|(\lambda I - A^c)^{-1} \xi\|$, the completions X_{-1} and $(X^c)_{-1}$ of X and X^c with respect to $\|\cdot\|_{-1}$ satisfy $(X_{-1})^c = (X^c)_{-1}$.

Assume now that (A1)–(A6) hold (for the complexifications X^c , A^c , \mathbf{T}^c , B^c , and C^c). Theorem 9 guarantees that there exists a (finite-dimensional) discrete-time controller (4.1) such that the sampled-data system (4.3) is exponentially L^q/l^q -input-to-state stable for every $q \in [2, \infty]$. We want to prove that this stabilizing discrete-time controller can be chosen to be real. An inspection of the proof of Theorem 9 (the argument proving that statement (1) implies statement (3)) reveals that it is sufficient to show that the matrix $\mathbf{Z} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p \times p})$ defined in (6.6) is real in the sense defined above. This in turn will be true if \mathbf{G}_τ^+ is real. Indeed, if the latter is the case, then, since \mathbf{G}_τ is real, $\mathbf{G}_\tau^- = \mathbf{G}_\tau - \mathbf{G}_\tau^+$ is real, and moreover, \mathbf{N}_+ and \mathbf{D}_+ , and therefore also \mathbf{Y}_+ and \mathbf{Z}_+ , can be chosen to be real. The realness of \mathbf{G}_τ^+ is an immediate consequence of the following lemma.

LEMMA 12. *Assume that (A1) holds for A^c . Let Φ be a smooth, positively oriented, and simple closed curve in \mathbb{C} not intersecting $\sigma(A^c)$, enclosing $\sigma(A^c) \cap \overline{\mathbb{C}_0}$ in its interior, and having $\sigma(A^c) \cap (\mathbb{C} \setminus \overline{\mathbb{C}_0})$ in its exterior. Then the spectral projection $\Pi : X^c \rightarrow X^c$ defined by*

$$\Pi = \frac{1}{2\pi i} \int_\Phi (sI - A^c)^{-1} ds$$

is real. Furthermore, Π extends to a projection on $(X^c)_{-1}$ and $\Pi X_{-1} = \Pi X \subset X$.

Proof. By (A1), there exists $\delta \in (0, \varepsilon)$ such that $\sigma(A^c) \cap \overline{\mathbb{C}}_0 = \sigma(A^c) \cap \overline{\mathbb{C}}_{-\delta}$ (this means in particular that there is no spectrum of A^c in the vertical strip $-\delta \leq \operatorname{Re} s < 0$). For $r > 0$, let S_r denote the open disk with center $r - \delta$ and radius r . Choose $r > 0$ sufficiently large such that

$$\sigma(A^c) \cap \overline{\mathbb{C}}_0 \subset S_r.$$

Define the curve $\varphi : [0, 2\pi] \rightarrow \mathbb{C}$ by $\varphi(t) = (r - \delta) + re^{it}$, that is, φ parametrizes ∂S_r with positive orientation. By Cauchy's theorem,

$$\int_{\Phi} (sI - A^c)^{-1} ds = \int_{\varphi} (sI - A^c)^{-1} ds.$$

Therefore, using the realness of A^c and the fact that $\overline{\varphi}(t) = \varphi(2\pi - t) =: \varphi_-(t)$, we conclude that, for all $\xi \in X^c$,

$$\overline{\Pi\xi} = \frac{-1}{2\pi i} \int_0^{2\pi} (\overline{\varphi}(t)I - A^c)^{-1} \overline{\varphi}'(t) \overline{\xi} dt = \frac{-1}{2\pi i} \int_0^{2\pi} (\varphi_-(t)I - A^c)^{-1} \varphi'_-(t) \overline{\xi} dt = \Pi\overline{\xi}.$$

Therefore, Π is real and consequently, $\Pi X \subset X$. As was already pointed out earlier, Π extends to a projection on $(X^c)_{-1}$. As a consequence of (A1), ΠX is a finite-dimensional subspace of X_{-1} and hence is closed. On the other hand ΠX_{-1} is the closure of ΠX , showing that $\Pi X_{-1} = \Pi X \subset X$. \square

