SIAM J. CONTROL OPTIM.
Vol. 48, No. 2, pp. 1031–1054

INPUT-TO-STATE STABILITY OF DIFFERENTIAL INCLUSIONS WITH APPLICATIONS TO HYSTERETIC AND QUANTIZED FEEDBACK SYSTEMS*

BAYU JAYAWARDHANA†, HARTMUT LOGEMANN‡, AND EUGENE P. RYAN‡

Abstract. Input-to-state stability (ISS) of a class of differential inclusions is proved. Every system in the class is of Lur'e type: a feedback interconnection of a linear system and a set-valued nonlinearity. Applications of the ISS results, in the context of feedback interconnections with a hysteresis operator or a quantization operator in the feedback path, are developed.

Key words. absolute stability, circle criterion, differential inclusions, input-to-state stability, hysteresis, nonlinear systems, quantization

AMS subject classifications. 34A60, 34C55, 34D05, 47J40, 93C10, 93D05, 93D10

DOI. 10.1137/070711323

1. Introduction.

Classical absolute stability theory, with origins in [18], is concerned with the analysis of systems of Lur'e type, that is, feedback interconnections of the form shown in Figure 1.1, consisting of a linear system L in the forward path and a static sector-bounded nonlinearity f in the (negative) feedback path. The methodology seeks to conclude stability of the overall system through the interplay or reciprocation of inherent frequency-domain properties of the linear component L and sector data for the nonlinearity f. Accounts of the classical theory can be found in, e.g., [7, 10, 13, 19, 21, 23]. The present paper adopts a similar standpoint but differs from the classical framework in three fundamental aspects: (i) in contrast with the literature, wherein the focus is on global asymptotic stability and L^2 or L^∞ stability, input-to-state stability (ISS) issues are addressed here; (ii) nonlinearities of considerably greater generality are permitted in the feedback path; (iii) the sector conditions of the classical theory are significantly weakened. With reference to (i), conditions on the linear and nonlinear components are identified under which ISS of the interconnection is guaranteed. With reference to (ii), a framework is developed of sufficient generality to encompass not only static nonlinearities but also causal operators (and hysteresis, in particular) and quantization operators in the feedback path. With reference to (iii), through the concept of a generalized sector condition, the investigation is extended to include nonlinearities which satisfy a sector condition only in the complement of a compact set: a theory is developed pertaining to ISS *with bias*. We proceed to outline these features more precisely.

With reference to Figure 1.2, the focus of the paper is a study of absolute stability, ISS, and boundedness properties of a feedback interconnection of a finite-dimensional, linear, m-input, m-output system (A, B, C) and a set-valued nonlinearity Φ. Throughout, we assume that Δ is a set-valued map in which input or disturbance signals are

*Received by the editors December 19, 2007; accepted for publication (in revised form) October 7, 2008; published electronically March 6, 2009. Supported by the UK Engineering & Physical Sciences Research Council grant GR/S94582/01.
http://www.siam.org/journals/sicon/48-2/71132.html
†Department of Discrete Technology and Production Automation, University of Groningen, 9747 AG Groningen, The Netherlands (B.Jayawardhana@rug.nl).
‡Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK (hl@maths.bath.ac.uk, epr@maths.bath.ac.uk).

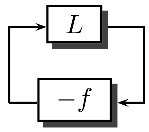

Fig. 1.1. *Classical feedback interconnection.*

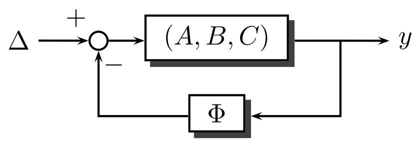

Fig. 1.2. *Interconnection of a linear system (A, B, C) and a set-valued nonlinearity Φ.*

embedded. We seek an analytical framework of sufficient generality to encompass inter alia feedback systems with causal operators (and, in particular, hysteresis operators) in the feedback loop. To illustrate this, let F be a causal operator from $\mathrm{dom}(F) \subset L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ to $L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$, where $\mathbb{R}_+ := [0, \infty)$ and consider the feedback system (structurally of Lur'e type), with input $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$, given by the functional differential equation

$$\dot{x}(t) = Ax(t) + B\big(d(t) - (F(Cx))(t)\big). \tag{1.1}$$

By causality of F, we mean that, for all $y, z \in \mathrm{dom}(F)$ and all $\alpha > 0$,

$$y|_{[0,\alpha]} = z|_{[0,\alpha]} \implies F(y)|_{[0,\alpha]} = F(z)|_{[0,\alpha]}.$$

To associate (1.1) with the structure of Figure 1.2, assume that F can be embedded in a set-valued map Φ in the sense that

$$y \in \mathrm{dom}(F) \quad \implies \quad (F(y))(t) \in \Phi(y(t)) \ \text{ for a.e. } t \in \mathbb{R}_+.$$

If the input d is such that $d(t) \in \Delta(t)$ for almost all t, then any solution of (1.1) is a fortiori a solution of the feedback interconnection in Figure 1.2. In this sense, properties of solutions of the feedback interconnection are inherited by solutions of (1.1). Under particular regularity assumptions on Δ and Φ, generalized sector conditions on Φ, and positive-real conditions related to the linear component (A, B, C), we establish ISS (in the sense of [20] but extended to differential inclusions) and boundedness properties of solutions of the system in Figure 1.2. The approach is partially based on that of Arcak & Teel [1]. In particular, some of the arguments adopted in the proof of Lemma 5.1 of the present paper are generalizations, to a differential inclusions setting, of arguments in [1]. The paper is structured as follows. In section 2, we make precise the nature of the maps Φ and Δ and state an existence theorem which underpins the stability analysis of the differential inclusion formulation implicit in Figure 1.2. The main results, Theorems 3.4 and 3.5 (and Corollaries 3.6 and 3.7), are assembled in section 3. For clarity of presentation, the proof of Theorem 3.4 (respectively, Theorem 3.5) is presented separately in section 4 (respectively, section 5). In section 6, the results in Theorem 3.4/Corollary 3.6 are applied in the context of single-input, single-output feedback interconnections with a hysteresis operator F in the feedback loop. New absolute stability and boundedness results are obtained for Lur'e systems with Preisach hysteresis (see, e.g., [3, 9, 12, 16, 17] for earlier stability results for hysteretic feedback systems). In the final section, quantized feedback systems are considered: these constitute an area of growing importance (see, e.g., [4, 8] in a linear systems context). Specifically, in section 7, nonlinear feedback systems with uniform output quantization (parameterized by $\gamma \geq 0$) are investigated. Through an application of Theorem 3.5/Corollary 3.7, we establish robustness with respect to quantization in the following sense: if, in the absence of quantization ($\gamma = 0$), the feedback system is ISS, then, in the presence of quantization ($\gamma > 0$), the feedback system is ISS *with bias* and is such that the unbiased ISS property of the unquantized system is "approached" as $\gamma \downarrow 0$.

Notation and terminology. The open right-half complex plane is denoted by $\mathbb{C}_+$. For nonempty $S \subset \mathbb{R}^m$, we define $|S| := \sup\{\|s\| \mid s \in S\}$. If H is a proper real-rational matrix of format $m \times m$, then we say that H is *positive real* if

$$H(s) + H^*(s) \geq 0\,, \quad \forall\, s \in \mathbb{C}_+,\ s \text{ not a pole of } H,$$

where $H^*(s) := (H(s))^*$. Moreover, if $H \in H^\infty := H^\infty(\mathbb{C}_+, \mathbb{C}^{m\times m})$ (and so H does not have any poles in $\overline{\mathbb{C}}_+$), then

$$\|H\|_{H^\infty} := \sup\nolimits_{s\in\mathbb{C}_+} \|H(s)\|\,,$$

where $\|H(s)\|$ is the matrix norm induced by the 2-norm on $\mathbb{C}^m$. Let $\mathcal{K}$ denote the set of all continuous and strictly increasing functions $f : \mathbb{R}_+ \to \mathbb{R}_+$, with $f(0) = 0$. We say that a function f is in $\mathcal{K}_\infty$ if $f \in \mathcal{K}$ and $f(s) \to \infty$ as $s \to \infty$. Finally, $\mathcal{KL}$ denotes the class of all continuous functions $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that, for each $r \in \mathbb{R}_+$, the function $s \mapsto f(r, s)$ is in $\mathcal{K}$ and, for each $s \in \mathbb{R}_+$, the function $r \mapsto f(r, s)$ is nonincreasing with $f(r, s) \to 0$ as $r \to \infty$.

2. Set-valued nonlinearities and differential inclusions. A set-valued map $y \mapsto \Phi(y) \subset \mathbb{R}^m$, with nonempty values and defined on $\mathbb{R}^m$, is said to be *upper semicontinuous at* $y \in \mathbb{R}^m$ if, for every open set U containing $\Phi(y)$, there exists an open neighborhood Y of y such that $\Phi(Y) := \cup_{z\in Y}\Phi(z) \subset U$; the map Φ is said to be *upper semicontinuous* if it is upper semicontinuous at every $y \in \mathbb{R}^m$. The set of upper semicontinuous compact–convex-valued maps

$$\Phi : \mathbb{R}^m \to \{S \subset \mathbb{R}^m \mid S \text{ nonempty, compact, and convex}\}$$

is denoted by $\mathcal{U}$. Let $\Delta : \mathbb{R}_+ \to \{S \subset \mathbb{R}^m \,|\, S \neq \emptyset\}$ be a set-valued map. The map Δ is said to be *measurable* if the preimage $\Delta^{-1}(U) := \{t \in \mathbb{R}_+ \,|\, \Delta(t) \cap\ U \neq \emptyset\}$ of every open set $U \subset \mathbb{R}^m$ is Lebesgue measurable; Δ is said to be *locally essentially bounded* if Δ is measurable and the function $t \mapsto |\Delta(t)|$ is in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$. The set of all locally essentially bounded set-valued maps $\mathbb{R}_+ \to \{S \subset \mathbb{R}^m \,|\, S \neq \emptyset\}$ is denoted by $\mathcal{B}$. For $\Delta \in \mathcal{B}$, $I \subset \mathbb{R}_+$ an interval, and $1 \leq p \leq \infty$, the L^p-norm of the restriction of the function $t \mapsto |\Delta(t)|$ to the interval I is denoted by $\|\Delta\|_{L^p(I)}$. For later use, we record a technicality.

LEMMA 2.1. *Assume that $\Phi \in \mathcal{U}$, $\Phi(0) = \{0\}$, and there exists $\varphi \in \mathcal{K}_\infty$, with*

$$\varphi(\|y\|)\|y\| \leq \langle y, v\rangle \quad \forall\, v \in \Phi(y)\ \forall\ y \in \mathbb{R}^m.$$

Then there exists $\psi \in \mathcal{K}_\infty$ such that

$$\|v\| \leq \psi(\|y\|) \quad \forall\, v \in \Phi(y)\ \forall\ y \in \mathbb{R}^m.$$

Proof. By upper semicontinuity of Φ and compactness of its values, for every compact set $K \subset \mathbb{R}^m$, the set $\Phi(K)$ is compact (see, for example, [2, Proposition 3,

p. 42]), and so the function $s \mapsto \psi_0(s) := \max\{\|v\| \,|\, v \in \Phi(y),\ \|y\| \le s\}$ is well defined and nondecreasing on $\mathbb{R}_+$, with $\psi_0(0) = 0$. Clearly, $\varphi(s) \le \psi_0(s)\ \forall\ s \in \mathbb{R}_+$, and so $\psi_0(s) \to \infty$ as $s \to \infty$. Let $\psi \in \mathcal{K}_\infty$ be such that $\psi(s) \ge \psi_0(s)\ \forall\ s \in \mathbb{R}_+$, for example, the function $\psi \in \mathcal{K}_\infty$ given by

$$\psi(0) := 0, \quad \psi(s) := \frac{1}{s}\int_s^{2s} \psi_0(\sigma)\mathrm{d}\sigma \quad \forall\ s > 0$$

suffices. $\square$

The feedback system shown in Figure 1.2 corresponds to the initial-value problem

$$\dot{x}(t) - Ax(t) \in B\left(\Delta(t) - \Phi(Cx(t))\right), \quad x(0) = x^0 \in \mathbb{R}^n,\ \ \Delta \in \mathcal{B}, \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $\Phi \in \mathcal{U}$. By a solution of (2.1), we mean an absolutely continuous function $x : [0,\omega) \to \mathbb{R}^n$, $0 < \omega \le \infty$, such that $x(0) = x^0$ and the differential inclusion in (2.1) is satisfied almost everywhere on $[0,\omega)$; a solution is *maximal* if it has no proper right extension that is also a solution; a solution is *global* if it exists on $[0,\infty)$. Before developing a stability theory for systems of the form (2.1), we briefly digress to record an existence result.

