International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713393989

Indirect sampled-data control with sampling period adaptation

Achim Ilchmann[a]; Zhenqing Ke[bc]; Hartmut Logemann[b]
[a] Institute of Mathematics, Technical University Ilmenau, 98693 Ilmenau, Germany [b] Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK [c] Department of Mathematics, Jinan University, Guangzhou 510632, China

Online publication date: 16 March 2011

To cite this Article Ilchmann, Achim , Ke, Zhenqing and Logemann, Hartmut(2011) 'Indirect sampled-data control with sampling period adaptation', International Journal of Control, 84: 2, 424 — 431
To link to this Article: DOI: 10.1080/00207179.2011.557782
URL: http://dx.doi.org/10.1080/00207179.2011.557782

International Journal of Control
Vol. 84, No. 2, February 2011, 424–431

Taylor & Francis
Taylor & Francis Group

Indirect sampled-data control with sampling period adaptation

Achim Ilchmann[a], Zhenqing Ke[bc] and Hartmut Logemann[b*]

[a]*Institute of Mathematics, Technical University Ilmenau, Weimarer Straße 25, 98693 Ilmenau, Germany;* [b]*Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK;* [c]*Department of Mathematics, Jinan University, Guangzhou 510632, China*

(*Received 4 July 2010; final version received 22 January 2011*)

It is known that if a continuous-time feedback system is exponentially stable, then the corresponding sampled-data system obtained by sample-hold discretisation with constant sampling period is also exponentially stable, provided that the sampling period $\tau > 0$ is sufficiently small. In general, it is difficult to estimate how small the sampling period has to be in order to achieve the stability of the sampled-data system. In this article, we present an adaptive mechanism for adjusting the sampling period. This mechanism has the properties that, for every initial state, (i) the adaptation of the sampling period terminates after finitely many time steps and (ii) the state of the adaptive sampled-data system is integrable and converges to zero as time goes to infinity.

Keywords: adaptive control; feedback stabilisation; indirect sampled-data control; variable sampling period

1. Introduction

Consider the finite-dimensional continuous-time static output feedback system

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t); \quad x(0) = x^0, \\ y(t) &= Cx(t), \\ u(t) &= Fy(t), \end{aligned}\right\} \qquad (1.1)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $F \in \mathbb{R}^{m\times p}$ and $x^0 \in \mathbb{R}^n$. System (1.1) is exponentially stable if, and only if, the matrix $A + BFC$ is exponentially stable, that is, all eigenvalues of $A + BFC$ have negative real parts.

Digital implementation of the output feedback in (1.1) requires the application of sampling and (zero-order) hold, leading to the sampled-data feedback system

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t); \quad x(0) = x^0, \\ y(t) &= Cx(t), \\ u(t) &= Fy(j\tau), \qquad \forall t \in [j\tau, (j+1)\tau), \end{aligned}\right\} \qquad (1.2)$$

where $\tau > 0$ is the sampling period. It is well known that if system (1.1) is exponentially stable and if sampling period τ is sufficiently small, then system (1.2) is also exponentially stable in the sense that there exist an $M \geq 1$ and an $\alpha > 0$ such that

$$\|x(t; x^0, \tau)\| \leq Me^{-\alpha t}\|x^0\|, \quad \forall x^0 \in \mathbb{R}^n \ \forall t \geq 0,$$

where $x(\cdot; x^0, \tau)$ denotes the solution of (1.2) (for the proof and for related results, see Dragan (1990), Chen and Francis (1991), Logemann, Rebarber, and Townley (2003) and Ke (2008).

Given that the continuous-time system (1.1) is exponentially stable, it is in general difficult to estimate how small the sampling period has to be in order to achieve the stability of the sampled-data system (1.2) (Tokarzewski and Olbrot 1995). In this article, we develop an adaptive strategy for adjusting the sampling period, so that, for every initial condition x^0, the adaptation of the sampling period terminates after finitely many time steps and the corresponding solution of (1.2) is integrable and tends to 0 as $t \to \infty$.

The idea to invoke sampling period adaptation in the synthesis of stable sampled-data feedback systems seems to have been introduced in Owens (1996), where it is used in a high-gain control context. The approach in Owens (1996), developed for single-input–single-output minimum phase systems with relative degree one, was extended in Ilchmann and Townley (1999) to include multi-input–multi-output systems. Additionally, a number of other assumptions imposed in Owens (1996) were relaxed in Ilchmann and Townley (1999). Furthermore, sampling period adaptation has also been used in Özdemir and Townley (2003) in a low-gain integral control context. However, the results in Ilchmann and Townley (1999), Owens (1996) and in Özdemir and Townley (2003) are specific to high-gain stabilisation and low-gain tracking, respectively, and have little overlap with the general

*Corresponding author. Email: hl@maths.bath.ac.uk

ISSN 0020–7179 print/ISSN 1366–5820 online

DOI: 10.1080/00207179.2011.557782
http://www.informaworld.com

result on adaptive sampling in indirect sampled-data control presented in this article.

The rest of this article is structured as follows. Section 2 is devoted to the statement, discussion and illustration (by means of an example) of Theorem 2.2, the main result of this article. Whilst Theorem 2.2 is restricted to static output feedback, it is shown in Section 3 how it can be extended to indirect sampled-data control involving dynamic feedback. All proofs can be found in Section 4. Finally, some conclusions are drawn in Section 5.

