A Sampled-Data Servomechanism for Stable Well-Posed Systems

Zhenqing Ke, Hartmut Logemann, and Richard Rebarber

***Abstract*—In this technical note, an approximate tracking and disturbance rejection problem is solved for the class of exponentially stable well-posed infinite-dimensional systems by invoking a simple sampled-data low-gain controller (suggested by the internal model principle). The reference signals are finite sums of sinusoids and the disturbance signals are asymptotic to finite sums of sinusoids.**

***Index Terms*—Disturbance rejection, infinite-dimensional systems, low-gain control, sampled-data control, tracking.**

I. Introduction

There has been much interest in low-gain integral control over the last thirty years. The following principle (tuning integrator) has become well established (see, for example, Davison [1], Lunze [6] and Morari [7]): closing the loop around an asymptotically stable, finite-dimensional, continuous-time plant, with square transfer function matrix $\mathbf{G}$, compensated by an integral controller $(\varepsilon/s)I$, will result in a stable closed-loop system that achieves asymptotic tracking of arbitrary constant reference signals, provided that the gain parameter $\varepsilon > 0$ is sufficiently small and the eigenvalues of the steady-state matrix $\mathbf{G}(0)$ have positive real parts. This principle has been extended to various classes of infinite-dimensional systems (see Logemann and Townley [5] and the references therein). Moreover, discrete-time and sampled-data versions of the tuning integrator have been developed (for infinite-dimensional systems) by Logemann and Townley in [4].

Hämäläinen and Pohjolainen [2] succeeded in generalizing the above principle to the multi-frequency case in which the reference and disturbance signals to be tracked and rejected, respectively, are (finite) linear combinations of sinusoids having prespecified frequencies. Their solution, inspired by the internal model principle, is a simple low-gain tuning controller and it is shown to work for exponentially stable infinite-dimensional systems with impulse response in the Callier-Desoer algebra. Rebarber and Weiss [8] proved a similar result for the (more general) class of exponentially stable well-posed systems. Ke, Logemann and Rebarber [3] developed a sampled-data version of the tuning regulator presented in [2], [8]. The main result in [3] guarantees approximate tracking and disturbance rejection for stable infinite-dimensional systems which have the property that their impulse responses are exponentially bounded matrix-valued Borel measures. We mention in this context that systems with measure impulse responses are necessarily regular (see [9] and the references therein for details on well-posed and regular systems): whilst verification of the regularity property can sometimes be difficult, it is usually even more difficult to show that the impulse response is a measure.

In this technical note, we consider essentially the same problem as in [2], [3], [8], but we seek a sampled-data solution which applies to exponentially stable well-posed systems (avoiding the assumption that the impulse response of the system is a measure). Adopting an input-output approach, we do not invoke any results from the state-space theory of well-posed systems, so that, for the purposes of this technical note, an exponentially stable well-posed system is simply a system with the property that its transfer function is holomorphic and bounded in a half-plane $\{s \in \mathbb{C} : \operatorname{Re} s > \alpha\}$ for some $\alpha < 0$. The essence of the main result of the technical note can be described as follows: low-gain sampled-data control based on a discrete-time version of the continuous-time controller given in [2], [8], in conjunction with suitable low-pass filters, achieves approximate tracking and disturbance rejection for exponentially stable well-posed systems.

The technical note is structured as follows. In Section II, we state a number of preliminary technical results used in the technical note. In Section III, we first prove a discrete-time result which is a crucial tool for the proof of the main result of the technical note. We consider a feedback controller with transfer function of the form

$$\varepsilon\left(\mathbf{K}^0(z) + \sum_{j=1}^{N} \frac{K_j}{z - \lambda_j}\right) \tag{1.1}$$

where $\mathbf{K}^0$ is holomorphic and bounded on $\{z \in \mathbb{C} : |z| > \alpha\}$ for some $\alpha \in (0,1)$, $K_j \in \mathbb{C}^{m\times p}$ and $\lambda_j \in \mathbb{C}$ with $|\lambda_j| = 1$. Applying this controller to a discrete-time plant with transfer function $\mathbf{P}$ which is holomorphic and bounded on $\{z \in \mathbb{C} : |z| > \alpha\}$, we show that the transfer function of the closed-loop feedback system is holomorphic and bounded on $\{z \in \mathbb{C} : |z| > \beta\}$ for some $\beta \in (\alpha, 1)$, provided that (i) all the eigenvalues of $\overline{\lambda}_j \mathbf{P}(\lambda_j) K_j$ have positive real parts, and (ii) the gain parameter ε is sufficiently small. This result is an extension of a result in [4] on low-gain discrete-time integral control.

In Section IV, the main result of Section III is then used in the context of approximate tracking and disturbance rejection for infinite-dimensional sampled-data feedback systems. The continuous-time plant is assumed to have a transfer function $\mathbf{G}$ which is holomorphic and bounded on $\{s \in \mathbb{C} : \operatorname{Re} s > \alpha\}$ for some $\alpha < 0$. The sampled-data servomechanism consists of a discrete-time feedback controller of the form (1.1) with $\lambda_j = e^{\xi_j \tau}$, where $\xi_j \in i\mathbb{R}$ for $j = 1, \ldots, N$ and $\tau > 0$ is the sampling period, in conjunction with two filters with transfer functions $\mathbf{F}_1$ and $\mathbf{F}_2$. The reference signal r is given by $r(t) = \sum_{j=1}^{N} e^{\xi_j t} \mathfrak{r}_j$, $\mathfrak{r}_j \in \mathbb{C}^p$ and the disturbance signals are assumed to be asymptotically equal (in a suitable sense) to signals of the same form. If all the eigenvalues of $\mathbf{G}(\xi_j)K_j$ have positive real parts and $\mathbf{F}_1(\xi_j)$ and $\mathbf{F}_2(\xi_j)$ are equal to the identity for all $j = 1, \ldots, N$, then it is shown that, for every $\delta > 0$, there exists $\tau_\delta > 0$ such that, for every sampling period $\tau \in (0, \tau_\delta)$, there exists $\varepsilon_\tau > 0$ such that, for every $\varepsilon \in (0, \varepsilon_\tau)$, the output y of the closed-loop sampled-data system can be decomposed as $y = y_1 + y_2$, where $y_1(\cdot)e^{-\gamma\cdot} \in L^2(\mathbb{R}_+, \mathbb{C}^p)$ for some $\gamma < 0$ and y_2 satisfies $\limsup_{t\to\infty} \|y_2(t) - r(t)\| \leq \delta$.

