APPROXIMATE TRACKING AND DISTURBANCE REJECTION FOR STABLE INFINITE-DIMENSIONAL SYSTEMS USING SAMPLED-DATA LOW-GAIN CONTROL*

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Abstract. In this paper we solve tracking and disturbance rejection problems for stable infinite-dimensional systems using a simple low-gain controller suggested by the internal model principle. For stable discrete-time systems, it is shown that the application of a low-gain controller (depending on only one gain parameter) leads to a stable closed-loop system which asymptotically tracks reference signals r of the form $r(k) = \sum_{j=1}^N \lambda_j^k \mathfrak{r}_j$, where $\mathfrak{r}_j \in \mathbb{C}^p$ and $\lambda_j \in \mathbb{C}$ with $|\lambda_j| = 1$ for $j = 1, \ldots, N$. The closed-loop system also rejects disturbance signals which are asymptotically of this form. The discrete-time result is used to derive results on approximate tracking and disturbance rejection for a large class of infinite-dimensional sampled-data feedback systems, with reference signals which are finite sums of sinusoids, and disturbance signals which are asymptotic to finite sums of sinusoids. The results are given for both input-output systems and state-space systems.

Key words. discrete-time systems, disturbance rejection, infinite-dimensional systems, internal model principle, low-gain control, sampled-data control, tracking

AMS subject classifications. 93C25, 93C55, 93C57, 93C80, 93D15, 93D25

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1. Introduction. The synthesis of low-gain integral controllers for uncertain stable continuous-time plants has received considerable attention in the last thirty years. Let **G** be a stable proper rational continuous-time transfer function matrix. The main existence result for robust low-gain integral control states that if all of the eigenvalues of **G**(0) have positive real parts, then there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$, the controller $(\varepsilon/s)I$ stabilizes **G**. Moreover, the resulting closed-loop system asymptotically tracks arbitrary constant reference signals. This result has been proved by Davison [2] using state-space methods and Morari [11] using frequency-domain methods. This low-gain controller allows stabilization and tracking with very little information about the plant, and it is not based on system identification. The above regulator result has been extended to various classes of (abstract) infinite-dimensional continuous-time systems: in [12] for exponentially stable parabolic systems, in [7] for systems in the Callier–Desoer algebra (CD-algebra), and in [9] for exponentially stable regular systems.

In the case that the reference and disturbance signals are of the form

$$\sum_{j=1}^N e^{i\omega_j t} \mathfrak{w}_j, \quad \omega_j \in \mathbb{R}, \ \mathfrak{w}_j \in \mathbb{C}^m,$$

Hämäläinen and Pohjolainen [3] solved the tracking and disturbance rejection problem for stable infinite-dimensional systems in the CD-algebra. (In their paper, reference

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and disturbance signals are more general, containing polynomial parts.) Rebarber and Weiss [13] proved similar results for the more general class of exponentially stable well-posed systems.

In this paper, we consider low-gain control for infinite-dimensional discrete-time and sampled-data feedback systems. In section 2, we give preliminary technical results. In section 3, we develop a frequency-domain approach to discrete-time low-gain control. We consider a feedback controller of the form

$$
(1.1) \qquad \varepsilon \left(\mathbf{K}^0(z) + \sum_{j=1}^{N} \frac{K_j}{z - \lambda_j} \right),
$$

where $\mathbf{K}^0$ has impulse response in $\ell^1(\mathbb{Z}_+, \mathbb{C}^{m \times p})$, $K_j \in \mathbb{C}^{m \times p}$, and $\lambda_j \in \mathbb{C}$ with $|\lambda_j| = 1$. We assume that the plant has a transfer function $\mathbf{G}$ which has impulse response in $\ell^1(\mathbb{Z}_+, \mathbb{C}^{p \times m})$. We show that the application of this controller to the plant will result in an ℓ^q-stable closed-loop system for $1 \le q \le \infty$, provided that

  (i) all the eigenvalues of $\bar{\lambda}_j \mathbf{G}(\lambda_j) K_j$ have positive real parts;
  (ii) $\limsup_{z \to \lambda_j,\ |z| > 1} \|(\mathbf{G}(z) - \mathbf{G}(\lambda_j))/(z - \lambda_j)\| < \infty$;
  (iii) the gain parameter ε is sufficiently small.

Moreover, the closed-loop system achieves asymptotic tracking and disturbance rejection for reference signals r of the form $r(k) := \sum_{j=1}^{N} \lambda_j^k \mathfrak{r}_j$ and disturbance signals d satisfying $\lim_{k \to \infty} (d(k) - \sum_{j=1}^{N} \lambda_j^k \mathfrak{d}_j) = 0$, where $\mathfrak{r}_j \in \mathbb{C}^p$ and $\mathfrak{d}_j \in \mathbb{C}^m$. The results are first proved for input-output systems, and then for state-space systems. Our results are an extension of results by Logemann and Townley [10]. In their paper, the reference and disturbance signals are constants.

In section 4, the discrete-time results in section 3 are used to derive results on approximate tracking and disturbance rejection for input-output and state-space sampled-data systems. The input-output operator G of the continuous-time plant is assumed to be a convolution operator of the form $Gu = \mu \star u$, where μ is a $\mathbb{C}^{p \times m}$-valued Borel measure such that $\int_{\mathbb{R}_+} e^{-\alpha t} |\mu|(dt) < \infty$ for some $\alpha < 0$, where $|\mu|$ is the total variation of μ. The discrete-time controller underlying the sampled-data feedback scheme is given by (1.1) with $\lambda_j = e^{\xi_j \tau}$, where $\xi_j \in i\mathbb{R}$ for $j = 1, \dots, N$ and $\tau > 0$ is the sampling period. The reference signals r are given by $r(t) = \sum_{j=1}^{N} e^{\xi_j t} \mathfrak{r}_j$, where $\mathfrak{r}_j \in \mathbb{C}^p$. Invoking both time-domain and frequency-domain methods, we prove that if all the eigenvalues of $\mathbf{G}(\xi_j) K_j$ have positive real parts, then, for every $\delta > 0$, there exists $\tau_\delta > 0$ such that, for every sampling period $\tau \in (0, \tau_\delta)$, there exists $\varepsilon_\tau > 0$ such that, for every $\varepsilon \in (0, \varepsilon_\tau)$, the output y of the closed-loop sampled-data system satisfies

$$
\limsup_{t \to \infty} \|y(t) - r(t)\| \le \delta
$$

in the presence of disturbance signals d satisfying $\lim_{t \to \infty} (d(t) - \sum_{j=1}^{N} e^{\xi_j t} \mathfrak{d}_j) = 0$, where $\mathfrak{d}_j \in \mathbb{C}^m$. At the end of the section we give an application to a heat equation.

To the best of our knowledge, the main results in sections 3 and 4 are new even for finite-dimensional systems.

Notation. Let X and Y be Banach spaces. The set of all bounded linear operators from X to Y is denoted by $\mathcal{B}(X, Y)$; we write $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$. Moreover, $F(\mathbb{Z}_+, X)$ denotes all X-valued sequences defined on $\mathbb{Z}_+$, and $L_b(\mathbb{R}_+, X)$ denotes the set of bounded X-valued Lebesgue measurable functions with the sup-norm $\| \cdot \|_\infty$.

The z-transform of $v \in F(\mathbb{Z}_+, X)$ is denoted by $\mathscr{Z}(v)$. Sometimes we write $\hat{v}$ for $\mathscr{Z}(v)$. For $\alpha > 0$ and $\beta \in \mathbb{R}$, define $\mathbb{E}_\alpha := \{z \in \mathbb{C} : |z| > \alpha\}$ and $\mathbb{C}_\beta := \{z \in \mathbb{C} : \operatorname{Re} z > \beta\}$.

Let $\Omega \subset \mathbb{C}$ be open. We define

$$H^\infty(\Omega, \mathbb{C}^{p \times m}) := \{f : \Omega \to \mathbb{C}^{p \times m} \,|\, f \text{ is holomorphic and bounded}\},$$

$$H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p \times m}) := \bigcup_{0 < \gamma < 1} H^\infty(\mathbb{E}_\gamma, \mathbb{C}^{p \times m}).$$

We write $H^\infty(\Omega) := H^\infty(\Omega, \mathbb{C})$. Let $\mathcal{Q}$ denote the quotient field of $H^\infty(\mathbb{E}_1)$, i.e., $\mathcal{Q} = \{n/d : n, d \in H^\infty(\mathbb{E}_1), d \neq 0\}$. Furthermore, let $\mathcal{R}_s$ denote the ring of discrete-time stable proper complex rational functions, i.e., rational functions with complex coefficients which are bounded at ∞ and have all their poles in $\{z \in \mathbb{C} : |z| < 1\}$.

For $\alpha > 0$, define the weighted ℓ^1-space $\ell^1_\alpha(\mathbb{Z}_+, \mathbb{C}^{p \times m})$ by

$$\ell^1_\alpha(\mathbb{Z}_+, \mathbb{C}^{p \times m}) := \{v \in F(\mathbb{Z}_+, \mathbb{C}^{p \times m}) : (v(k)\alpha^{-k})_{k \in \mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{C}^{p \times m})\}$$

and set

$$\hat{\ell}^1_\alpha(\mathbb{C}^{p \times m}) := \{\mathscr{Z}(g) : g \in \ell^1_\alpha(\mathbb{Z}_+, \mathbb{C}^{p \times m})\} \subset H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{p \times m}).$$

We write $\hat{\ell}^1(\mathbb{C}^{p \times m}) := \hat{\ell}^1_1(\mathbb{C}^{p \times m})$. For $A \in \mathcal{B}(X)$, let $\sigma(A)$ denote the spectrum of A. For $N \in \mathbb{N}$, set $\underline{N} := \{1, 2, \ldots, N\}$. Finally, throughout, the symbol $\star$ denotes convolution (in discrete and continuous time).

2. Preliminaries. Let $\mathcal{F}(\mathbf{G}, \mathbf{K})$ denote the (discrete-time) feedback system shown in Figure 2.1, where $\mathbf{G} \in \mathcal{Q}^{p \times m}$ and $\mathbf{K} \in \mathcal{Q}^{m \times p}$. For $(\mathbf{G}, \mathbf{K}) \in \mathcal{Q}^{p \times m} \times \mathcal{Q}^{m \times p}$ such that $\det(I + \mathbf{GK}) \neq 0$, we set

$$(2.1) \qquad F(\mathbf{G}, \mathbf{K}) := \begin{pmatrix} (I + \mathbf{GK})^{-1} & \mathbf{G}(I + \mathbf{KG})^{-1} \\ \mathbf{K}(I + \mathbf{GK})^{-1} & (I + \mathbf{KG})^{-1} \end{pmatrix}.$$

The feedback system $\mathcal{F}(\mathbf{G}, \mathbf{K})$ is called ℓ^q-*stable* (where $1 \leq q \leq \infty$) if there exists $M \geq 0$ such that, for all $r, d_2 \in \ell^q(\mathbb{Z}_+, \mathbb{C}^p)$ and all $d_1 \in \ell^q(\mathbb{Z}_+, \mathbb{C}^m)$,

$$\|y_p\|_{\ell^q} + \|y_c\|_{\ell^q} \leq M(\|r\|_{\ell^q} + \|d_1\|_{\ell^q} + \|d_2\|_{\ell^q}).$$

It is easy to see that $\mathcal{F}(\mathbf{G}, \mathbf{K})$ is ℓ^q-stable if $F(\mathbf{G}, \mathbf{K}) \in \hat{\ell}^1(\mathbb{C}^{(m+p) \times (m+p)})$, and it is a standard result that $\mathcal{F}(\mathbf{G}, \mathbf{K})$ is ℓ^2-stable if and only if $F(\mathbf{G}, \mathbf{K}) \in H^\infty(\mathbb{E}_1, \mathbb{C}^{(m+p) \times (m+p)})$.
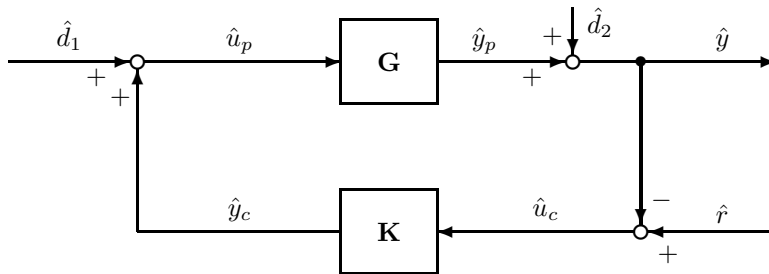

FIG. 2.1. *Discrete-time closed-loop system* $\mathcal{F}(\mathbf{G}, \mathbf{K})$.

DEFINITION 2.1. *A left-coprime factorization of* $\mathbf{G} \in \mathcal{Q}^{p \times m}$ *(over* $H^\infty(\mathbb{E}_1)$*) is a pair* $(\mathbf{D}, \mathbf{N}) \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p \times p}) \times H^\infty(\mathbb{E}_1, \mathbb{C}^{p \times m})$ *such that* $\det \mathbf{D} \neq 0$, $\mathbf{G} = \mathbf{D}^{-1}\mathbf{N}$ *and* $\mathbf{D}$ *and* $\mathbf{N}$ *are left coprime; i.e., there exist* $\mathbf{X} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p \times p})$, $\mathbf{Y} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{m \times p})$ *satisfying* $\mathbf{DX} + \mathbf{NY} = I$.

A right-coprime factorization of $\mathbf{G} \in \mathcal{Q}^{p \times m}$ *(over* $H^\infty(\mathbb{E}_1)$*) is a pair* $(\mathbf{N}, \mathbf{D}) \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p \times m}) \times H^\infty(\mathbb{E}_1, \mathbb{C}^{m \times m})$ *such that* $\det \mathbf{D} \neq 0$, $\mathbf{G} = \mathbf{ND}^{-1}$ *and* $\mathbf{N}$ *and* $\mathbf{D}$ *are right coprime; i.e., there exist* $\mathbf{X} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{m \times p})$, $\mathbf{Y} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{m \times m})$ *satisfying* $\mathbf{XN} + \mathbf{YD} = I$.

Remark 2.2. It follows from [14] that $\mathbf{G}$ and $\mathbf{K}$ admit left- and right-coprime factorizations (over $H^\infty(\mathbb{E}_1)$) if $\mathcal{F}(\mathbf{G}, \mathbf{K})$ is ℓ^2-stable.

An application of a standard result in fractional representation theory (see [17, Lemma 3.1]) gives the following necessary and sufficient algebraic condition for closed-loop stability in terms of coprime factors.

PROPOSITION 2.3. *Let* $\mathbf{G} \in \mathcal{Q}^{p \times m}$ *and* $\mathbf{K} \in \mathcal{Q}^{m \times p}$. *Assume that there exist a left-coprime factorization* $(\mathbf{D_G}, \mathbf{N_G})$ *of* $\mathbf{G}$ *and a right-coprime factorization* $(\mathbf{N_K}, \mathbf{D_K})$ *of* $\mathbf{K}$ *(both over* $H^\infty(\mathbb{E}_1)$*). Then the feedback system* $\mathcal{F}(\mathbf{G}, \mathbf{K})$ *is* ℓ^2-*stable if and only if the matrix* $\mathbf{N_G}\mathbf{N_K} + \mathbf{D_G}\mathbf{D_K}$ *has an inverse in* $H^\infty(\mathbb{E}_1, \mathbb{C}^{p \times p})$, *i.e., if and only if*

$$\inf_{z \in \mathbb{E}_1} |\det[\mathbf{N_G}(z)\mathbf{N_K}(z) + \mathbf{D_G}(z)\mathbf{D_K}(z)]| > 0.$$

PROPOSITION 2.4 (see [1, Lemma 3.1]). *Assume that* $\mathbf{G} \in \hat{\ell}^1(\mathbb{C}^{m \times m})$. *Then* $\mathbf{G}$ *has an inverse in* $\hat{\ell}^1(\mathbb{C}^{m \times m})$ *if and only if*

$$\inf_{z \in \mathbb{E}_1} |\det \mathbf{G}(z)| > 0.$$

The next result will be an important tool in the proof of our main theorem in section 3, and it is also interesting in its own right.