Nomenclature and terminology

$\lfloor\sigma\rfloor$	:=	$\max\{n\in\mathbb{N}_0 \mid n\le\sigma\}$, $\sigma\in\mathbb{R}_+$
$\ell^\infty(\mathbb{N}_0, \mathbb{R}^n)$		space of bounded $\mathbb{R}^n$-valued sequences $(s_j)_{j\in\mathbb{N}_0}$
$\ell^1(\mathbb{N}_0, \mathbb{R}^n)$		space of $\mathbb{R}^n$-valued sequences $(s_j)_{j\in\mathbb{N}_0}$ with $\sum_{j=0}^\infty \|s_j\| < \infty$
$L^1(\mathbb{R}_+, \mathbb{R}^n)$		vector space of all measurable functions $f:\mathbb{R}_+\to\mathbb{R}^n$ with $\int_0^\infty \|f(t)\|\mathrm{d}t<\infty$

A sequence $(s_j)_{j\in\mathbb{N}_0}$ is said to be *ultimately constant* if, and only if, there exists an $N\in\mathbb{N}_0$ such that $s_{N+j}=s_N$ for all $j\in\mathbb{N}_0$.

2. Adaptation of the sampling period

The purpose of this section is to develop an adaptive feedback mechanism for adjusting the sampling period. The use of sampling and hold in (1.1), corresponding to the sampling points $(t_j)_{j\in\mathbb{N}_0}$, leads to the following sampled-data feedback system:

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t); \quad x(0)=x^0,\\ y(t) &= Cx(t),\\ u(t) &= Fy(t_j), \qquad \forall t\in[t_j,t_{j+1}). \end{aligned}\right\} \tag{2.1}$$

The sampling points t_j, or, equivalently, the sampling periods $\tau_j := t_{j+1}-t_j$, are determined by the following adaptive strategy:

$$\left.\begin{aligned} &\text{for given } \alpha\in(0,1) \text{ and } (\eta_j)_{j\in\mathbb{N}_0}\in\ell^\infty(\mathbb{N}_0,\mathbb{R})\\ &\quad\text{with } \inf_{j\in\mathbb{N}_0}\eta_j>0\,,\\ &\text{set } t_0=0, \text{ let } \sigma_0\ge 0,\\ &\text{and, for } j=0,1,2,\ldots, \text{ set}\\ &\quad k_j=\lfloor\sigma_j\rfloor,\\ &\quad \tau_j=\max\{\eta_j/(j+1)^\alpha,\ \eta_{k_j}/(k_j+1)^\alpha\},\\ &\quad t_{j+1}=t_j+\tau_j,\\ &\quad \sigma_{j+1}=\sigma_j+\|y(t_j)\|. \end{aligned}\right\} \tag{2.2}$$

The rationale for the adaptive strategy (2.2) is described in the following remark.

Remark 2.1:

(i) The choice of $(\eta_j)_{j\in\mathbb{N}_0}$ and α in (2.2) allows to influence the size of the sampling periods τ_j in the transient phase where j is 'small': for example, the larger the η_j, the larger the τ_j and similarly, the smaller the α, the larger τ_j. Moreover, we emphasise that the sequence $(\eta_j)_{j\in\mathbb{N}_0}$ plays a further role which will become clear later (see part (iii) of Remark 2.8).

(ii) Obviously, the last line in (2.2) (the recursion for σ_j) is a discrete-time integrator with initial state σ_0 and input $(\|y(t_j)\|)_{j\in\mathbb{N}_0}$, so that

$$\sigma_j=\sigma_0+\sum_{l=0}^{j-1}\|y(t_l)\|, \quad \forall j\in\mathbb{N}. \tag{2.3}$$

It is immediate that the following properties are equivalent:

(a) $(\tau_j)_{j\in\mathbb{N}_0}$ is ultimately constant;
(b) $(k_j)_{j\in\mathbb{N}_0}$ is ultimately constant;
(c) $(\sigma_j)_{j\in\mathbb{N}_0}\in\ell^\infty(\mathbb{N}_0,\mathbb{R})$;
(d) $\big(y(t_j)\big)_{j\in\mathbb{N}_0}\in\ell^1(\mathbb{N}_0,\mathbb{R}^p)$.

We note that if $(\tau_j)_{j\in\mathbb{N}_0}$ is not ultimately constant, then $\lim_{j\to\infty}\tau_j=0$ (as follows from the equivalence of (a) and (b)). Furthermore, we see that if the sequence $(\eta_j)_{j\in\mathbb{N}_0}$ is non-increasing (a natural choice), then $(\tau_j)_{j\in\mathbb{N}_0}$ is non-increasing. The idea behind (2.2) is to drive τ_j to zero as long as the norm of the sampled output values $y(t_j)$ is 'large' in the sense that the partial sum σ_j has not 'started to converge'.

For the following, it is convenient to define

$$\delta_l := \eta_l/(l+1)^\alpha, \quad \forall l\in\mathbb{N}_0. \tag{2.4}$$

Note that, for each sampling period τ_j generated by (2.2), there exists an $l_j\in\mathbb{N}_0$ such that $\tau_j=\delta_{l_j}$. We introduce the following detectability hypothesis.