LEMMA 2.2. *Let $\Phi \in \mathcal{U}$. For each $x^0 \in \mathbb{R}^n$ and each $\Delta \in \mathcal{B}$, initial-value problem (2.1) has a solution. Moreover, every solution can be extended to a maximal solution $x : [0,\omega) \to \mathbb{R}^n$, and if x is bounded, then x is global.*

Proof. Let $x^0 \in \mathbb{R}^n$ and $\Delta \in \mathcal{B}$ be arbitrary. By [6, Corollary 5.2], initial-value problem (2.1) has a solution, and every solution can be extended to a solution $x : [0,\omega) \to \mathbb{R}^n$ with the property that the graph of x is unbounded. Evidently, x is maximal, and if x is bounded, then $\omega = \infty$. $\square$

3. ISS: The main results.

In the context of differential inclusion (2.1), the transfer-function matrix of the linear system given by (A, B, C) is denoted by G, i.e., $G(s) = C(sI - A)^{-1}B$.

We assemble four hypotheses which will be variously invoked in the theory developed below.

(H1) There exist numbers $a < b$ and $\delta > 0$ such that

$$\langle ay - v, by - v\rangle \le 0 \quad \forall\, v \in \Phi(y)\ \forall\, y \in \mathbb{R}^m, \tag{3.1}$$

$G(I + aG)^{-1} \in H^\infty$, and $(I + bG)(I + aG)^{-1} - \delta I$ is positive real.

(H2) $\Phi(0) = \{0\}$ and there exist numbers $a > 0$, $\delta \in [0, 1)$, and $\theta \ge 0$ such that

$$a\|y\|^2 \le \langle y, v\rangle \quad \forall\, v \in \Phi(y)\ \forall\, y \in \mathbb{R}^m, \tag{3.2}$$

$$\|v - a\delta y\| \le \langle y, v - a\delta y\rangle \quad \forall\, v \in \Phi(y)\ \forall\, y \in \mathbb{R}^m,\ \text{with } \|y\| \ge \theta, \tag{3.3}$$

and $G(I + \delta aG)^{-1}$ is positive real.

(H3) There exist $\varphi \in \mathcal{K}_\infty$ and numbers $b > 0$ and $\delta \in [0, 1)$ such that

$$\max\left\{\varphi(\|y\|)\|y\|, \|v\|^2/b\right\} \le \langle y, v\rangle \quad \forall\, v \in \Phi(y)\ \forall\, y \in \mathbb{R}^m, \tag{3.4}$$

and $(\delta/b)I + G$ is positive real.

(H4) $\Phi(0) = \{0\}$ and there exist $\varphi \in \mathcal{K}_\infty$ and a number $\theta \ge 0$ such that

$$\varphi(\|y\|)\|y\| \le \langle y, v\rangle \quad \forall\, v \in \Phi(y)\ \forall\, y \in \mathbb{R}^m, \tag{3.5}$$

$$\|v\| \le \langle y, v\rangle \quad \forall\, v \in \Phi(y)\ \forall\, y \in \mathbb{R}^m,\ \text{with } \|y\| \ge \theta, \tag{3.6}$$

and G is positive real.

Remark 3.1. (a) (H1) is a set-valued version of the familiar multivariable sector condition. A routine calculation shows that (3.1) holds if and only if

$$\left\| v - \frac{a+b}{2} y \right\| \le \frac{b-a}{2} \|y\| \quad \forall\, v \in \Phi(y)\ \forall\, y \in \mathbb{R}^m.$$

(b) If $m = 1$ (the single-input, single-output case), then the combined frequency-domain assumptions in (H1), namely, the condition $G(I+aG)^{-1} \in H^\infty$ together with the positive realness of $(I+bG)(I+aG)^{-1} - \delta I$, admit a graphical characterization in terms of the Nyquist diagram of G (see, e.g., [13, pp. 268]).
(c) Conditions (3.2) and (3.5) can be viewed as the limits of (3.1) and (3.4), respectively, as $b \to \infty$.
(d) A sufficient condition for (3.4) to hold is the "nonlinear" sector condition

$$\big\langle \varphi(y)\|y\|^{-1} y - v\,,\, by - v \big\rangle \le 0 \quad \forall\, v \in \Phi(y)\ \forall\, y \in \mathbb{R}^m, \tag{3.7}$$

which is (3.1) with the term ay replaced by $\varphi(y)\|y\|^{-1}y$ (which should be interpreted as taking the value 0 for $y = 0$). It is easy to construct counterexamples which show that (3.7) is not necessary for (3.4) to hold.
(e) If $m = 1$ and (3.2) holds, then (3.3) is trivially satisfied for any $\theta \ge 1$ and any $\delta \in [0, 1)$. Similarly, if $m = 1$ and (3.5) holds, then (3.6) is satisfied for every $\theta \ge 1$.
(f) If (3.4) holds for some $\varphi \in \mathcal{K}_\infty$ and for some $b > 0$, then $\Phi(0) = \{0\}$, and furthermore, (3.6) is satisfied for any $\theta > 0$ satisfying $\varphi(\theta) \ge b$.

DEFINITION 3.2. *System* (2.1) *is said to be* input-to-state stable with bias $c \ge 0$ *if every maximal solution of* (2.1) *is global and there exist* $\beta_1 \in \mathcal{KL}$ *and* $\beta_2 \in \mathcal{K}_\infty$ *such that, for all* $x^0 \in \mathbb{R}^n$ *and all* $\Delta \in \mathcal{B}$, *every global solution* x *satisfies*

$$\|x(t)\| \le \max\big\{\beta_1(t, \|x^0\|), \beta_2(\|\Delta\|_{L^\infty[0,t]} + c)\big\} \quad \forall\, t \in \mathbb{R}_+. \tag{3.8}$$

System (2.1) *is* input-to-state stable *if it is input-to-state stable with bias* 0.

System (2.1) has the *converging-input-converging-state property* if, for all $x^0 \in \mathbb{R}^n$ and all $\Delta \in \mathcal{B}$ with $\|\Delta\|_{L^\infty[t,\infty)} \to 0$ as $t \to \infty$, every maximal solution x of (2.1) is global and satisfies $x(t) \to 0$ as $t \to \infty$. The following lemma shows in particular that if system (2.1) is input-to-state stable, then it has the converging-input-converging-state property.

LEMMA 3.3. *Assume that system* (2.1) *is input-to-state stable with bias* $c \ge 0$, *and let* β_1 *and* β_2 *be as in Definition* 3.2. *Let* $x^0 \in \mathbb{R}^n$ *and* $\Delta \in \mathcal{B}$. *If* Δ *is essentially bounded* ($\|\Delta\|_{L^\infty[0,\infty)} < \infty$), *then every global solution* x *of* (2.1) *satisfies*

$$\limsup_{t\to\infty} \|x(t)\| \le \limsup_{t\to\infty} \beta_2(\|\Delta\|_{L^\infty[t,\infty)} + c).$$

Proof. Let $x^0 \in \mathbb{R}^n$, and let $\Delta \in \mathcal{B}$ be essentially bounded. Let x be a global solution of (2.1), let $\tau \ge 0$ be arbitrary, and set $x_\tau(t) := x(t+\tau)$ and $\Delta_\tau(t) := \Delta(t+\tau)\ \forall\, t \ge 0$. Then, $\Delta_\tau \in \mathcal{B}$ and x_τ satisfies the initial-value problem

$$\dot{x}_\tau(t) - Ax_\tau(t) \in B(\Delta_\tau(t) - \Phi(Cx_\tau(t))), \quad x_\tau(0) = x(\tau).$$

By ISS with bias c,

$$\begin{aligned} \|x(t+\tau)\| = \|x_\tau(t)\| &\le \max\big\{\beta_1(t, \|x(\tau)\|), \beta_2(\|\Delta_\tau\|_{L^\infty[0,t]} + c)\big\} \\ &= \max\big\{\beta_1(t, \|x(\tau)\|), \beta_2(\|\Delta\|_{L^\infty[\tau,t+\tau]} + c)\big\} \quad \forall\, t \in \mathbb{R}_+. \end{aligned}$$

Therefore, $\limsup_{t\to\infty} \|x(t)\| \le \beta_2\big(\|\Delta\|_{L^\infty[\tau,\infty)} + c\big)$ $\forall\, \tau \ge 0$, from which the claim follows. $\square$

We now state the two main results on ISS. The proofs can be found in sections 4 and 5.

THEOREM 3.4. *Let linear system* (A, B, C) *be stabilizable and detectable. Assume that* (H1) *holds. Then, every maximal solution of* (2.1) *is global and there exist positive constants* c_1, c_2, *and* ε *such that, for all* $x^0 \in \mathbb{R}^n$ *and* $\Delta \in \mathcal{B}$, *every global solution* x *satisfies*

$$\|x(t)\| \le c_1 e^{-\varepsilon t}\|x^0\| + c_2\|\Delta\|_{L^\infty[0,t]} \quad \forall\, t \in \mathbb{R}_+.$$

In particular, system (2.1) *is input-to-state stable.*

THEOREM 3.5. *Let linear system* (A, B, C) *be minimal. Assume that at least one of hypotheses* (H2), (H3), *or* (H4) *holds. Then system* (2.1) *is input-to-state stable.*

In [1] it is has been proved, for single-valued Φ and Δ, that if (H4) holds, then (2.1) is input-to-state stable. Therefore, Theorem 3.5 can be considered as a generalization of the main result in [1].

In the following two corollaries (to Theorem 3.4 and Theorem 3.5, respectively), we will consider not only nonlinearities satisfying at least one of the conditions (3.1), (3.2), (3.4), and (3.5) *for all* arguments $y \in \mathbb{R}^m$, but also nonlinearities $\Phi \in \mathcal{U}$ with the property that there exist a set-valued map $\tilde{\Phi} \in \mathcal{U}$ satisfying at least one of the conditions (3.1), (3.2), (3.4), and (3.5) and a compact set $K \subset \mathbb{R}^m$ such that

$$y \in \mathbb{R}^m \backslash K \quad \Longrightarrow \quad \Phi(y) \subset \tilde{\Phi}(y). \tag{3.9}$$

For example, single-input, single-output hysteretic elements can be subsumed by this set-valued formulation provided that the "characteristic diagram" of the hysteresis is contained in the graph of some $\Phi \in \mathcal{U}$; see section 6 for details.

COROLLARY 3.6. *Let linear system* (A, B, C) *be stabilizable and detectable. Let* $\Phi \in \mathcal{U}$ *be such that there exist a set-valued map* $\tilde{\Phi} \in \mathcal{U}$ *and a compact set* $K \subset \mathbb{R}^m$ *such that* (3.9) *holds. Assume that* (H1) *holds with* Φ *replaced by* $\tilde{\Phi}$. *Then, every maximal solution of* (2.1) *is global and there exist positive constants* c_1, c_2, *and* ε *such that, for all* $x^0 \in \mathbb{R}^n$ *and* $\Delta \in \mathcal{B}$, *every global solution* x *satisfies*

$$\|x(t)\| \le c_1 e^{-\varepsilon t}\|x^0\| + c_2(\|\Delta\|_{L^\infty[0,t]} + E) \quad \forall\, t \in \mathbb{R}_+,$$

where

$$E := \sup_{y\in K} \sup_{v\in\Phi(y)} \inf_{\tilde{v}\in\tilde{\Phi}(y)} \|v - \tilde{v}\|. \tag{3.10}$$

Proof. First, we remark that, by upper semicontinuity of Φ and $\tilde{\Phi} \in \mathcal{U}$, together with compactness of their values and compactness of K, E is finite. Let $x^0 \in \mathbb{R}^n$ and $\Delta \in \mathcal{B}$. By Lemma 2.2, (2.1) has a solution, and every solution can be maximally extended. Let $x : [0, \omega) \to \mathbb{R}^n$ be a maximal solution of (2.1) and write $y := Cx$. Define $z \in L^1_{\text{loc}}([0, \omega), \mathbb{R}^n)$ by $z := \dot{x} - Ax$. Since $z(t) \in B\big(\Delta(t) - \Phi(Cx(t))\big)$ for almost every $t \in [0, \omega)$, there exist functions $d, v : [0, \omega) \to \mathbb{R}^m$ such that

$$(d(t), v(t)) \in \Delta(t) \times \Phi(y(t)) \;\; \forall\, t \in [0, \omega), \quad z(t) = B\big(d(t) - v(t)\big) \;\text{ for a.e. } t \in [0, \omega).$$