PROPOSITION 2.5. *Let* $\mathbf{G} \in \mathcal{Q}^{p \times m}$ *and* $\mathbf{K} \in \mathcal{Q}^{m \times p}$. *Assume that the feedback system* $\mathcal{F}(\mathbf{G}, \mathbf{K})$ *is* ℓ^2-*stable. Let* $(\mathbf{D_G}, \mathbf{N_G})$ *be a left-coprime factorization of* $\mathbf{G}$ *and* $(\mathbf{N_K}, \mathbf{D_K})$ *be a right-coprime factorization of* $\mathbf{K}$ *(both over* $H^\infty(\mathbb{E}_1)$*). Assume that* $\mathbf{D_G}, \mathbf{D_K} \in \hat{\ell}^1(\mathbb{C}^{p \times p})$, $\mathbf{N_G} \in \hat{\ell}^1(\mathbb{C}^{p \times m})$, *and* $\mathbf{N_K} \in \hat{\ell}^1(\mathbb{C}^{m \times p})$. *Then* $F(\mathbf{G}, \mathbf{K}) \in \hat{\ell}^1(\mathbb{C}^{(m+p) \times (m+p)})$. *In particular,* $\mathcal{F}(\mathbf{G}, \mathbf{K})$ *is* ℓ^q-*stable for* $1 \leq q \leq \infty$.

Proof. By hypothesis, it is clear that $\mathbf{N_G}\mathbf{N_K} + \mathbf{D_G}\mathbf{D_K} \in \hat{\ell}^1(\mathbb{C}^{p \times p})$. Since $\mathcal{F}(\mathbf{G}, \mathbf{K})$ is ℓ^2-stable, by Proposition 2.3,

$$\inf_{z \in \mathbb{E}_1} |\det[\mathbf{N_G}(z)\mathbf{N_K}(z) + \mathbf{D_G}(z)\mathbf{D_K}(z)]| > 0.$$

Then it follows from Proposition 2.4 that $(\mathbf{N_G}\mathbf{N_K} + \mathbf{D_G}\mathbf{D_K})^{-1} \in \hat{\ell}^1(\mathbb{C}^{p \times p})$. It is easy to see that

$$(I + \mathbf{GK})^{-1} = \mathbf{D_K}(\mathbf{N_G}\mathbf{N_K} + \mathbf{D_G}\mathbf{D_K})^{-1}\mathbf{D_G},$$

so that $(I + \mathbf{GK})^{-1} \in \hat{\ell}^1(\mathbb{C}^{p \times p})$. By simple calculations, we obtain

$$\mathbf{K}(I + \mathbf{GK})^{-1} = \mathbf{N_K}(\mathbf{N_G}\mathbf{N_K} + \mathbf{D_G}\mathbf{D_K})^{-1}\mathbf{D_G},$$

$$\mathbf{G}(I + \mathbf{KG})^{-1} = (I + \mathbf{GK})^{-1}\mathbf{G} = \mathbf{D_K}(\mathbf{N_G}\mathbf{N_K} + \mathbf{D_G}\mathbf{D_K})^{-1}\mathbf{N_G},$$

$$(I + \mathbf{KG})^{-1} = I - \mathbf{K}(I + \mathbf{GK})^{-1}\mathbf{G} = I - \mathbf{N_K}(\mathbf{N_G}\mathbf{N_K} + \mathbf{D_G}\mathbf{D_K})^{-1}\mathbf{N_G},$$

showing that $\mathbf{K}(I + \mathbf{GK})^{-1}$, $\mathbf{G}(I + \mathbf{KG})^{-1}$, and $(I + \mathbf{KG})^{-1}$ have all their entries in $\hat{\ell}^1(\mathbb{C})$. Hence $F(\mathbf{G}, \mathbf{K}) \in \hat{\ell}^1(\mathbb{C}^{(m+p) \times (m+p)})$. ◻

The following frequency-response result for transfer functions in $\hat{\ell}^1(\mathbb{C}^{p \times m})$ will be useful for understanding the asymptotic behavior of the closed-loop system.

LEMMA 2.6. *Let* $g \in F(\mathbb{Z}_+, \mathbb{C}^{p \times m})$, $u \in F(\mathbb{Z}_+, \mathbb{C}^m)$, $\lambda \in \overline{\mathbb{E}}_1$, $\mathfrak{v} \in \mathbb{C}^m$ *and set* $\mathbf{G} := \mathscr{Z}(g)$.

1. *If* $g \in \ell^1(\mathbb{Z}_+, \mathbb{C}^{p \times m})$ *and* $\lim_{n \to \infty}(u(n) - \lambda^n \mathfrak{v}) = 0$, *then*

$$\lim_{n \to \infty}[(g \star u)(n) - \lambda^n \mathbf{G}(\lambda)\mathfrak{v}] = 0\,.$$

2. *If there exist* $\beta \in (0,1)$ *and* $M \geq 0$ *such that* $g \in \ell_\beta^1(\mathbb{Z}_+, \mathbb{C}^{p \times m})$ *and*

$$\|u(n) - \lambda^n \mathfrak{v}\| \leq M\beta^n \quad \forall n \in \mathbb{Z}_+\,,$$

then there exists $L \geq 0$ *such that*

$$\|(g \star u)(n) - \mathbf{G}(\lambda)\lambda^n \mathfrak{v}\| \leq L\beta^n \quad \forall n \in \mathbb{Z}_+\,.$$

Proof. Since $g \in \ell^1(\mathbb{Z}_+, \mathbb{C}^{p \times m})$,

$$\|\mathbf{G}(z)\| = \left\|\sum_{k=0}^\infty g(k)z^{-k}\right\| \leq \sum_{k=0}^\infty \|g(k)\||z|^{-k} \leq \sum_{k=0}^\infty \|g(k)\| < \infty \quad \forall z \in \overline{\mathbb{E}}_1\,,$$

so that $\mathbf{G}(z)$ is well defined for $z \in \overline{\mathbb{E}}_1$. Define $v \in F(\mathbb{Z}_+, \mathbb{C}^m)$ by $v(k) := \lambda^k \mathfrak{v}$. Since $\lambda \in \overline{\mathbb{E}}_1$, $|\lambda|^{-k} \leq 1$ for all $k \in \mathbb{Z}_+$. Therefore,

$$\|(g \star u)(n) - \lambda^n \mathbf{G}(\lambda)\mathfrak{v}\| = \left\|\sum_{k=0}^n g(k)u(n-k) - \sum_{k=0}^\infty \lambda^{n-k} g(k)\mathfrak{v}\right\|$$

$$\leq \left\|\sum_{k=0}^n g(k)(u(n-k) - v(n-k))\right\| + \|\mathfrak{v}\| \sum_{k=n+1}^\infty |\lambda|^{n-k}\|g(k)\|$$

$$(2.2) \qquad\qquad \leq \|(g \star (u-v))(n)\| + \|\mathfrak{v}\| \sum_{k=n}^\infty \|g(k)\| \quad \forall n \in \mathbb{Z}_+\,.$$

We proceed to prove statement 1. Let $M_1 \geq 0$ be such that $\|u(k) - v(k)\| \leq M_1$ for all $k \in \mathbb{Z}_+$. By hypothesis, $\lim_{k \to \infty} \|u(k) - v(k)\| = 0$ and $g \in \ell^1(\mathbb{Z}_+, \mathbb{C}^{p \times m})$. Therefore, for $\varepsilon > 0$, there exists $k_0 \in \mathbb{Z}_+$ such that

$$\|u(k) - v(k)\| \leq \frac{\varepsilon}{2\|g\|_{\ell^1}}\,, \quad \sum_{j=k}^\infty \|g(j)\| \leq \frac{\varepsilon}{2M_1}\,; \quad \forall k \geq k_0.$$

Then, for $n \geq 2k_0$,

$$\|(g \star (u-v))(n)\| \leq \sum_{k=0}^{k_0} \|g(k)\|\|(u-v)(n-k)\| + \sum_{k=k_0+1}^n \|g(k)\|\|(u-v)(n-k)\|$$

$$\leq \frac{\varepsilon}{2\|g\|_{\ell^1}} \sum_{k=0}^{k_0} \|g(k)\| + M_1 \sum_{k=k_0+1}^n \|g(k)\|$$

$$\leq \varepsilon\,,$$

showing that

$$\lim_{n \to \infty} \|g \star (u - v)(n)\| = 0. \tag{2.3}$$

A combination of (2.2), (2.3), and the fact that $\lim_{n \to \infty} \sum_{k=n}^{\infty} \|g(k)\| = 0$ yields statement 1.

To prove statement 2, we set $M_2 := \sum_{k=0}^{\infty} \beta^{-k} \|g(k)\| < \infty$. By hypothesis, there exists $M \geq 0$ such that

$$\|(u - v)(n)\| \leq M\beta^n \quad \forall n \in \mathbb{Z}_+.$$

Since $\beta \in (0, 1)$ and by (2.2), we have

$$\beta^{-n} \|(g \star u)(n) - \mathbf{G}(\lambda)\lambda^n \mathfrak{v}\| \leq \beta^{-n} \sum_{k=0}^{n} \|g(k)\| \|(u - v)(n - k)\| + \beta^{-n} \|\mathfrak{v}\| \sum_{k=n}^{\infty} \|g(k)\|$$

$$\leq \beta^{-n} \sum_{k=0}^{n} \|g(k)\| M\beta^{n-k} + \|\mathfrak{v}\| \sum_{k=n}^{\infty} \beta^{-k} \|g(k)\|$$

$$\leq MM_2 + \|\mathfrak{v}\|M_2 \quad \forall n \in \mathbb{Z}_+.$$

Hence $\|(g \star u)(n) - \mathbf{G}(\lambda)\lambda^n \mathfrak{v}\| \leq M_2(M + \|\mathfrak{v}\|)\beta^n$ for all $n \in \mathbb{Z}_+$. $\square$

The next result shows that Lemma 2.6 applies in particular to input-output operators with transfer functions in $H_{\lessgtr}^{\infty}(\mathbb{E}_1, \mathbb{C}^{p \times m})$. We omit the routine proof.

PROPOSITION 2.7. *For* $0 < \alpha < \beta$, $H^{\infty}(\mathbb{E}_{\alpha}, \mathbb{C}^{p \times m}) \subset \hat{\ell}_{\beta}^1(\mathbb{C}^{p \times m})$.

The following remark shows that Lemma 2.6 also applies to power stable state-space systems.

Remark 2.8. Consider a discrete-time state-space system

$$x_p(k + 1) = Ax_p(k) + Bu_p(k), \tag{2.4a}$$

$$y_p(k) = Cx_p(k) + Du_p(k), \tag{2.4b}$$

evolving on a Banach space X, where $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(\mathbb{C}^m, X)$, $C \in \mathcal{B}(X, \mathbb{C}^p)$, and $D \in \mathcal{B}(\mathbb{C}^m, \mathbb{C}^p)$. The transfer function $\mathbf{G}$ of (2.4) is given by

$$\mathbf{G}(z) = C(zI - A)^{-1}B + D.$$

System (2.4) is called *power stable* if A is power stable, i.e., there exist $M \geq 1$ and $\rho \in (0, 1)$ such that

$$\|A^k\| \leq M\rho^k \quad \forall k \in \mathbb{Z}_+.$$

Clearly, if (2.4) is power stable, then $\sigma(A) \subset \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbf{G} \in H_{\lessgtr}^{\infty}(\mathbb{E}_1, \mathbb{C}^{p \times m})$. Hence, by Proposition 2.7, Lemma 2.6 applies to power stable systems of the form (2.4).

3. Low-gain control of discrete-time systems. Let $\mathcal{F}(\mathbf{G}, \mathbf{K}_\varepsilon)$ denote the discrete-time feedback system shown in Figure 2.1 and given by (2.1), with $\mathbf{K}$ replaced with $\mathbf{K}_\varepsilon$. The following asymptotic tracking theorem is the main result of this section. It is the discrete-time counterpart of the continuous-time result due to Rebarber and Weiss [13], which is a partial extension of the main results in Hämäläinen and Pohjolainen [3].

THEOREM 3.1. *Let $N \in \mathbb{N}$. For $j \in \underline{N}$, let $\lambda_j \in \mathbb{C}$ be such that $|\lambda_j| = 1$ and $\lambda_j \neq \lambda_k$ for $j \neq k$. Assume that $\mathbf{G} \in \hat{\ell}^1(\mathbb{C}^{p \times m})$ and $\mathbf{K}_\varepsilon$ is given by*

$$(3.1) \qquad \mathbf{K}_\varepsilon(z) := \varepsilon \left(\mathbf{K}^0(z) + \sum_{j=1}^{N} \frac{K_j}{z - \lambda_j} \right),$$

where $\mathbf{K}^0 \in \hat{\ell}^1(\mathbb{C}^{m \times p})$ and $K_j \in \mathbb{C}^{m \times p}$. If

$$(3.2) \qquad \sigma[\bar{\lambda}_j \mathbf{G}(\lambda_j) K_j] \subset \mathbb{C}_0 \quad \forall j \in \underline{N}$$

and

$$(3.3) \qquad \limsup_{z \to \lambda_j, \, z \in \mathbb{E}_1} \left\| \frac{\mathbf{G}(z) - \mathbf{G}(\lambda_j)}{z - \lambda_j} \right\| < \infty \quad \forall j \in \underline{N},$$

then there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*)$, we have $F(\mathbf{G}, \mathbf{K}_\varepsilon) \in \hat{\ell}^1$ $(\mathbb{C}^{(m+p) \times (m+p)})$ (thus $\mathcal{F}(\mathbf{G}, \mathbf{K}_\varepsilon)$ is ℓ^q-stable for every $1 \leq q \leq \infty$).

Moreover, if the reference signal r is given by

$$(3.4) \qquad r(k) := \sum_{j=1}^{N} \lambda_j^k \mathfrak{r}_j, \; \mathfrak{r}_j \in \mathbb{C}^p, \quad \forall k \in \mathbb{Z}_+,$$

and the disturbance signals d_1, d_2 satisfy

$$(3.5) \qquad \lim_{k \to \infty} \left(d_1(k) - \sum_{j=1}^{N} \lambda_j^k \mathfrak{d}_{1j} \right) = 0, \quad \lim_{k \to \infty} \left(d_2(k) - \sum_{j=1}^{N} \lambda_j^k \mathfrak{d}_{2j} \right) = 0$$

$$\text{for some } \mathfrak{d}_{1j} \in \mathbb{C}^m \text{ and } \mathfrak{d}_{2j} \in \mathbb{C}^p,$$

then, for every $\varepsilon \in (0, \varepsilon^*)$, the output of the closed-loop system y asymptotically tracks r in the presence of d_1, d_2, that is, $\lim_{k \to \infty}(y(k) - r(k)) = 0$.

Remark 3.2. (i) If condition (3.2) does not hold, then there is no guarantee that there exists an $\varepsilon > 0$ such that the feedback system $\mathcal{F}(\mathbf{G}, \mathbf{K}_\varepsilon)$ is ℓ^2-stable. Indeed, if $N = m = p = 1$, $\lambda_1 = 1$, $K_1 = 1$, and $\mathbf{G} \in \hat{\ell}^1(\mathbb{C})$ with $\mathbf{G}(1) \in (-\infty, 0]$, then an application of Proposition 2.3 shows that $\mathcal{F}(\mathbf{G}, \mathbf{K}_\varepsilon)$ is not ℓ^2-stable for every $\varepsilon > 0$. Furthermore, if $N = 1$ and $\lambda_1 = 1$, then it can be shown that the existence of an $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*)$, $\mathcal{F}(\mathbf{G}, \mathbf{K}_\varepsilon)$ is ℓ^2-stable implies that $\sigma(\mathbf{G}(1)K_1) \subset \overline{\mathbb{C}}_0$ (this follows from a suitable modification of an argument used in [11, Theorem 3]). Consequently, at least in the case $N = 1$ and $\lambda_1 = 1$, condition (3.2) is "close" to being necessary for the stability conclusion of Theorem 3.1 to hold.

(ii) Condition (3.3) is not very restrictive. It is, for example, satisfied if, for every $j \in \underline{N}$, the transfer function $\mathbf{G}$ has a holomorphic extension to an open neighborhood of λ_j (which is trivially the case if $\mathbf{G} \in H_\gtrless^\infty(\mathbb{E}_1, \mathbb{C}^{p \times m})$).

(iii) Note that only very little plant information is required in order to apply Theorem 3.1, namely, stability of the system to be controlled, condition (3.3), and some information on $\mathbf{G}(\lambda_j)$, where the latter is required for the computation of K_j such that (3.2) holds. The spectral condition (3.2) is robust with respect to "sufficiently small" plant perturbations, while (3.3) is robust with respect to all plant perturbation in $H_\gtrless^\infty(\mathbb{E}_1, \mathbb{C}^{p \times m})$.