(D) The pair $(C, e^{A\delta_l})$ is discrete-time detectable for every $l\in\mathbb{N}_0$.

We are now ready to state the main result of this contribution. The proof can be found in Section 4.

Theorem 2.2: *Assume that the continuous-time feedback system* (1.1) *is exponentially stable and let* $x(\cdot;x^0)$ *denote the solution of the adaptive sampled-data system given by* (2.1) *and* (2.2). *Then, for every initial state* $x^0\in\mathbb{R}^n$, *the following statements hold*:

(i) *the sequence* $(\tau_j)_{j\in\mathbb{N}_0}$ *is ultimately constant, that is, the adaptation of the sampling period terminates in finite time*;

(ii) *if, additionally, hypothesis* (D) *is satisfied, then* $\lim_{t\to\infty} x(t; x^0) = 0$, $x(\cdot; x^0) \in L^1(\mathbb{R}_+, \mathbb{R}^n)$ *and* $(x(t_j; x^0))_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^n)$.

Note that, by part (i) of Theorem 2.2, the limit $\tau := \lim_{j\to\infty} \tau_j$ exists. Whilst Theorem 2.2 guarantees that, for every $x^0 \in \mathbb{R}^n$, $x(t; x^0) \to 0$ as $t \to \infty$, it does not ensure that the sampled-data feedback system (2.1) with constant sampling period τ is asymptotically stable or, equivalently, that the spectral radius of the matrix

$$\Delta_\tau := e^{A\tau} + \int_0^\tau e^{As}\mathrm{d}sBFC \tag{2.5}$$

is smaller than 1, as the following trivial example shows.

Example 2.3: Let $C = I$ (in which case, for every sequence $(\eta_j)_{j\in\mathbb{N}_0}$ and every α, hypothesis (D) is trivially satisfied). Choose A, B, F and $\tau > 0$ such that $A + BF$ is Hurwitz and Δ_τ has at least one eigenvalue λ_u with $|\lambda_u| \geq 1$ and at least one eigenvalue λ_s with $|\lambda_s| < 1$. Let v_s be in the λ_s-eigenspace of Δ_τ with

$$\|v_s\| < 1 - |\lambda_s|. \tag{2.6}$$

With $\eta_j = \tau$ for all $j \in \mathbb{N}_0$, $\alpha \in (0, 1)$ arbitrary, $x^0 = v_s$ and $\sigma_0 = 0$, it follows easily that the adaptive sampled-data system given by (2.1) and (2.2) has the following properties: $\tau_j = \tau$ for all $j \in \mathbb{N}_0$, $y(t_j) = x(t_j; x^0) = x(j\tau; x^0) = \lambda_s^j v_s$ and $k_j = 0$ for all $j \in \mathbb{N}_0$. This can be shown by an elementary induction argument combined with the observation that

$$\sigma_j = \sum_{l=0}^{j-1} |\lambda_s|^l \|v_s\| \leq \frac{1}{1 - |\lambda_s|}\|v_s\| < 1, \quad \forall j \in \mathbb{N},$$

which is a consequence of (2.3) and (2.6).

The phenomenon described in Example 2.3 is reminiscent of the well-known fact that, in adaptive stabilisation, the limiting feedback controller is not necessarily stabilising (see Townley (1996, 1999) and the references therein for more details).

Remark 2.4: As has already been indicated in Section 1: given a feedback matrix F rendering the continuous-time system (1.1) exponentially stable, it is a difficult task to derive conditions (in terms of A, B, C and F) for a sampling period τ^* guaranteeing that the sampled-data system (1.2) is asymptotically stable for every (fixed) sampling period $\tau \in (0, \tau^*)$, or equivalently, such that the spectral radius of the matrix Δ_τ given by (2.5) is smaller than 1 for every $\tau \in (0, \tau^*)$ (see Tokarzewski and Olbrot (1995), one of the very few papers addressing this issue). To the best of our knowledge, no satisfactory solution of this problem is available in the literature. Naturally, whilst this problem becomes even more difficult in the presence of plant uncertainty, the adaptive strategy (2.2) 'handles' plant uncertainty easily. More precisely, assume that the plant is not exactly known, but that it is known to be contained in a (known) set $\mathcal{P}$ of plants and that (by using methods from robust control) a feedback F has been designed which stabilises all plants in $\mathcal{P}$ in continuous time (i.e. (1.1) is exponentially stable for every system (A, B, C) in $\mathcal{P}$). Then the conclusions of Theorem 2.2 are valid for every (A, B, C) in $\mathcal{P}$.

As we have already noted in Example 2.3: in the case of state feedback (that is, $p = n$ and $C = I$), hypothesis (D) is trivially satisfied (for every sequence $(\eta_j)_{j\in\mathbb{N}_0}$ and every α). In general, however, the appearance of hypothesis (D) in statement (ii) of Theorem 2.2 is somewhat unsatisfactory, because it is formulated in discrete-time terms and not in terms of the original continuous-time data. The following definition will be useful in addressing this issue.