8. Appendix: Proofs of Proposition 7 and Theorem 11. We start by considering the series interconnection of (2.1) and (5.15) which is obtained by setting $u = x_p$. The series interconnection is a regular system $\tilde{\Sigma}$ with input u_p , the input of (5.15), and output y , the output of (2.1). Its transfer function $\tilde{\mathbf{G}}$ is given by $\tilde{\mathbf{G}}(s) = \mathbf{G}(s)/(s + a)$. The state space of $\tilde{\Sigma}$ is $\tilde{X} := X \times \mathbb{C}^m$ and the generating operators \tilde{A} , \tilde{B} , and \tilde{C} are

$$\tilde{A} := \begin{pmatrix} A & B \\ 0 & -aI \end{pmatrix}, \quad \operatorname{dom}(\tilde{A}) := \{(\xi_1, \xi_2) \in \tilde{X} : A\xi_1 + B\xi_2 \in X\}, \quad \tilde{B} := \begin{pmatrix} 0 \\ I \end{pmatrix},$$

and

$$\tilde{C} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} := C(\xi_1 - (\lambda I - A)^{-1} B\xi_2) + \mathbf{G}(\lambda)\xi_2 \quad \forall \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \operatorname{dom}(\tilde{A}).$$

The semigroup generated by \tilde{A} is denoted by $\tilde{\mathbf{T}} = (\tilde{\mathbf{T}}_t)_{t \geq 0}$. The regular system $\tilde{\Sigma}$ can be written as

$$(8.1) \quad \begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u_p, & \tilde{x}(0) = \tilde{x}^0 \in \tilde{X}, \\ y = \tilde{C}_\lambda \tilde{x}, \end{cases}$$

where

$$\tilde{x} = \begin{pmatrix} x \\ x_p \end{pmatrix} \quad \text{and} \quad \tilde{x}^0 = \begin{pmatrix} x^0 \\ x_p^0 \end{pmatrix}.$$

Therefore, the sampled-data system (5.17) can be expressed in the form

$$(8.2) \quad \begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}(v - \mathcal{H}_\tau y_d), & \tilde{x}(0) = \tilde{x}^0 \in \tilde{X}, \\ y = \tilde{C}_\lambda \tilde{x}, \\ x_d^\nabla = Px_d + Q(v_d + \mathcal{S}_\tau y), & x_d(0) = x_d^0 \in X_d, \\ y_d = Rx_d. \end{cases}$$

Since $\sigma(\tilde{A}) = \sigma(A) \cup \{-a\}$, it follows trivially that (A1) holds for (2.1) if and only if (A1) holds for (8.1). Assume that (A1) holds for (2.1). Without loss of generality we may assume that the curve Φ used in the definition of the spectral projection (5.1) has $-a < 0$ in its exterior. Then Φ does not intersect $\sigma(\tilde{A})$, encloses $\sigma(\tilde{A}) \cap \overline{\mathbb{C}_0}$ in its interior, and $\sigma(\tilde{A}) \cap (\mathbb{C} \setminus \overline{\mathbb{C}_0})$ is contained in its exterior. Defining the spectral projection

$$(8.3) \quad \tilde{\Pi} := \frac{1}{2\pi i} \int_{\Phi} (sI - \tilde{A})^{-1} ds$$

and setting $\tilde{X}^+ := \tilde{\Pi}\tilde{X}$ and $\tilde{X}^- := (I - \tilde{\Pi})\tilde{X}$, we have that $\tilde{X}^+ \subset \text{dom}(\tilde{A})$, $\dim \tilde{X}^+ < \infty$, and $\tilde{X} = \tilde{X}^+ \oplus \tilde{X}^-$. We can now decompose system (8.1) accordingly, yielding an infinite-dimensional regular system with generating operators $(\tilde{A}^-, \tilde{B}^-, \tilde{C}^-)$ and a finite-dimensional system $(\tilde{A}^+, \tilde{B}^+, \tilde{C}^+)$ with corresponding semigroups $\tilde{\mathbf{T}}^-$ and $\tilde{\mathbf{T}}^+$; cf. (5.4)–(5.6). Consider the operator

$$(8.4) \quad S := \frac{1}{2\pi i} \int_{\Phi} \frac{1}{s+a} (sI - A)^{-1} ds \in \mathcal{B}(X) \cap \mathcal{B}(X_{-1}).$$