(iv) If, in Theorem 3.1, we replace the controller $\mathbf{K}_\varepsilon$ by

$$\tilde{\mathbf{K}}_\varepsilon(z) := \varepsilon \left(\tilde{\mathbf{K}}^0(z) + \sum_{j=1}^{N} \frac{z\tilde{K}_j}{z - \lambda_j} \right),$$

where $\tilde{\mathbf{K}}^0 \in \hat{\ell}^1(\mathbb{C}^{m \times p})$ and $\tilde{K}_j \in \mathbb{C}^{m \times p}$, and condition (3.2) by

(3.6) $\sigma(\mathbf{G}(\lambda_j)\tilde{K}_j) \subset \mathbb{C}_0 \quad \forall j \in \underline{N},$

while all the other conditions in the theorem remain the same, then the conclusions on stability, tracking, and disturbance rejection in Theorem 3.1 are still valid. This follows directly from Theorem 3.1, since

$$\tilde{\mathbf{K}}_\varepsilon(z) = \varepsilon \left(\tilde{\mathbf{K}}^0(z) + \sum_{j=1}^{N} \tilde{K}_j + \sum_{j=1}^{N} \frac{\lambda_j \tilde{K}_j}{z - \lambda_j} \right)$$

is of the form (3.1) with

$$\mathbf{K}^0(z) = \tilde{\mathbf{K}}^0(z) + \sum_{j=1}^{N} \tilde{K}_j, \quad K_j = \lambda_j \tilde{K}_j,$$

and $\sigma(\bar{\lambda}_j \mathbf{G}(\lambda_j) K_j) = \sigma(\mathbf{G}(\lambda_j)\tilde{K}_j) \subset \mathbb{C}_0$.

(v) The spectral condition (3.2) (or, alternatively, (3.6)) is the discrete-time analogue of the continuous-time condition in [13]; see [13, equation (1.5)]. Moreover, in the continuous-time result [13, Theorem 1.1], it is assumed that the transfer function of the plant is holomorphic and bounded in a half-plane of the form $\operatorname{Re} s > -\alpha$ for some $\alpha > 0$; the discrete-time analogue of this condition is, in the terminology of the present paper, $\mathbf{G} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p \times m})$, which implies (3.3) (cf. part (ii) of this remark). Consequently, condition (3.3) is weaker than the corresponding continuous-time condition in [13, Theorem 1.1].

To facilitate the proof of Theorem 3.1, we first state and prove the following key lemma, which shows that the transfer function $(I + \mathbf{GK}_\varepsilon)^{-1}$, the so-called sensitivity function, is in $H^\infty(\mathbb{E}_1, \mathbb{C}^{p \times p})$ for sufficiently small $\varepsilon > 0$.

LEMMA 3.3. *Let $N \in \mathbb{N}$ and let $\lambda_j \in \mathbb{C}$ be such that $|\lambda_j| = 1$ and $\lambda_j \neq \lambda_k$ for $j, k \in \underline{N}, j \neq k$. Let $\mathbf{G} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p \times m})$ be such that the limit $\mathbf{G}(\lambda_j) := \lim_{z \to \lambda_j, \, z \in \mathbb{E}_1} \mathbf{G}(z)$ exists for every $j \in \underline{N}$. Let $\mathbf{K}_\varepsilon$ be given by (3.1), where $\mathbf{K}^0 \in H^\infty(\mathbb{E}_1, \mathbb{C}^{m \times p})$ and $K_j \in \mathbb{C}^{m \times p}$. Assume that (3.2) and (3.3) hold. Then there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*)$, $(I + \mathbf{GK}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p \times p})$. Moreover, if the additional assumptions that $\mathbf{G} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p \times m})$ and $\mathbf{K}^0 \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{m \times p})$ are satisfied, then, for every $\varepsilon \in (0, \varepsilon^*)$, $(I + \mathbf{GK}_\varepsilon)^{-1} \in H^\infty_<(\mathbb{E}_1, \mathbb{C}^{p \times p})$.*

Proof. Before proceeding to the technical details, we summarize the idea of the proof. We wish to show that $(I + \mathbf{GK}_\varepsilon)^{-1}$ is bounded in $\mathbb{E}_1$, the complement of the closed unit disc, for all sufficiently small $\varepsilon > 0$. Roughly speaking, we decompose $\mathbb{E}_1$ in the form $\mathbb{E}_1 = \Omega \cup \left(\cup_{j=1}^{n} \Omega_j \right)$, where Ω is bounded away from all of the λ_j's and Ω_j is the "part of $\mathbb{E}_1$ near λ_j." We prove that, for sufficiently small $\varepsilon > 0$, $(I + \mathbf{GK}_\varepsilon)^{-1}$ is bounded on each of these sets. Special care is required for the analysis on the sets Ω_j.
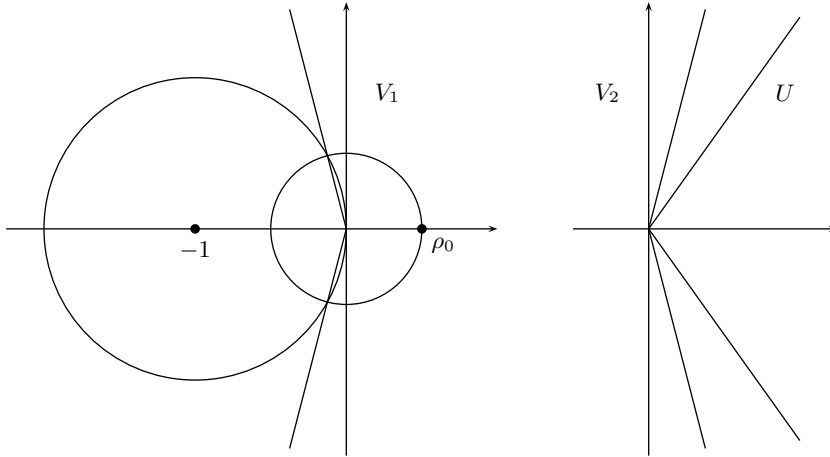
FIG. 3.1. *An illustration of the sets U, V_1, and V_2.*

Returning to the technical details of the proof, we first note that, since $\sigma[\bar{\lambda}_j \mathbf{G}(\lambda_j)K_j] \subset \mathbb{C}_0$ for all $j \in \underline{N}$, there exists $\theta \in (0, \pi/2)$ such that

$$(3.7) \qquad \bigcup_{j=1}^{N} \sigma[\bar{\lambda}_j \mathbf{G}(\lambda_j)K_j] \subset \{z \in \mathbb{C} \setminus \{0\} : \arg z \in (-\theta, \theta)\} =: U.$$

Let $\rho \in (0, 1)$ and consider Figure 3.1. The circles $\{z \in \mathbb{C} : |z| = \rho\}$ and $\{z \in \mathbb{C} : |z + 1| = 1\}$ intersect at two points, denoted by $\rho e^{i\phi(\rho)}$ and $\rho e^{-i\phi(\rho)}$, where $\phi(\rho) \in (\pi/2, \pi)$. Note that $\phi(\rho) \to \pi/2$ monotonically as $\rho \to 0$. Hence there exists $\rho_0 \in (0, 1)$ such that $\pi - \phi(\rho) > \theta$ for all $\rho \in (0, \rho_0]$. Set

$$V_1 := \{z \in \mathbb{C} \setminus \{0\} : \arg z \in (-\phi(\rho_0), \phi(\rho_0))\}$$

and

$$V_2 := -V_1 = \{z \in \mathbb{C} \setminus \{0\} : \arg z \in (\pi - \phi(\rho_0), \pi + \phi(\rho_0))\}.$$

Clearly,

$$(3.8) \qquad\qquad\qquad U \cap \overline{V}_2 = \emptyset.$$

There exists $\rho_1 \in (0, \rho_0]$ such that $|\lambda_j - \lambda_k| > 2\rho_1$ for all $j, k \in \underline{N}$, $j \neq k$. Defining

$$\Omega_j := \mathbb{E}_1 \bigcap \{z \in \mathbb{C} : |z - \lambda_j| < \rho_1\},$$

we have that $\Omega_j \cap \Omega_k = \emptyset$ for $j, k \in \underline{N}$, $j \neq k$. Moreover, set $\Omega := \mathbb{E}_1 \setminus \bigcup_{j=1}^{N} \Omega_j$. Assume that $\mathbf{G} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p \times m})$ and $\mathbf{K}^0 \in H^\infty(\mathbb{E}_1, \mathbb{C}^{m \times p})$. It is clear that

$$\sup_{z \in \Omega} \left\| \mathbf{G}(z) \left(\mathbf{K}^0(z) + \sum_{j=1}^{N} \frac{K_j}{z - \lambda_j} \right) \right\| < \infty.$$

Therefore, there exists $\varepsilon_\infty > 0$ such that

$$\mathbf{S}(z) := [I + \mathbf{G}(z)\mathbf{K}_\varepsilon(z)]^{-1} = \left[I + \varepsilon \mathbf{G}(z) \left(\mathbf{K}^0(z) + \sum_{j=1}^{N} \frac{K_j}{z - \lambda_j} \right) \right]^{-1}$$

is uniformly bounded for all $z \in \Omega$ and for all $\varepsilon \in (0, \varepsilon_\infty)$. Fix $j \in \underline{N}$. To analyze $\mathbf{S}$ on Ω_j, we define

$$\mathbf{S}_j(z) := \left(I + \frac{\varepsilon \mathbf{G}(\lambda_j) K_j}{z - \lambda_j} \right)^{-1} = \left(I + \frac{\varepsilon \bar{\lambda}_j \mathbf{G}(\lambda_j) K_j}{\bar{\lambda}_j z - 1} \right)^{-1}$$

and

$$\mathbf{Q}_j(z) := \frac{\mathbf{G}(z) - \mathbf{G}(\lambda_j)}{z - \lambda_j} K_j + \mathbf{G}(z) \mathbf{K}^0(z) + \sum_{k \in \underline{N}, \, k \neq j} \frac{\mathbf{G}(z) K_k}{z - \lambda_k} \, .$$

By (3.3), we see that $\mathbf{Q}_j$ is bounded on Ω_j, with a bound that is independent of ε. For convenience, we set $G_j := \bar{\lambda}_j \mathbf{G}(\lambda_j) K_j$. Moreover, since $\rho_1 \in (0, \rho_0]$, it follows that $\bar{\lambda}_j \Omega_j - 1 \subset V_1$. Together with the implication that if $w \in V_1$, then $\gamma w \in V_1$ for all $\gamma \geq 0$, this yields

$$\sup_{z \in \Omega_j} \|\mathbf{S}_j(z)\| = \sup \left\{ \left\| \left(I + \varepsilon \frac{G_j}{w} \right)^{-1} \right\| : w \in \bar{\lambda}_j \Omega_j - 1 \right\}$$

$$\leq \sup_{s \in V_1} \|s(sI + G_j)^{-1}\| = \sup_{s \in V_2} \|s(sI - G_j)^{-1}\| \, .$$

By (3.7) and (3.8), the function $s \mapsto s(sI - G_j)^{-1}$ is holomorphic on an open set $W \supset \overline{V}_2$ (where $\overline{V}_2$ denotes the closure of V_2). Furthermore,

$$\lim_{|s| \to \infty} s(sI - G_j)^{-1} = I \, .$$

Hence $s \mapsto s(sI - G_j)^{-1}$ is bounded on $\overline{V}_2$. Therefore, $\mathbf{S}_j$ is bounded on Ω_j with bound independent of ε. We have $\mathbf{S}^{-1} - \mathbf{S}_j^{-1} = \varepsilon \mathbf{Q}_j$, so that we can write

$$\mathbf{S}(z) = \mathbf{S}_j(z)(I + \varepsilon \mathbf{Q}_j(z) \mathbf{S}_j(z))^{-1} \, .$$

Hence there exists $\varepsilon_j \in (0, \varepsilon_\infty)$ such that $\mathbf{S}$ is bounded on Ω_j for all $\varepsilon \in (0, \varepsilon_j)$. Setting

$$\varepsilon^* := \min \{ \varepsilon_j : j \in \underline{N} \} \, ,$$

it follows that

(3.9) $$(I + \mathbf{G} \mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p \times p}) \quad \forall \varepsilon \in (0, \varepsilon^*) \, .$$

Finally, let $\varepsilon \in (0, \varepsilon^*)$ and assume that $\mathbf{G} \in H_\lesssim^\infty(\mathbb{E}_1, \mathbb{C}^{p \times m})$ and $\mathbf{K}^0 \in H_\lesssim^\infty(\mathbb{E}_1, \mathbb{C}^{m \times p})$. It is clear that $(I + \mathbf{G} \mathbf{K}_\varepsilon)^{-1}$ is meromorphic on $\mathbb{E}_\gamma$ for some $\gamma \in (0, 1)$. Letting $\beta \in (\gamma, 1)$, it follows that $(I + \mathbf{G} \mathbf{K}_\varepsilon)^{-1}$ has at most finitely many poles in the compact annulus $\overline{\mathbb{E}}_\beta \setminus \mathbb{E}_1$. By (3.9), $(I + \mathbf{G} \mathbf{K}_\varepsilon)^{-1}$ does not have any poles on $\partial \mathbb{E}_1$ and so there exists $\alpha \in (\beta, 1)$ such that $(I + \mathbf{G} \mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{p \times p})$. $\quad \square$

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.3, we know that there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$, $(I + \mathbf{G} \mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p \times p})$. In the following, let $\varepsilon \in (0, \varepsilon^*)$.

We first show that the other block entries of $F(\mathbf{G}, \mathbf{K}_\varepsilon)$ are also H^∞-functions. Due to the stability of $\mathbf{G}$, it suffices to show that $\mathbf{K}_\varepsilon(I + \mathbf{G} \mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{m \times p})$. In the remainder of the proof, when we write $z \to \lambda_j$, it is assumed that $z \in \mathbb{E}_1$. By

assumption, $\lambda_j \neq \lambda_k$ for $j, k \in \underline{N}$, $j \neq k$. Note that, by (3.2), $\mathbf{G}(\lambda_j)K_j$ is invertible. Consequently,

$$\lim_{z \to \lambda_j} \frac{1}{z - \lambda_j} (I + \mathbf{G}(z)\mathbf{K}_\varepsilon(z))^{-1}$$

$$= \lim_{z \to \lambda_j} \left[\varepsilon \mathbf{G}(z)K_j + (z - \lambda_j) \left(I + \varepsilon \mathbf{G}(z)\mathbf{K}^0(z) + \varepsilon \sum_{k \in \underline{N},\ k \neq j} \frac{\mathbf{G}(z)K_k}{z - \lambda_k} \right) \right]^{-1}$$

$$(3.10) \quad = (\varepsilon \mathbf{G}(\lambda_j)K_j)^{-1} \quad \forall j \in \underline{N}.$$

By (3.1) and (3.10), we conclude that $\mathbf{K}_\varepsilon(z)(I + \mathbf{GK}_\varepsilon)^{-1}$ has a finite limit at λ_j, so that $\mathbf{K}_\varepsilon(I + \mathbf{GK}_\varepsilon)^{-1}$ is bounded on $\mathbb{E}_1 \cap \Lambda$, where Λ is a neighborhood of the set $\{\lambda_j : j \in \underline{N}\}$. Since $(I + \mathbf{GK}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p \times p})$ and $\mathbf{K}_\varepsilon$ is uniformly bounded on $\mathbb{E}_1 \setminus \Lambda$, it follows that $\mathbf{K}_\varepsilon(I + \mathbf{GK}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{m \times p})$. Consequently, $F(\mathbf{G}, \mathbf{K}_\varepsilon) \in H^\infty(\mathbb{E}_1, \mathbb{C}^{(m+p) \times (m+p)})$, showing that $\mathcal{F}(\mathbf{G}, \mathbf{K}_\varepsilon)$ is ℓ^2-stable.