Definition 2.5: A number $\delta > 0$ is said to be *pathological* relative to $A \in \mathbb{R}^{n\times n}$ if, and only if, there exist $q \in \mathbb{Z}\backslash\{0\}$ and λ, $\mu \in \sigma(A) \cap \{s \in \mathbb{C} : \mathrm{Re}\ s \geq 0\}$ such that $\delta(\lambda - \mu) = 2q\pi i$. Otherwise, δ is said to be non-pathological relative to A.

We shall see that, in Theorem 2.2, hypothesis (D) can be replaced by the following hypothesis.

(D′) For every $l \in \mathbb{N}_0$, δ_l is non-pathological relative to A.

Lemma 2.6: *If the pair* (C, A) *is detectable in continuous time and hypothesis* (D′) *is satisfied, then* (D) *holds.*

The proof of Lemma 2.6 can be found in Section 4.

The assumption of exponential stability of the continuous-time feedback system (1.1) in Theorem 2.2 trivially implies that (C, A) is detectable in continuous time. Therefore the following corollary is an immediate consequence of Lemma 2.6.

Corollary 2.7: *The conclusions of Theorem* 2.2 *remain valid if, in the statement of Theorem* 2.2, *hypothesis* (D) *is replaced by hypothesis* (D′).

The following remark contains some commentary on hypotheses (D) and (D′).

Remark 2.8:

(i) The converse of Lemma 2.6 is not correct. Whilst hypothesis (D) implies the continuous-time detectability of (C, A), it does, in general, not imply (D′). Consequently, in the context of Theorem 2.2, hypothesis (D) is weaker than hypothesis (D′).

(ii) Let α and $(\eta_l)_{l\in\mathbb{N}_0}$ be given as in (2.2) and define $(\delta_l)_{l\in\mathbb{N}_0}$ by (2.4). Then it can be shown that the set

$$\{A \in \mathbb{R}^{n\times n} : \delta_l \text{ is non-pathological relative to } A \text{ for every } l \in \mathbb{N}_0\}$$

is open and dense in $\mathbb{R}^{n\times n}$ [Ke (2008), Appendix A.1]. Consequently, the probability that, for a randomly chosen matrix $A \in \mathbb{R}^{n\times n}$, there exists $l \in \mathbb{N}_0$ such that δ_l is pathological relative to A is zero.

(iii) Let $A \in \mathbb{R}^{n\times n}$ and $\alpha \in (0, 1)$ be given and let $NP(A,\alpha)$ denote the set of all bounded sequences $(\eta_l)_{l\in\mathbb{N}_0}$ with $\inf_{l\in\mathbb{N}_0}\eta_l > 0$ and such that δ_l (defined in (2.4)) is non-pathological relative to A for every $l \in \mathbb{N}_0$ (i.e. hypothesis (D′) holds). It is easy to show that $NP(A,\alpha)$ is open and dense (with respect to the ℓ^∞-norm) in the set of all bounded sequences $(\eta_l)_{l\in\mathbb{N}_0}$ with $\inf_{j\in\mathbb{N}_0}\eta_l > 0$. As a consequence, the probability that a randomly chosen sequence $(\eta_l)_{l\in\mathbb{N}_0}$ with $\inf_{l\in\mathbb{N}_0}\eta_l > 0$ is not contained in $NP(A,\alpha)$ is zero.

Part (ii) of Remark 2.8 shows that, if α and $(\eta_l)_{l\in\mathbb{N}_0}$ are fixed, then, with respect to A, (D′) is generically satisfied. Similarly, if α and A are fixed, then part (iii) of Remark 2.8 shows that, with respect to $(\eta_l)_{l\in\mathbb{N}_0}$, (D′) holds generically. The same comment applies to hypothesis (D), provided that (C, A) is detectable in continuous time (the latter is trivially satisfied if the continuous-time feedback system (1.1) is exponentially stable). Consequently, assumptions (D) and (D′) imposed in Theorem 2.2 and Corollary 2.7, respectively, are not very restrictive.

We illustrate Theorem 2.2 by an example (including a numerical simulation).

Example 2.9: Assume that A, B, C and F in system (2.1) are given by

$$A = \begin{pmatrix} -a_1 & 1 & a_2 \\ -1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$C = I, \quad F = -B^T.$$

Then, for all $(a_1, a_2, a_3) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$, the matrix A is dissipative (that is $\langle Az, z\rangle \le 0$ for all $z \in \mathbb{R}^3$) and the pair (A, B) is continuous-time controllable. Consequently, as is well known, the corresponding continuous-time feedback system (1.1) is exponentially stable (for all $(a_1, a_2, a_3) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$). Hypothesis (D) is trivially satisfied (for every sequence $(\eta_j)_{j\in\mathbb{N}_0}$ and every α) and therefore the conclusions of Theorem 2.2 hold (for all $(a_1, a_2, a_3) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$).

Consider A with specific parameter values given by $a_1 = 0$, $a_2 = 1/2$ and $a_3 = 1$, in which case A has eigenvalues 0 and $\pm i3/2$ and the eigenvalues of $A - BB^T$ are approximately -0.8836 and $-0.5582 \pm i1.3971$. Moreover, with α, (η_j), x^0 and σ_0 given by

$$\alpha = 0.3, \quad \eta_j = 1 \ \forall j \in \mathbb{N}_0, \quad x^0 = (1, 2, 1)^T, \quad \sigma_0 = 0,$$

the evolution of the sampled-data system given by (2.1) and (2.2) is illustrated in Figure 1.