As has been mentioned earlier, Π extends to a projection on X_{-1} and $\Pi X_{-1} = \Pi X = X^+$. It can be shown that $\Pi S = S\Pi = S$ on X and X_{-1} (see the proof of Theorem 1.5.4 in [2]). Consequently, S is in $\mathcal{B}(X_{-1}, X)$ satisfying $SX_{-1} \subset X^+$. Noting that

$$(8.5) \quad (sI - \tilde{A})^{-1} = \begin{pmatrix} (sI - A)^{-1} & \frac{1}{s+a}(sI - A)^{-1}B \\ 0 & \frac{1}{s+a}I \end{pmatrix},$$

it follows that, for all $(\xi_1, \xi_2) \in \tilde{X}$,

$$(8.6) \quad \tilde{\Pi} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \Pi\xi_1 + SB\xi_2 \\ 0 \end{pmatrix} \quad \text{and} \quad (I - \tilde{\Pi}) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} (I - \Pi)\xi_1 - SB\xi_2 \\ \xi_2 \end{pmatrix}.$$

In particular, since $SBC^m \subset X^+$, we conclude that

$$(8.7) \quad \tilde{X}^+ = X^+ \times \{0\}.$$

Proof of Proposition 7. To prove (5.18), it is sufficient to show that (A1) and (A2) hold in the context of (8.1) (in which case we may apply Proposition 6 to (8.2)). It is clear that (A1) holds for (8.1). To prove that (A2) holds for (8.1), that is, $\tilde{\mathbf{T}}^-$ is exponentially stable, it is sufficient to show that the resolvent of \tilde{A} is in $H^\infty(\mathbb{C}_0, \mathcal{B}(\tilde{X}^-))$ (see, for example, Theorem 5.1.5 in [1], Theorem 10.6.4 in [2], or Theorem 1.11 on p. 302 in [3]). To this end, note that, for all $(\xi_1, \xi_2) \in \tilde{X}^-$,

$$(sI - \tilde{A}^-)^{-1} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = (I - \tilde{\Pi})(sI - \tilde{A})^{-1} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

Therefore, invoking (8.5) and (8.6), we obtain

$$(8.8) \quad (sI - \tilde{A}^-)^{-1} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} (sI - A^-)^{-1}(I - \Pi)\xi_1 + \frac{1}{s+a}((sI - A^-)^{-1}B^-\xi_2 - SB\xi_2) \\ \frac{1}{s+a}\xi_2 \end{pmatrix} \\ \forall \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \tilde{X}^-.$$

By (A2), the resolvent of A^- is in $H^\infty(\mathbb{C}_0, \mathcal{B}(X^-))$, and thus the resolvent of \tilde{A}^- is in $H^\infty(\mathbb{C}_0, \mathcal{B}(\tilde{X}^-))$.

Now assume that $v \in L^2(\mathbb{R}_+, \mathbb{C}^m)$, $v_d \in l^2(\mathbb{Z}_+, \mathbb{C}^p)$, and $\mathbf{T}_{t_0}(Ax^0 + Bx_p^0) \in X$ for some $t_0 \geq 0$. The input of (2.1) is x_p and it follows from Lemma 4 that

$$y = y^- + C^+\Pi x, \quad \text{where } y^- := (C^-)_\Lambda \mathbf{T}^-(I - \Pi)x^0 + G^-x_p.$$