For each $t \in [0, \omega)$, let $\tilde{v}(t) \in \tilde{\Phi}(y(t))$ be the unique point of the closed convex set $\tilde{\Phi}(y(t))$ closest to $v(t) \in \Phi(y(t))$. Then

$$y(t) \in K \;\Longrightarrow\; \|v(t) - \tilde{v}(t)\| \le E, \qquad y(t) \in \mathbb{R}^m \backslash K \;\Longrightarrow\; \|v(t) - \tilde{v}(t)\| = 0.$$

Define $\tilde{\Delta} \in \mathcal{B}$ by $\tilde{\Delta}(t) := \Delta(t) + \mathbb{B}_E$ (where $\mathbb{B}_E$ denotes the ball of radius $E > 0$ centered at 0 in $\mathbb{R}^m$) and $\tilde{d} : [0, \omega) \to \mathbb{R}^m$ by $\tilde{d}(t) := d(t) - v(t) + \tilde{v}(t)$. Then

$$z(t) = B(\tilde{d}(t) - \tilde{v}(t)), \quad \tilde{d}(t) \in \tilde{\Delta}, \quad \tilde{v}(t) \in \tilde{\Phi}(y(t)) \text{ for a.e. } t \in [0, \omega),$$

and so the solution x of (2.1) is also a solution of

$$\dot{x}(t) - Ax(t) \in B\big(\tilde{\Delta}(t) - \tilde{\Phi}(Cx(t))\big), \quad x(0) = x^0. \tag{3.11}$$

An application of Theorem 3.4 to (3.11) yields the claim. ∎

COROLLARY 3.7. *Let linear system (A, B, C) be minimal, and let $\Phi \in \mathcal{U}$ be such that there exist a set-valued map $\tilde{\Phi} \in \mathcal{U}$ and a compact set $K \subset \mathbb{R}^m$ such that* (3.9) *holds. Assume that at least one of the hypotheses* (H2)*,* (H3)*, or* (H4) *holds with Φ replaced by $\tilde{\Phi}$. Then system* (2.1) *is input-to-state stable with bias E, where the constant E is given by* (3.10)*.*

Proof. The proof is identical to that of Corollary 3.6 with one exception: instead of invoking Theorem 3.4 at the end of the proof, an application of Theorem 3.5 to (3.11) completes the argument here. ∎

Remark 3.8. If the hypotheses of Corollary 3.6 (respectively, Corollary 3.7) hold, then there exist positive constants c_1, c_2, ε (respectively, functions $\beta_1 \in \mathcal{KL}$ and $\beta_2 \in \mathcal{K}_\infty$) such that (3.8) holds, with $c = E$ given by (3.10). We emphasize that c_1, c_2, ε (respectively, β_1 and β_2) are determined by data associated with only (A, B, C) and $\tilde{\Phi}$. In particular, they do not depend on Φ. This observation is of importance in the analysis of quantized feedback systems in section 7.

4. Proof of Theorem 3.4.

The following lemma will play an essential role in the proof of Theorem 3.4

LEMMA 4.1. *Let $a < b$ and set $\kappa := (a+b)/2$ and $\lambda := (b-a)/2$. If $G(I+aG)^{-1} \in H^\infty$ and there exists $\delta > 0$ such that $(I + bG)(I + aG)^{-1} - \delta I$ is positive real, then $G(I + \kappa G)^{-1} \in H^\infty$ and $\|G(I + \kappa G)^{-1}\|_{H^\infty} < 1/\lambda$.*

Proof. Setting $\eta := \|G(I + aG)^{-1}\|_{H^\infty}$, we have that

$$(I + aG^*(s))^{-1} G^*(s) G(I + aG(s))^{-1} \leq \eta^2 I \quad \forall\, s \in \mathbb{C}_+.$$

By hypothesis,

$$(I + bG(s))(I + aG(s))^{-1} + (I + aG^*(s))^{-1}(I + bG^*(s)) \geq 2\delta I \quad \forall s \in \mathbb{C}_+.$$

Setting $\varepsilon := \delta/\eta^2$, we obtain that

$$\begin{aligned} &2\varepsilon(I + aG^*(s))^{-1} G^*(s) G(s)(I + aG(s))^{-1} \\ &\qquad \leq (I + bG(s))(I + aG(s))^{-1} + (I + aG^*(s))^{-1}(I + bG^*(s)) \ \ \forall s \in \mathbb{C}_+. \end{aligned}$$

Therefore,

$$2\varepsilon G^*(s)G(s) \leq 2I + (a + b)G^*(s) + (a + b)G(s) + 2abG^*(s)G(s) \quad \forall s \in \Gamma,$$

where $\Gamma := \{s \in \mathbb{C}_+ \mid s \text{ not a pole of } G\}$. Consequently,

$$-(ab - \varepsilon)G^*(s)G(s) \leq I + \kappa G^*(s) + \kappa G(s) \quad \forall s \in \Gamma.$$

Setting $\rho := \sqrt{1 + \varepsilon/\lambda^2}$, it follows that

$$\begin{aligned} \lambda^2\rho^2 G^*(s)G(s) &\leq I + \kappa G^*(s) + \kappa G(s) + \kappa^2 G^*(s)G(s) \\ &= (I + \kappa G^*(s))(I + \kappa G(s)) \ \ \forall s \in \Gamma, \end{aligned}$$

which, in turn, implies that

$$\rho^2(I+\kappa G^*(s))^{-1}G^*(s)G(s)(I+\kappa G(s))^{-1} \le \lambda^{-2}I \quad \forall s\in\Gamma_0,$$

where $\Gamma_0 := \{s\in\Gamma \mid \det(sI+\kappa G(s))\neq 0\}$. We may now infer that $G(I+\kappa G)^{-1}\in H^\infty$, and since $\rho>1$, $\|G(I+\kappa G)^{-1}\|_{H^\infty} < 1/\lambda$. ∎

Proof of Theorem 3.4. Let x be a maximal solution of (2.1) defined on the maximal interval of existence $[0,\omega)$, where $0<\omega\le\infty$. We first show that $\omega=\infty$. Seeking a contradiction, suppose that $\omega<\infty$. A routine application of the generalized Filippov selection theorem (see [22], p. 72) shows that there exists a measurable function $w:[0,\omega)\to\mathbb{R}^m$ such that $w(t)\in\Delta(t)-\Phi(Cx(t))$ for a.e. $t\in[0,\omega)$ and

$$\dot x(t) = Ax(t)+Bw(t) \quad \text{a.e. } t\in[0,\omega).$$

Setting $\kappa := (a+b)/2$ and $A_\kappa := A-\kappa BC$, we have

$$x(t) = e^{A_\kappa t}x^0 + \int_0^t e^{A_\kappa(t-\tau)}B(w(\tau)+\kappa Cx(\tau))\mathrm{d}\tau \quad \forall\, t\in[0,\omega). \tag{4.1}$$

Since $w(t)\in\Delta(t)-\Phi(Cx(t))$ for a.e. $t\in[0,\omega)$, there exist functions $d,v:[0,\omega)\to\mathbb{R}^m$ (not necessarily measurable) such that $w(t)=d(t)-v(t)$, $d(t)\in\Delta(t)$ and $v(t)\in\Phi(Cx(t))$ for a.e. $t\in[0,\omega)$. Setting $\lambda:=(b-a)/2$ and invoking the sector condition (3.1) combined with part (a) of Remark 3.1, we may infer that

$$\begin{aligned}\|w(\tau)+\kappa Cx(\tau)\| &= \|d(\tau)-(v(\tau)-\kappa Cx(\tau))\| \\ &\le \|d(\tau)\| + \|(v(\tau)-\kappa Cx(\tau))\| \le |\Delta(\tau)|+\lambda\|Cx(\tau)\| \quad \text{for a.e. } \tau\in[0,\omega).\end{aligned} \tag{4.2}$$

Therefore,

$$\begin{aligned}\|x(t)\| \le \|e^{A_\kappa t}x^0\| + \|B\|\int_0^t \|e^{A_\kappa(t-\tau)}\||\Delta(\tau)|\mathrm{d}\tau \\ + \lambda\|B\|\|C\|\int_0^t \|e^{A_\kappa(t-\tau)}\|\|x(\tau)\|\mathrm{d}\tau \quad \forall\, t\in[0,\omega).\end{aligned}$$

Since (by supposition) ω is finite, we conclude that, for some constant $c>0$,

$$\|x(t)\| \le c\left(1+\int_0^t \|x(\tau)\|\mathrm{d}\tau\right) \quad \forall\, t\in[0,\omega).$$

By Gronwall's lemma, it follows that the maximal solution x is bounded on $[0,\omega)$, contradicting (via Lemma 2.2) the supposition that $\omega<\infty$. Consequently, $\omega=\infty$.

Defining $G_\kappa(s) := G(I+\kappa G(s))^{-1} = C(sI-A_\kappa)^{-1}B$, it follows from (H1), via Lemma 4.1, that $G_\kappa\in H^\infty$ and $\|G_\kappa\|_{H^\infty}<1/\lambda$. Moreover, by stabilizability and detectability, A_κ is Hurwitz. Let $\varepsilon>0$ be sufficiently small so that $A_\kappa+\varepsilon I$ is Hurwitz and

$$\gamma := \sup_{\operatorname{Re}s\ge-\varepsilon}\|G_\kappa(s)\| < 1/\lambda. \tag{4.3}$$

Set $y:=Cx$ and, for all $t\in\mathbb{R}_+$, define $y_\varepsilon(t):=e^{\varepsilon t}y(t)$ and $w_\varepsilon(t):=e^{\varepsilon t}w(t)$. It follows from (4.1) that

$$y_\varepsilon(t) = Ce^{(A_\kappa+\varepsilon I)t}x^0 + \int_0^t Ce^{(A_\kappa+\varepsilon I)(t-\tau)}B(w_\varepsilon(\tau)+\kappa y_\varepsilon(\tau))\mathrm{d}\tau \quad \forall\, t\in\mathbb{R}_+.$$

Setting $k_0 := \left(\int_0^\infty \|Ce^{(A_\kappa+\varepsilon I)\tau}\|^2 \mathrm{d}\tau\right)^{1/2} < \infty$, we obtain that

$$\|y_\varepsilon\|_{L^2[0,t]} \le k_0\|x^0\| + \gamma\|w_\varepsilon + \kappa y_\varepsilon\|_{L^2[0,t]} \quad \forall\, t \in \mathbb{R}_+. \tag{4.4}$$

By (4.2),

$$\|w_\varepsilon(\tau) + \kappa y_\varepsilon(\tau))\| \le |\Delta_\varepsilon(\tau)| + \lambda\|y_\varepsilon(\tau))\| \quad \text{for a.e. } \tau \in \mathbb{R}_+, \tag{4.5}$$

where $\Delta_\varepsilon(\tau) := e^{\varepsilon\tau}\Delta(\tau)\ \forall\ \tau \in \mathbb{R}_+$. From (4.3), we see that $\gamma\lambda < 1$: setting $k_1 := 1/(1-\gamma\lambda)$ and invoking (4.4) and (4.5), we have

$$\|y_\varepsilon\|_{L^2[0,t]} \le k_1\left(k_0\|x^0\| + \gamma\|\Delta_\varepsilon\|_{L^2[0,t]}\right) \quad \forall\, t \in \mathbb{R}_+. \tag{4.6}$$

By (4.1),

$$e^{\varepsilon t}x(t) = e^{(A_\kappa+\varepsilon I)t}x^0 + \int_0^t e^{(A_\kappa+\varepsilon I)(t-\tau)}B(w_\varepsilon(\tau) + \kappa y_\varepsilon(\tau))\mathrm{d}\tau \quad \forall\, t \in \mathbb{R}_+,$$

which together with (4.5) yields

$$\|x(t)\|e^{\varepsilon t} \le k_2\|x^0\| + \|B\|\int_0^t \|e^{(A_\kappa+\varepsilon I)(t-\tau)}\|(|\Delta_\varepsilon(\tau)| + \lambda\|y_\varepsilon(\tau)\|)\mathrm{d}\tau \quad \forall\, t \in \mathbb{R}_+,$$

where $k_2 := \sup_{t\ge 0}\|e^{(A_\kappa+\varepsilon I)t}\|$. Invoking Hölder's inequality to estimate the integral on the right-hand side of the above inequality, we conclude that there exists a constant $k_3 > 0$ such that

$$\|x(t)\|e^{\varepsilon t} \le k_2\|x^0\| + k_3(\|\Delta_\varepsilon\|_{L^2[0,t]} + \lambda\|y_\varepsilon\|_{L^2[0,t]}) \quad \forall\, t \in \mathbb{R}_+.$$

Combining this with (4.6), we conclude that

$$\|x(t)\|e^{\varepsilon t} \le (k_2 + \lambda k_0 k_1 k_3)\|x^0\| + k_3(1 + \lambda\gamma k_1)\|\Delta_\varepsilon\|_{L^2[0,t]} \quad \forall\, t \in \mathbb{R}_+.$$

Noting that $\|\Delta_\varepsilon\|_{L^2[0,t]} \le (e^{\varepsilon t}/\sqrt{2\varepsilon})\|\Delta\|_{L^\infty[0,t]}\ \forall\ t \in \mathbb{R}_+$, setting $c_1 := k_2 + \lambda k_0 k_1 k_3$ and $c_2 := k_3(1 + \lambda\gamma k_1)/\sqrt{2\varepsilon}$, we conclude that

$$\|x(t)\| \le c_1 e^{-\varepsilon t}\|x^0\| + c_2\|\Delta\|_{L^\infty[0,t]} \quad \forall\, t \in \mathbb{R}_+.$$

This completes the proof. ∎

Remark 4.2. Theorem 3.4 can be considered as a refinement of the classical circle criterion (see, for example, [7, 13, 21]). In particular, it shows that, under the standard assumptions of the circle criterion, ISS is guaranteed. The exponential weighting technique used in the proof of Theorem 3.4 is well known and has been used to prove stability results of input-output type (see [7, section V.3] and the references therein). The application of this technique in an ISS context seems to be new. In particular, whilst the standard textbook version of the circle criterion for state-space systems is usually proved using Lyapunov techniques combined with the positive-real lemma (see, for example, [13, Theorem 7.1] or [21, p. 227]), the above proof of Theorem 3.4 provides an alternative, more elementary approach. Moreover, the methodology can be extended to an infinite-dimensional setting; see [11].