3. Generalisation to dynamic output feedback

Consider a dynamic output feedback system with plant given by

$$\dot{x}_p = A_p x_p + B_p u_p; \quad x_p(0) = x_p^0, \qquad y_p = C_p x_p, \tag{3.1}$$

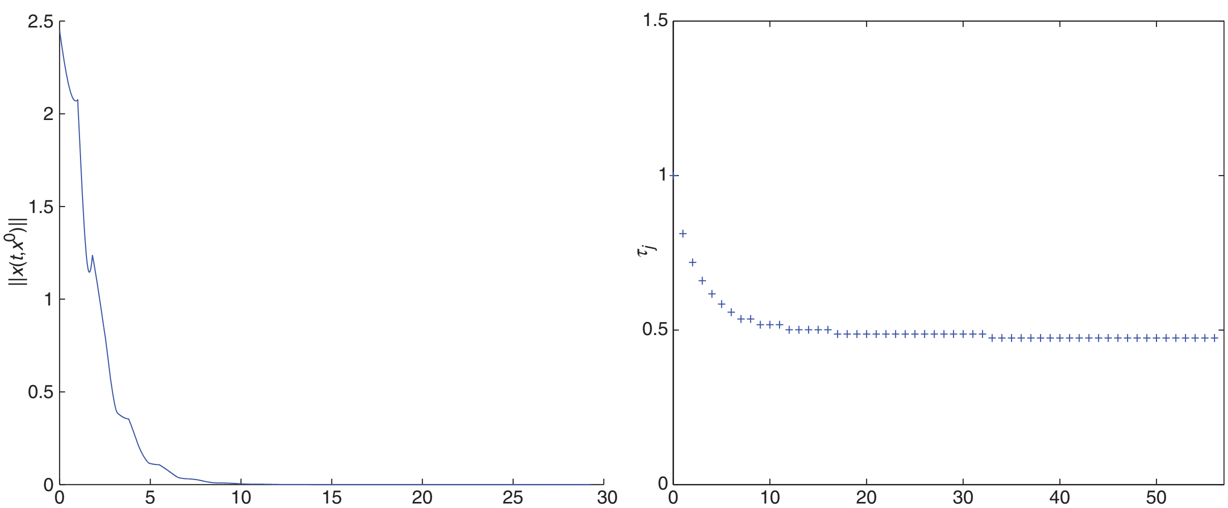

Figure 1. Sampled-data control with adaptive sampling period.

controller given by

$$\dot{x}_{\rm c} = A_{\rm c}x_{\rm c} + B_{\rm c}u_{\rm c}; \quad x_{\rm c}(0) = x_{\rm c}^0, \\ y_{\rm c} = C_{\rm c}x_{\rm c} + D_{\rm c}u_{\rm c}, \tag{3.2}$$

and feedback interconnection equations

$$u_{\rm c} = y_{\rm p}, \quad u_{\rm p} = y_{\rm c}, \tag{3.3}$$

where $A_{\rm p} \in \mathbb{R}^{n_{\rm p}\times n_{\rm p}}$, $B_{\rm p} \in \mathbb{R}^{n_{\rm p}\times m}$, $C_{\rm p} \in \mathbb{R}^{p\times n_{\rm p}}$, $A_{\rm c} \in \mathbb{R}^{n_{\rm c}\times n_{\rm c}}$, $B_{\rm c} \in \mathbb{R}^{n_{\rm c}\times p}$, $C_{\rm c} \in \mathbb{R}^{m\times n_{\rm c}}$, $D_{\rm c} \in \mathbb{R}^{m\times p}$, $x_{\rm p}^0 \in \mathbb{R}^{n_{\rm p}}$ and $x_{\rm c}^0 \in \mathbb{R}^{n_{\rm c}}$. Defining

$$A := \mathrm{diag}(A_{\rm p}, A_{\rm c}), \quad B := \mathrm{diag}(B_{\rm p}, B_{\rm c}), \\ C := \begin{pmatrix} C_{\rm p} & 0 \\ D_{\rm c}C_{\rm p} & C_{\rm c} \end{pmatrix}, \quad F := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{3.4}$$

a routine calculation shows that the continuous-time dynamic feedback system given by (3.1)–(3.3) can be written as

$$\dot{x} = (A + BFC)x; \quad x(0) = x^0 = \begin{pmatrix} x_{\rm p}^0 \\ x_{\rm c}^0 \end{pmatrix}, \quad \text{where } x := \begin{pmatrix} x_{\rm p} \\ x_{\rm c} \end{pmatrix}. \tag{3.5}$$