To prove that $F(\mathbf{G}, \mathbf{K}_\varepsilon) \in \hat{\ell}^1(\mathbb{C}^{(m+p) \times (m+p)})$, we set

$$\mathbf{K}^1(z) := \sum_{j=1}^N \frac{K_j}{z - \lambda_j}.$$

We see that $\mathbf{K}^1$ is a (strictly proper) rational matrix function. By a standard result (see [16, p. 75, Theorem 4.1.43]), $\mathbf{K}^1$ has a right-coprime factorization over $\mathcal{R}_s$, i.e., $\mathbf{K}^1 = \mathbf{ND}^{-1}$, where $\mathbf{N} \in \mathcal{R}_s^{m \times p}$, $\mathbf{D} \in \mathcal{R}_s^{p \times p}$, and there exist $\mathbf{X} \in \mathcal{R}_s^{p \times m}$, $\mathbf{Y} \in \mathcal{R}_s^{p \times p}$ such that $\mathbf{XN} + \mathbf{YD} = I$. Therefore,

$$\mathbf{K}_\varepsilon = \varepsilon(\mathbf{K}^0 + \mathbf{K}^1) = \varepsilon(\mathbf{K}^0\mathbf{D} + \mathbf{N})\mathbf{D}^{-1},$$

showing that $\mathbf{K}_\varepsilon$ has right-coprime factorization $(\varepsilon(\mathbf{K}^0\mathbf{D} + \mathbf{N}), \mathbf{D})$, since

$$(\varepsilon^{-1}\mathbf{X})\varepsilon(\mathbf{K}^0\mathbf{D} + \mathbf{N}) + (\mathbf{Y} - \mathbf{XK}^0)\mathbf{D} = \mathbf{XD} + \mathbf{YD} = I.$$

Since $\mathbf{K}^0, \mathbf{N} \in \hat{\ell}^1(\mathbb{C}^{m \times p})$ and $\mathbf{D} \in \hat{\ell}^1(\mathbb{C}^{p \times p})$, we have that $\mathbf{K}^0\mathbf{D} + \mathbf{N} \in \hat{\ell}^1(\mathbb{C}^{m \times p})$. Moreover, $(I, \mathbf{G})$ is a left-coprime factorization of $\mathbf{G}$ over $H^\infty(\mathbb{E}_1)$ and, by assumption, $\mathbf{G} \in \hat{\ell}^1(\mathbb{C}^{m \times p})$. Therefore, invoking Proposition 2.5, it follows that $F(\mathbf{G}, \mathbf{K}_\varepsilon) \in \hat{\ell}^1(\mathbb{C}^{(m+p) \times (m+p)})$.

To prove tracking and disturbance rejection, we note first that, since $\mathbf{G}(\lambda_j)K_j$ is invertible,

$$(3.11) \qquad (I + \mathbf{GK}_\varepsilon)^{-1}(\lambda_j) = \lim_{z \to \lambda_j} (I + \mathbf{G}(z)\mathbf{K}_\varepsilon(z))^{-1} = 0 \quad \forall j \in \underline{N}$$

and

$$(3.12) \qquad ((I + \mathbf{GK}_\varepsilon)^{-1}\mathbf{G})(\lambda_j) = \lim_{z \to \lambda_j} (I + \mathbf{G}(z)\mathbf{K}_\varepsilon(z))^{-1}\mathbf{G}(z) = 0 \quad \forall j \in \underline{N}.$$

Let r be given by (3.4) and let d_1, d_2 satisfy (3.5). For $j \in \underline{N}$, define $a_j \in F(\mathbb{Z}_+, \mathbb{C}^p)$ and $b_j \in F(\mathbb{Z}_+, \mathbb{C}^m)$ by

$$a_j(k) := \lambda_j^k \mathfrak{r}_j, \quad b_j(k) := \lambda_j^k \mathfrak{d}_{1j},$$

and define $\tilde{d}_1$ by

$$\tilde{d}_1(k) := d_1(k) - \sum_{j=1}^{N} b_j = d_1(k) - \sum_{j=1}^{N} \lambda_j^k \mathfrak{d}_{1j} \,.$$

Obviously, $r = \sum_{j=1}^{N} a_j$ and $\lim_{n\to\infty} \tilde{d}_1(k) = 0$. Then, by Lemma 2.6, (3.11), and (3.12), we obtain

$$\lim_{k\to\infty} [\mathscr{Z}^{-1}((I + \mathbf{GK}_\varepsilon)^{-1}) \star r](k)$$

$$= \sum_{j=1}^{N} \lim_{k\to\infty} \{[\mathscr{Z}^{-1}((I + \mathbf{GK}_\varepsilon)^{-1}) \star a_j](k) - ((I + \mathbf{GK}_\varepsilon)^{-1})(\lambda_j)\lambda_j^k \mathfrak{r}_j\}$$

(3.13) $= 0$

and

$$\lim_{k\to\infty} [\mathscr{Z}^{-1}((I + \mathbf{GK}_\varepsilon)^{-1}\mathbf{G}) \star d_1](k)$$

$$= \sum_{j=1}^{N} \lim_{k\to\infty} \{[\mathscr{Z}^{-1}((I + \mathbf{GK}_\varepsilon)^{-1}\mathbf{G}) \star b_j](k) - ((I + \mathbf{GK}_\varepsilon)^{-1}\mathbf{G})(\lambda_j)\lambda_j^k \mathfrak{d}_{1j}\}$$

$$+ \lim_{k\to\infty} [\mathscr{Z}^{-1}((I + \mathbf{GK}_\varepsilon)^{-1}\mathbf{G}) \star \tilde{d}_1](k)$$

(3.14) $= 0 \,.$

Similarly, by Lemma 2.6 and (3.11),

(3.15) $$\lim_{k\to\infty} [\mathscr{Z}^{-1}((I + \mathbf{GK}_\varepsilon)^{-1}) \star d_2](k) = 0 \,.$$

By Figure 2.1 (with $\mathbf{K}$ replaced by $\mathbf{K}_\varepsilon$), it is clear that

(3.16) $$\hat{r} - \hat{y} = \hat{u}_c = (I + \mathbf{GK}_\varepsilon)^{-1}(\hat{r} - \hat{d}_2) - (I + \mathbf{GK}_\varepsilon)^{-1}\mathbf{G}\hat{d}_1 \,.$$

Therefore, by (3.13)–(3.16),

$$\lim_{k\to\infty} (r - y)(k) = \lim_{k\to\infty} \{[\mathscr{Z}^{-1}((I + \mathbf{GK}_\varepsilon)^{-1}) \star (r - d_2)](k)$$

$$- [\mathscr{Z}^{-1}((I + \mathbf{GK}_\varepsilon)^{-1}\mathbf{G}) \star d_1](k)\} = 0 \,.$$

This completes the proof.   ∎

Next we show that, under a mild extra assumption on $\mathbf{G}$, $\mathbf{K}^0$, d_1, and d_2, the convergence of $y(k)$ to $r(k)$ as $k \to \infty$ is exponentially fast.

THEOREM 3.4. *Consider the discrete-time feedback system $\mathcal{F}(\mathbf{G}, \mathbf{K}_\varepsilon)$ shown in Figure 2.1 (with $\mathbf{K}$ replaced by $\mathbf{K}_\varepsilon$). Assume that $\mathbf{G} \in H^\infty_{\mathbb{Z}}(\mathbb{E}_1, \mathbb{C}^{p\times m})$ and $\mathbf{K}_\varepsilon$ is given by (3.1), where $\mathbf{K}^0 \in H^\infty_{\mathbb{Z}}(\mathbb{E}_1, \mathbb{C}^{m\times p})$, $K_j \in \mathbb{C}^{m\times p}$, and $|\lambda_j| = 1$ for $j \in \underline{N}$ with $\lambda_j \neq \lambda_k$ for $j \neq k$. If (3.2) holds, then there exists $\varepsilon^* > 0$ such that, for every $\varepsilon \in (0, \varepsilon^*)$, $F(\mathbf{G}, \mathbf{K}_\varepsilon) \in H^\infty_{\mathbb{Z}}(\mathbb{E}_1, \mathbb{C}^{(m+p)\times(m+p)})$.*

Moreover, if the reference signal r is given by (3.4) and there exist $M \geq 0$ and $\rho \in (0,1)$ such that the disturbance signals d_1, d_2 satisfy

$$(3.17) \qquad \left\| d_1(k) - \sum_{j=1}^{N} \lambda_j^k \mathfrak{d}_{1j} \right\| \leq M\rho^k, \ \left\| d_2(k) - \sum_{j=1}^{N} \lambda_j^k \mathfrak{d}_{2j} \right\| \leq M\rho^k \ \forall k \in \mathbb{Z}_+,$$

$$\text{where } \mathfrak{d}_{1j} \in \mathbb{C}^m, \ \mathfrak{d}_{2j} \in \mathbb{C}^p,$$

then, for every $\varepsilon \in (0, \varepsilon^*)$, there exist $L \geq 0$ and $\beta \in (\rho, 1)$ such that

$$\| y(k) - r(k) \| \leq L\beta^k \quad \forall k \in \mathbb{Z}_+.$$

Proof. By Lemma 3.3 and the hypotheses on $\mathbf{G}$, $\mathbf{K}^0$, we know that there exists $\varepsilon^* > 0$ such that, for every $\varepsilon \in (0, \varepsilon^*)$, there exists $\alpha \in (\rho, 1)$ such that

$$(I + \mathbf{G}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{p\times p}), \ \mathbf{G} \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{p\times m}), \ \mathbf{K}^0 \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{m\times p}).$$

To prove that $F(\mathbf{G}, \mathbf{K}_\varepsilon) \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{(m+p)\times(m+p)})$, it suffices to show that $\mathbf{K}_\varepsilon(I + \mathbf{G}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{m\times p})$. By (3.10), we conclude that $\mathbf{K}_\varepsilon(z)(I + \mathbf{G}(z)\mathbf{K}_\varepsilon(z))^{-1}$ has a finite limit as $z \to \lambda_j$ for every $j \in \underline{N}$, so that $\mathbf{K}_\varepsilon(I + \mathbf{G}\mathbf{K}_\varepsilon)^{-1}$ is bounded on a neighborhood Λ of the set $\{\lambda_j : j \in \underline{N}\}$. Since $(I + \mathbf{G}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{p\times p})$ and $\mathbf{K}_\varepsilon$ is uniformly bounded on $\mathbb{E}_\alpha \setminus \Lambda$, it follows that

$$\mathbf{K}_\varepsilon(I + \mathbf{G}\mathbf{K}_\varepsilon)^{-1} \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{m\times p}).$$

Hence $F(\mathbf{G}, \mathbf{K}_\varepsilon) \in H^\infty(\mathbb{E}_\alpha, \mathbb{C}^{(m+p)\times(m+p)})$. Therefore, it follows from Proposition 2.7 that, for every $\beta \in (\alpha, 1)$, we have

$$(I + \mathbf{G}\mathbf{K}_\varepsilon)^{-1} \in \hat{\ell}_\beta^1(\mathbb{C}^{p\times p}), \quad (I + \mathbf{G}\mathbf{K}_\varepsilon)^{-1}\mathbf{G} \in \hat{\ell}_\beta^1(\mathbb{C}^{p\times m}).$$

Finally, invoking Lemma 2.6, (3.4), (3.11), (3.12), and (3.17), we conclude that there exists $M_1 \geq 0$ such that

$$\| [\mathscr{Z}^{-1}((I + \mathbf{G}\mathbf{K}_\varepsilon)^{-1}) \star r](k) \| \leq M_1\beta^k \quad \forall k \in \mathbb{Z}_+,$$

$$\| [\mathscr{Z}^{-1}((I + \mathbf{G}\mathbf{K}_\varepsilon)^{-1}\mathbf{G}) \star d_1](k) \| \leq M_1\beta^k \quad \forall k \in \mathbb{Z}_+,$$

$$\| [\mathscr{Z}^{-1}((I + \mathbf{G}\mathbf{K}_\varepsilon)^{-1}) \star d_2](k) \| \leq M_1\beta^k \quad \forall k \in \mathbb{Z}_+.$$

Consequently, by (3.16), we have

$$\| y(k) - r(k) \| \leq 3M_1\beta^k \quad \forall k \in \mathbb{Z}_+,$$

completing the proof. ∎

Application to state-space systems. We now apply Theorem 3.1 to obtain tracking results for discrete-time state-space systems. Let X be a Banach space and let the plant Σ_p be given by

$$(3.18a) \qquad x_p(k+1) = Ax_p(k) + Bu_p(k); \quad x_p(0) = x_p^0 \in X,$$

$$(3.18b) \qquad y_p(k) = Cx_p(k) + Du_p(k),$$

where $A \in \mathcal{B}(X,X)$, $B \in \mathcal{B}(\mathbb{C}^m, X)$, $C \in \mathcal{B}(X, \mathbb{C}^p)$, and $D \in \mathcal{B}(\mathbb{C}^m, \mathbb{C}^p)$. The transfer function $\mathbf{G}$ of Σ_p is given by

$$\mathbf{G}(z) = C(zI - A)^{-1}B + D.$$

Next we construct a state-space realization of the controller transfer function (3.1). Let $\mathbf{K}^0 \in \mathcal{R}_s^{m \times p}$ and let $(A_0, B_0, C_0, D_0) \in \mathbb{C}^{n_0 \times n_0} \times \mathbb{C}^{n_0 \times p} \times \mathbb{C}^{m \times n_0} \times \mathbb{C}^{m \times p}$ be a stabilizable and detectable realization of $\mathbf{K}^0$; i.e., $\mathbf{K}^0(z) = C_0(zI - A_0)^{-1}B_0 + D_0$, (A_0, B_0) is stabilizable, and (C_0, A_0) is detectable. Since $\mathbf{K}^0$ is ℓ^2-stable, A_0 is power stable. Let $K_j \in \mathbb{C}^{m \times p}$ and $|\lambda_j| = 1$ for $j \in \underline{N}$ with $\lambda_j \neq \lambda_k$ for $j \neq k$. Moreover, let $A_c \in \mathbb{C}^{(Np+n_0) \times (Np+n_0)}$, $B_c \in \mathbb{C}^{(Np+n_0) \times p}$, $C_c \in \mathbb{C}^{m \times (Np+n_0)}$, and $D_c \in \mathbb{C}^{m \times p}$ be given by

$$(3.19a) \qquad A_c := \begin{pmatrix} A_0 & & & & \\ & \lambda_1 I_p & & & \\ & & \ddots & & \\ & & & \lambda_N I_p \end{pmatrix}, \quad B_c := \begin{pmatrix} B_0 \\ I_p \\ \vdots \\ I_p \end{pmatrix},$$

$$(3.19b) \qquad C_c := (C_0, K_1, \ldots, K_N), \quad D_c := D_0,$$

where I_p is the $p \times p$ identity matrix. We define the controller Σ_c by

$$(3.20a) \qquad x_c(k+1) = A_c x_c(k) + B_c u_c(k); \quad x_c(0) = x_c^0 \in \mathbb{C}^{Np+n_0},$$

$$(3.20b) \qquad y_c(k) = \varepsilon C_c x_c(k) + \varepsilon D_c u_c(k).$$

Obviously, the transfer function $\mathbf{K}_\varepsilon$ of Σ_c is given by

$$\mathbf{K}_\varepsilon(z) = \varepsilon(C_c(zI - A_c)^{-1}B_c + D_c) = \varepsilon \left(\mathbf{K}^0(z) + \sum_{j=1}^N \frac{K_j}{z - \lambda_j} \right).$$

Consider the feedback interconnection of (3.18) and (3.20) given by

$$(3.21) \qquad u_c = r - y_p - d_2, \quad u_p = y_c + d_1, \quad y = y_p + d_2,$$

where r is a reference signal and d_1 and d_2 are disturbance signals. Let $\mathcal{F}(\Sigma_p, \Sigma_c)$ denote the feedback system given by (3.18)–(3.21). The state-space system $\mathcal{F}(\Sigma_p, \Sigma_c)$ is a state-space realization of the system $\mathcal{F}(\mathbf{G}, \mathbf{K}_\varepsilon)$ shown in Figure 2.1 (with $\mathbf{K}$ replaced by $\mathbf{K}_\varepsilon$).

THEOREM 3.5. *Assume that* (3.18) *is power stable and that* (3.2) *holds, i.e.,* $\sigma(\bar{\lambda}_j \mathbf{G}(\lambda_j) K_j) \subset \mathbb{C}_0$ *for every* $j = \underline{N}$. *Then there exists* $\varepsilon^* > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon^*)$, *the following statements hold:*

1. *$\mathcal{F}(\Sigma_p, \Sigma_c)$ is power stable. Moreover, $\mathcal{F}(\Sigma_p, \Sigma_c)$ is input-to-state stable in the sense that there exist $M_1 \geq 1$ and $\gamma \in (0,1)$ such that, for all $x_p^0 \in X$, $x_c^0 \in \mathbb{C}^{Np+n_0}$, $r, d_2 \in \ell^\infty(\mathbb{Z}_+, \mathbb{C}^p)$, and all $d_1 \in \ell^\infty(\mathbb{Z}_+, \mathbb{C}^m)$,*

$$\left\| \begin{pmatrix} x_p \\ x_c \end{pmatrix} \right\|_{\ell^\infty} \leq M_1 \left(\gamma^k \left\| \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} \right\| + \|r\|_{\ell^\infty} + \|d_1\|_{\ell^\infty} + \|d_2\|_{\ell^\infty} \right).$$

2. *If r is given by (3.4) and d_1, d_2 satisfy (3.5), then for all initial conditions $x_p^0 \in X$ and $x_c^0 \in \mathbb{C}^{Np+n_0}$, the output $y = y_p + d_2$ asymptotically tracks r, that is, $\lim_{k \to \infty}(y(k) - r(k)) = 0$. Additionally, if (3.17) holds with $M \geq 0$ and $\rho \in (0,1)$, then the convergence is exponentially fast.*

We omit the proof, which is based on a routine argument involving a combination of Theorem 3.1 and a result on the equivalence of input-output and power stability [5, Theorem 2]; see [4] for details.