5. Proof of Theorem 3.5. In this section, we provide a proof of Theorem 3.5. In contrast to the proof of Theorem 3.4, we adopt a Lyapunov argument. In particular, we prove Theorem 3.5 by establishing the existence of a Lyapunov function with special properties (a so-called ISS Lyapunov function) if any one of hypotheses (H2), (H3), or (H4) hold. This we do in two preliminary lemmas.

LEMMA 5.1. *Let linear system* (A, B, C) *be minimal. Assume that either* (H3) *or* (H4) *holds. Then there exist* $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{K}_\infty$, *and a continuously differentiable function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *such that*

$$\left.\begin{aligned} &\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \quad \forall\, x \in \mathbb{R}^n, \\ &\max_{v\in\Phi(Cx)} \langle \nabla V(x), Ax + B(d - v)\rangle \le -\alpha_3(\|x\|) + \alpha_4(\|d\|) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \forall\, (x, d) \in \mathbb{R}^n \times \mathbb{R}^m. \end{aligned}\right\} \tag{5.1}$$

Proof. By Lemma 2.1, there exists $\psi \in \mathcal{K}_\infty$ such that

$$\|v\| \le \psi(\|y\|) \quad \forall\, y \in \mathbb{R}^m \quad \forall\, v \in \Phi(y). \tag{5.2}$$

(If (H3) holds, then we may take $\psi : s \mapsto bs$ in (5.2).) Combining (5.2) with either (H3) or (H4) yields

$$\varphi(\|y\|)\|y\| \le \langle y, v\rangle \le \psi(\|y\|)\|y\| \quad \forall\, y \in \mathbb{R}^m \quad \forall\, v \in \Phi(y). \tag{5.3}$$

If (H3) holds, then $(\delta/b)I + G$ is positive real for some $\delta \in [0, 1)$; if (H4) holds, then G is positive real. Introducing the following notational convenience

$$\lambda := \begin{cases} 1/b & \text{if (H3) holds,} \\ 0 & \text{otherwise,} \end{cases}$$

both possibilities are captured by the statement that $\delta\lambda I + G$ is positive real for some $\delta \in [0, 1)$. This implies, via the positive-real lemma, the existence of a real matrix L and a symmetric, positive-definite real matrix P such that

$$PA + A^TP = -L^TL, \quad PB = C^T - \sqrt{\kappa}\, L^T, \quad \kappa := 2\delta\lambda. \tag{5.4}$$

We also record that

$$\lambda\|v\|^2 \le \langle y, v\rangle \quad \forall\, v \in \Phi(y) \quad \forall\, y \in \mathbb{R}^m. \tag{5.5}$$

Now, define $V_0 : \mathbb{R}^n \to \mathbb{R}_+,\ x \mapsto \langle x, Px\rangle$. Then, invoking (5.4),

$$\begin{aligned} \langle \nabla V_0(x), Ax + B(d - v)\rangle &= 2\langle Px, Ax\rangle + 2\langle B^TPx, (d - v)\rangle \\ &\le -\|Lx\|^2 + 2\langle Cx, (d - v)\rangle - 2\sqrt{\kappa}\langle Lx, (d - v)\rangle \\ &= -\|Lx + \sqrt{\kappa}(d - v)\|^2 + \kappa\|d - v\|^2 + 2\langle Cx, (d - v)\rangle \\ &\qquad\qquad \forall\, x \in \mathbb{R}^n, \quad \forall\, (d, v) \in \mathbb{R}^m \times \Phi(Cx)\,, \end{aligned}$$

from which, together with (5.5), we may infer

$$\begin{aligned} \langle \nabla V_0(x), Ax + B(d - v)\rangle &\le \kappa\|d\|^2 + 2\kappa\|v\|\|d\| + \kappa\|v\|^2 + 2\|y\|\|d\| - 2\langle y, v\rangle \\ &\le 2(1 + 2\delta)\|y\|\|d\| + \kappa\|d\|^2 - 2(1 - \delta)\langle y, v\rangle \\ &\qquad \forall\, x \in \mathbb{R}^n, \quad \forall\, (d, v) \in \mathbb{R}^m \times \Phi(y), \ y = Cx. \end{aligned} \tag{5.6}$$

Observe that, for all $y \in \mathbb{R}^m$ and all $(d,v) \in \mathbb{R}^m \times \Phi(y)$,

$$\begin{aligned}
&2(1+2\delta)\|d\| \le (1-\delta)\varphi(\|y\|) \implies \\
&\qquad\qquad 2(1+2\delta)\|d\|\|y\| \le (1-\delta)\varphi(\|y\|)\|y\| \le (1-\delta)\langle y, v\rangle, \\
&2(1+2\delta)\|d\| > (1-\delta)\varphi(\|y\|) \implies \\
&\qquad\qquad 2(1+2\delta)\|d\|\|y\| < 2(1+2\delta)\|d\|\varphi^{-1}(2(1+2\delta)\|d\|/(1-\delta))
\end{aligned}$$

and so, defining $\gamma \in \mathcal{K}_\infty$ by $\gamma(s) := 2(1+2\delta)s\,\varphi^{-1}\,(2(1+2\delta)s/(1-\delta))$, we have

$$2(1+2\delta)\|d\|\|y\| \le (1-\delta)\langle y, v\rangle + \gamma(\|d\|) \quad \forall\, y \in \mathbb{R}^m \quad \forall\, (d,v) \in \mathbb{R}^m \times \Phi(y). \tag{5.7}$$

The conjunction of (5.6) and (5.7) gives

$$\begin{aligned}
&\langle \nabla V_0(x), Ax + B(d-v)\rangle \le -(1-\delta)\langle y, v\rangle + \gamma(\|d\|) + \kappa\|d\|^2 \\
&\qquad\qquad \forall\, x \in \mathbb{R}^n,\ \forall\, (d,v) \in \mathbb{R}^m \times \Phi(y),\ y = Cx.
\end{aligned} \tag{5.8}$$

Let $H \in \mathbb{R}^{n\times m}$ be such that $A - HC$ is Hurwitz. Let $Q = Q^T > 0$ be such that

$$Q(A-HC) + (A-HC)^T Q = -3I$$

and define $W : \mathbb{R}^n \to \mathbb{R}_+$ by $W(x) := \langle x, Qx\rangle$.

Writing $k_0 := \max\{2\|QB\|\,,\,2\|QH\|\,,\,\|QB\|^2\}$, we have

$$\begin{aligned}
\langle \nabla W(x), Ax + B(d-v)\rangle &= 2\langle Qx, (A-HC)x + Hy + B(d-v)\rangle \\
&= -3\|x\|^2 + 2\langle H^T Qx, y\rangle + 2\langle B^T Qx, d-v\rangle \\
&\le -2\|x\|^2 + k_0\|x\|\big(\|y\| + \|v\|\big) + k_0\|d\|^2 \\
&\qquad \forall\, x \in \mathbb{R}^n, \forall\, (d,v) \in \mathbb{R}^m \times \Phi(y),\ y = Cx.
\end{aligned} \tag{5.9}$$

Since either (H3) or (H4) holds and invoking part (f) of Remark 3.1 in the former case, we may infer the existence of $\theta \ge 1/2$ such that

$$y \in \mathbb{R}^m,\quad \|y\| \ge \theta \quad \implies \quad \langle y, v\rangle \ge \|v\| \quad \forall\, v \in \Phi(y). \tag{5.10}$$

Define $f_0 \in \mathcal{K}_\infty$ by $f_0(s) := s + \psi(s)$, the continuous, nondecreasing function $f_1 : (0,\theta] \to (0,\infty)$ by

$$f_1(s) := \min_{t\in[s,\theta]} \frac{t\varphi(t)}{(f_0(t))^2},$$

and $f_2 : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f_2(s) := \begin{cases} 0, & s = 0, \\ \min\{s, f_1(s)\}, & s \in (0,\theta], \\ f_1(\theta) + (s-\theta), & s > \theta. \end{cases}$$

Observe that

$$f_1(\theta) = \frac{\theta\varphi(\theta)}{(\theta + \psi(\theta))^2} < \frac{\theta\varphi(\theta)}{2\theta\psi(\theta)} \le \frac{\theta\varphi(\theta)}{\psi(\theta)} \le \theta\,,$$

where we have used that $\theta \ge 1/2$. It follows that $f_2(\theta) = f_1(\theta)$, and therefore, f_2 is continuous. Clearly, f_2 is unbounded, and moreover, it is readily verified that f_2

is nondecreasing. Write $f_3 := f_2 \circ f_0^{-1}$ (continuous, nondecreasing, and unbounded, with $f_3(0) = 0$) and observe (for later use) that

$$\text{(5.11)}\quad \|y\| < \theta \implies f_3(\|y\| + \psi(\|y\|))(\|y\| + \psi(\|y\|))^2 = (f_3 \circ f_0)(\|y\|)(f_0(\|y\|))^2 \\ = f_2(\|y\|)(f_0(\|y\|))^2 \le f_1(\|y\|)(f_0(\|y\|))^2 \le \|y\|\varphi(\|y\|).$$

Next, we introduce functions $\eta \in \mathcal{K}_\infty$ and σ (continuous, nondecreasing, and unbounded, with $\sigma(0) = 0$) given by

$$\eta : s \mapsto \frac{1}{k_0}\sqrt{\frac{s}{\|Q\|}}\,, \quad \sigma := f_3 \circ \eta.$$

Let $s^* > 0$ be the unique point with the property $\eta(s^*)\sigma(s^*) = 1$ and define the continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\rho(s) := \begin{cases} \sigma(s), & 0 \le s \le s^*, \\ 1/\eta(s), & s > s^*\,. \end{cases}$$

Finally, define $R \in \mathcal{K}_\infty$ by

$$R(s) := \int_0^s \rho(\tau)\, d\tau,$$

and $V_1 : \mathbb{R}^n \to \mathbb{R}_+,\ \ x \mapsto R(W(x))$. Note that

$$\text{(5.12)}\quad \left.\begin{array}{ll} \text{(a)} & \rho(s) \le \sigma(s) \le \sigma(s^*) =: k_1 \ \ \forall\, s \in \mathbb{R}_+\,, \\ \text{(b)} & \rho(W(x))\|x\| \le k_0\sqrt{\|Q\|\|Q^{-1}\|} =: k_2 \ \ \forall\, x \in \mathbb{R}^n, \\ \text{(c)} & \rho(W(x))\|x\|^2 \ge \|x\| \min\left\{\|x\| f_3\big(\|x\|/k_2\big)\,,\, k_0\right\} \ \ \forall\, x \in \mathbb{R}^n. \end{array}\right\}$$

Invoking (5.9) and (5.12)(a), we have

$$\text{(5.13)}\quad \langle \nabla V_1(x), Ax + B(d - v)\rangle \le -2\rho(W(x))\|x\|^2 + \rho(W(x))k_0\|x\|\big(\|y\| + \|v\|\big) + k_0 k_1\|d\|^2 \\ \forall\, x\ \in \mathbb{R}^n\ \forall\, (d, v) \in \mathbb{R}^m \times \Phi(Cx).$$

We proceed to obtain a convenient estimate of the term $\rho(W(x))k_0\|x\|\big(\|y\| + \|v\|\big)$.