Notation: For $\alpha > 0$, $\beta \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, define $\mathbb{E}_\alpha := \{z \in \mathbb{C} : |z| > \alpha\}$, $\mathbb{C}_\beta := \{z \in \mathbb{C} : \operatorname{Re} z > \beta\}$ and $\mathbb{B}(\lambda, \alpha) := \{z \in \mathbb{C} : |z - \lambda| < \alpha\}$. For a set $U \subseteq \mathbb{C}$, let $\operatorname{cl}(U)$ denote the closure of U. In the following definitions, let $\Omega \subset \mathbb{C}$ be open and let $X = \mathbb{C}^m$ or $X = \mathbb{C}^{p\times m}$. We define

$$H^\infty(\Omega, X) := \{f : \Omega \to X \,|\, f \text{ is holomorphic and bounded}\}$$

$$H^\infty_<(\mathbb{E}_1, X) := \bigcup_{0<\gamma<1} H^\infty(\mathbb{E}_\gamma, X)$$

$$H^2(\mathbb{C}_\beta, X) := \left\{f : \mathbb{C}_\beta \to X \,|\, f \text{ is holomorphic and } \sup_{x>\beta} \int_{-\infty}^{\infty} \|f(x + i\sigma)\|^2 \, d\sigma < \infty\right\}.$$

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Z. Ke and H. Logemann are with the Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY, U.K. (e-mail: kezhenqing@hotmail.com; hl@maths.bath.ac.uk).

R. Rebarber is with the Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588-0130 USA (e-mail: rrebarber1@unl.edu).

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For $\beta \in \mathbb{R}$ and $1 \leq q < \infty$, we define the exponentially weighted L^q-space $L^q_\beta(\mathbb{R}_+, X)$ by

$$L^q_\beta(\mathbb{R}_+, X) := \left\{ f \in L^q_{\mathrm{loc}}(\mathbb{R}_+, X) : f(\cdot)e^{-\beta \cdot} \in L^q(\mathbb{R}_+, X) \right\}.$$

We write $H^\infty(\Omega) := H^\infty(\Omega, \mathbb{C})$, $H^2(\mathbb{C}_\beta) := H^2(\mathbb{C}_\beta, \mathbb{C})$ and $L^q_\beta(\mathbb{R}_+) := L^q_\beta(\mathbb{R}_+, \mathbb{C})$. For $A \in \mathbb{C}^{m\times m}$, let $\sigma(A)$ denote the spectrum of A. For $N \in \mathbb{N}$, set $\underline{N} := \{1, 2, \ldots, N\}$.

II. Preliminaries

In this section, we present some technical results for both discrete-time and continuous-time systems. The proofs of Lemmas II.1, II.2 and II.3 are straightforward: they can be found, for example, in [3].

Lemma II.1: Assume that H is a discrete-time input-output operator with impulse response in $\ell^1(\mathbb{Z}_+, \mathbb{C}^{p\times m})$ and transfer function $\mathbf{H}$. If $v : \mathbb{Z}_+ \to \mathbb{C}^m$ satisfies $\lim_{k\to\infty}(v(k) - \lambda^k \mathfrak{v}) = 0$, where $\lambda \in \overline{\mathbb{E}}_1$ and $\mathfrak{v} \in \mathbb{C}^m$, then $\lim_{k\to\infty}((Hv)(k) - \lambda^k \mathbf{H}(\lambda)\mathfrak{v}) = 0$.

Lemma II.2: Assume that H is a continuous-time input-output operator with impulse response $h \in L^1(\mathbb{R}_+, \mathbb{C}^{p\times m})$ and transfer function $\mathbf{H}$. Let $u : \mathbb{R}_+ \to \mathbb{C}^m$ be measurable. If u is bounded, then

$$\limsup_{t\to\infty} \|(Hu)(t)\| \leq \|h\|_{L^1} \limsup_{t\to\infty} \|u(t)\|.$$

Moreover, if u satisfies $\lim_{t\to\infty}(u(t) - e^{\xi t}\mathfrak{u}) = 0$, where $\xi \in \overline{\mathbb{C}}_0$, $\mathfrak{u} \in \mathbb{C}^m$, then

$$\lim_{t\to\infty} \left[(Hu)(t) - e^{\xi t}\mathbf{H}(\xi)\mathfrak{u}\right] = 0.$$

Let $\tau > 0$ denote the sampling period and let $F(\mathbb{Z}_+, \mathbb{C}^m)$ and $F(\mathbb{R}_+, \mathbb{C}^m)$ denote all $\mathbb{C}^m$-valued functions defined on $\mathbb{Z}_+$ and $\mathbb{R}_+$, respectively. The *ideal sampling operator* $\mathcal{S}_\tau : F(\mathbb{R}_+, \mathbb{C}^m) \to F(\mathbb{Z}_+, \mathbb{C}^m)$ is defined by

$$(\mathcal{S}_\tau u)(k) := u(k\tau), \quad \forall k \in \mathbb{Z}_+.$$

The (zero-order) *hold operator* $\mathcal{H}_\tau : F(\mathbb{Z}_+, \mathbb{C}^m) \to F(\mathbb{R}_+, \mathbb{C}^m)$ is defined by

$$(\mathcal{H}_\tau v)(t) := v(k), \quad \forall t \in [k\tau, (k+1)\tau).$$

Lemma II.3: Assume that H is a continuous-time input-output operator with impulse response in $L^1_\alpha(\mathbb{R}_+, \mathbb{C}^{p\times m})$ for some $\alpha \leq 0$ and let H_τ be the sample-hold discretization of H, defined by $H_\tau := \mathcal{S}_\tau H \mathcal{H}_\tau$. Set $\rho := e^{\alpha\tau}$ and let $\mathbf{H}$ and $\mathbf{H}_\tau$ denote the transfer functions of H and H_τ, respectively. Then $\mathbf{H}_\tau \in H^\infty(\mathbb{E}_\rho, \mathbb{C}^{p\times m})$ and

$$\lim_{\tau\to 0} \mathbf{H}_\tau(e^{\xi\tau}) = \mathbf{H}(\xi), \quad \forall \xi \in \overline{\mathbb{C}}_0.$$

For the purposes of this technical note, it is convenient to the define the concept of a (finite-dimensional) filter as follows.

Definition II.4: A (finite-dimensional) *filter* is an exponentially stable, strictly causal, finite-dimensional system. ◇

We note that a filter has impulse response of the form $t \mapsto Ce^{At}B$, where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$ and all eigenvalues of A have negative real parts.