Let $(t_j)_{j\in\mathbb{N}_0}$ be the sampling points to be determined adaptively. As before, we define the associated sampling periods $\tau_j := t_{j+1} - t_j$ for $j \in \mathbb{N}_0$. Consider the corresponding sample-hold discretisation of (3.2):

$$\left.\begin{aligned} x_{\rm c}^{\rm d}(j+1) &= e^{A_{\rm c}\tau_j}x_{\rm c}^{\rm d}(j) + \int_0^{\tau_j} e^{A_{\rm c}s}\mathrm{d}sB_{\rm c}u_{\rm c}^{\rm d}(j); \\ x_{\rm c}^{\rm d}(0) &= x_{\rm c}^0 \in \mathbb{R}^{n_{\rm c}}, \\ y_{\rm c}^{\rm d}(j) &= C_{\rm c}x_{\rm c}^{\rm d}(j) + D_{\rm c}u_{\rm c}^{\rm d}(j), \end{aligned}\right\} \tag{3.6}$$

together with the feedback interconnection equations

$$u_{\rm c}^{\rm d}(j) = y_{\rm p}(t_j), \quad u_{\rm p}(t_j + \theta) = y_{\rm c}^{\rm d}(j), \quad \forall \theta \in [0, \tau_j), \ \forall j \in \mathbb{N}_0. \tag{3.7}$$

The adaptive strategy for determining the sampling points is very similar to that in the case of static feedback, the only difference being in the equation for $(\sigma_j)_{j\in\mathbb{N}_0}$:

$$\left.\begin{aligned} \text{for given} \quad & \alpha \in (0,1) \text{ and } (\eta_j)_{j\in\mathbb{N}_0} \in \ell^\infty(\mathbb{N}_0, \mathbb{R}) \\ & \quad \text{with } \inf_{j\in\mathbb{N}_0} \eta_j > 0\ , \\ \text{set} \quad & t_0 = 0, \quad \text{let } \sigma_0 \geq 0, \\ \text{and, for} \quad & j = 0,1,2,\ldots, \text{ set} \\ & k_j = \lfloor \sigma_j \rfloor, \\ & \tau_j = \max\{\eta_j/(j+1)^\alpha, \eta_{k_j}/(k_j+1)^\alpha\}, \\ & t_{j+1} = t_j + \tau_j, \\ & \sigma_{j+1} = \sigma_j + \left\| \left(y_{\rm p}(t_j), y_{\rm c}^{\rm d}(j) \right) \right\|. \end{aligned}\right\} \tag{3.8}$$

Remark 3.1: Remark 2.1 remains true in the context of the adaptive strategy (3.8), provided that, in (2.3), $\|y(t_j)\|$ is replaced by $\|(y_{\rm p}(t_j), y_{\rm c}^{\rm d}(j))\|$ and, in item (d) of part (ii), $(y(t_j))_{j\in\mathbb{N}_0}$ and $\ell^1(\mathbb{N}_0, \mathbb{R}^p)$ are replaced by $(y_{\rm p}(t_j), y_{\rm c}^{\rm d}(j))_{j\in\mathbb{N}_0}$ and $\ell^1(\mathbb{N}_0, \mathbb{R}^{p+m})$, respectively.

The sampled-data feedback system given by (3.1), (3.6), (3.7) and (3.8) has a unique solution which will be denoted by

$$\begin{pmatrix} x_{\rm p}(t_j + \theta; x^0) \\ x_{\rm c}^{\rm d}(j; x^0) \end{pmatrix}, \quad \forall \theta \in [0, \tau_j), \ \forall j \in \mathbb{N}_0. \tag{3.9}$$

The following corollary is the main result of this section. The proof can be found in Section 4.

Corollary 3.2: *Assume that the continuous-time dynamic feedback system given by* (3.1)–(3.3) (*or, equivalently, system* (3.5)) *is exponentially stable. Then, for every initial state* $x^0 \in \mathbb{R}^{n_{\rm p}+n_{\rm c}}$, *the sampled-data feedback system given by* (3.1), (3.6)–(3.8) *has the following properties*:

(i) *the sequence* $(\tau_j)_{j\in\mathbb{N}_0}$ *is ultimately constant, that is, the adaptation of the sampling period terminates in finite time*;

(ii) *if, additionally,* $\eta_l/(l+1)^\alpha$ *is non-pathological relative to* $A = \mathrm{diag}(A_{\rm p}, A_{\rm c})$ *for every* $l \in \mathbb{N}_0$, *then* $\lim_{t\to\infty} x_{\rm p}(t; x^0) = 0$, $x_{\rm p}(\,\cdot\,; x^0) \in L^1(\mathbb{R}_+, \mathbb{R}^{n_{\rm p}})$, $(x_{\rm p}(t_j; x^0))_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^{n_{\rm p}})$ *and* $(x_{\rm c}^{\rm d}(j; x^0))_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^{n_{\rm c}})$.

4. Proofs

To facilitate the proofs of the results in Sections 2 and 3, it is convenient to first state and prove a technical lemma. To this end, consider the sampled-data feedback system (2.1) with a prespecified sequence $\boldsymbol{t} := (t_j)_{j\in\mathbb{N}_0}$ of sampling points satisfying

$$t_0 = 0, \quad t_{j+1} > t_j \ \ \forall j \in \mathbb{N}_0, \quad t_j \to \infty \ \text{ as } \ j \to \infty.$$

Let $x(\cdot;\ x^0, \boldsymbol{t})$ denote the corresponding solution of system (2.1).