Write $k_3 := \frac{1}{2}\min\{1, \varphi(\theta)\}$. By (5.3) and (5.10), we have

$$\|y\| \ge \theta \implies 2\langle y, v\rangle \ge \|v\| + \|y\|\varphi(\|y\|) \ge \|v\| + \|y\|\varphi(\theta) \ge 2k_3(\|v\| + \|y\|) \\ \forall\, v \in \Phi(y),$$

which, in conjunction with (5.12)(b), gives

$$\text{(5.14)}\quad x \in \mathbb{R}^n,\ y = Cx,\ \|y\| \ge \theta \implies \\ \rho(W(x))k_0\|x\|\big(\|y\| + \|v\|\big) \le \frac{k_0 k_2}{k_3}\langle y, v\rangle \ \ \forall\, v \in \Phi(y).$$

Invoking (5.2), (5.3), and (5.11), we have

$$\|y\| < \theta \implies f_3(\|y\| + \|v\|)(\|y\| + \|v\|)^2 \\ \le f_3(\|y\| + \psi(\|y\|))(\|y\| + \psi(\|y\|))^2 \le \|y\|\varphi(\|y\|) \le \langle y, v\rangle \ \ \forall\, v \in \Phi(y)$$

from which, together with the observation that

$$x \in \mathbb{R}^n,\ y = Cx,\ v \in \Phi(y),\ k_0(\|y\| + \|v\|) \geq \|x\| \implies \\ \rho(W(x)) \leq \sigma(\|Q\|\|x\|^2)) \leq \sigma(k_0^2\|Q\|(\|y\| + \|v\|)^2) = f_3(\|y\| + \|v\|),$$

we may infer

$$\text{(5.15)}\quad x \in \mathbb{R}^n,\ y = Cx,\ v \in \Phi(y),\ k_0(\|y\| + \|v\|) \geq \|x\|,\ \|y\| < \theta \implies \\ \rho(W(x))k_0\|x\|\big(\|y\| + \|v\|\big) \leq \rho(W(x))\|x\|^2 + \frac{k_0^2}{4}\rho(W(x))(\|y\| + \|v\|)^2 \\ \leq \rho(W(x))\|x\|^2 + \frac{k_0^2}{4}\langle y, v\rangle\,.$$

Clearly,

$$\text{(5.16)}\quad x \in \mathbb{R}^n,\ y = Cx,\ v \in \Phi(y),\ k_0(\|y\| + \|v\|) \leq \|x\|,\ \|y\| < \theta \implies \\ \rho(W(x))k_0\|x\|\big(\|y\| + \|v\|\big) \leq \rho(W(x))\|x\|^2.$$

Combining (5.15) and (5.16), we have

$$\text{(5.17)}\quad x \in \mathbb{R}^n,\ y = Cx,\ \|y\| < \theta \implies \\ \rho(W(x))k_0\|x\|\big(\|y\| + \|v\|\big) \leq \rho(W(x))\|x\|^2 + \frac{k_0^2}{4}\langle y, v\rangle \quad \forall\ v \in \Phi(y).$$

Writing $k_4 := \max\big\{k_0k_2/k_3\,,\ k_0^2/4\big\}$, we conclude from (5.13), (5.14), (5.17) that

$$\text{(5.18)}\quad \langle \nabla V_1(x), Ax + B(d - v)\rangle \leq -\rho(W(x))\|x\|^2 + k_4\langle y, v\rangle + k_0k_1\|d\|^2 \\ \forall\ x \in \mathbb{R}^n\ \forall\ (d, v) \in \mathbb{R}^m \times \Phi(Cx).$$

Now define $V := k_4V_0 + (1 - \delta)V_1$. Then, combining (5.8) and (5.18), we arrive at

$$\text{(5.19)}\quad \langle \nabla V(x), Ax + B(d - v)\rangle \\ \leq -(1 - \delta)\rho(W(x))\|x\|^2 + \big((1 - \delta)k_0k_1 + \kappa k_4\big)\|d\|^2 + k_4\gamma(\|d\|) \\ \forall\ x \in \mathbb{R}^n,\ \ \forall\ (d, v) \in \mathbb{R}^m \times \Phi(y),\ y = Cx.$$

Finally, defining $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4 \in \mathcal{K}_\infty$ by

$$\alpha_1(s) := k_4\|P^{-1}\|^{-1}s^2, \quad \alpha_2 := k_4\|P\|s^2 + (1 - \delta)R(\|Q\|s^2),$$

$$\alpha_3(s) := (1 - \delta)s\min\{sf_3(s/k_2)\,,\ k_0\}, \quad \alpha_4(s) := \big((1 - \delta)k_0k_1 + \kappa k_4\big)s^2 + k_4\gamma(s),$$

and invoking (5.12)(c), we conclude that (5.1) holds. This completes the proof. ∎

Lemma 5.2. *Let linear system* (A, B, C) *be minimal. Assume that* (H2) *holds. Then the assertions of Lemma* 5.1 *are valid.*

Proof. Let $a > 0$, $\delta \in [0, 1)$, and $\theta \geq 0$ be as in hypothesis (H2). Without loss of generality, we may assume $\theta \geq 1/2$. Note that linear system $(A_1,\ B,\ C)$, with $A_1 := A - \delta aBC$, is a minimal realization of $G(I + \delta aG)^{-1}$. Therefore, hypothesis (H2) implies, via the positive-real lemma, the existence of a real matrix L and a symmetric, positive-definite real matrix P such that

$$\text{(5.20)}\qquad PA_1 + A_1^TP = -L^TL, \quad PB = C^T.$$

Invoking Lemma 2.1, there exists $\psi \in \mathcal{K}_\infty$ such that (5.3) holds with $\varphi(s) = as$. Now define $\varphi_1, \psi_1 \in \mathcal{K}_\infty$, and $y \mapsto \Phi_1(y) \subset \mathbb{R}^m$ by

$$\varphi_1(s) := \varphi(s) - \delta as = (1-\delta)as, \quad \psi_1(s) := \psi(s) - \delta as \quad \forall\, s \in \mathbb{R}_+,$$

$$\Phi_1(y) := \{v - \delta ay \mid v \in \Phi(y)\} \quad \forall\, y \in \mathbb{R}^m.$$

In view of (5.3), we have

$$(5.21) \quad (1-\delta)a\|y\|^2 = \varphi_1(\|y\|)\|y\| \le \langle y, v\rangle \le \psi_1(\|y\|)\|y\| \quad \forall\, y \in \mathbb{R}^m \quad \forall\, v \in \Phi_1(y).$$

Moreover, by hypothesis (H2),

$$(5.22) \qquad y \in \mathbb{R}^m, \quad \|y\| \ge \theta \quad \Longrightarrow \quad \langle y, v\rangle \ge \|v\| \quad \forall\, v \in \Phi_1(y).$$

Recalling that $A_1 := A - \delta aBC$, we have

$$(5.23) \qquad \big\{Ax - Bv \mid v \in \Phi(Cx)\big\} = \big\{A_1x - Bv \mid v \in \Phi_1(Cx)\big\} \quad \forall\, x \in \mathbb{R}^n.$$

Now, define $V_0 : \mathbb{R}^n \to \mathbb{R}_+$, $x \mapsto \langle x, Px\rangle$. Then, invoking (5.20),

$$(5.24) \quad \begin{aligned} \langle \nabla V_0(x), A_1x + B(d-v)\rangle &= 2\langle Px, A_1x\rangle + 2\langle B^TPx, (d-v)\rangle \\ &\le -\|Lx\|^2 + 2\langle Cx, (d-v)\rangle \le 2\|y\|\|d\| - 2\langle y, v\rangle \\ &\qquad \forall\, x \in \mathbb{R}^n, \ \ \forall\, (d,v) \in \mathbb{R}^m \times \Phi_1(y), \ y = Cx. \end{aligned}$$

Observe that, for all $y \in \mathbb{R}^m$ and all $(d,v) \in \mathbb{R}^m \times \Phi_1(y)$,

$$2\|d\| \le \varphi_1(\|y\|) \implies 2\|d\|\|y\| \le \varphi_1(\|y\|)\|y\| \le \langle y, v\rangle,$$

$$2\|d\| > \varphi_1(\|y\|) \implies 2\|d\|\|y\| < 2\|d\|\varphi_1^{-1}(2\|d\|)$$

and so, defining $\gamma \in \mathcal{K}_\infty$ by $\gamma(s) := 2s\,\varphi_1^{-1}(2s)$, it follows from (5.24) that

$$(5.25) \quad \begin{aligned} &\langle \nabla V_0(x), A_1x + B(d-v)\rangle \le -\langle y, v\rangle + \gamma(\|d\|) \\ &\qquad \forall\, x \in \mathbb{R}^n, \ \forall\, (d,v) \in \mathbb{R}^m \times \Phi_1(y), \ y = Cx. \end{aligned}$$

The conjunction of (5.23) and (5.25) yields

$$(5.26) \quad \begin{aligned} &\langle \nabla V_0(x), Ax + B(d-v)\rangle \le -\langle y, v\rangle + \delta a\|y\|^2 + \gamma(\|d\|) \\ &\qquad \forall\, x \in \mathbb{R}^n, \ \forall\, (d,v) \in \mathbb{R}^m \times \Phi(y), \ y = Cx. \end{aligned}$$

Let $H \in \mathbb{R}^{n\times m}$ be such that $A_1 - HC$ is Hurwitz. Let $Q = Q^T > 0$ be such that

$$Q(A_1 - HC) + (A_1 - HC)^TQ = -3I$$

and define $W : \mathbb{R}^n \to \mathbb{R}_+$ by $W(x) := \langle x, Qx\rangle$. The same construction as in the proof of Lemma 5.1 (with A_1 replacing A and Φ_1 replacing Φ therein) yields a function f_3 (continuous, nondecreasing, and unbounded, with $f_3(0) = 0$), a continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$, with primitive $R \in \mathcal{K}_\infty$, and positive constants c_0, c_1, c_2, c_3 such that, on writing $V_1 : \mathbb{R}^n \to \mathbb{R}_+$, $x \mapsto R(W(x))$, the following counterparts of (5.12)(c) and

(5.18) hold:

$$\rho(W(x))\|x\|^2 \geq \|x\| \min\{c_0,\, \|x\| f_3(c_1\|x\|)\} \quad \forall\, x \in \mathbb{R}^n, \tag{5.27}$$

$$\langle \nabla V_1(x), A_1 x + B(d-v)\rangle \leq -\rho(W(x))\|x\|^2 + c_2\langle y, v\rangle + c_3\|d\|^2$$
$$\forall\, x \in \mathbb{R}^n, \forall\, (d,v) \in \mathbb{R}^m \times \Phi_1(y),\ y = Cx.$$

In view of (5.23), the latter yields

$$\langle \nabla V_1(x), Ax + B(d-v)\rangle \leq -\rho(W(x))\|x\|^2 + c_2\langle y, v\rangle - c_2\delta a\|y\|^2 + c_3\|d\|^2$$
$$\forall\, x \in \mathbb{R}^n, \forall\, (d,v) \in \mathbb{R}^m \times \Phi(y),\ y = Cx. \tag{5.28}$$

Now define $V := c_2 V_0 + V_1$. Then, combining (5.26) and (5.28), we have

$$\langle \nabla V(x), Ax + B(d-v)\rangle \leq -\rho(W(x))\|x\|^2 + c_2\gamma(\|d\|) + c_3\|d\|^2$$
$$\forall\, x \in \mathbb{R}^n\ \forall\, (d,v) \in \mathbb{R}^m \times \Phi(Cx). \tag{5.29}$$

Finally, defining α_1, α_2, α_3, $\alpha_4 \in \mathcal{K}_\infty$ by

$$\alpha_1(s) := c_2\|P^{-1}\|^{-1}s^2, \quad \alpha_2 := c_2\|P\|s^2 + R(\|Q\|s^2),$$
$$\alpha_3(s) := s\min\{c_0,\, s f_3(c_1 s)\}, \quad \alpha_4(s) := c_2\gamma(s) + c_3 s^2$$

and invoking (5.27), we may conclude that (5.1) holds. This completes the proof. □

We are now in a position to prove Theorem 3.5. The argument developed below is not new and can be found (usually in form of sketch proofs) in the literature (see [20] and the references therein). For completeness, we provide a detailed proof.