Lemma II.5: Let H be a continuous-time input-output operator with transfer function $\mathbf{H} \in H^\infty(\mathbb{C}_\alpha)$ for some $\alpha < 0$, and let F be a single-input-single-output filter. Then there exists $\beta \in (\alpha, 0)$ such that the impulse response of HF is in $L^1_\beta(\mathbb{R}_+)$.

Proof: Since the transfer function $\mathbf{F}$ of F is a strictly proper stable rational function, there exists $\gamma \in (\alpha, 0)$ such that $\mathbf{F} \in H^2(\mathbb{C}_\gamma)$, and hence, $\mathbf{HF} \in H^2(\mathbb{C}_\gamma)$. Let g denote the impulse response of HF. By the Paley-Wiener Theorem, $g \in L^2_\gamma(\mathbb{R}_+)$. Therefore, it follows easily from Hölder's inequality that $g \in L^1_\beta(\mathbb{R}_+)$ for every $\beta \in (\gamma, 0)$. □

We close this section with the statement of a result from the fractional representation theory of feedback systems. To this end, let $\Omega \subset \mathbb{C}$ be open and let $\mathcal{Q}$ denote the quotient field of $H^\infty(\Omega)$, i.e., $\mathcal{Q} = \{n/d : n, d \in H^\infty(\Omega), d \neq 0\}$.

Defintion II.6:

(i) A *left-coprime factorization* of $\mathbf{H} \in \mathcal{Q}^{p\times m}$ (over $H^\infty(\Omega)$) is a pair $(\mathbf{D}, \mathbf{N}) \in H^\infty(\Omega, \mathbb{C}^{p\times p}) \times H^\infty(\Omega, \mathbb{C}^{p\times m})$ such that $\det \mathbf{D} \neq 0$, $\mathbf{H} = \mathbf{D}^{-1}\mathbf{N}$ and $\mathbf{D}$, $\mathbf{N}$ are left coprime, i.e., there exist $\mathbf{X} \in H^\infty(\Omega, \mathbb{C}^{p\times p})$, $\mathbf{Y} \in H^\infty(\Omega, \mathbb{C}^{m\times p})$ satisfying $\mathbf{DX} + \mathbf{NY} = I$.

(ii) A *right-coprime factorization* of $\mathbf{H} \in \mathcal{Q}^{p\times m}$ (over $H^\infty(\Omega)$) is a pair $(\mathbf{N}, \mathbf{D}) \in H^\infty(\Omega, \mathbb{C}^{p\times m}) \times H^\infty(\Omega, \mathbb{C}^{m\times m})$ such that $\det \mathbf{D} \neq 0$, $\mathbf{H} = \mathbf{ND}^{-1}$ and $\mathbf{N}$, $\mathbf{D}$ are right coprime, i.e., there exist $\mathbf{X} \in H^\infty(\Omega, \mathbb{C}^{m\times p})$, $\mathbf{Y} \in H^\infty(\Omega, \mathbb{C}^{m\times m})$ satisfying $\mathbf{XN} + \mathbf{YD} = I$. ◇

Proposition II.7: Let $\mathbf{H} \in \mathcal{Q}^{p\times m}$ and $\mathbf{K} \in \mathcal{Q}^{m\times p}$. Assume that there exist a left-coprime factorization $(\mathbf{D_H}, \mathbf{N_H})$ of $\mathbf{H}$ and a right-coprime factorization $(\mathbf{N_K}, \mathbf{D_K})$ of $\mathbf{K}$ (both over $H^\infty(\Omega)$). If the matrix $\mathbf{N_H N_K} + \mathbf{D_H D_K}$ has an inverse in $H^\infty(\Omega, \mathbb{C}^{p\times p})$, i.e.,

$$\inf_{z\in\Omega} |\det(\mathbf{N_H}(z)\mathbf{N_K}(z) + \mathbf{D_H}(z)\mathbf{D_K}(z))| > 0$$

then $(I + \mathbf{HK})^{-1} \in H^\infty(\Omega, \mathbb{C}^{p\times p})$.

The proof is straightforward and is therefore omitted (see also [10, Lemma 3.1], of which Proposition II.7 is a special case).

III. A Discrete-Time Result

The following proposition will be crucial in the proof of Theorem IV.1, the main result of this technical note. It is also interesting in its own right.

Proposition III.1: Let $N \in \mathbb{N}$ and let $\lambda_j \in \mathbb{C}$, $|\lambda_j| = 1$ for all $j \in \underline{N}$ be such that $\lambda_j \neq \lambda_k$ for all $j, k \in \underline{N}$, $j \neq k$. Assume that $\mathbf{P} \in H^\infty_{<}(\mathbb{E}_1, \mathbb{C}^{p\times m})$ and that there exist $K_j \in \mathbb{C}^{m\times p}$ such that

$$\sigma\left(\bar{\lambda}_j \mathbf{P}(\lambda_j)K_j\right) \subset \mathbb{C}_0, \quad \forall\, j \in \underline{N}. \tag{3.1}$$

Let $\mathbf{K}^0 \in H^\infty_{<}(\mathbb{E}_1, \mathbb{C}^{m\times p})$ and set

$$\mathbf{K}_\varepsilon(z) := \varepsilon \left(\mathbf{K}^0(z) + \sum_{j=1}^{N} \frac{K_j}{z - \lambda_j} \right). \tag{3.2}$$

Then there exists $\varepsilon^* > 0$ such that

$$\mathbf{K}_\varepsilon (I + \mathbf{PK}_\varepsilon)^{-1} \in H^\infty_{<}(\mathbb{E}_1, \mathbb{C}^{m\times p}), \quad \forall \varepsilon \in (0, \varepsilon^*).$$

Although Proposition III.1 is contained as a special case in [3, Theorem 3.1], we prove this result to make the technical note self-contained. We emphasize that the proof given here is new, with coprime factorizations playing a pivotal role and thereby providing an alternative approach to that developed in [3]. It is convenient to first state and prove the following lemma which will facilitate the proof of Proposition III.1.