Obviously, x is continuous and, by Proposition 3 (applied to (8.2)), we have that $\tilde{x}(t) \rightarrow 0$, and hence $x(t) \rightarrow 0$ as $t \rightarrow \infty$. It remains to show that y^- is continuous on $[t_0, \infty)$ and $y^-(t) \rightarrow 0$ as $t \rightarrow \infty$. To this end, note that y^- is the output of an exponentially stable well-posed system corresponding to the initial condition $(I - \Pi)x^0$ and the input x_p . By Proposition 6 (applied to (8.2)), $\mathcal{H}_\tau y_d \in L^2(\mathbb{R}_+, \mathbb{C}^m)$. Since

$$x_p(t) = e^{-at}x_p^0 + \int_0^t e^{-a(t-s)}(v(s) - (\mathcal{H}_\tau y_d)(s))ds,$$

it follows that $x_p \in W^{1,2}(\mathbb{R}_+, \mathbb{C}^m)$. Furthermore,

$$\mathbf{T}_{t_0}^-(A^-(I - \Pi)x^0 + B^-x_p(0)) = (I - \Pi)\mathbf{T}_{t_0}(Ax^0 + Bx_p^0) \in X^-.$$

An application of Proposition 1 to the exponentially stable well-posed system with generating operators (A^-, B^-, C^-) and input-output operator G^- shows that y^- is continuous on $[t_0, \infty)$ and $y^-(t) \rightarrow 0$ as $t \rightarrow \infty$. \square

Proof of Theorem 11. As has been already noted, (A1) holds in the context of (8.1) if and only if it holds for (2.1). Assume that (A1) holds. By Theorem 9, it is sufficient to show that any of the assumptions (A2)–(A6) holds in the context of (8.1) if and only if it holds for (2.1). Since $\sigma(\tilde{A}) = \sigma(A) \cup \{-a\}$, this is certainly the case for (A5) and (A6). As for (A2), we have seen in the proof of Proposition 7 that if (A2) holds for (2.1), then it holds for (8.1). Formula (8.8) shows that the converse is also true.

Let us now consider (A3). Invoking (8.6), we obtain

$$\tilde{B}^+\xi = \tilde{\Pi} \begin{pmatrix} 0 \\ I \end{pmatrix} \xi = \begin{pmatrix} SB\xi \\ 0 \end{pmatrix} \quad \forall \xi \in \mathbb{C}^m.$$

Setting $S^+ := S|_{X^+}$, it follows from (8.4) and (finite-dimensional) functional calculus (see, for example, p. 44 in [5]) that

$$S^+ = \frac{1}{2\pi i} \int_{\Phi} \frac{1}{s+a} (sI - A^+)^{-1} ds = (aI + A^+)^{-1}.$$

Since $S = S\Pi$, it follows that $SB = S^+B^+$ and so

$$\tilde{B}^+ = \begin{pmatrix} (aI + A^+)^{-1}B^+ \\ 0 \end{pmatrix}.$$

Appealing to (8.7), we conclude

$$\tilde{A}^+ \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \tilde{A} \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} = \begin{pmatrix} A^+\xi_1 \\ 0 \end{pmatrix} \quad \forall \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \tilde{X}^+ = X^+ \times \{0\},$$

and thus,

$$\tilde{\mathbf{T}}_\tau^+ \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \mathbf{T}_\tau^+\xi_1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{A^+\tau}\xi_1 \\ 0 \end{pmatrix} \quad \forall \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \tilde{X}^+ = X^+ \times \{0\}.$$

Now $(aI + A^+)^{-1} \mathbf{T}_\tau^+ (aI + A^+) = \mathbf{T}_\tau^+$, showing that the pairs (\mathbf{T}_τ^+, B^+) and $(\mathbf{T}_\tau^+, (aI + A^+)^{-1} B^+)$ are similar. Consequently, by the above formulas for \tilde{B}^+ and $\tilde{\mathbf{T}}_\tau^+$, the reachability map of $(\tilde{\mathbf{T}}_\tau^+, \tilde{B}^+)$ has full rank (that is, rank equal to $\dim \tilde{X}^+ = \dim X^+$) if and only if the reachability map of (\mathbf{T}_τ^+, B^+) has full rank. This shows that (A3) holds in the context of (8.1) if and only if (A3) holds for (2.1).

Finally, noting that

$$\tilde{C}^+ \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \tilde{C} \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} = C^+ \xi_1 \quad \forall \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \tilde{X}^+ = X^+ \times \{0\},$$

it is clear that a similar argument can be used to establish that (A4) holds in the context of (8.1) if and only if (A4) holds for (2.1). \square

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