Proof of Theorem 3.5. If either (H3) or (H4) holds (respectively, if (H2) holds), then Lemma 5.1 (respectively, Lemma 5.2) ensures the existence of $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{K}_\infty$ and continuously differentiable V such that $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)\ \forall\, x \in \mathbb{R}^n$ and

$$\langle \nabla V(x), Ax + B(d-v)\rangle \leq -\alpha_3(\|x\|) + \alpha_4(\|d\|)$$
$$\forall\, x \in \mathbb{R}^n\ \ \forall\, (d,v) \in \mathbb{R}^m \times \Phi(Cx). \tag{5.30}$$

Let $x^0 \in \mathbb{R}^n$ and $\Delta \in \mathcal{B}$. By Lemma 2.2, (2.1) has a solution, and every solution can be maximally extended. Let $x : [0,\omega) \to \mathbb{R}^n$ be a maximal solution of (2.1). By (5.30), we have

$$(V \circ x)'(t) \leq \alpha_4(|\Delta(t)|) \quad \text{for a.e. } t \in [0,\omega). \tag{5.31}$$

Seeking a contradiction, suppose that $\omega < \infty$. Then, by local essential boundedness of Δ and continuity of α_4, there exists $c_0 > 0$ such that $\alpha_4(|\Delta(t)|) \leq c_0\ \forall\, t \in [0,\omega)$. Now, by the final assertion of Lemma 2.2, x is unbounded, which contradicts the fact that, by (5.31), $\alpha_1(\|x(t)\|) \leq V(x(t)) \leq V(x^0) + c_0\omega\ \forall\, t \in [0,\omega)$. Therefore, every maximal solution of (2.1) is global.

Write $\alpha_5 := \alpha_3 \circ \alpha_2^{-1} \in \mathcal{K}_\infty$ and define $\alpha_6 : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\alpha_6(s) := \frac{2}{s}\int_{s/2}^{s} \alpha_5(t)dt \ \ \forall\, s > 0, \quad \alpha_6(0) := \lim_{s\downarrow 0}\alpha_6(s) = 0.$$

Since $\alpha_5 \in \mathcal{K}_\infty$, we have $\alpha_5(s/2) \le \alpha_6(s) \le \alpha_5(s)$ $\forall\ s \in \mathbb{R}_+$, and moreover, α_6 is differentiable on $(0,\infty)$, with derivative $\alpha_6'(s) \ge 0$ $\forall\ s \in (0,\infty)$. Now define $\alpha_7 : \mathbb{R}_+ \to \mathbb{R}_+$ by $\alpha_7(s) := \min\{1, s\}\alpha_6(s)$. Clearly, α_7 is locally Lipschitz, $\alpha_7(0) = 0$, and $0 < \alpha_7(s) \le \alpha_5(s)$ $\forall\ s > 0$. Define the locally Lipschitz function

$$Z : \mathbb{R} \to \mathbb{R}, \quad \zeta \mapsto Z(\zeta) := \begin{cases} -\alpha_7(\zeta)/2, & \zeta \ge 0, \\ \alpha_7(-\zeta)/2, & \zeta < 0, \end{cases}$$

and consider the scalar system

$$\dot z(t) = Z(z(t)).$$

Since $Z(0) = 0$ and $\zeta Z(\zeta) = -|\zeta|\alpha_7(|\zeta|)/2 < 0$ $\forall\ \zeta \ne 0$, it follows that 0 is a globally asymptotically stable equilibrium of this system which, together with the local Lipschitz property of Z, ensures the existence of a continuous global semiflow $\beta : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ (and so, for each $z^0 \in \mathbb{R}$, $z : \mathbb{R}_+ \to \mathbb{R}$, $t \mapsto \beta(t, z^0)$, is the unique global solution of the initial-value problem $\dot z = Z(z)$, $z(0) = z^0$; moreover, $\beta(t, z^0) \to 0$ as $t \to \infty$). Let $\beta_0 := \beta|_{\mathbb{R}_+\times\mathbb{R}_+}$ be the restriction of β to $\mathbb{R}_+ \times \mathbb{R}_+$. Evidently, $\beta_0 \in \mathcal{KL}$. Now define $\beta_1 \in \mathcal{KL}$ by

$$\beta_1(t, s) := \alpha_1^{-1}(\beta_0(t, \alpha_2(s)))$$

and define $\beta_2 \in \mathcal{K}_\infty$ by

$$\beta_2(s) := \big(\alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}\big)(2\alpha_4(s)).$$

Let x^0 and $\Delta \in \mathcal{B}$ be arbitrary, and let x be a global solution of (2.1). Let $t \in \mathbb{R}_+$ be arbitrary. By (5.30), we have

$$(V \circ x)'(\tau) \le -\alpha_3(\|x(\tau)\|) + \alpha_4(|\Delta(\tau)|) \le -\alpha_3(\|x(\tau)\|) + \alpha_4(\|\Delta\|_{L^\infty[0,t]}) \quad \text{for a.e. } \tau \in [0, t]. \tag{5.32}$$

Clearly,

$$V(x(t)) \le \big(\alpha_2 \circ \alpha_3^{-1}\big)(2\alpha_4(\|\Delta\|_{L^\infty[0,t]})) \implies \|x(t)\| \le \beta_2(\|\Delta\|_{L^\infty[0,t]}).$$

Moreover,

$$V(x(t)) > \big(\alpha_2 \circ \alpha_3^{-1}\big)(2\alpha_4(\|\Delta\|_{L^\infty[0,t]})) \implies \alpha_3(\|x(t)\|) > 2\alpha_4(\|\Delta\|_{L^\infty[0,t]}),$$

which, together with (5.32), yields

$$\begin{aligned} &V(x(t)) > \big(\alpha_2 \circ \alpha_3^{-1}\big)(2\alpha_4(\|\Delta\|_{L^\infty[0,t]})) \\ &\quad\implies (V \circ x)'(\tau) < -\tfrac{1}{2}\alpha_3(\|x(\tau)\|) \le -\tfrac{1}{2}\alpha_5(V(x(\tau))) \le -\tfrac{1}{2}\alpha_7(V(x(\tau))) \\ &\qquad\qquad = Z(V(x(\tau))) \quad \text{for a.e. } \tau \in [0, t], \end{aligned}$$

and so

$$\begin{aligned} &V(x(t)) > \big(\alpha_2 \circ \alpha_3^{-1}\big)(2\alpha_4(\|\Delta\|_{L^\infty[0,t]}) \\ &\qquad \implies V(x(t)) \le \beta_0(t, V(x^0)) \implies \|x(t)\| \le \beta_1(t, \|x^0\|). \end{aligned}$$

Therefore, $\|x(t)\| \le \max\big\{\beta_1(t, \|x^0\|)\,,\, \beta_2(\|\Delta\|_{L^\infty[0,t]})\big\}$ $\forall\ t \in \mathbb{R}_+$. $\square$

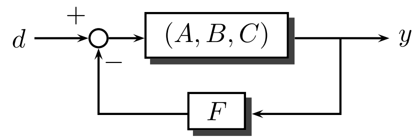

FIG. 6.1. *Interconnection of linear system (A, B, C) and hysteresis operator F.*

6. Hysteretic feedback systems. We return to the feedback interconnection of Figure 1.2, but now in a single-input ($t \mapsto d(t) \in \mathbb{R}$), single-output ($t \mapsto y(t) \in \mathbb{R}$) setting and with a hysteresis operator F in the feedback path, as shown in Figure 6.1. We deem an operator $F : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ to be a *hysteresis operator* if it is both causal and rate independent. By *rate independence*, we mean that $F(y \circ \zeta) = (Fy) \circ \zeta$ for every $y \in C(\mathbb{R}_+)$ and every time transformation $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ (that is, a continuous, nondecreasing, and surjective map). Conditions on F which ensure well-posedness of the feedback interconnection (existence and uniqueness of solutions of the associated initial-value problem) are expounded in, for example, [16] and [17]. Whilst, in principle, the ensuing analysis is applicable in the context of any causal operator F that can be embedded in a set-valued map $\Phi \in \mathcal{U}$, for clarity of presentation, we focus on the class of Preisach operators.

Preisach and Prandtl hysteresis. The Preisach operator described in this section encompasses both backlash and Prandtl operators. It can model complex hysteresis effects: For example, nested loops in input-output characteristics. A basic building block for these operators is the *backlash* operator. A discussion of the *backlash* operator (also called *play* operator) can be found in a number of references; see, for example, [5], [14], and [15]. Let $\sigma \in \mathbb{R}_+$ and introduce the function $b_\sigma \colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$b_\sigma(v_1, v_2) := \max\big\{v_1 - \sigma\,,\, \min\{v_1 + \sigma, v_2\}\big\} = \begin{cases} v_1 - \sigma, & \text{if } v_2 < v_1 - \sigma, \\ v_2, & \text{if } v_2 \in [v_1 - \sigma, v_1 + \sigma], \\ v_1 + \sigma, & \text{if } v_2 > v_1 + \sigma. \end{cases}$$

Let $C_{\mathrm{pm}}(\mathbb{R}_+)$ denote the space of continuous piecewise monotone functions defined on $\mathbb{R}_+$. For all $\sigma \in \mathbb{R}_+$ and $\zeta \in \mathbb{R}$, define the operator $\mathcal{B}_{\sigma,\,\zeta} : C_{\mathrm{pm}}(\mathbb{R}_+) \to C(\mathbb{R}_+)$ by

$$\mathcal{B}_{\sigma,\,\zeta}(y)(t) = \left\{ \begin{array}{ll} b_\sigma(y(0), \zeta) & \text{for } t = 0\,, \\ b_\sigma(y(t), (\mathcal{B}_{\sigma,\,\zeta}(u))(t_i)) & \text{for } t_i < t \le t_{i+1},\ i = 0, 1, 2, \ldots, \end{array} \right.$$

where $0 = t_0 < t_1 < t_2 < \ldots$, $\lim_{n\to\infty} t_n = \infty$, and u is monotone on each interval $[t_i, t_{i+1}]$. We remark that ζ plays the role of an "initial state." It is not difficult to show that the definition is independent of the choice of the partition (t_i). Figure 6.2 illustrates how $\mathcal{B}_{\sigma,\,\zeta}$ acts. It is well known that $\mathcal{B}_{\sigma,\,\zeta}$ extends to a Lipschitz continuous hysteresis operator on $C(\mathbb{R}_+)$ (with Lipschitz constant $L = 1$), the so-called backlash operator, which we shall denote by the same symbol $\mathcal{B}_{\sigma,\,\zeta}$.

Let $\xi : \mathbb{R}_+ \to \mathbb{R}$ be a compactly supported and globally Lipschitz function with Lipschitz constant 1. Let μ be a regular signed Borel measure on $\mathbb{R}_+$. Denoting Lebesgue measure on $\mathbb{R}$ by μ_L, let $w : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be a locally $(\mu_L \otimes \mu)$-integrable function, and let $w_0 \in \mathbb{R}$. The operator $\mathcal{P}_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by

$$(6.1)\quad (\mathcal{P}_\xi(y))(t) = \int_0^\infty \int_0^{(\mathcal{B}_{\sigma,\,\xi(\sigma)}(y))(t)} w(s, \sigma)\mu_L(\mathrm{d}s)\mu(\mathrm{d}\sigma) + w_0 \qquad \forall\, u \in C(\mathbb{R}_+)\ \ \forall\, t \in \mathbb{R}_+$$

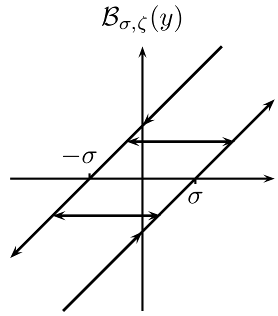

FIG. 6.2. *Backlash hysteresis.*

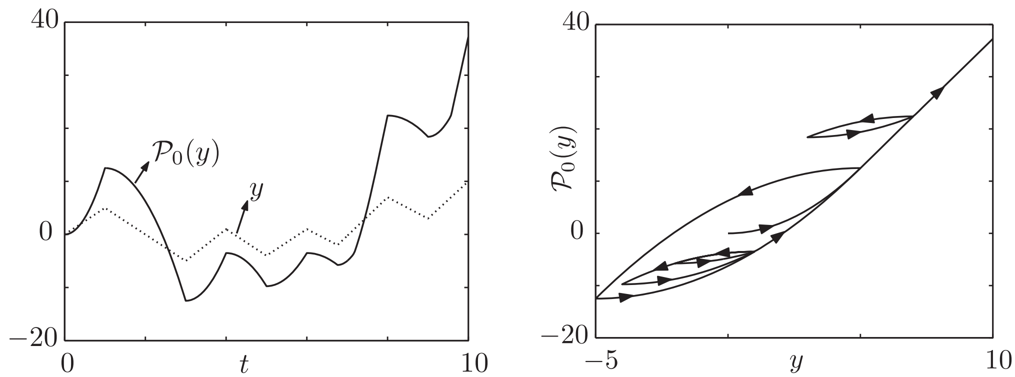

FIG. 6.3. *Example of Prandtl hysteresis.*

is called a *Preisach* operator: This definition is equivalent to that adopted in [5, section 2.4]. It is well known that $\mathcal{P}_\xi$ is a hysteresis operator (this follows from the fact that $\mathcal{B}_{\sigma,\,\xi(\sigma)}$ is a hysteresis operator for every $\sigma \geq 0$). Under the assumption that the measure μ is finite and w is essentially bounded, the operator $\mathcal{P}_\xi$ is Lipschitz continuous with Lipschitz constant $L = |\mu|(\mathbb{R}_+)\|w\|_\infty$ (see [15]) in the sense that

$$\sup_{t\in\mathbb{R}_+} |\mathcal{P}_\xi(y_1)(t) - \mathcal{P}_\xi(y_2)(t)| \leq L \sup_{t\in\mathbb{R}_+} |y_1(t) - y_2(t)| \quad \forall\, y_1, y_2 \in C(\mathbb{R}_+).$$

This property ensures the well-posedness of the feedback interconnection.