Lemma III.2: For $\rho > 0$, set $B_\rho := \mathbb{B}(1,\rho) \cap \mathbb{E}_1$ and let $U \supset \mathrm{cl}(B_\rho)$ be open. Let $\mathbf{Q} \in H^\infty(U, \mathbb{C}^{p\times m})$, $\mathbf{H} \in H^\infty(U, \mathbb{C}^{m\times p})$ and $K \in \mathbb{C}^{m\times p}$. If

$$\sigma\left(\mathbf{Q}(1)K\right) \subset \mathbb{C}_0 \tag{3.3}$$

then there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*)$,

$$z \mapsto \left(I + \varepsilon \mathbf{Q}(z)\left(\mathbf{H}(z) + \frac{K}{z-1}\right)\right)^{-1} \in H^\infty(B_\rho, \mathbb{C}^{p\times p}).$$

Proof: Note that, by (3.3), $\mathrm{rk}K = p$, so that K^*K is invertible. Setting

$$\mathbf{D}(z) := \frac{z-1}{z} I_p, \quad \mathbf{N}(z) := \mathbf{H}(z)\mathbf{D}(z) + \frac{1}{z}K$$

we conclude that $(\mathbf{N}, \mathbf{D})$ is a right coprime factorization of $\mathbf{H}(z) + K/(z-1)$ over $H^\infty(B_\rho)$, since $\mathbf{N}(z)\mathbf{D}^{-1}(z) = \mathbf{H}(z) + K/(z-1)$ and

$$(K^*K)^{-1}K^*\mathbf{N}(z) + \left(I_p - (K^*K)^{-1}K^*\mathbf{H}(z)\right)\mathbf{D}(z) = I_p.$$

By Proposition II.7, it is sufficient to show that there exists $\varepsilon^* > 0$ such that

$$\inf_{z \in B_\rho} \left|\det\left(\varepsilon \mathbf{Q}(z)\mathbf{N}(z) + \mathbf{D}(z)\right)\right| > 0, \quad \forall \varepsilon \in (0, \varepsilon^*).$$

Seeking a contradiction, suppose that such a constant ε^* does not exist. Then there exist $\varepsilon_n \downarrow 0$ and $z_n \in \mathrm{cl}(B_\rho)$ such that

$$\det\left(\varepsilon_n(z_n - 1)\mathbf{Q}(z_n)\mathbf{H}(z_n) + \varepsilon_n \mathbf{Q}(z_n)K + (z_n - 1)I_p\right) = 0, \quad \forall n \in \mathbb{Z}_+. \tag{3.4}$$

Since $\lim_{n\to\infty} \varepsilon_n = 0$, we may conclude from (3.4) that

$$\lim_{n\to\infty} z_n = 1. \tag{3.5}$$

Moreover, we obtain from (3.4) that

$$\frac{1 - z_n}{\varepsilon_n} \in \sigma\left((z_n - 1)\mathbf{Q}(z_n)\mathbf{H}(z_n) + \mathbf{Q}(z_n)K\right), \quad \forall n \in \mathbb{Z}_+. \tag{3.6}$$

Consequently, by (3.3) and (3.5), there exists $\beta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\frac{1 - z_n}{\varepsilon_n} \in \mathbb{C}_\beta, \quad \forall n \ge n_0. \tag{3.7}$$

Furthermore, since the function $z \mapsto (z-1)\mathbf{Q}(z)\mathbf{H}(z) + \mathbf{Q}(z)K$ is bounded on $\mathrm{cl}(B_\rho)$, it follows from (3.6) that there exists a constant $M > 0$ such that

$$\frac{|1 - z_n|}{\varepsilon_n} \le M, \quad \forall n \in \mathbb{Z}_+. \tag{3.8}$$

As a trivial consequence of (3.7), $\mathrm{Re}z_n < 1$ for $n \ge n_0$. Invoking this, together with (3.8) and the fact that $|z_n| \ge 1$ for all $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}\frac{1 - \mathrm{Re}z_n}{\varepsilon_n} &\le \frac{M(1 - \mathrm{Re}z_n)}{|1 - z_n|} \\ &= \frac{M(1 - \mathrm{Re}z_n)}{\sqrt{2(1 - \mathrm{Re}z_n) + |z_n|^2 - 1}} \\ &\le \frac{M}{\sqrt{2}}\sqrt{1 - \mathrm{Re}z_n}, \quad \forall n \ge n_0.\end{aligned}$$

By (3.5), $\mathrm{Re}z_n \to 1$ as $n \to \infty$, and thus, $(1 - \mathrm{Re}z_n)/\varepsilon_n \to 0$ as $n \to \infty$, contradicting (3.7). □

We are now in the position to prove Proposition III.1.

Proof of Proposition III.1: We first show that $(I + \mathbf{PK}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ for sufficiently small ε. Since $\lambda_j \ne \lambda_k$ for all $j, k \in \underline{N}$, $j \ne k$, we can choose $\rho > 0$ sufficiently small such that

$$\mathrm{cl}\left(\mathbb{B}(\lambda_j, \rho)\right) \cap \mathrm{cl}\left(\mathbb{B}(\lambda_k, \rho)\right) = \emptyset, \quad \forall j, k \in \underline{N}, j \ne k.$$

Setting $\Omega_j := \mathbb{E}_1 \cap \mathbb{B}(\lambda_j, \rho)$ and $\Omega := \bigcup_{j=1}^N \Omega_j$, it is clear that the function

$$z \mapsto \mathbf{P}(z)\left(\mathbf{K}^0(z) + \sum_{j=1}^N \frac{K_j}{z - \lambda_j}\right)$$

is bounded on $\mathbb{E}_1 \setminus \Omega$. Thus, there exists $\varepsilon^\infty > 0$ such that

$$(I + \mathbf{PK}_\varepsilon)^{-1} \text{ is bounded on } \mathbb{E}_1 \setminus \Omega \text{ for all } \varepsilon \in (0, \varepsilon^\infty). \tag{3.9}$$

Fix $j \in \underline{N}$ and set

$$\mathbf{H}(z) := \mathbf{K}^0(z) + \sum_{k \in \underline{N}, k \ne j} \frac{K_k}{z - \lambda_k}.$$

Then there exists an open set $V_j \supset \mathrm{cl}(\Omega_j)$ such that $\mathbf{H} \in H^\infty(V_j, \mathbb{C}^{m\times p})$ and, furthermore,

$$\mathbf{P}(z)\mathbf{K}_\varepsilon(z) = \varepsilon \mathbf{P}(\lambda_j w)\left(\mathbf{H}(\lambda_j w) + \frac{\bar{\lambda}_j K_j}{w - 1}\right)$$

where $w := \bar{\lambda}_j z$. Setting

$$\tilde{\mathbf{H}}(w) := \mathbf{H}(\lambda_j w), \quad \mathbf{Q}(w) := \mathbf{P}(\lambda_j w), \quad \tilde{K}_j := \bar{\lambda}_j K_j$$

it follows that

$$\mathbf{P}(z)\mathbf{K}_\varepsilon(z) = \varepsilon \mathbf{Q}(w)\left(\tilde{\mathbf{H}}(w) + \frac{\tilde{K}_j}{w - 1}\right).$$