4. Low-gain sampled-data control. In the following, let $\mathfrak{B}(\mathbb{R}_+)$ denote the Borel-σ-algebra on $\mathbb{R}_+$. For a $\mathbb{C}^{p \times m}$-valued Borel measure μ on $\mathbb{R}_+$, the total variation $|\mu| : \mathfrak{B}(\mathbb{R}_+) \to [0, \infty]$ of μ is defined by

$$|\mu|(E) := \sup \left\{ \sum_{j=1}^{\infty} \|\mu(E_j)\| : E_j \in \mathfrak{B}(\mathbb{R}_+), \ E_j \cap E_k = \emptyset \text{ if } j \neq k, \ E = \bigcup_{j=1}^{\infty} E_j \right\}.$$

It is clear that

$$\|\mu(E)\| \leq |\mu|(E) \quad \forall E \in \mathfrak{B}(\mathbb{R}_+).$$

The following theorem, for which the proof is omitted, shows that a $\mathbb{C}^{p \times m}$-valued Borel measure is necessarily bounded.

THEOREM 4.1. *The total variation $|\mu|$ of a $\mathbb{C}^{p \times m}$-valued Borel measure μ is a finite nonnegative Borel measure on $\mathbb{R}_+$.*

The following technical result, for which we omit the routine proof, is used later.

PROPOSITION 4.2. *Let μ be a $\mathbb{C}^{p \times m}$-valued Borel measure on $\mathbb{R}_+$. For every $\varepsilon > 0$, there exists $T > 0$ such that*

$$\int_t^{\infty} |\mu|(ds) < \varepsilon \quad \forall t \geq T.$$

Let μ be a $\mathbb{C}^{p \times m}$-valued Borel measure on $\mathbb{R}_+$. Then the continuous-time input-output operator G defined by

$$(4.1) \qquad (Gu)(t) := (\mu \star u)(t) = \int_0^t \mu(ds)u(t-s), \quad t \geq 0, \ u \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^m),$$

is L^q-stable for $1 \leq q \leq \infty$. The transfer function $\mathbf{G}$ of G is the Laplace transform of μ, that is,

$$(4.2) \qquad\qquad \mathbf{G}(s) = \int_{\mathbb{R}_+} e^{-st} \mu(dt) \quad \forall s \in \overline{\mathbb{C}}_0.$$

Trivially, by Theorem 4.1, $\|\mathbf{G}(s)\| \leq \int_0^{\infty} |\mu|(dt) < \infty$ for all $s \in \overline{\mathbb{C}}_0$. It follows that $\mathbf{G} \in H^{\infty}(\mathbb{C}_0, \mathbb{C}^{p \times m})$.

LEMMA 4.3. *Let the operator G be given by (4.1), where μ is a $\mathbb{C}^{p \times m}$-valued Borel measure on $\mathbb{R}_+$. Then*

$$\limsup_{t \to \infty} \|(Gu)(t)\| \leq |\mu|(\mathbb{R}_+) \limsup_{t \to \infty} \|u(t)\| \quad \forall u \in L_b(\mathbb{R}_+, \mathbb{C}^m).$$

Proof. Let $\varepsilon > 0$. By Proposition 4.2, there exists $T > 0$ such that

$$\int_T^{\infty} |\mu|(ds) \leq \frac{\varepsilon}{2\|u\|_{\infty}} \quad \text{and} \quad \|u(t)\| \leq \sigma + \frac{\varepsilon}{2M} \quad \forall t \geq T,$$

where $M := |\mu|(\mathbb{R}_+)$ and $\sigma := \limsup_{t\to\infty} \|u(t)\|$. Hence, for $t \geq 2T$,

$$\|(Gu)(t)\| \leq \int_0^{t/2} \|u(t-s)\| |\mu|(ds) + \int_{t/2}^t \|u(t-s)\| |\mu|(ds)$$

$$\leq \left(\sigma + \frac{\varepsilon}{2M}\right) \int_0^{t/2} |\mu|(ds) + \|u\|_\infty \int_{t/2}^t |\mu|(ds)$$

$$\leq \left(\sigma + \frac{\varepsilon}{2M}\right) \int_0^\infty |\mu|(ds) + \|u\|_\infty \int_T^\infty |\mu|(ds)$$

$$\leq \left(\sigma + \frac{\varepsilon}{2M}\right) M + \|u\|_\infty \frac{\varepsilon}{2\|u\|_\infty}$$

$$\leq M\sigma + \varepsilon.$$

Since this holds for all $\varepsilon > 0$, the claim follows. $\square$

LEMMA 4.4. *Let $\xi \in \overline{\mathbb{C}}_0$, $\mathfrak{v} \in \mathbb{C}^m$, $u \in L_b(\mathbb{R}_+, \mathbb{C}^m)$ and let G be given by (4.1), where μ is a $\mathbb{C}^{p\times m}$-valued Borel measure on $\mathbb{R}_+$.*

1. *If $\lim_{t\to\infty}(u(t) - e^{\xi t}\mathfrak{v}) = 0$, then*

$$\lim_{t\to\infty}[(Gu)(t) - \mathbf{G}(\xi)e^{\xi t}\mathfrak{v}] = 0.$$

2. *If there exist $\alpha < 0$ and $M \geq 0$ such that*

$$\int_0^\infty e^{-\alpha s} |\mu|(ds) < \infty \quad and \quad \|u(t) - e^{\xi t}\mathfrak{v}\| \leq Me^{\alpha t} \quad \forall t \geq 0,$$

then there exists $L \geq 0$ such that

$$\|(Gu)(t) - \mathbf{G}(\xi)e^{\xi t}\mathfrak{v}\| \leq Le^{\alpha t} \quad \forall t \geq 0.$$

Proof. Define $v : \mathbb{R}_+ \to \mathbb{C}^m$ by $v(t) := e^{\xi t}\mathfrak{v}$. By (4.1) and (4.2), using $\xi \in \overline{\mathbb{C}}_0$, we have

$$\|(Gu)(t) - \mathbf{G}(\xi)e^{\xi t}\mathfrak{v}\| = \left\|\int_0^t \mu(ds)u(t-s) - \int_0^\infty e^{\xi(t-s)}\mu(ds)\mathfrak{v}\right\|$$

$$\leq \left\|\int_0^t \mu(ds)(u(t-s) - e^{\xi(t-s)}\mathfrak{v})\right\| + \|\mathfrak{v}\| \int_t^\infty |e^{\xi(t-s)}| |\mu|(ds)$$

(4.3) $$\leq \|(G(u-v))(t)\| + \|\mathfrak{v}\| \int_t^\infty |\mu|(ds) \quad \forall t \geq 0.$$

By hypothesis, $\lim_{t\to\infty} \|u(t) - v(t)\| = 0$, and so, by Lemma 4.3,

(4.4) $$\lim_{t\to\infty} \|G(u-v)(t)\| = 0.$$

Moreover, it follows from Proposition 4.2 that $\lim_{t\to\infty} \int_t^\infty |\mu|(ds) = 0$. Hence, invoking (4.3) and (4.4) completes the proof of statement 1.

To prove statement 2, assume that there exist $\alpha < 0$ and $M \geq 0$ such that $M_1 := \int_0^\infty e^{-\alpha s} |\mu|(ds) < \infty$ and $\|u(t) - e^{\xi t}\mathfrak{v}\| \leq Me^{\alpha t}$ for all $t \geq 0$. Since $\alpha < 0$, it

follows from (4.3) that

$$e^{-\alpha t}\|(Gu)(t) - \mathbf{G}(\xi)e^{\xi t}\mathfrak{v}\| \le e^{-\alpha t}\int_0^t \|(u-v)(t-s)\|\,|\mu|(ds) + \|\mathfrak{v}\|e^{-\alpha t}\int_t^\infty |\mu|(ds)$$

$$\le M\int_0^t e^{-\alpha s}|\mu|(ds) + \|\mathfrak{v}\|\int_t^\infty e^{-\alpha s}|\mu|(ds)$$

$$\le MM_1 + \|\mathfrak{v}\|M_1 \quad \forall t \ge 0\,.$$

Hence $\|(Gu)(t) - \mathbf{G}(\xi)e^{\xi t}\mathfrak{v}\| \le M_1(M + \|\mathfrak{v}\|)e^{\alpha t}$ for all $t \ge 0$.   □

DEFINITION 4.5. *Let $\tau > 0$ denote the sampling period and let $F(\mathbb{R}_+, \mathbb{C}^m)$ denote the space of all $\mathbb{C}^m$-valued functions defined on $\mathbb{R}_+$. We define the* ideal sampling *operator $\mathcal{S}_\tau : F(\mathbb{R}_+, \mathbb{C}^m) \to F(\mathbb{Z}_+, \mathbb{C}^m)$ by*

$$(\mathcal{S}_\tau u)(k) := u(k\tau) \quad \forall k \in \mathbb{Z}_+\,.$$

The (zero-order) hold operator *$\mathcal{H}_\tau : F(\mathbb{Z}_+, \mathbb{C}^m) \to F(\mathbb{R}_+, \mathbb{C}^m)$ is defined by*

$$(\mathcal{H}_\tau v)(t) := v(k) \quad \forall t \in [k\tau, (k+1)\tau)\,.$$

Define the sample-hold discretization G_τ of G by

$$(4.5) \qquad\qquad\qquad\qquad G_\tau := \mathcal{S}_\tau G \mathcal{H}_\tau$$

and define $g_\tau \in F(\mathbb{Z}_+, \mathbb{C}^{p\times m})$ by

$$(4.6) \qquad g_\tau(k) := \mu(E_k), \quad \text{where} \quad E_k := \begin{cases} \{0\}, & k = 0, \\ ((k-1)\tau, k\tau], & k \in \mathbb{N}\,. \end{cases}$$

PROPOSITION 4.6. *Assume that G is given by (4.1) and g_τ is defined by (4.6), where μ is a $\mathbb{C}^{p\times m}$-valued Borel measure on $\mathbb{R}_+$. Then g_τ is in $\ell^1(\mathbb{Z}_+, \mathbb{C}^{p\times m})$ and the operator G_τ defined by (4.5) satisfies*

$$G_\tau v = g_\tau \star v \quad \forall v \in F(\mathbb{Z}_+, \mathbb{C}^m)\,.$$

Consequently, $G_\tau \in \mathcal{B}(\ell^q(\mathbb{Z}_+, \mathbb{C}^m), \ell^q(\mathbb{Z}_+, \mathbb{C}^p))$ for $1 \le q \le \infty$.

Proof. Clearly,

$$\sum_{k=0}^\infty \|g_\tau(k)\| = \sum_{k=0}^\infty \|\mu(E_k)\| \le \sum_{k=0}^\infty |\mu|(E_k) = |\mu|(\mathbb{R}_+) < \infty\,,$$

showing that $g_\tau \in \ell^1(\mathbb{Z}_+, \mathbb{C}^{p\times m})$. For any discrete-time input $v \in F(\mathbb{Z}_+, \mathbb{C}^m)$, we have

$$(G_\tau v)(k) = ((\mathcal{S}_\tau G \mathcal{H}_\tau)v)(k) = (G(\mathcal{H}_\tau v))(k\tau) = \int_0^{k\tau} \mu(ds)(\mathcal{H}_\tau v)(k\tau - s)$$

$$= \sum_{j=0}^k \int_{E_j} \mu(ds)v(k-j) = \sum_{j=0}^k g_\tau(k)v(k-j) = (g_\tau \star v)(k) \quad \forall k \in \mathbb{Z}_+\,.$$

Hence $G_\tau \in \mathcal{B}(\ell^q(\mathbb{Z}_+, \mathbb{C}^m), \ell^q(\mathbb{Z}_+, \mathbb{C}^p))$ for $1 \le q \le \infty$.   □

Let $\mathbf{G}_\tau$ denote the transfer function of G_τ. Note that, since $g_\tau \in \ell^1(\mathbb{Z}_+, \mathbb{C}^{p\times m})$, $\mathbf{G}_\tau(z)$ is well defined for $z \in \overline{\mathbb{E}}_1$.

Remark 4.7. Let $\alpha < 0$, assume that $\int_0^\infty e^{-\alpha t}|\mu|(dt) < \infty$, and set $\rho := e^{\alpha\tau} \in (0,1)$. Then

$$\sum_{k=0}^\infty \|g_\tau(k)\|\rho^{-k} \leq e^{-\alpha\tau} \sum_{k=0}^\infty \int_{E_k} e^{-\alpha t}|\mu|(dt) = e^{-\alpha\tau} \int_0^\infty e^{-\alpha t}|\mu|(dt) < \infty\,,$$

so that $g_\tau \in \ell^1_\rho(\mathbb{Z}_+, \mathbb{C}^{p\times m})$, or, equivalently, $\mathbf{G}_\tau \in \hat{\ell}^1_\rho(\mathbb{C}^{p\times m}) \subset H^\infty(\mathbb{E}_\rho, \mathbb{C}^{p\times m})$.

LEMMA 4.8. *Let* $\xi \in \overline{\mathbb{C}}_0$. *Then* $\lim_{\tau\to 0} \mathbf{G}_\tau(e^{\xi\tau}) = \mathbf{G}(\xi)$.

Proof. Clearly,

$$\mathbf{G}_\tau(e^{\xi\tau}) = \sum_{k=0}^\infty g_\tau(k)e^{-\xi\tau k} = \sum_{k=0}^\infty \mu(E_k)e^{-\xi\tau k} = \sum_{k=0}^\infty \int_{E_k} e^{-\xi\tau k}\mu(dt)$$

and

$$\mathbf{G}(\xi) = \int_{\mathbb{R}_+} e^{-\xi t}\mu(dt) = \sum_{k=0}^\infty \int_{E_k} e^{-\xi t}\mu(dt)\,,$$

so that

$$\|\mathbf{G}_\tau(e^{\xi\tau}) - \mathbf{G}(\xi)\| = \left\|\sum_{k=0}^\infty \int_{E_k} (e^{-\xi\tau k} - e^{-\xi t})\mu(dt)\right\| \leq \sum_{k=0}^\infty \int_{E_k} |e^{-\xi\tau k} - e^{-\xi t}||\mu|(dt)\,.$$

Using the fact that $\xi \in \overline{\mathbb{C}}_0$, we obtain

$$\|\mathbf{G}_\tau(e^{\xi\tau}) - \mathbf{G}(\xi)\| \leq \sum_{k=0}^\infty \int_{E_k} |1 - e^{-\xi(t-\tau k)}||\mu|(dt) \leq \sup_{t\in[0,\tau]} |1 - e^{\xi t}||\mu|(\mathbb{R}_+)\,.$$

Since $\lim_{\tau\to 0} \sup_{t\in[0,\tau]} |1 - e^{\xi t}| = 0$ and $|\mu|(\mathbb{R}_+)$ is finite, the claim follows. $\square$

Remark 4.9. The convergence of $\mathbf{G}_\tau(e^{\xi\tau})$ to $\mathbf{G}(\xi)$ as $\tau \to 0$ is uniform for all $\xi \in U$ if $U \subset \overline{\mathbb{C}}_0$ is compact. Moreover, it is obvious that $\mathbf{G}_\tau(1) = \mathbf{G}(0)$ for all $\tau > 0$.

The following theorem is the main result of this section.