Setting $w(\cdot,\cdot) = 1$ and $w_0 = 0$ in (6.1), we obtain the *Prandtl* operator $\mathcal{P}_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by

$$\mathcal{P}_\xi(y)(t) = \int_0^\infty (\mathcal{B}_{\sigma,\,\xi(\sigma)}(y))(t)\mu(\mathrm{d}\sigma) \quad \forall\, u \in C(\mathbb{R}_+) \;\; \forall\, t \in \mathbb{R}_+. \tag{6.2}$$

For $\xi \equiv 0$ and μ given by $\mu(E) = \int_E \chi_{[0,5]}(\sigma)\mathrm{d}\sigma$ (where $\chi_{[0,5]}$ denotes the indicator function of the interval $[0, 5]$), the Prandtl operator is illustrated in Figure 6.3.

The next proposition identifies conditions under which Preisach operator (6.1) satisfies a generalized sector bound. For simplicity, we assume that the measure μ and the function w are nonnegative (an important case in applications), although the proposition can be extended to signed measures μ and sign-indefinite functions w.

PROPOSITION 6.1. *Let $\mathcal{P}_\xi$ be the Preisach operator defined in* (6.1). *Assume that the measure μ is nonnegative, $a_1 := \mu(\mathbb{R}_+) < \infty$, $a_2 := \int_0^\infty \sigma\mu(\mathrm{d}\sigma) < \infty$, $b_1 := \operatorname{ess\,inf}_{(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+} w(s,\sigma) \geq 0$, $b_2 := \operatorname{ess\,sup}_{(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+} w(s,\sigma) < \infty$, and set*

$$a_{\mathcal{P}} := a_1 b_1, \quad b_{\mathcal{P}} := a_1 b_2\,, \quad c_{\mathcal{P}} := a_2 b_2 + |w_0|. \tag{6.3}$$

Then, for all $y \in C(\mathbb{R}_+)$ and all $t \in \mathbb{R}_+$,

$$y(t) \geq 0 \implies a_{\mathcal{P}} y(t) - c_{\mathcal{P}} \leq (\mathcal{P}_\xi(y))(t) \leq b_{\mathcal{P}} y(t) + c_{\mathcal{P}}, \tag{6.4}$$

$$y(t) \leq 0 \implies b_{\mathcal{P}} y(t) - c_{\mathcal{P}} \leq (\mathcal{P}_\xi(y))(t) \leq a_{\mathcal{P}} y(t) + c_{\mathcal{P}}, \tag{6.5}$$

and furthermore, for every $\eta > 0$,

$$|y(t)| \geq c_{\mathcal{P}}/\eta \implies (a_{\mathcal{P}} - \eta) y^2(t) \leq (\mathcal{P}_\xi(y))(t) y(t) \leq (b_{\mathcal{P}} + \eta) y^2(t). \tag{6.6}$$

Proof. Let $y \in C(\mathbb{R}_+)$ and $t \in \mathbb{R}_+$ be arbitrary. Note initially that, by the definition of the backlash operator,

$$\big(\mathcal{B}_{\sigma,\xi(\sigma)}(y)\big)(t) \in [y(t) - \sigma, y(t) + \sigma] \ \ \forall\, \sigma \in \mathbb{R}_+.$$

Case 1. Assume $y(t) \geq 0$. Writing $E_1 := [0, y(t)]$ and $E_2 := (y(t), \infty)$, we have

$$\begin{aligned}
\big(\mathcal{P}_\xi y\big)(t) &\geq \left(\int_{E_1} + \int_{E_2}\right) \int_0^{y(t)-\sigma} w(s,\sigma)\mu_L(\mathrm{d}s)\mu(\mathrm{d}\sigma) - |w_0| \\
&\geq b_1 \int_{E_1} (y(t) - \sigma)\mu(\mathrm{d}\sigma) + b_2 \int_{E_2} (y(t) - \sigma)\mu(\mathrm{d}\sigma) - |w_0| \\
&= \big(b_1\mu(E_1) + b_2\mu(E_2)\big) y(t) - b_1 \int_{E_1} \sigma\,\mu(\mathrm{d}\sigma) - b_2 \int_{E_2} \sigma\,\mu(\mathrm{d}\sigma) - |w_0| \\
&\geq a_1 b_1 y(t) - a_2 b_2 - |w_0| = a_{\mathcal{P}} y(t) - c_{\mathcal{P}}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\big(\mathcal{P}_\xi y\big)(t) &\leq \int_0^\infty \int_0^{y(t)+\sigma} w(s,\sigma)\mu_L(\mathrm{d}s)\mu(\mathrm{d}\sigma) + |w_0| \\
&\leq b_2 \int_0^\infty (y(t) + \sigma)\mu(\mathrm{d}\sigma) + |w_0| \leq a_1 b_2 y(t) + a_2 b_2 + |w^0| = b_{\mathcal{P}} y(t) + c_{\mathcal{P}}.
\end{aligned}$$

This establishes (6.4).

Case 2. Now assume $y(t) \leq 0$. The argument used in Case 1 applies mutatis mutandis to conclude (6.5).

Finally, inequality (6.6) is a straightforward consequence of (6.4) and(6.5). ◻

For example, the Prandtl operator in Figure 6.3 satisfies the hypotheses of Proposition 6.1.

Let $\mathcal{P}_\xi$ be a Preisach operator satisfying the hypotheses of Proposition 6.1. Let $a_{\mathcal{P}}$, $b_{\mathcal{P}}$, and $c_{\mathcal{P}}$ be given by (6.3) and define Φ, $\tilde{\Phi} \in \mathcal{U}$ by

$$\Phi(y) := \begin{cases} \{v \in \mathbb{R} \mid a_{\mathcal{P}} y - c_{\mathcal{P}} \leq v \leq b_{\mathcal{P}} y + c_{\mathcal{P}}\}, & y \geq 0, \\ \{v \in \mathbb{R} \mid b_{\mathcal{P}} y - c_{\mathcal{P}} \leq v \leq a_{\mathcal{P}} y + c_{\mathcal{P}}\}, & y < 0. \end{cases}$$

$$\tilde{\Phi}(y) := \{v \in \mathbb{R} \mid (a_{\mathcal{P}} - \eta) y^2 \leq vy \leq (b_{\mathcal{P}} + \eta) y^2\},$$

where $\eta > 0$. In view of (6.4) and (6.5),

$$y \in C(\mathbb{R}_+) \quad \Longrightarrow \quad (\mathcal{P}_\xi(y))(t) \in \Phi(y(t)) \ \ \forall\, t \in \mathbb{R}_+.$$

Moreover, writing $K := [-c_{\mathcal{P}}/\eta\,,\, c_{\mathcal{P}}/\eta]$, we have

$$\Phi(y) \subset \tilde{\Phi}(y) \quad \forall\, y \in \mathbb{R}\backslash K \quad \text{and} \quad E := \sup_{y\in K}\sup_{v\in\Phi(y)}\inf_{\tilde{v}\in\tilde{\Phi}(y)}|v-\tilde{v}| = c_{\mathcal{P}}.$$

Let linear system (A,B,C) (with transfer function G) be stabilizable and detectable. Write $a := a_{\mathcal{P}} - \eta$, $b := b_{\mathcal{P}} + \eta$, and assume that $G/(1+aG) \in H^\infty$, and for some $\delta \in (0,1)$, $(1+bG)/(1+aG) - \delta$ is positive real. Then hypothesis (H1) holds with $m = 1$ and $\tilde{\Phi}$ replacing Φ.

Example. As a concrete example, consider a mechanical system with damping coefficient $\gamma > 0$ and a hysteretic restoring force in the form of backlash, with real parameters $\sigma > 0$ and ζ:

$$\ddot{y}(t) + \gamma\dot{y}(t) + \mathcal{B}_{\sigma,\zeta}(y)(t) = d(t). \tag{6.7}$$

Setting $w(\cdot,\cdot) := 1$, $w^0 = 0$, $\mu := \delta_\sigma$ (the Dirac measure with support $\{\sigma\}$), and $\xi(\cdot) := \zeta$ in (6.1), we see that $\mathcal{B}_{\sigma,\zeta} = \mathcal{P}_\xi$. In this case and in the notation of Proposition 6.1, we have $a_1 = b_1 = b_2 = a_{\mathcal{P}} = b_{\mathcal{P}} = 1$ and $a_2 = c_{\mathcal{P}} = \sigma$. Choosing $\eta \in (0,1)$, we have $0 < a < b$, where, as before, $a = a_{\mathcal{P}} - \eta$ and $b = b_{\mathcal{P}} + \eta$ and by Proposition 6.1,

$$|y(t)| \geq \sigma/\eta \implies ay^2(t) \leq \big(\mathcal{B}_{\sigma,\xi}(y)\big)(t)y(t) \leq by^2(t).$$

The transfer function G is given by $G(s) = 1/(s^2+\gamma s)$, $G/(1+aG)$ is given by $1/(s^2+\gamma s+a)$, and $(1+bG)/(1+aG) - \delta$ is given by $(1-\delta) + 2\eta/(s^2+\gamma s+a)$. Clearly, $G/(1+aG) \in H^\infty$ and a straightforward calculation reveals that, for all $\eta > 0$ sufficiently small, $(1+bG)/(1+aG) - \delta$ is positive real.

Returning to the general setting, we are now in a position to invoke Corollary 3.6 to conclude properties of solutions of the single-input, single-output functional differential equation

$$\dot{x}(t) = Ax(t) + B\big[d(t) - (\mathcal{P}_\xi(Cx))(t)\big], \quad x(0) = x^0. \tag{6.8}$$

We reiterate that, for each $x^0 \in \mathbb{R}^n$ and $d \in L^\infty_{\text{loc}}(\mathbb{R}_+)$, (6.8) has a unique global solution. An application of Corollary 3.6 (with $\Delta(t) = \{d(t)\}$ for all $t \in \mathbb{R}_+$) yields the existence of constants $\varepsilon, c_1, c_2 > 0$ such that, for every global solution x,

$$\|x(t)\| \leq c_1 e^{-\varepsilon t}\|x^0\| + c_2\left(\|d\|_{L^\infty[0,t]} + c_{\mathcal{P}}\right) \quad \forall\, t \in \mathbb{R}_+, \tag{6.9}$$

showing, in particular, that (6.8) is input-to-state stable with bias $c_{\mathcal{P}}$. Furthermore, by Lemma 3.3,

$$\lim_{t\to\infty} d(t) = 0 \quad \Longrightarrow \quad \limsup_{t\to\infty}\|x(t)\| \leq c_2 c_{\mathcal{P}}. \tag{6.10}$$

We emphasize that the convergence $d(t) \to 0$ as $t \to \infty$ does, in general, not imply convergence of $x(t)$ as $t \to \infty$. To see this, consider again mechanical example (6.7). Then, for every $\gamma > 0$, there exist constants $\varepsilon, c_1, c_2 > 0$ such that (6.9) and (6.10) hold (with $x(t) = (y(t), \dot{y}(t))$ and $c_{\mathcal{P}} = \sigma$). However, we know from [17, Example 4.8] that if $d = 0$ and $\gamma \in (1,2)$, then for all initial conditions, $\limsup_{t\to\infty} y(t) = \sigma$ and $\liminf_{t\to\infty} y(t) = -\sigma$ (equivalently, y has ω-limit set $[-\sigma,\sigma]$), showing, in particular, that $x(t) = (y(t), \dot{y}(t))$ does not converge as $t \to \infty$.