Since $\mathbf{P} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times m})$ and $\mathbf{K}^0 \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})$, we see that $\mathbf{Q} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times m})$ and $\tilde{\mathbf{H}} \in H^\infty(U_j, \mathbb{C}^{m\times p})$, where

$$U_j := \bar{\lambda}_j V_j \supset \bar{\lambda}_j \mathrm{cl}(\Omega_j) = \mathrm{cl}\left(\mathbb{E}_1 \cap \mathbb{B}(1, \rho)\right) = \mathrm{cl}(B_\rho)$$

with $B_\rho := \mathbb{E}_1 \cap \mathbb{B}(1, \rho)$ (as in Lemma III.2). Moreover, by (3.1)

$$\sigma\left(\mathbf{Q}(1)\tilde{K}_j\right) = \sigma\left(\bar{\lambda}_j \mathbf{P}(\lambda_j)K_j\right) \subset \mathbb{C}_0, \quad \forall j \in \underline{N}.$$

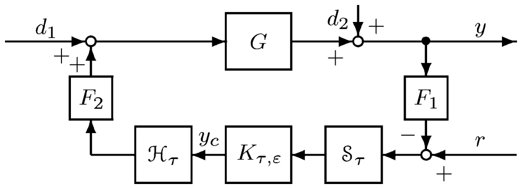

Fig. 1. Sampled-data low-gain control with filters.

It follows from Lemma III.2 that, for every $j \in \underline{N}$, there exists $\varepsilon_j \in (0, \varepsilon^\infty)$ such that, for all $\varepsilon \in (0, \varepsilon_j)$, the function

$$w \mapsto \left[I + \varepsilon \mathbf{Q}(w)\left(\tilde{\mathbf{H}}(w) + \frac{\tilde{K}_j}{w-1}\right)\right]^{-1}$$

is in $H^\infty(B_\rho, \mathbb{C}^{p\times p})$. Consequently,

$$(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\Omega_j, \mathbb{C}^{p\times p}), \quad \forall \varepsilon \in (0, \varepsilon_j), \forall j \in \underline{N}. \quad (3.10)$$

Setting $\varepsilon^* := \min\{\varepsilon_j : j \in \underline{N}\}$ and invoking (3.9) and (3.10), we conclude that

$$(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p}), \quad \forall \varepsilon \in (0, \varepsilon^*). \quad (3.11)$$

Next we prove that $(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times p})$ for all $\varepsilon \in (0, \varepsilon^*)$. Since $\mathbf{P} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times m})$ and $\mathbf{K}^0 \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})$, it is clear that $(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}$ is meromorphic on $\mathbb{E}_\alpha$ for some $\alpha \in (0,1)$. Letting $\gamma \in (\alpha, 1)$, it follows that $(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}$ has at most finitely many poles in the compact annulus $\mathrm{cl}(\mathbb{E}_\gamma) \setminus \mathbb{E}_1$. By (3.11), $(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}$ does not have any poles on the unit circle $\partial\mathbb{E}_1$ and so there exists $\beta \in (\gamma, 1)$ such that $(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_\beta, \mathbb{C}^{p\times p})$, where β depends on ε.

Finally, to show that $\mathbf{K}_\varepsilon(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})$, note that, by (3.1), $\mathbf{P}(\lambda_j)K_j$ is invertible for all $j \in \underline{N}$. Therefore, using (3.2)

$$\lim_{z\to\lambda_j} \frac{1}{z-\lambda_j}(I + \mathbf{P}(z)\mathbf{K}_\varepsilon(z))^{-1} = (\varepsilon\mathbf{P}(\lambda_j)K_j)^{-1}.$$

Hence, $\mathbf{K}_\varepsilon(z)(I + \mathbf{P}(z)\mathbf{K}_\varepsilon(z))^{-1}$ has a finite limit as $z \to \lambda_j$ for every $j \in \underline{N}$, so that $\mathbf{K}_\varepsilon(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1}$ is bounded on a neighbourhood Λ of the set $\{\lambda_j : j \in \underline{N}\}$. Since $(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p\times p})$ and, for some $\beta \in (0,1)$, $\mathbf{K}_\varepsilon$ is bounded on $\mathbb{E}_\beta \setminus \Lambda$, it follows that $\mathbf{K}_\varepsilon(I + \mathbf{P}\mathbf{K}_\varepsilon)^{-1} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})$. □

IV. A Low-Gain Sampled-Data Controller

Consider the sampled-data system shown in Fig. 1, where G is the input-output operator of the continuous-time plant, $K_{\tau,\varepsilon}$ is the input-output operator of the discrete-time controller, F_1 and F_2 are filters, r is a reference signal and d_1 and d_2 are disturbance signals. We assume that the transfer function $\mathbf{G}$ of G is in $H^\infty(\mathbb{C}_\alpha, \mathbb{C}^{p\times m})$ for some $\alpha < 0$, or equivalently, that G is the input-output operator of an exponentially stable well-posed state-space system. Mathematically, Fig. 1 can be expressed as

$$y = G(F_2\mathcal{H}_\tau y_c + d_1) + d_2, \quad y_c = K_{\tau,\varepsilon}\mathcal{S}_\tau(r - F_1 y). \quad (4.1)$$

The following theorem is the main result of this technical note.