THEOREM 4.10. *Let* $N \in \mathbb{N}$ *and* $\xi_j \in i\mathbb{R}$ *for all* $j \in \underline{N}$ *with* $\xi_j \neq \xi_k$ *for* $j \neq k$. *Let* G *be given by* (4.1), *where* μ *is a* $\mathbb{C}^{p\times m}$*-valued Borel measure on* $\mathbb{R}_+$ *such that* $\int_0^\infty e^{-\alpha t}|\mu|(dt) < \infty$ *for some* $\alpha < 0$. *Let the discrete-time controller* $K_{\tau,\varepsilon}$ *be such that its transfer function* $\mathbf{K}_{\tau,\varepsilon}$ *is given by*

$$(4.7) \qquad \mathbf{K}_{\tau,\varepsilon}(z) = \varepsilon\left(\mathbf{K}^0(z) + \sum_{j=1}^N \frac{K_j}{z - e^{\xi_j\tau}}\right),$$

where $\mathbf{K}^0 \in \hat{\ell}^1(\mathbb{C}^{m\times p})$ *and* $K_j \in \mathbb{C}^{m\times p}$. *Assume that*

$$(4.8) \qquad \sigma(\mathbf{G}(\xi_j)K_j) \subset \mathbb{C}_0 \quad \forall j \in \underline{N}\,.$$

The following statements hold for the output y *of the sampled-data system shown in Figure* 4.1:
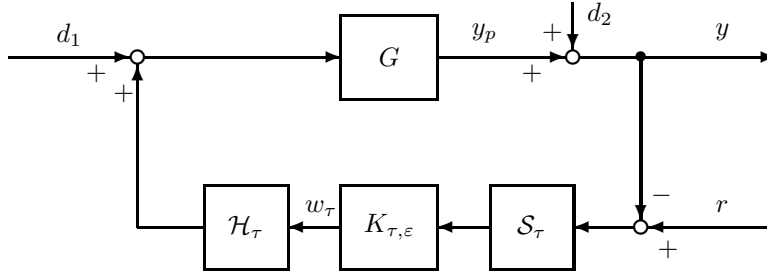
FIG. 4.1. *Sampled-data low-gain control.*

1. *There exists $\tau^* > 0$ such that, for every sampling period $\tau \in (0, \tau^*)$, there exists $\varepsilon_\tau > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\tau)$, the feedback system is L^∞-stable, in the sense that there exists $N_1 \geq 0$ such that, for all $r, d_2 \in L_b(\mathbb{R}_+, \mathbb{C}^p)$ and all $d_1 \in L_b(\mathbb{R}_+, \mathbb{C}^m)$,*

$$\|y\|_\infty \leq N_1(\|r\|_\infty + \|d_1\|_\infty + \|d_2\|_\infty) \,.$$

If r is given by

$$(4.9) \qquad r(t) := \sum_{j=1}^{N} e^{\xi_j t} \mathfrak{r}_j \ \ \forall t \geq 0 \ \ where \ \mathfrak{r}_j \in \mathbb{C}^p \,,$$

and $d_1 \in L_b(\mathbb{R}_+, \mathbb{C}^m)$, $d_2 \in L_b(\mathbb{R}_+, \mathbb{C}^p)$ satisfy

$$(4.10) \quad \lim_{t \to \infty} \left(d_1(t) - \sum_{j=1}^{N} e^{\xi_j t} \mathfrak{d}_{1j} \right) = 0 \,, \ \lim_{t \to \infty} \left(d_2(t) - \sum_{j=1}^{N} e^{\xi_j t} \mathfrak{d}_{2j} \right) = 0 \,,$$

$$where \ \mathfrak{d}_{1j} \in \mathbb{C}^m, \ \mathfrak{d}_{2j} \in \mathbb{C}^p,$$

then, for every $\delta > 0$, there exists $\tau_\delta \in (0, \tau^*)$ such that, for every sampling period $\tau \in (0, \tau_\delta)$ and every $\varepsilon \in (0, \varepsilon_\tau)$,

$$(4.11) \qquad \limsup_{t \to \infty} \|y(t) - r(t)\| \leq \delta \,.$$

2. *Under the additional assumptions that $\mathbf{K}^0 \in H^\infty_{\lessgtr}(\mathbb{E}_1, \mathbb{C}^{m \times p})$ and that there exist $\gamma \in (\alpha, 0)$ and $N_2 \geq 0$ such that*

(4.12)

$$\left\| d_1(t) - \sum_{j=1}^{N} e^{\xi_j t} \mathfrak{d}_{1j} \right\| \leq N_2 e^{\gamma t} \,, \quad \left\| d_2(t) - \sum_{j=1}^{N} e^{\xi_j t} \mathfrak{d}_{2j} \right\| \leq N_2 e^{\gamma t} \quad \forall t \geq 0 \,,$$

(4.11) can be replaced by

$$\|y(t) - r(t)\| \leq \delta + N_3 e^{\beta t} \quad \forall t \geq 0$$

for suitable $\beta \in (\gamma, 0)$ and $N_3 \geq 0$ (both depending on τ and ε).

Only very little plant information is required in order to apply Theorem 4.10, namely, stability of the system to be controlled and some information on $\mathbf{G}(\xi_j)$, where the latter is required for the computation of K_j such that (4.8) holds. The spectral condition (4.8) is robust with respect to "sufficiently small" plant perturbations.

Proof of Theorem 4.10. To prove statement 1, set $\tau_0 := 2\pi/\sup\{|\xi_j - \xi_k| : j, k \in \underline{N}, j \neq k\}$ and note that if $\tau \in (0, \tau_0)$, then $e^{\xi_j \tau} \neq e^{\xi_k \tau}$ for all $j, k \in \underline{N}$, $j \neq k$. It follows from Lemma 4.8 that

$$\lim_{\tau \to 0} e^{\bar{\xi}_j \tau} \mathbf{G}_\tau(e^{\xi_j \tau}) K_j = \mathbf{G}(\xi_j) K_j \quad \forall j \in \underline{N}.$$

Hence, by hypothesis (4.8), there exists $\tau^* \in (0, \tau_0)$ such that

(4.13) $$\sigma(e^{\bar{\xi}_j \tau} \mathbf{G}_\tau(e^{\xi_j \tau}) K_j) \subset \mathbb{C}_0 \quad \forall j \in \underline{N}, \ \forall \tau \in (0, \tau^*).$$

By assumption, there exists $\alpha < 0$ such that $\int_0^\infty e^{-\alpha t} |\mu|(dt) < \infty$. Therefore, by Remark 4.7, $\mathbf{G}_\tau \in H_<^\infty(\mathbb{E}_1, \mathbb{C}^{p \times m})$. Clearly,

$$\limsup_{z \to e^{\xi_j \tau}, \, z \in \mathbb{E}_1} \left\| \frac{\mathbf{G}_\tau(z) - \mathbf{G}_\tau(e^{\xi_j \tau})}{z - e^{\xi_j \tau}} \right\| < \infty$$

holds for every $j \in \underline{N}$. Moreover, by assumption, $\mathbf{K}^0 \in \hat{\ell}^1(\mathbb{C}^{m \times p})$. It follows from Theorem 3.1 that, for every $\tau \in (0, \tau^*)$, there exists $\varepsilon_\tau > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\tau)$,

$$\mathbf{K}_{\tau,\varepsilon}(I + \mathbf{G}_\tau \mathbf{K}_{\tau,\varepsilon})^{-1} \in \hat{\ell}^1(\mathbb{C}^{m \times p}).$$

Consequently, for all such τ and ε, the convolution operator $K_{\tau,\varepsilon}(I + G_\tau K_{\tau,\varepsilon})^{-1}$ has impulse response in $\ell^1(\mathbb{Z}_+, \mathbb{C}^{m \times p})$.

In the following, let $\tau \in (0, \tau^*)$ and $\varepsilon \in (0, \varepsilon_\tau)$. Set

(4.14) $$M := |\mu|(\mathbb{R}_+) \quad \text{and} \quad M_1 := \|K_{\tau,\varepsilon}(I + G_\tau K_{\tau,\varepsilon})^{-1}\|.$$

Let $d_1 \in L_b(\mathbb{R}_+, \mathbb{C}^m)$ and $d_2, r \in L_b(\mathbb{R}_+, \mathbb{C}^p)$. It is well known that $\|Gd_1\|_\infty \leq M\|d_1\|_\infty$. Furthermore, set

(4.15) $$d := Gd_1 + d_2.$$

Trivially,

(4.16) $$\|\mathcal{S}_\tau d\|_{\ell^\infty} \leq \|d\|_\infty \leq M\|d_1\|_\infty + \|d_2\|_\infty \quad \text{and} \quad \|\mathcal{S}_\tau r\|_{\ell^\infty} \leq \|r\|_\infty.$$

The discrete-time signal w_τ in Figure 4.1 is given by

$$w_\tau = K_{\tau,\varepsilon} \mathcal{S}_\tau [r - (G\mathcal{H}_\tau w_\tau + d)] = K_{\tau,\varepsilon}[\mathcal{S}_\tau r - (G_\tau w_\tau + \mathcal{S}_\tau d)].$$

It follows that

(4.17) $$w_\tau = K_{\tau,\varepsilon}(I + G_\tau K_{\tau,\varepsilon})^{-1}(\mathcal{S}_\tau r - \mathcal{S}_\tau d).$$

Invoking (4.14) and (4.16), we have

(4.18) $$\|w_\tau\|_{\ell^\infty} \leq M_1(\|\mathcal{S}_\tau r\|_{\ell^\infty} + \|\mathcal{S}_\tau d\|_{\ell^\infty}) \leq M_1(\|r\|_\infty + M\|d_1\|_\infty + \|d_2\|_\infty).$$

Clearly, the continuous-time signal y in Figure 4.1 satisfies

(4.19) $$y = G\mathcal{H}_\tau w_\tau + Gd_1 + d_2 = G\mathcal{H}_\tau w_\tau + d.$$

Since $\|\mathcal{H}_\tau w_\tau\|_\infty = \|w_\tau\|_{\ell^\infty}$, it follows from (4.18) and (4.19) that

$$\begin{aligned}
\|y\|_\infty &\le \|G\mathcal{H}_\tau w_\tau\|_\infty + \|Gd_1\|_\infty + \|d_2\|_\infty \\
&\le M\|\mathcal{H}_\tau w_\tau\|_\infty + M\|d_1\|_\infty + \|d_2\|_\infty \\
&= M\|w_\tau\|_{\ell^\infty} + M\|d_1\|_\infty + \|d_2\|_\infty \\
&\le MM_1(\|r\|_\infty + M\|d_1\|_\infty + \|d_2\|_\infty) + M\|d_1\|_\infty + \|d_2\|_\infty \\
&\le N_1(\|r\|_\infty + \|d_1\|_\infty + \|d_2\|_\infty),
\end{aligned}$$

with $N_1 := (M+1)(MM_1+1)$. This completes the proof of the L^∞-stability of the feedback system.

To prove approximate tracking (see (4.11)), note that, by (4.13), $\mathbf{G}_\tau(e^{\xi_j \tau})K_j$ is invertible for every $j \in \underline{N}$ and every $\tau \in (0, \tau^*)$. In the following, we take limits as $z \to e^{\xi_j \tau}$ for $z \in \mathbb{E}_1$. It is assumed that $\tau \in (0, \tau^*)$ and $\varepsilon \in (0, \varepsilon_\tau)$. A straightforward calculation yields that

$$\lim_{z \to e^{\xi_j \tau}} (I + \mathbf{G}_\tau(z)\mathbf{K}_{\tau,\varepsilon}(z))^{-1} = 0 \quad \forall j \in \underline{N}$$

and

$$\lim_{z \to e^{\xi_j \tau}} \frac{1}{z - e^{\xi_j \tau}}(I + \mathbf{G}_\tau(z)\mathbf{K}_{\tau,\varepsilon}(z))^{-1} = (\varepsilon\mathbf{G}_\tau(e^{\xi_j \tau})K_j)^{-1} \quad \forall j \in \underline{N}.$$

Consequently,

$$\begin{aligned}
(\mathbf{K}_{\tau,\varepsilon}(I + \mathbf{G}_\tau\mathbf{K}_{\tau,\varepsilon})^{-1})(e^{\xi_j \tau}) &= \lim_{z \to e^{\xi_j \tau}} \varepsilon\mathbf{K}^0(z)(I + \mathbf{G}_\tau(z)\mathbf{K}_{\tau,\varepsilon}(z))^{-1} \\
&\quad + \lim_{z \to e^{\xi_j \tau}} \sum_{k=1}^N \left(\frac{\varepsilon K_k}{z - e^{\xi_k \tau}}(I + \mathbf{G}_\tau(z)\mathbf{K}_{\tau,\varepsilon}(z))^{-1} \right) \\
&= \lim_{z \to e^{\xi_j \tau}} \frac{\varepsilon K_j}{z - e^{\xi_j \tau}}(I + \mathbf{G}_\tau(z)\mathbf{K}_{\tau,\varepsilon}(z))^{-1} \\
&= K_j(\mathbf{G}_\tau(e^{\xi_j \tau})K_j)^{-1} \quad \forall j \in \underline{N}.
\end{aligned}$$

(4.20)

Setting

(4.21) $$\mathfrak{d}_j := \mathbf{G}(\xi_j)\mathfrak{d}_{1j} + \mathfrak{d}_{2j} \quad \forall j \in \underline{N},$$

it follows from the definition of d (see (4.15)) that

$$d(t) - \sum_{j=1}^N e^{\xi_j t}\mathfrak{d}_j = (Gd_1)(t) - \sum_{j=1}^N e^{\xi_j t}\mathbf{G}(\xi_j)\mathfrak{d}_{1j} + d_2(t) - \sum_{j=1}^N e^{\xi_j t}\mathfrak{d}_{2j}.$$

Invoking Lemma 4.4 and (4.10), we obtain that

(4.22) $$\lim_{t \to \infty} \left\| d(t) - \sum_{j=1}^N e^{\xi_j t}\mathfrak{d}_j \right\| = 0.$$

It follows trivially from (4.9) and (4.22) that

$$(4.23) \quad (\mathcal{S}_\tau r)(k) = \sum_{j=1}^{N} e^{\xi_j k\tau} \mathfrak{r}_j \ \forall k \in \mathbb{Z}_+ \ \text{ and } \ \lim_{k\to\infty} \left\| (\mathcal{S}_\tau d)(k) - \sum_{j=1}^{N} e^{\xi_j k\tau} \mathfrak{d}_j \right\| = 0.$$

Define $a_\tau, b_\tau \in F(\mathbb{Z}_+, \mathbb{C}^m)$ by

(4.24)

$$a_\tau(k) := \sum_{j=1}^{N} e^{\xi_j \tau k} K_j (\mathbf{G}_\tau(e^{\xi_j \tau}) K_j)^{-1} \mathfrak{r}_j, \quad b_\tau(k) := \sum_{j=1}^{N} e^{\xi_j \tau k} K_j (\mathbf{G}_\tau(e^{\xi_j \tau}) K_j)^{-1} \mathfrak{d}_j.$$

It follows from Lemma 2.6, (4.17), (4.20), and (4.23) that

$$(4.25) \qquad\qquad \lim_{k\to\infty} [w_\tau(k) - a_\tau(k) + b_\tau(k)] = 0.$$

By (4.8), $\mathbf{G}(\xi_j) K_j$ is invertible for every $j \in \underline{N}$. Define $v_1, v_2 : \mathbb{R}_+ \to \mathbb{C}^m$ by

$$(4.26) \quad v_1(t) := \sum_{j=1}^{N} e^{\xi_j t} K_j (\mathbf{G}(\xi_j) K_j)^{-1} \mathfrak{r}_j, \quad v_2(t) := \sum_{j=1}^{N} e^{\xi_j t} K_j (\mathbf{G}(\xi_j) K_j)^{-1} \mathfrak{d}_j.$$

We conclude from Lemma 4.4 and (4.9) that

$$(4.27) \ \lim_{t\to\infty} [(Gv_1)(t) - r(t)] = \sum_{j=1}^{N} \lim_{t\to\infty} [(G(e^{\xi_j \cdot} K_j (\mathbf{G}(\xi_j) K_j)^{-1} \mathfrak{r}_j))(t) - e^{\xi_j t} \mathfrak{r}_j] = 0.$$