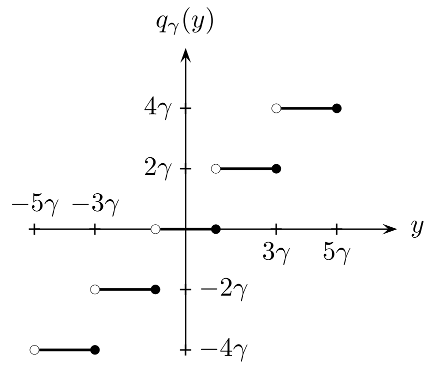

FIG. 7.1. *Uniform quantizer.*

7. Quantized feedback systems. Let (A, B, C) be a minimal realization of a linear, single-input, single-output system with transfer function G. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous static nonlinearity with the following property:

(Q1) There exist $\varphi \in \mathcal{K}_\infty$ and a number $b > 0$ such that

$$\varphi(|y|)|y| \le f(y)y \le by^2 \quad \forall\, y \in \mathbb{R}.$$

Furthermore, we impose the following assumption:

(Q2) There exists $\kappa \in [0\,,\,1/b)$ such that $\kappa + G$ is positive real.

From (Q1) and (Q2), it follows that (H3) holds with $\Phi(y) = \{f(y)\}$ and $\delta = \kappa b \in [0, 1)$. Consequently, by Theorem 3.5, the system

$$\dot{x}(t) = Ax(t) + B(d(t) - f(Cx(t))), \quad x(0) = x^0 \in \mathbb{R}^n, \quad d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+) \tag{7.1}$$

is input-to-state stable. Now consider (7.1) subject to quantization of the output $y = Cx$, that is, the system

$$\dot{x}(t) = Ax(t) + B\big(d(t) - (f \circ q_\gamma)(Cx(t))\big), \quad x(0) = x^0 \in \mathbb{R}^n, \quad d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+), \tag{7.2}$$

where $q_\gamma : \mathbb{R} \to \mathbb{R}$, parameterized by $\gamma > 0$, is a uniform quantizer (see Figure 7.1) given by

$$q_\gamma(y) = 2(m+1)\gamma \quad \forall\, y \in \big((2m+1)\gamma,\, (2m+3)\gamma\big] \quad \forall\, m \in \mathbb{Z}.$$

We interpret the differential equation (with discontinuous right-hand side) in (7.1) in a set-valued sense by embedding the quantizer q_γ in the set-valued map $Q_\gamma \in \mathcal{U}$ defined by

$$Q_\gamma(y) := \begin{cases} \{q_\gamma(y)\}, & y \in \big((2m+1)\gamma\,,\,(2m+3)\gamma\big), \quad m \in \mathbb{Z}, \\ [2m\gamma\,,\,2(m+1)\gamma], & y = (2m+1)\gamma, \quad m \in \mathbb{Z}, \end{cases}$$

and subsuming (7.2) in the differential inclusion

$$\dot{x}(t) - Ax(t) \in B\big(\Delta(t) - \Phi_\gamma(Cx(t))\big), \quad x(0) = x^0 \in \mathbb{R}^n, \quad \Delta \in \mathcal{B}, \tag{7.3}$$

where $\Delta : t \mapsto \{d(t)\}$ and $\Phi_\gamma \in \mathcal{U}$ is given by

$$\Phi_\gamma(y) := f(Q_\gamma(y)) = \{f(\xi) \mid \ \xi \in Q_\gamma(y)\}.$$

Choose $\varepsilon \in (0,1)$ sufficiently small so that $(1+\varepsilon)\kappa < 1/b$. Write $\tilde b := (1+\varepsilon)b$ and define $\tilde\varphi \in \mathcal{K}_\infty$ by $\tilde\varphi(s) := \varphi((1-\varepsilon)s)\ \forall\, s \in \mathbb{R}_+$.

LEMMA 7.1. *There exists $M \in \mathbb{N}$ such that, for every $\gamma > 0$,*

$$y \in \mathbb{R},\ |y| \ge \gamma M,\ v \in \Phi_\gamma(y) \quad \Longrightarrow \quad \tilde\varphi(|y|)|y| \le yv \le \tilde b y^2.$$

Proof. Observe that, for all $m \in \mathbb{N}$,

$$\frac{2m+2}{2m+3} \le \frac{w}{y} \le \frac{2m+4}{2m+1} \quad \forall\, w \in Q_\gamma(y) \ \ \forall\, y \in \big((2m+1)\gamma,\, (2m+3)\gamma\big].$$

Therefore, there exists $M \in \mathbb{N}$ such that

$$(1-\varepsilon)y^2 \le wy \le (1+\varepsilon)y^2 \quad \forall\, w \ \in Q_\gamma(y) \ \ \forall\, y \ge \gamma M.$$

Since Q_γ has odd symmetry ($Q_\gamma(y) = -Q_\gamma(-y)$), it immediately follows that

$$(1-\varepsilon)y^2 \le wy \le (1+\varepsilon)y^2 \quad \forall\, w \ \in Q_\gamma(y) \ \ \forall\, |y| \ge \gamma M. \tag{7.4}$$

Let y be such that $|y| \ge \gamma M$, and let $v \in \Phi_\gamma(y)$. Then $v = f(w)$ for some $w \in Q_\gamma(y)$. Invoking (Q1) and (7.4), it follows that

$$\varphi(|w|)|y| = \varphi(|w|)|w|\frac{y}{w} \le f(w)w\frac{y}{w} = f(w)y = vy \le bwy \le (1+\varepsilon)by^2 = \tilde b y^2. \tag{7.5}$$

Since $\varphi(|w|) = \varphi(|w||y|/|y|) = \varphi(wy/|y|)$ and invoking (7.4) and (7.5), we have

$$\tilde\varphi(|y|)|y| = \varphi((1-\varepsilon)|y|)|y| \le \varphi(|w|)|y| \le vy \le \tilde b y^2.$$

This completes the proof. $\square$

Let $M \in \mathbb{N}$ be as in Lemma 7.1 and define $\tilde\Phi \in \mathcal{U}$ by

$$\tilde\Phi(y) := \begin{cases} [\tilde\varphi(|y|)\,,\, \tilde b|y|], & y \ge 0, \\ [-\tilde b|y|\,,\, -\tilde\varphi(|y|)], & y < 0. \end{cases} \tag{7.6}$$

Clearly,

$$y \in \mathbb{R},\ v \in \tilde\Phi(y) \quad \Longrightarrow \quad \max\big\{\tilde\varphi(|y|)|y|,\, v^2/\tilde b\big\} \le yv,$$

and, by Lemma 7.1, we also have $\Phi_\gamma(y) \subset \tilde\Phi(y)\ \forall\, y \in \mathbb{R} \setminus [-\gamma M, \gamma M]$. Moreover, by (Q2) (and recalling that $\kappa\tilde b < 1$), $(\delta/\tilde b) + G$ is positive real for every $\delta \in [\kappa\tilde b\,,\,1)$. We are now in a position to invoke Corollary 3.7 (with $K = [-\gamma M, \gamma M]$) to conclude the existence of $\beta_1 \in \mathcal{KL}$ and $\beta_2 \in \mathcal{K}_\infty$, which do not depend on $\gamma > 0$ (recall Remark 3.8) such that, for all $\gamma > 0$, all $x^0 \in \mathbb{R}^n$, and all $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$, every global solution of (7.3), with $\Delta : t \mapsto \{d(t)\}$ satisfies

$$\|x(t)\| \le \max\big\{\beta_1(t, \|x^0\|)\,,\, \beta_2(\|d\|_{L^\infty[0,t]} + E_\gamma)\big\} \quad \forall\, t \in \mathbb{R}_+,$$

where $E_\gamma := \sup_{|y|\le\gamma M}\sup_{v\in\Phi_\gamma(y)}\inf_{\tilde v\in\tilde\Phi_\gamma(y)}|v-\tilde v|$. Noting that $E_\gamma\to 0$ as $\gamma\downarrow 0$ (if f is locally Lipschitz, then $E_\gamma = O(\gamma)$ as $\gamma\downarrow 0$), we may conclude robustness with respect to quantization in the sense that the quantized feedback system is such that the unbiased ISS property of unquantized system (7.1) is approached as $\gamma\downarrow 0$.

8. Conclusion. Feedback interconnections consisting of a linear system in the forward path and a nonlinearity in the feedback path have been considered. Adopting a differential inclusions framework, nonlinearities of considerable generality are encompassed, including inter alia both hysteresis operators and quantization operators. Conditions on the linear and nonlinear components have been identified (in Theorems 3.4 and 3.5) under which ISS (and a fortiori global asymptotic stability of the zero state) of the feedback interconnection is assured. The results of this paper are in the spirit of absolute stability theory: in particular, when specialized appropriately, classical absolute stability results pertaining to the circle criterion are recovered. In Corollaries 3.6 and 3.7, hypotheses are imposed on the nonlinearities (namely, generalized sector conditions) considerably weaker than those posited in Theorems 3.4 and 3.5, under which ISS with bias (and a fortiori asymptotic stability of a compact neighborhood of the zero state) may be concluded. Applications of the results to systems with hysteresis and to systems with output quantization have been detailed.

REFERENCES

[1] M. Arcak and A. Teel, *Input-to-state stability for a class of Lurie systems*, Automatica, 38 (2002), pp. 1945–1949.

[2] J.P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin, 1984.

[3] N.A. Barabanov and V.A.Yakubovich, *Absolute stability of control systems with one hysteresis nonlinearity*, Autom. Remote Control, 12 (1979), pp. 5–12.

[4] R.W. Brockett and D. Liberzon, *Quantized feedback stabilization of linear systems*, IEEE Trans. Automat. Control, 45 (2000), pp. 1279–1289.

[5] M. Brokate and J. Sprekels, *Hysteresis and Phase Transitions*, Springer-Verlag, New York, 1996.

[6] K. Deimling, *Multivalued Differential Equations*, de Gruyter, Berlin, 1992.

[7] C.A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, Academic Press, New York, 1975.

[8] M. Fu and L. Xie, *The sector bound approach to quantized feedback control*, IEEE Trans. Automat. Control, 50 (2005), pp. 1698–1711.

[9] R.B. Gorbet, K.A. Morris, and D.W.L. Wang, *Passivity-based stability and control of hysteresis in smart actuators*, IEEE Trans. Control Syst. Technol., 9 (2001), pp. 5–16.

[10] W. Hahn, *Stability of Motion*, Springer-Verlag, Berlin, 1967.

[11] B. Jayawardhana, H. Logemann, and E.P. Ryan, *Infinite-dimensional feedback systems: The circle criterion and input-to-state stability*, Commun. Inf. Syst., to appear.

[12] U. Jönson, *Stability of uncertain systems with hysteresis nonlinearities.* Internat. J. Robust Nonlinear Control, 8 (1998), pp. 279–293.

[13] H.K. Khalil, *Nonlinear Systems*, 3rd ed., Prentice-Hall, Upper Saddle River, NJ, 2002.

[14] M.A. Krasnosel'skii and A.V. Pokrovskii, *Systems with Hysteresis*, Springer-Verlag, Berlin, 1989.

[15] H. Logemann and A.D. Mawby, *Low-gain integral control of infinite-dimensional regular linear systems subject to input hysteresis*, in Advances in Mathematical Systems Theory, F. Colonius et al., eds., Birkhäuser, Boston, 2001, pp. 255–293.

[16] H. Logemann and E.P. Ryan, *Systems with hysteresis in the feedback loop: Existence, regularity and asymptotic behavior of solutions*, ESAIM Control, Optim. Calc. Var., 9 (2003), pp. 169–196.

[17] H. Logemann, E.P. Ryan, and I. Shvartsman, *A class of differential-delay systems with hysteresis: Asymptotic behavior of solutions*, Nonlinear Anal., 69 (2008), pp. 363–391.

[18] A.I. Lur'e and V.N. Postnikov, *On the theory of stability of control systems*, Appl. Math. Mech., 8 (1944), pp. 246–248 (in Russian).

[19] K.S. Narendra and J.H. Taylor, *Frequency Criteria for Absolute Stability*, Academic Press, New York, 1973.
[20] E.D. Sontag, *Input to state stability: Basic concepts and results*, in Nonlinear and Optimal Control Theory, P. Nistri and G. Stefani, eds., Springer-Verlag, Berlin, 2006, pp. 163–220.
[21] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed., Prentice-Hall, Englewood Cliffs, NJ, 1993.
[22] R. Vinter, *Optimal Control*, Birkhäuser, Boston, 2000.
[23] V.A. Yakubovich, G.A. Leonov, and A.Kh. Gelig, *Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities*, World Scientific, Singapore, 2004.