Theorem IV.1: Let $N \in \mathbb{N}$ and let $\xi_j \in i\mathbb{R}$ for all $j \in \underline{N}$ be such that $\xi_j \neq \xi_k$ for $j,k \in \underline{N}$, $j \neq k$. Assume that the transfer function $\mathbf{G}$ of G is in $H^\infty(\mathbb{C}_\alpha, \mathbb{C}^{p\times m})$ for some $\alpha < 0$ and there exist $K_j \in \mathbb{C}^{m\times p}$ such that

$$\sigma(\mathbf{G}(\xi_j)K_j) \subset \mathbb{C}_0, \quad \forall j \in \underline{N}. \quad (4.2)$$

Let $\tau > 0$ be the sampling period and assume that the transfer function $\mathbf{K}_{\tau,\varepsilon}$ of $K_{\tau,\varepsilon}$ is of the form

$$\mathbf{K}_{\tau,\varepsilon}(z) = \varepsilon\left(\mathbf{K}^0(z) + \sum_{j=1}^{N} \frac{K_j}{z - e^{\xi_j\tau}}\right)$$

where $\mathbf{K}^0 \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m\times p})$. Assume that the transfer functions $\mathbf{F}_1$ and $\mathbf{F}_2$ of the filters F_1 and F_2 satisfy

$$\mathbf{F}_1(\xi_j) = I_p \quad \text{and} \quad \mathbf{F}_2(\xi_j) = I_m, \qquad \forall j \in \underline{N}. \quad (4.3)$$

Suppose that r is given by $r(t) := \sum_{j=1}^{N} e^{\xi_j t}\mathfrak{r}_j$, $\mathfrak{r}_j \in \mathbb{C}^p$, and d_1, d_2 are given by

$$d_1(t) := \sum_{j=1}^{N} e^{\xi_j t}\mathfrak{d}_{1j} + p_1(t), \ \mathfrak{d}_{1j} \in \mathbb{C}^m, \quad (4.4a)$$

$$d_2(t) := \sum_{j=1}^{N} e^{\xi_j t}\mathfrak{d}_{2j} + p_{21}(t) + p_{22}(t), \ \mathfrak{d}_{2j} \in \mathbb{C}^p, \quad (4.4b)$$

where $p_1 \in L^2_\gamma(\mathbb{R}_+, \mathbb{C}^m)$, $p_{21} \in L^2_\gamma(\mathbb{R}_+, \mathbb{C}^p)$ for some $\gamma \in (\alpha, 0)$, and $p_{22} \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^p)$ with $\lim_{t\to\infty} p_{22}(t) = 0$. Then, for every $\delta > 0$, there exists $\tau_\delta > 0$ such that, for every sampling period $\tau \in (0, \tau_\delta)$, there exists $\varepsilon_\tau > 0$ such that, for every $\varepsilon \in (0, \varepsilon_\tau)$, the output y of the sampled-data feedback system (4.1) can be decomposed as $y = y_1 + y_2$, where $y_1 \in L^2_\gamma(\mathbb{R}_+, \mathbb{C}^p)$ and y_2 satisfies

$$\limsup_{t\to\infty} \|y_2(t) - r(t)\| < \delta.$$

Before we prove the theorem, we provide some commentary in the following remark.

Remark IV.2:

(i) Theorem IV.1 says that the output y can be decomposed in the form $y = y_1 + y_2$, where
- the signal y_1 is "small" in the sense that $y_1 \in L^2_\gamma(\mathbb{R}_+, \mathbb{C}^p)$, implying that the "energy" of the restriction $y_1|_{[t,\infty)}$ converges to zero exponentially fast (with exponential rate γ) as $t \to \infty$.
- the signal y_2 is "persistent" and, for all sufficiently large $t \geq 0$, $y_2(t)$ is close to $r(t)$ in the sense that $\|y_2(t) - r(t)\| < \delta$.

(ii) Denoting the Lebesgue measure on $\mathbb{R}_+$ by μ_L, the conclusions of Theorem IV.1 imply that

$$\lim_{T\to\infty} \mu_L(\{t \geq T : \|y(t) - r(t)\| \geq \delta\}) = 0$$

that is, as $t \to \infty$, the error $y(t) - r(t)$ is "bounded in measure" by δ.

(iii) An inspection of the proof of Theorem IV.1 (see (4.22)) shows that if the impulse response of G is a $\mathbb{C}^{p\times m}$-valued Borel measure on $\mathbb{R}_+$, $p_1(t) \to 0$ and $p_{21}(t) \to 0$ as $t \to \infty$, then $y_1(t) \to 0$ as $t \to \infty$, so that $\limsup_{t\to\infty} \|y(t) - r(t)\| < \delta$.

(iv) One of the motivations for including the term $p_{21} \in L^2_\gamma(\mathbb{R}_+, \mathbb{C}^p)$ in the disturbance d_2 is that it can be used to model non-zero initial conditions in an exponentially-stable well-posed state-space realization of G.

(v) An inspection of the proof of Theorem IV.1 (see the argument guaranteeing the existence of τ_δ) shows that, for given $\{\xi_j : j \in \underline{N}\}$, τ_δ and ε_τ can be chosen to be uniform for all signals r, d_1 and d_2 with $\mathfrak{r}_j$, $\mathfrak{d}_{1j}$ and $\mathfrak{d}_{2j}$, $j \in \underline{N}$, satisfying a pre-specified bound.

(vi) A filter with transfer function $\mathbf{F}$ satisfying $\mathbf{F}(\xi_j) = I$ for all $j \in \underline{N}$ can be constructed in the following way:

$$\mathbf{F}(s) := \frac{1}{h(s)} \sum_{l=1}^{N} \left[h(\xi_l) \prod_{k \in \underline{N}, k \neq l} \frac{(s - \xi_k)}{\xi_l - \xi_k} \right] I$$

where $h(s)$ is a real Hurwitz polynomial, the degree of which is greater or equal to N. It is clear that $\mathbf{F}$ is a strictly proper stable rational function. Moreover, if the ξ_j occur in complex conjugate pairs, then it is easy to see that $\mathbf{F}$ has real coefficients. ◇

Proof of Theorem IV.1: Setting $\tau_0 := 2\pi / \sup\{|\xi_j - \xi_k| : j, k \in \underline{N}, j \neq k\}$, it follows that if $\tau \in (0, \tau_0)$, then $e^{\xi_j \tau} \neq e^{\xi_k \tau}$ for all $j, k \in \underline{N}$, $j \neq k$. Define

$$H := F_1 G F_2, \quad H_\tau := \mathcal{S}_\tau H \mathcal{H}_\tau = \mathcal{S}_\tau F_1 G F_2 \mathcal{H}_\tau.$$

The transfer functions of H and H_τ are denoted by $\mathbf{H}$ and $\mathbf{H}_\tau$, respectively. By Lemma II.5, there exists $\beta \in (\alpha, 0)$ such that the impulse responses of H, $F_1 G$ and $G F_2$ are in $L^1_\beta(\mathbb{R}_+, \mathbb{C}^{p \times m})$. Hence, by Lemma II.3 and (4.3), $\mathbf{H}_\tau \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p \times m})$ and

$$\lim_{\tau \to 0} \mathbf{H}_\tau(e^{\xi_j \tau}) = \mathbf{H}(\xi_j) = \mathbf{G}(\xi_j), \quad \forall j \in \underline{N}. \tag{4.5}$$