The following lemma shows that if the continuous-time system (1.1) is exponentially stable and if the sampling periods $\tau_j := t_{j+1} - t_j$ converge to 0 as $j \to \infty$, with rate of convergence sufficiently small, then the sequence $(x(t_j; x^0, \boldsymbol{t}))_{j\in\mathbb{N}_0}$ is summable. Here 'sufficiently small' means that there exist constants $M > 0$ and $\alpha \in (0, 1)$ such that $\tau_j > Mj^{-\alpha}$ for all $j \in \mathbb{N}$.

Lemma 4.1: *Assume that the continuous-time feedback system* (1.1) *is exponentially stable. Let the sequence* $\boldsymbol{t} = (t_j)_{j\in\mathbb{N}_0}$ *be such that* $t_0 = 0$ *and* $t_{j+1} > t_j$

for all $j \in \mathbb{N}_0$. *Set* $\tau_j := t_{j+1} - t_j$ *and assume that*

$$\lim_{j\to\infty} \tau_j = 0 \quad \text{and} \quad \inf_{j\in\mathbb{N}} \tau_j j^\alpha > 0 \quad \text{for some } \alpha \in (0,1). \tag{4.1}$$

Then, for every $x^0 \in \mathbb{R}^n$, *the sequence* $(x(t_j; x^0, \boldsymbol{t}))_{j\in\mathbb{N}_0}$ *is in* $\ell^1(\mathbb{N}_0, \mathbb{R}^n)$.

Proof: The variation-of-parameters formula yields

$$x(t_{j+1}; x^0, \boldsymbol{t}) = \left(e^{A\tau_j} + \int_0^{\tau_j} e^{As}\mathrm{d}s\, BFC\right) x(t_j; x^0, \boldsymbol{t}), \quad \forall j \in \mathbb{N}_0. \tag{4.2}$$

Considering

$$\Delta_j := e^{A\tau_j} + \int_0^{\tau_j} e^{As}\mathrm{d}s\, BFC \quad \text{and} \quad x_j := x(t_j; x^0, \boldsymbol{t}); \quad \forall j \in \mathbb{N}_0,$$

Equation (4.2) becomes

$$x_{j+1} = \Delta_j x_j, \quad \forall j \in \mathbb{N}_0; \quad x_0 = x^0. \tag{4.3}$$

It follows from the exponential stability of (1.1) that there exists a unique matrix $P = P^T > 0$, such that (see, e.g. Sontag (1998), Theorem 18, p. 231)

$$(A + BFC)^T P + P(A + BFC) = -I. \tag{4.4}$$

Let $\|\cdot\|_P$ be the norm on $\mathbb{R}^n$ defined by

$$\|z\|_P^2 := \langle z, Pz\rangle, \quad \forall z \in \mathbb{R}^n.$$

Using the power series expansion of e^{At}, we may decompose

$$\Delta_j = I + \tau_j(A + BFC) + \tau_j^2 \Gamma(\tau_j), \quad \forall j \in \mathbb{N}_0, \tag{4.5}$$

where

$$\Gamma(\tau) := \sum_{l=0}^{\infty} \frac{\tau^l}{(l+2)!} A^{l+1}(A + BFC), \quad \forall \tau \geq 0.$$

The boundedness of $(\tau_j)_{j\in\mathbb{N}_0}$ implies the boundedness of the sequence $(\Gamma(\tau_j))_{j\in\mathbb{N}_0}$ and hence, invoking (4.3) and (4.5), we conclude that there exists a constant $L \geq 0$ such that

$$\begin{aligned} \|x_{j+1}\|_P^2 - \|x_j\|_P^2 &= \langle \Delta_j x_j, P\Delta_j x_j\rangle - \langle x_j, Px_j\rangle \\ &\leq \tau_j \big\langle x_j, \big[(A+BFC)^T P + P(A+BFC)\big] x_j \big\rangle \\ &\quad + L\tau_j^2 \|x_j\|^2, \quad \forall j \in \mathbb{N}_0. \end{aligned}$$

Combining this with (4.4), we have

$$\|x_{j+1}\|_P^2 - \|x_j\|_P^2 \leq (-\tau_j + L\tau_j^2)\|x_j\|^2, \quad \forall j \in \mathbb{N}_0,$$

and therefore, in view of $\lim_{j\to 0} \tau_j = 0$, we obtain that there exists an $N \in \mathbb{N}$ such that

$$\|x_{j+1}\|_P^2 - \|x_j\|_P^2 \leq -\frac{\tau_j}{2}\|x_j\|^2, \quad \forall j \geq N.$$

Consequently,

$$\|x_{j+1}\|_P^2 \leq \|x_j\|_P^2 - \frac{\tau_j}{2}\|x_j\|^2 \leq \left(1 - \frac{\tau_j}{2\|P\|}\right)\|x_j\|_P^2, \quad \forall j \geq N, \tag{4.6}$$

and hence,

$$\|x_j\|_P^2 \leq \left[\prod_{l=N}^{j-1}\left(1 - \frac{\tau_l}{2\|P\|}\right)\right]\|x_N\|_P^2, \quad \forall j \geq N+1. \tag{4.7}$$