Furthermore, writing

$$(Gv_2)(t) - d(t) = \sum_{j=1}^{N} \left[(G(e^{\xi_j \cdot} K_j (\mathbf{G}(\xi_j) K_j)^{-1} \mathfrak{d}_j))(t) - e^{\xi_j t} \mathfrak{d}_j \right] + \sum_{j=1}^{N} e^{\xi_j t} \mathfrak{d}_j - d(t),$$

an application of Lemma 4.4, and (4.22) yields that

$$(4.28) \qquad\qquad \lim_{t\to\infty} [(Gv_2)(t) - d(t)] = 0.$$

Let $\delta > 0$. Invoking Lemma 4.8 and the fact that $\xi_j \in i\mathbb{R}$, there exists $\tau_\delta \in (0, \tau^*)$ such that if $\tau \in (0, \tau_\delta)$, then

$$\sup_{t \in [k\tau, (k+1)\tau)} \|v_1(t) - (\mathcal{H}_\tau a_\tau)(t)\|$$

$$= \sup_{t \in [k\tau, (k+1)\tau)} \left\| \sum_{j=1}^{N} e^{\xi_j t} K_j (\mathbf{G}(\xi_j) K_j)^{-1} \mathfrak{r}_j - \sum_{j=1}^{N} e^{\xi_j \tau k} K_j (\mathbf{G}_\tau(e^{\xi_j \tau}) K_j)^{-1} \mathfrak{r}_j \right\|$$

$$\leq \sup_{t \in [k\tau, (k+1)\tau)} \sum_{j=1}^{N} |e^{\xi_j(t-k\tau)} - 1| \|K_j (\mathbf{G}(\xi_j) K_j)^{-1} \mathfrak{r}_j\|$$

$$+ \sup_{t \in [k\tau, (k+1)\tau)} \sum_{j=1}^{N} (\|K_j\| \|(\mathbf{G}(\xi_j) K_j)^{-1} - (\mathbf{G}_\tau(e^{\xi_j \tau}) K_j)^{-1}\| \|\mathfrak{r}_j\|)$$

$$\leq \frac{\delta}{2M} \quad \forall k \in \mathbb{Z}_+,$$

and, similarly,

$$\sup_{t\in[k\tau,(k+1)\tau)} \|v_2(t) - (\mathcal{H}_\tau b_\tau)(t)\| \le \frac{\delta}{2M} \quad \forall k \in \mathbb{Z}_+ \,,$$

where M is defined in (4.14). Hence,

$$(4.29) \qquad \sup_{t\ge 0}\|v_1(t) - (\mathcal{H}_\tau a_\tau)(t)\| + \sup_{t\ge 0}\|v_2(t) - (\mathcal{H}_\tau b_\tau)(t)\| \le \frac{\delta}{M}\,.$$

Let $\tau \in (0,\tau_\delta)$ and $\varepsilon \in (0,\varepsilon_\tau)$. Then, writing

$$\mathcal{H}_\tau w_\tau - v_1 + v_2 = \mathcal{H}_\tau(w_\tau - a_\tau + b_\tau) + (\mathcal{H}_\tau a_\tau - v_1) + (v_2 - \mathcal{H}_\tau b_\tau)$$

and invoking (4.25) and (4.29), we obtain

$$(4.30) \qquad \limsup_{t\to\infty}\|(\mathcal{H}_\tau w_\tau)(t) - v_1(t) + v_2(t)\| \le \frac{\delta}{M}\,.$$

By (4.19),

$$y - r = G(\mathcal{H}_\tau w_\tau - v_1 + v_2) + (d - Gv_2) + (Gv_1 - r)\,,$$

so that it follows from (4.27) and (4.28) that

$$\limsup_{t\to\infty}\|y(t) - r(t)\| \le \limsup_{t\to\infty}\|(G(\mathcal{H}_\tau w_\tau - v_1 + v_2))(t)\|\,.$$

Finally, $\mathcal{H}_\tau w_\tau - v_1 + v_2$ is bounded and thus, by Lemma 4.3 and (4.30),

$$\limsup_{t\to\infty}\|y(t) - r(t)\| \le M \limsup_{t\to\infty}\|(\mathcal{H}_\tau w_\tau)(t) - v_1(t) + v_2(t)\| \le \delta\,,$$

completing the proof of statement 1.

To prove statement 2 of Theorem 4.10, let $\tau \in (0,\tau_\delta)$ and $\varepsilon \in (0,\varepsilon_\tau)$. Assume that $\mathbf{K}^0 \in H_{\not\gtrless}^\infty(\mathbb{E}_1, \mathbb{C}^{m\times p})$ and that there exist $N_2 \ge 0$ and $\gamma \in (\alpha, 0)$ such that (4.12) holds. Invoking Remark 4.7, we conclude that $\mathbf{G}_\tau \in H_{\not\gtrless}^\infty(\mathbb{E}_1, \mathbb{C}^{p\times m})$. Therefore, by Theorem 3.4, $\mathbf{K}_{\tau,\varepsilon}(I + \mathbf{G}_\tau \mathbf{K}_{\tau,\varepsilon})^{-1} \in H_{\not\gtrless}^\infty(\mathbb{E}_1, \mathbb{C}^{m\times p})$. Hence, by Proposition 2.7, there exists $\rho \in (e^{\gamma\tau}, 1)$ such that

$$\mathbf{K}_{\tau,\varepsilon}(I + \mathbf{G}_\tau \mathbf{K}_{\tau,\varepsilon})^{-1} \in \hat{\ell}_\rho^1(\mathbb{C}^{m\times p})\,.$$

By Lemma 4.4 and (4.12), there exists $M_2 \ge 0$ such that

$$\left\|(Gd_1)(t) - \sum_{j=1}^N e^{\xi_j t}\mathbf{G}(\xi_j)\mathfrak{d}_{1j}\right\| \le M_2 e^{\gamma t} \quad \forall t \ge 0\,.$$

Invoking (4.12), it follows that

$$\left\|d(t) - \sum_{j=1}^N e^{\xi_j t}\mathfrak{d}_j\right\| \le \left\|(Gd_1)(t) - \sum_{j=1}^N e^{\xi_j t}\mathbf{G}(\xi_j)\mathfrak{d}_{1j}\right\| + \left\|d_2(t) - \sum_{j=1}^N e^{\xi_j t}\mathfrak{d}_{2j}\right\|$$

$$(4.31) \qquad\qquad\qquad \le (M_2 + N_2)e^{\gamma t} \quad \forall t \ge 0\,,$$

where d and $\mathfrak{d}_j$ are defined in (4.15) and (4.21), respectively. Trivially,

$$\left\| (\mathcal{S}_\tau d)(k) - \sum_{j=1}^{N} e^{\xi_j k \tau} \mathfrak{d}_j \right\| \le (M_2 + N_2)(e^{\gamma \tau})^k \le (M_2 + N_2)\rho^k \quad \forall k \in \mathbb{Z}_+ \,.$$

It follows from (4.20) and Lemma 2.6 that there exists $M_3 \ge 0$ such that

$$(4.32) \qquad \|w_\tau(k) - a_\tau(k) + b_\tau(k)\| \le M_3 \rho^k \quad \forall k \in \mathbb{Z}_+ \,,$$

where w_τ and a_τ, b_τ are defined in (4.17) and (4.24), respectively. We conclude from Lemma 4.4, (4.9), and (4.31) that there exists $M_4 \ge 0$ such that

$$(4.33) \qquad \|(Gv_1)(t) - r(t)\| \le M_4 e^{\gamma t} \,, \quad \|(Gv_2)(t) - d(t)\| \le M_4 e^{\gamma t} \,; \quad \forall t \ge 0 \,,$$

where v_1 and v_2 are defined in (4.26). Since $\rho \in (0,1)$, we have

$$\rho^k \le \rho^{-1} \rho^{(k\tau + \theta)/\tau} = \rho^{-1} e^{\beta(k\tau + \theta)} \quad \forall \theta \in [0,\tau) \,, \; \forall k \in \mathbb{Z}_+ \,,$$

where $\beta := (\ln \rho)/\tau$. Consequently, by (4.32) and (4.29),

$$\|(\mathcal{H}_\tau w_\tau)(t) - v_1(t) + v_2(t)\| \le \|(\mathcal{H}_\tau w_\tau - \mathcal{H}_\tau a_\tau + \mathcal{H}_\tau b_\tau)(t)\|$$
$$+ \|(\mathcal{H}_\tau a_\tau)(t) - v_1(t)\| + \|v_2(t) - (\mathcal{H}_\tau b_\tau)(t)\|$$
$$\le M_3 \rho^{-1} e^{\beta t} + \frac{\delta}{M} \quad \forall t \ge 0 \,.$$

Since $\rho \in (e^{\gamma \tau}, 1)$, we have that $\beta \in (\gamma, 0) \subset (\alpha, 0)$, and hence

$$\|(G(\mathcal{H}_\tau w_\tau - v_1 + v_2))(t)\| \le \int_0^t \|(\mathcal{H}_\tau w_\tau - v_1 + v_2)(t - s)\| |\mu|(ds)$$
$$\le \int_0^t M_3 \rho^{-1} e^{\beta(t-s)} |\mu|(ds) + \frac{\delta}{M} \int_0^\infty |\mu|(ds)$$
$$\le M_3 \rho^{-1} e^{\beta t} \int_0^\infty e^{-\beta s} |\mu|(ds) + \delta$$
$$\le M_3 M_5 \rho^{-1} e^{\beta t} + \delta \quad \forall t \ge 0 \,,$$

where $M_5 := \int_0^\infty e^{-\beta s} |\mu|(ds) \le \int_0^\infty e^{-\alpha s} |\mu|(ds) < \infty$. Therefore, by (4.19) and (4.33), it follows that

$$\|y(t) - r(t)\| \le \|(G(\mathcal{H}_\tau w_\tau - v_1 + v_2))(t)\| + \|(d(t) - Gv_2(t))\| + \|(Gv_1)(t) - r(t)\|$$
$$\le \|(G(\mathcal{H}_\tau w_\tau - v_1 + v_2))(t)\| + 2M_4 e^{\gamma t}$$
$$\le \delta + (M_3 M_5 \rho^{-1} + 2M_4) e^{\beta t} \quad \forall t \ge 0 \,.$$

This completes the proof. $\quad \square$

Remark 4.11. The proof of Theorem 4.10 shows that, for fixed $\{\xi_j : j \in \underline{N}\}$, τ_δ and ε_τ can be chosen to be uniform for all signals r, d_1, and d_2 with $\mathfrak{r}_j$, $\mathfrak{d}_{1j}$, and $\mathfrak{d}_{2j}$, $j \in \underline{N}$, satisfying a prespecified bound.

Application to state-space systems. In the following, we apply the input-output results in this paper to a class of infinite-dimensional state-space systems.

Let X be a Hilbert space and assume that the plant is given by

$$(4.34a) \qquad \dot{x}_p(t) = Ax_p(t) + Bu_p(t); \quad x_p(0) = x_p^0 \in X,$$

$$(4.34b) \qquad y_p(t) = Cx_p(t) + Du_p(t),$$

where $A : D(A) \to X$ is the generator of a strongly continuous semigroup $\mathbf{T}(t)$ on X, $B \in \mathcal{B}(\mathbb{C}^m, X_{-1})$ is the control operator, $C \in \mathcal{B}(X, \mathbb{C}^p)$ is the (bounded) observation operator, and $D \in \mathbb{C}^{p \times m}$ is the feedthrough matrix. Here X_{-1} is the completion of X with respect to the norm $\|x\|_{-1} := \|(\beta I - A)^{-1}x\|_X$, where β is in the resolvent set A. It is known that X_{-1} does not depend on the choice of β. Moreover, $X \hookrightarrow X_{-1}$ and $\mathbf{T}(t)$ extends to a C_0-semigroup on X_{-1}. The generator of the extended semigroup is a bounded operator from X to X_{-1} which extends A. The extended semigroup and its generator will be denoted by the same symbols $\mathbf{T}(t)$ and A, respectively. We assume that B is admissible for $\mathbf{T}(t)$, that is, for every $t \geq 0$, there exists $b_t \geq$ such that

$$\left\| \int_0^t \mathbf{T}(t-s)Bu(s) \right\|_X \leq b_t \|u\|_{L^2} \quad \forall u \in L^2([0,t], \mathbb{C}^m).$$

The admissibility assumption implies, in particular, that system (4.34) is regular (see [15, 18] for more details on admissible control operators and regular systems). For $u_p \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^m)$, the mild solution x_p of (4.34a), given by

$$(4.35) \qquad x_p(t) = \mathbf{T}(t)x_p^0 + \int_0^t \mathbf{T}(t-\sigma)Bu_p(\sigma)d\sigma,$$

is a continuous X-valued function, satisfying the differential equation (4.34a) in X_{-1} for almost every $t \in \mathbb{R}_+$. The transfer function $\mathbf{G}$ of (4.34) is given by

$$\mathbf{G}(s) = C(sI - A)^{-1}B + D \quad \forall s \in \mathbb{C}_{\omega(\mathbf{T})},$$

where

$$\omega(\mathbf{T}) := \lim_{t \to \infty} \frac{1}{t} \ln \|\mathbf{T}(t)\|.$$

We say that (4.34) is *exponentially stable* if $\omega(\mathbf{T}) < 0$. Let $\mathbf{K}^0 \in \mathcal{R}_s^{m \times p}$ and let $(A_0, B_0, C_0, D_0) \in \mathbb{C}^{n_0 \times n_0} \times \mathbb{C}^{n_0 \times p} \times \mathbb{C}^{m \times n_0} \times \mathbb{C}^{m \times p}$ be a stabilizable and detectable realization of $\mathbf{K}^0$; i.e., $\mathbf{K}^0(z) = C_0(zI - A_0)^{-1}B_0 + D_0$, (A_0, B_0) is stabilizable, and (C_0, A_0) is detectable. Since $\mathbf{K}^0$ is ℓ^2-stable, it follows that A_0 is power stable. Let $A_c \in \mathbb{C}^{(Np+n_0) \times (Np+n_0)}$, $B_c \in \mathbb{C}^{(Np+n_0) \times p}$, $C_c \in \mathbb{C}^{m \times (Np+n_0)}$, and $D_c \in \mathbb{C}^{m \times p}$ be given by (3.19) with $\lambda_j = e^{\xi_j \tau}$, $\xi_j \in i\mathbb{R}$ for $j \in \underline{N}$. We define the controller by

$$(4.36a) \qquad x_c(k+1) = A_c x_c(k) + B_c u_c(k); \quad x_c(0) = x_c^0 \in \mathbb{C}^{Np+n_0},$$

$$(4.36b) \qquad y_c(k) = \varepsilon C_c x_c(k) + \varepsilon D_c u_c(k).$$

The transfer function $\mathbf{K}_{\tau,\varepsilon}$ of (4.36) is given by

$$\mathbf{K}_{\tau,\varepsilon}(z) = \varepsilon \left(\mathbf{K}^0(z) + \sum_{j=1}^N \frac{K_j}{z - e^{\xi_j \tau}} \right).$$

We consider the following feedback interconnection of (4.34) and (4.36):

(4.37) $$u_p = \mathcal{H}_\tau y_c + d_1 \,, \quad y = y_p + d_2 \,, \quad u_c = \mathcal{S}_\tau (r - y) \,,$$

where r is a reference signal and d_1 and d_2 are disturbance signals.