By (4.2) and (4.5), there exists $\tau_1 \in (0, \tau_0)$ such that

$$\sigma\left(e^{\bar{\xi}_j \tau} \mathbf{H}_\tau(e^{\xi_j \tau}) K_j\right) \subset \mathbb{C}_0, \quad \forall (\tau, j) \in (0, \tau_1) \times \underline{N}. \tag{4.6}$$

In particular,

$$\mathbf{H}_\tau(e^{\xi_j \tau}) K_j \text{ is invertible}, \ \forall\, (\tau, j) \in (0, \tau_1) \times \underline{N}. \tag{4.7}$$

For $j \in \underline{N}$, set $L_j := K_j (\mathbf{G}(\xi_j) K_j)^{-1}$, where we have used that, by (4.2), $\mathbf{G}(\xi_j) K_j$ is invertible for every $j \in \underline{N}$. Define the functions $v_1, v_2, v_3 \in F(\mathbb{R}_+, \mathbb{C}^m)$ by

$$v_1(t) := \sum_{j=1}^{N} e^{\xi_j t} L_j \mathfrak{r}_j, \quad v_2(t) := \sum_{j=1}^{N} e^{\xi_j t} L_j \mathbf{G}(\xi_j) \mathfrak{d}_{1j}$$
$$v_3(t) := \sum_{j=1}^{N} e^{\xi_j t} L_j \mathfrak{d}_{2j}.$$

Let $\tau \in (0, \tau_1)$ and set $L^\tau_j := K_j (\mathbf{H}_\tau(e^{\xi_j \tau}) K_j)^{-1}$ for $j \in \underline{N}$ ($\mathbf{H}_\tau(e^{\xi_j \tau}) K_j$ is invertible by (4.7)). Define the sequences $v^\tau_1, v^\tau_2, v^\tau_3 \in F(\mathbb{Z}_+, \mathbb{C}^m)$ by

$$v^\tau_1(k) := \sum_{j=1}^{N} e^{\xi_j k \tau} L^\tau_j \mathfrak{r}_j, \quad v^\tau_2(k) := \sum_{j=1}^{N} e^{\xi_j k \tau} L^\tau_j \mathbf{G}(\xi_j) \mathfrak{d}_{1j}$$
$$v^\tau_3(k) := \sum_{j=1}^{N} e^{\xi_j k \tau} L^\tau_j \mathfrak{d}_{2j}.$$

Since $\xi_j \in i\mathbb{R}$ for $j \in \underline{N}$, a routine calculation yields

$$\begin{aligned} \|v_1(t) - (\mathcal{H}_\tau v^\tau_1)(t)\| &\leq \sum_{j=1}^{N} \left\| (\mathbf{G}(\xi_j) K_j)^{-1} - \left(\mathbf{H}_\tau(e^{\xi_j \tau}) K_j\right)^{-1} \right\| \|K_j\| \|\mathfrak{r}_j\| \\ &\quad + \sum_{j=1}^{N} \max_{s \in [0, \tau]} |e^{\xi_j s} - 1| \|L_j \mathfrak{r}_j\|, \forall t \geq 0. \end{aligned} \tag{4.8}$$

Let $\delta > 0$. By (4.5) and (4.8), there exists $\tilde{\tau}_\delta \in (0, \tau_1)$ such that

$$\|v_1(t) - (\mathcal{H}_\tau v^\tau_1)(t)\| \leq \frac{\delta}{4M}, \quad \forall t \geq 0, \forall\, \tau \in (0, \tilde{\tau}_\delta) \tag{4.9}$$

where M denotes the L^1-norm of the impulse response of $G F_2$. Similarly, there exists $\tau_\delta \in (0, \tilde{\tau}_\delta) \subset (0, \tau_1)$ such that

$$\|v_2(t) - (\mathcal{H}_\tau v^\tau_2)(t)\|, \|v_3(t) - (\mathcal{H}_\tau v^\tau_3)(t)\| \leq \frac{\delta}{4M}; \quad \forall t \geq 0, \forall \tau \in (0, \tau_\delta). \tag{4.10}$$

In the following, let $\tau \in (0, \tau_\delta)$. Invoking the fact that $\mathbf{H}_\tau \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p \times m})$ together with (4.6) and Proposition III.1, we conclude that there exists $\varepsilon_\tau > 0$ such that

$$\mathbf{K}_{\tau,\varepsilon}(I + \mathbf{H}_\tau \mathbf{K}_{\tau,\varepsilon})^{-1} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m \times p}), \quad \forall \varepsilon \in (0, \varepsilon_\tau). \tag{4.11}$$

Let $\varepsilon \in (0, \varepsilon_\tau)$. Using (4.7) and exploiting the structure of $\mathbf{K}_{\tau,\varepsilon}$, we obtain

$$\left(\mathbf{K}_{\tau,\varepsilon}(I + \mathbf{H}_\tau \mathbf{K}_{\tau,\varepsilon})^{-1}\right)(e^{\xi_j \tau}) = K_j \left(\mathbf{H}_\tau(e^{\xi_j \tau}) K_j\right)^{-1}. \tag{4.12}$$

The output y_c of the discrete-time controller (see (4.1)) is given by

$$\begin{aligned} y_c &= K_{\tau,\varepsilon} \mathcal{S}_\tau \left[r - F_1(G F_2 \mathcal{H}_\tau y_c + G d_1 + d_2) \right] \\ &= K_{\tau,\varepsilon} \mathcal{S}_\tau r - K_{\tau,\varepsilon} H_\tau y_c - K_{\tau,\varepsilon} \mathcal{S}_\tau F_1 G d_1 - K_{\tau,\varepsilon} \mathcal{S}_\tau F_1 d_2 \end{aligned}$$

and thus,

$$y_c = K_{\tau,\varepsilon}(I + H_\tau K_{\tau,\varepsilon})^{-1}(\mathcal{S}_\tau r - \mathcal{S}_\tau F_1 G d_1 - \mathcal{S}_\tau F_1 d_2). \tag{4.13}$$

Since p_1, p_{21} and the impulse responses of $F_1 G$ and F_1 are L^2-functions, we conclude that

$$\lim_{t \to \infty} (F_1 G p_1)(t) = 0, \quad \lim_{t \to \infty} (F_1 p_{21})(t) = 0. \tag{4.14}$$

Invoking the fact that the impulse responses of $F_1 G$ and F_1 are L^1-functions, together with Lemma II.2, (4.3) and (4.14), we obtain

$$\lim_{t \to \infty} \left((F_1 G d_1)(t) - \sum_{j=1}^{N} e^{\xi_j t} \mathbf{G}(\xi_j) \mathfrak{d}_{1j} \right) = 0$$
$$\lim_{t \to \infty} \left((F_1 d_2)(t) - \sum_{j=1}^{N} e^{\xi_j t} \mathfrak{d}_{2j} \right) = 0$$

showing that

$$\lim_{k \to \infty} \left((\mathcal{S}_\tau F_1 G d_1)(k) - \sum_{j=1}^{N} e^{\xi_j k \tau} \mathbf{G}(\xi_j) \mathfrak{d}_{1j} \right) = 0, \tag{4.15}$$

$$\lim_{k \to \infty} \left((\mathcal{S}_\tau F_1 d_2)(k) - \sum_{j=1}^{N} e^{\xi_j k \tau} \mathfrak{d}_{2j} \right) = 0. \tag{4.16}$$

By (4.11), the impulse response of $K_{\tau,\varepsilon}(I + H_\tau K_{\tau,\varepsilon})^{-1}$ is in $\ell^1(\mathbb{Z}_+, \mathbb{C}^{m\times p})$, so that it follows from Lemma II.1, (4.12), (4.13), (4.15), and (4.16) that

$$\lim_{k\to\infty} \left(y_c(k) - v_1^\tau(k) + v_2^\tau(k) + v_3^\tau(k)\right) = 0. \tag{4.17}$$

Then, by (4.9), (4.10) and (4.17),

$$\begin{aligned}\limsup_{t\to\infty} &\|(\mathcal{H}_\tau y_c)(t) - v_1(t) + v_2(t) + v_3(t)\| \\ &\le \limsup_{t\to\infty} \|(\mathcal{H}_\tau (y_c - v_1^\tau + v_2^\tau + v_3^\tau))(t)\| \\ &\quad + \limsup_{t\to\infty} \|(\mathcal{H}_\tau v_1^\tau)(t) - v_1(t)\| \\ &\quad + \limsup_{t\to\infty} \|v_2(t) - (\mathcal{H}_\tau v_2^\tau)(t)\| \\ &\quad + \limsup_{t\to\infty} \|v_3(t) - (\mathcal{H}_\tau v_3^\tau)(t)\| \\ &\le \frac{3\delta}{4M}. \end{aligned}\tag{4.18}$$

Moreover, we conclude from Lemma II.2 and (4.3) that

$$\lim_{t\to\infty} \left((GF_2 v_1)(t) - r(t)\right) = 0, \tag{4.19}$$

$$\lim_{t\to\infty} \left((GF_2 v_2)(t) - \sum_{j=1}^{N} e^{\xi_j t}\mathbf{G}(\xi_j)\mathfrak{d}_{1j}\right) = 0, \tag{4.20}$$

and

$$\begin{aligned}\lim_{t\to\infty} &\left((GF_2 v_3)(t) - d_2(t) + p_{21}(t)\right) \\ &= \lim_{t\to\infty} \left((GF_2 v_3)(t) - \sum_{j=1}^{N} e^{\xi_j t}\mathfrak{d}_{2j} - p_{22}(t)\right) = 0.\end{aligned}\tag{4.21}$$

Setting

$$y_1(t) := (Gd_1)(t) - \sum_{j=1}^{N} \mathbf{G}(\xi_j)e^{\xi_j t}\mathfrak{d}_{1j} + p_{21}(t) \tag{4.22}$$

and

$$y_2(t) := (GF_2\mathcal{H}_\tau y_c)(t) + \sum_{j=1}^{N} \mathbf{G}(\xi_j)e^{\xi_j t}\mathfrak{d}_{1j} + d_2(t) - p_{21}(t)$$

it follows that $y = y_1 + y_2$. Denoting the Laplace transform by $\mathcal{L}$ and invoking (4.4), we obtain that

$$(\mathcal{L}(y_1))(s) = \sum_{j=1}^{N} \frac{(\mathbf{G}(s) - \mathbf{G}(\xi_j))\mathfrak{d}_{1j}}{s - \xi_j} + \mathbf{G}(s)(\mathcal{L}p_1)(s) + (\mathcal{L}p_{21})(s).$$

Since $\mathbf{G} \in H^\infty(\mathbb{C}_\alpha, \mathbb{C}^{p\times m})$, $p_1 \in L^2_\gamma(\mathbb{R}_+, \mathbb{C}^m)$ and $p_{21} \in L^2_\gamma(\mathbb{R}_+, \mathbb{C}^p)$ with $\alpha < \gamma < 0$, it follows that $\mathcal{L}(y_1) \in H^2(\mathbb{C}_\gamma, \mathbb{C}^p)$. Hence, the Paley-Wiener Theorem implies that $y_1 \in L^2_\gamma(\mathbb{R}_+, \mathbb{C}^p)$. Furthermore, since

$$\begin{aligned}\|y_2(t) - r(t)\| &\le \|(GF_2(\mathcal{H}_\tau y_c - v_1 + v_2 + v_3))(t)\| \\ &\quad + \|(GF_2 v_1)(t) - r(t)\| \\ &\quad + \left\|\sum_{j=1}^{N} \mathbf{G}(\xi_j)e^{\xi_j t}\mathfrak{d}_{1j} - (GF_2 v_2)(t)\right\| \\ &\quad + \|d_2(t) - p_{21}(t) - (GF_2 v_3)(t)\|, \quad \forall t \ge 0\end{aligned}$$

it follows from (4.19)–(4.21) that

$$\limsup_{t\to\infty} \|y_2(t) - r(t)\| \le \limsup_{t\to\infty} \|(GF_2(\mathcal{H}_\tau y_c - v_1 + v_2 + v_3))(t)\|.$$

Finally, recalling that M denotes the L^1-norm of the impulse response of GF_2, Lemma II.2 and (4.18) yield

$$\begin{aligned}\limsup_{t\to\infty} &\|y_2(t) - r(t)\| \\ &\le M \limsup_{t\to\infty} \|(\mathcal{H}_\tau y_c)(t) - v_1(t) + v_2(t) + v_3(t)\| \\ &\le \frac{3\delta}{4} < \delta\end{aligned}$$

completing the proof. □

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