THEOREM 4.12. *Consider the sampled-data state-space system given by* (4.34), (4.36), *and* (4.37). *Assume that* (4.34) *is exponentially stable and* $\sigma(\mathbf{G}(\xi_j)K_j) \subset \mathbb{C}_0$ *for all* $j = \underline{N}$. *The following statements hold:*

1. *There exists* $\tau^* > 0$ *such that, for every sampling period* $\tau \in (0, \tau^*)$, *there exists* $\varepsilon_\tau > 0$ *such that if* $\varepsilon \in (0, \varepsilon_\tau)$, *then the sampled-data system is exponentially stable; i.e., for every* $\varepsilon \in (0, \varepsilon_\tau)$, *there exist* $N_1 \geq 0$ *and* $\beta < 0$ *such that*

$$\left\| \begin{pmatrix} x_p(k\tau + \theta) \\ x_c(k) \end{pmatrix} \right\| \leq N_1 \left(e^{\beta(k\tau + \theta)} \left\| \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} \right\| + \|r\|_\infty + \|d_1\|_\infty + \|d_2\|_\infty \right)$$

$$\forall \theta \in [0, \tau) \,, \ \forall k \in \mathbb{Z}_+ \,, \ \forall x_p^0 \in X \,, \ \forall x_c^0 \in \mathbb{C}^{Np + n_0} \,,$$

$$\forall r, d_2 \in L_b(\mathbb{R}_+, \mathbb{C}^p) \,, \ \forall d_1 \in L_b(\mathbb{R}_+, \mathbb{C}^m) \,.$$

2. *If* r *is of the form* (4.9) *and* $d_1 \in L_b(\mathbb{R}_+, \mathbb{C}^m)$, $d_2 \in L_b(\mathbb{R}_+, \mathbb{C}^p)$ *satisfy* (4.10), *then, for every* $\delta > 0$, *there exists* $\tau_\delta > 0$ *such that, for every sampling period* $\tau \in (0, \tau_\delta)$, *there exists* $\varepsilon_\tau > 0$, *such that, for every* $\varepsilon \in (0, \varepsilon_\tau)$,

$$\limsup_{t \to \infty} \|y(t) - r(t)\| \leq \delta \quad \forall x_p^0 \in X \,, \ x_c^0 \in \mathbb{C}^{Np + n_0} \,.$$

Proof. The sample-hold discretization of (4.34) is given by the quadruple

(4.38) $$\left(\mathbf{T}(\tau) \,, \ \int_0^\tau \mathbf{T}(s) B ds \,, \ C \,, \ D \right) \,.$$

Clearly, since $\mathbf{T}(t)$ is exponentially stable, $\mathbf{T}(\tau)$ is power stable. Since admissibility of B for $\mathbf{T}(t)$ implies that $A^{-1}B \in \mathcal{B}(\mathbb{C}^m, X)$ and

$$\int_0^\tau \mathbf{T}(s) B v ds = (\mathbf{T}(\tau) - I) A^{-1} B v \quad \forall v \in \mathbb{C}^m \,,$$

we see that $\int_0^\tau \mathbf{T}(s) B ds \in \mathcal{B}(\mathbb{C}^m, X)$ for every $\tau > 0$. The transfer function of (4.38) is denoted by $\mathbf{G}_\tau$. By Lemma 4.8 and the assumption that $\sigma(\mathbf{G}(\xi_j)K_j) \subset \mathbb{C}_0$, there exists $\tau^* > 0$ such that if $\tau \in (0, \tau^*)$, then $e^{\xi_j \tau} \neq e^{\xi_k \tau}$ for all $j, k \in \underline{N}$, $j \neq k$, and

(4.39) $$\sigma(e^{\bar{\xi}_j \tau} \mathbf{G}_\tau(e^{\xi_j \tau}) K_j) \subset \mathbb{C}_0 \quad \forall j \in \underline{N} \,.$$

Define

$$E := (I + \varepsilon D_c D)^{-1} \,, \quad E_c := (I + \varepsilon D D_c)^{-1} \,,$$

and $\Delta : [0, \tau] \to \mathcal{B}(X \times \mathbb{C}^{Np + n_0})$ by

$$\Delta(\theta) := \begin{pmatrix} \mathbf{T}(\theta) & 0 \\ 0 & A_c \end{pmatrix} + \begin{pmatrix} \int_0^\theta \mathbf{T}(s) B ds & 0 \\ 0 & B_c \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & E_c \end{pmatrix} \begin{pmatrix} -\varepsilon D_c & \varepsilon I \\ -I & -\varepsilon D \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & C_c \end{pmatrix} \,.$$

For $\theta \in [0,\tau]$ and $k \in \mathbb{Z}_+$, define $R(k,\theta) : L_b(\mathbb{R}_+, \mathbb{C}^m) \times L_b(\mathbb{R}_+, \mathbb{C}^p) \times L_b(\mathbb{R}_+, \mathbb{C}^p) \to X \times \mathbb{C}^{Np+n_0}$ by

$$R(k,\theta)\begin{pmatrix} d_1 \\ d_2 \\ r \end{pmatrix} := \begin{pmatrix} \int_{k\tau}^{k\tau+\theta} \mathbf{T}(k\tau+\theta-s)Bd_1(s)ds + \varepsilon \int_0^\theta \mathbf{T}(s)Bdsf(k\tau; d_1, d_2, r) \\ B_c E_c[-Dd_1(k\tau) + r(k\tau) - d_2(k\tau)] \end{pmatrix},$$

where

$$f(k\tau; d_1, d_2, r) := -D_c DEd_1(k\tau) + ED_c[r(k\tau) - d_2(k\tau)].$$

By (4.35)–(4.37) and a routine calculation, we obtain

(4.40) $\qquad \begin{pmatrix} x_p(k\tau+\theta) \\ x_c(k+1) \end{pmatrix} = \Delta(\theta)\begin{pmatrix} x_p(k\tau) \\ x_c(k) \end{pmatrix} + R(k,\theta)\begin{pmatrix} d_1 \\ d_2 \\ r \end{pmatrix} \quad \forall k \in \mathbb{Z}_+, \ \theta \in [0,\tau).$

It follows from (4.40) with $\theta = \tau$ that

(4.41) $\qquad \begin{pmatrix} x_p((k+1)\tau) \\ x_c(k+1) \end{pmatrix} = \Delta(\tau)\begin{pmatrix} x_p(k\tau) \\ x_c(k) \end{pmatrix} + R(k,\tau)\begin{pmatrix} d_1 \\ d_2 \\ r \end{pmatrix} \quad \forall k \in \mathbb{Z}_+.$

In the following, let $\tau \in (0,\tau^*)$. Applying statement 1 of Theorem 3.5 to the feedback interconnection of discrete-time systems (4.38) and (4.36), we conclude that there exists $\varepsilon_\tau > 0$ such that, for every $\varepsilon \in (0, \varepsilon_\tau)$, $\Delta(\tau)$ is power stable.

By the admissibility of B, there exists $M_1 \geq 0$ such that

$$\left\| \int_{k\tau}^{k\tau+\theta} \mathbf{T}(k\tau+\theta-s)Bd_1(s)ds \right\|_X = M_1 \|d_1\|_{L^2((k\tau, k\tau+\theta), \mathbb{C}^m)} \leq M_1 \sqrt{\tau} \|d_1\|_\infty$$

$$\forall k \in \mathbb{Z}_+, \ \forall \theta \in [0,\tau], \ \forall d_1 \in L_b(\mathbb{R}_+, \mathbb{C}^m).$$

Therefore, there exists $M_2 \geq 0$ such that

(4.42) $\qquad \left\| R(k,\theta)\begin{pmatrix} d_1 \\ d_2 \\ r \end{pmatrix} \right\| \leq M_2(\|r\|_\infty + \|d_1\|_\infty + \|d_2\|_\infty) \ \forall k \in \mathbb{Z}_+, \ \forall \theta \in [0,\tau],$

$$\forall r, d_2 \in L_b(\mathbb{R}_+, \mathbb{C}^p), \ \forall d_1 \in L_b(\mathbb{R}_+, \mathbb{C}^m).$$

Hence, it follows from the discrete-time variation-of-parameters formula, the power stability of $\Delta(\tau)$, (4.41), and (4.42) that there exist $M_3 \geq 1$ and $\rho \in (0,1)$ such that

$$\left\| \begin{pmatrix} x_p(k\tau) \\ x_c(k) \end{pmatrix} \right\| \leq M_3 \left(\rho^k \left\| \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} \right\| + \|r\|_\infty + \|d_1\|_\infty + \|d_2\|_\infty \right) \quad \forall k \in \mathbb{Z}_+, \ \forall x_p^0 \in X,$$

(4.43) $\qquad \forall x_c^0 \in \mathbb{C}^{Np+n_0}, \ \forall r, d_2 \in L_b(\mathbb{R}_+, \mathbb{C}^p), \ \forall d_1 \in L_b(\mathbb{R}_+, \mathbb{C}^m).$

Setting $M_4 := \max_{\theta \in [0,\tau]} \|\Delta(\theta)\|$, it follows from (4.40), (4.42), and (4.43) that, for all $\theta \in [0,\tau)$, $k \in \mathbb{Z}_+$, $x_p^0 \in X$, $x_c^0 \in \mathbb{C}^{Np+n_0}$, r, $d_2 \in L_b(\mathbb{R}_+, \mathbb{C}^p)$, and $d_1 \in L_b(\mathbb{R}_+, \mathbb{C}^m)$,

$$\left\| \begin{pmatrix} x_p(k\tau + \theta) \\ x_c(k+1) \end{pmatrix} \right\| \leq M_4 \left\| \begin{pmatrix} x_p(k\tau) \\ x_c(k) \end{pmatrix} \right\| + M_2(\|r\|_\infty + \|d_1\|_\infty + \|d_2\|_\infty)$$

$$\leq M_3 M_4 \rho^k \left\| \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} \right\| + (M_2 + M_3 M_4)(\|r\|_\infty + \|d_1\|_\infty + \|d_2\|_\infty)$$

$$\leq N_1 \left(e^{\beta(k\tau + \theta)} \left\| \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} \right\| + \|r\|_\infty + \|d_1\|_\infty + \|d_2\|_\infty \right),$$

where $\beta := (\ln \rho)/\tau < 0$ and $N_1 := \max\{M_3 M_4 \rho^{-1}, M_2 + M_3 M_4\}$. This completes the proof of statement 1.

To prove the approximate tracking and disturbance rejection result claimed in statement 2, note that, by exponential stability of (4.34) and boundedness of C, the impulse response of (4.34) is a $\mathbb{C}^{p \times m}$-valued Borel measure μ of the form $\mu(ds) = g(s)ds + D\delta_0(ds)$, where $g(\cdot)e^{\alpha \cdot} \in L^1(\mathbb{R}_+, \mathbb{C}^{p \times m})$ for some $\alpha > 0$, and δ_0 is the Dirac measure (see [8, Lemma 2.3]). By (4.34)–(4.37) and a routine calculation, we obtain

$$(4.44) \qquad \begin{pmatrix} y(k\tau + \theta) \\ y_c(k) \end{pmatrix} = Q(\theta)\Delta^k(\tau) \begin{pmatrix} x_p^0 \\ x_c^0 \end{pmatrix} + \begin{pmatrix} \tilde{y}(k\tau + \theta) \\ \tilde{y}_c(k) \end{pmatrix} \qquad \forall \theta \in [0,\tau), \ \forall k \in \mathbb{Z}_+,$$

where

$$Q(\theta) := \begin{pmatrix} C\mathbf{T}(\theta) - \varepsilon(F(\theta) + DE)D_cC & \varepsilon F(\theta)C_c + \varepsilon DEC_c \\ -\varepsilon D_c E_c C & \varepsilon C_c - \varepsilon^2 D_c E_c DC_c \end{pmatrix},$$

$$\text{with } F(\theta) := C\int_0^\theta \mathbf{T}(s)B\,ds\,E\,,$$

and $\tilde{y}$, $\tilde{y}_c$ satisfy

$$(4.45) \qquad \tilde{y} = G(d_1 + \mathcal{H}_\tau \tilde{y}_c) + d_2\,, \quad \tilde{y}_c = K_{\tau,\varepsilon}\mathcal{S}_\tau(r - \tilde{y})\,.$$

An application of Theorem 4.10 to system (4.45), with r given by (4.9) and d_1, d_2 satisfying (4.10), shows that for every $\delta > 0$, there exists $\tau_\delta \in (0, \tau^*)$ such that, for every sampling period $\tau \in (0, \tau_\delta)$, there exists $\varepsilon_\tau > 0$, such that, for every $\varepsilon \in (0, \varepsilon_\tau)$,

$$\limsup_{t \to \infty} \|\tilde{y}(t) - r(t)\| \leq \delta\,.$$

Therefore, by power stability of $\Delta(\tau)$ and (4.44),

$$\limsup_{t \to \infty} \|y(t) - r(t)\| \leq \delta \quad \forall x_p^0 \in X\,, \ \forall x_c^0 \in \mathbb{C}^{Np+n_0}\,,$$

completing the proof. $\square$

Example 4.13. For purposes of illustration, we consider the heat equation for a bar of length 1. We keep both endpoints at zero temperature and inject heat of magnitude u_p at the point $\eta_1 \in (0,1)$. The measurement is generated by a spatial averaging of the state over an σ-neighborhood of a point $\eta_2 \in (\eta_1, 1)$. The system to be controlled can be formulated as follows:

$$z_t(\eta, t) = z_{\eta\eta}(\eta, t) + \delta(\eta - \eta_1)u_p(t)\,,$$

$$y_p(t) = \frac{1}{2\sigma} \int_{\eta_2 - \sigma}^{\eta_2 + \sigma} z(\lambda, t)d\lambda\,,$$

with boundary conditions

$$z(0,t) = z(1,t) = 0 \quad \forall t > 0\,.$$

For simplicity, we assume zero initial conditions

$$z(\eta, 0) = 0 \quad \forall \eta \in [0,1]\,.$$

Sampled-data low-gain integral control of this system (in the presence of input hysteresis) was studied in [6].

With input u_p and output y_p, it is not hard to show that this system is a regular linear system with state space $X = L^2(0,1)$ and bounded observation. In particular, the corresponding semigroup $\mathbf{T}(t)$, given by

$$(\mathbf{T}(t)x)(\eta) = \sum_{n=1}^{\infty} 2\exp(-n^2\pi^2 t)\sin(n\pi\eta)\int_0^1 \sin(n\pi\lambda)x(\lambda)d\lambda$$

$$\forall x \in L^2(0,1), \ \ \forall \eta \in [0,1]\,,$$

is exponentially stable. The transfer function $\mathbf{G}$ is given by

$$\mathbf{G}(s) = \frac{\sinh(\sigma\sqrt{s})\sinh(\eta_1\sqrt{s})\sinh((1-\eta_2)\sqrt{s})}{\sigma s \sinh(\sqrt{s})}\,.$$

The aim is to design a robust controller such that the closed-loop system approximately tracks the reference signal $r(t) = \sin t$ in the presence of disturbance signals d_1, d_2 given by

$$d_1(t) = \frac{1}{5}\cos(5t) + \frac{1}{t+1}\,, \quad d_2(t) = \frac{1}{5}\sin(5t) - \frac{1}{2}\ln\left(1 + \frac{1}{t+1}\right)\,, \quad t \geq 0\,.$$

Set

$$K_1 := 1/\mathbf{G}(i)\,, \quad K_2 := \overline{K_1}\,, \quad K_3 := 1/\mathbf{G}(5i)\,, \quad K_4 := \overline{K_3}\,,$$

and $\mathbf{K}^0(z) \equiv 10$, so that the transfer function $\mathbf{K}_{\tau,\varepsilon}$ of the controller $K_{\tau,\varepsilon}$ (see (4.7)) is given by

$$\mathbf{K}_{\tau,\varepsilon}(z) := \varepsilon\left(10 + \frac{K_1}{z - e^{i\tau}} + \frac{K_2}{z - e^{-i\tau}} + \frac{K_3}{z - e^{5i\tau}} + \frac{K_4}{z - e^{-5i\tau}}\right)$$

$$= \varepsilon\left(10 + \frac{2\mathrm{Re}\,(K_1)z - 2\mathrm{Re}\,(K_1 e^{-i\tau})}{z^2 - (2\cos\tau)z + 1} + \frac{2\mathrm{Re}\,(K_3)z - 2\mathrm{Re}\,(K_3 e^{-5i\tau})}{z^2 - (2\cos 5\tau)z + 1}\right)\,.$$

Since all the relevant hypotheses are satisfied, the conclusions of Theorem 4.10 are valid. In Figure 4.2, simulations are shown for the specific values

$$\eta_1 = 0.2\,, \quad \eta_2 = 0.6\,, \quad \sigma = 0.01\,, \quad \tau = 0.1\,, \quad \varepsilon = 0.1\,,$$

with zero initial conditions for the controller. The error signal $e = r - y_p - d_2$ and the output of the sampled-data system $y = y_p + d_2$ are shown in Figure 4.2. Asymptotically, the error is bounded by 0.0028, that is, $\limsup_{t\geq 0}|e(t)| \leq 0.0028$. Simulations show that, for the sampling period $\tau = 0.1$, instability occurs at $\varepsilon \approx 0.22$.
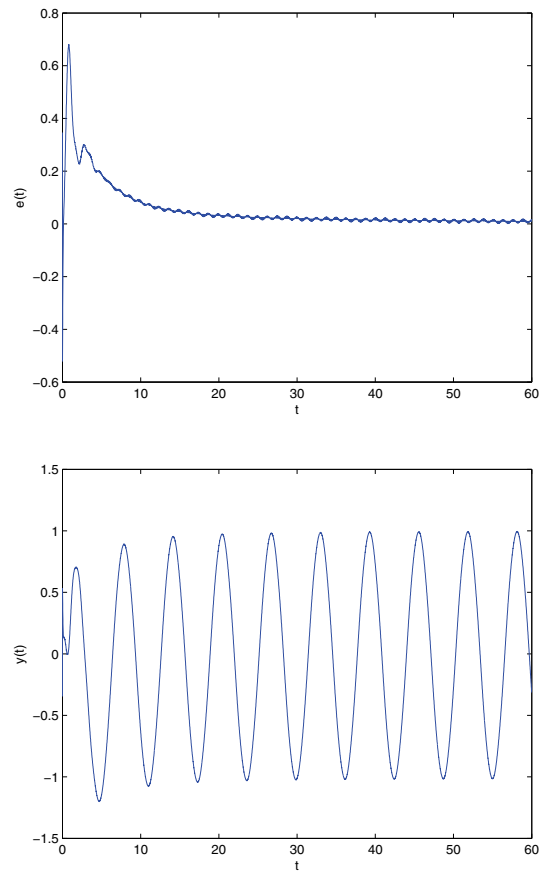
FIG. 4.2. *Error signal e and output y.*

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