If $x_{j_0} = 0$ for some $j_0 \geq N$, then it follows from (4.6) that $x_j = 0$ for all $j \geq j_0$, and thus $(x_j)_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^n)$. Now assume that $x_j \neq 0$ for all $j \geq N$. Then, by (4.6), $1 - \tau_j/(2\|P\|) > 0$ for all $j \geq N$. Moreover, since (4.1) yields $M := \inf_{j\in\mathbb{N}}\{\tau_j j^\alpha\} > 0$, we have $\tau_j \geq M/j^\alpha$ for all $j \in \mathbb{N}$, and thus

$$0 < 1 - \frac{\tau_j}{2\|P\|} \leq 1 - \frac{M}{2\|P\| j^\alpha}, \quad \forall j \geq N.$$

Combining this with (4.7) yields

$$\|x_j\|_P \leq \left[\prod_{l=N}^{j-1}\left(1 - \frac{M}{2\|P\| l^\alpha}\right)^{1/2}\right]\|x_N\|_P, \quad \forall j \geq N+1. \tag{4.8}$$

Define a positive sequence $(v_j)_{j\in\mathbb{N}_0}$ by

$$v_j := \prod_{l=N}^{N+j}\left(1 - \frac{M}{2\|P\| l^\alpha}\right)^{1/2} = \prod_{l=N}^{N+j}\left(1 - \frac{\gamma}{l^\alpha}\right)^{1/2},$$

where $\gamma := M/(2\|P\|)$. By (4.8), to show that $(x_j)_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^n)$, it suffices to prove that $(v)_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R})$. Invoking the inequality $1 - t \leq e^{-t}$ (which holds for all $t \in \mathbb{R}$), we have

$$\begin{aligned} \sum_{j=0}^{k} v_j &\leq \sum_{j=0}^{k} \exp\left(-\frac{\gamma}{2}\sum_{l=N}^{N+j}\frac{1}{l^\alpha}\right) \\ &\leq \sum_{j=0}^{k} \exp\left(-\frac{\gamma(j+1)}{2(N+j)^\alpha}\right), \quad \forall k \in \mathbb{N}_0. \end{aligned} \tag{4.9}$$

Since, by (4.1), we have $\alpha \in (0, 1)$, it follows that

$$\exp\left(-\frac{\gamma(j+1)}{2(N+j)^\alpha}\right) \leq \frac{1}{j^2}, \quad \text{for all sufficiently large } j.$$

Hence, the right-hand side of (4.9) converges to a finite limit as $k \to \infty$, showing that $(v_j)_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R})$. □

Proof of Theorem 2.2: Let $x^0 \in \mathbb{R}^n$ be fixed, but arbitrary.

To prove statement (i), we adopt a contradiction argument and suppose that the sequence of sampling periods $(\tau_j)_{j\in\mathbb{N}_0}$ is not ultimately constant. Then, by Remark 2.1, $\lim_{j\to\infty}\tau_j = 0$. Moreover, invoking the definition of τ_j in (2.2), we obtain

$$\tau_j j^\alpha \geq \eta_j\left(\frac{j}{j+1}\right)^\alpha, \quad \forall j\in\mathbb{N}.$$

By assumption, $\inf_{j\in\mathbb{N}_0}\eta_j > 0$, and thus,

$$\inf_{j\in\mathbb{N}}\tau_j j^\alpha > 0.$$

Therefore, (4.1) is satisfied and Lemma 4.1 yields that $(x(t_j;x^0))_{j\in\mathbb{N}_0}\in\ell^1(\mathbb{N}_0,\mathbb{R}^n)$, and hence, $(y(t_j))_{j\in\mathbb{N}_0}\in\ell^1(\mathbb{N}_0,\mathbb{R}^p)$. Invoking again Remark 2.1 shows that $(\tau_j)_{j\in\mathbb{N}_0}$ is ultimately constant, contradicting the supposition that $(\tau_j)_{j\in\mathbb{N}_0}$ is not ultimately constant.

To prove statement (ii), we first note that, by the variation-of-parameter formula,

$$x(t_j+\theta;x^0) = \left(e^{A\theta} + \int_0^\theta e^{As}\mathrm{d}s\,BFC\right)x(t_j;x^0), \quad \forall\theta\in[0,\tau_j],\ \forall j\in\mathbb{N}_0. \tag{4.10}$$

By statement (i), there exists an $N\in\mathbb{N}_0$ such that

$$\tau_j = \tau_N =: \tau, \quad \forall j\geq N.$$

Hypothesis (D) guarantees that the pair $(C, e^{A\tau})$ is discrete-time detectable. Hence, there exists $H\in\mathbb{R}^{n\times p}$ such that $e^{A\tau}+\mathrm{HC}$ is power stable, i.e. all eigenvalues of $e^{A\tau}+\mathrm{HC}$ are in the open unit disc $\{s\in\mathbb{C} : |s|<1\}$. Setting $B_\tau := \int_0^\tau e^{As}\mathrm{d}s\,B$, it follows from (4.10) with $\theta=\tau$ that

$$\begin{aligned}x(t_{j+1};x^0) &= e^{A\tau}x(t_j;x^0) + B_\tau FCx(t_j;x^0)\\ &= (e^{A\tau}+HC)x(t_j;x^0) + (B_\tau F - H)y(t_j),\\ &\quad \forall j\geq N.\end{aligned}$$

Combining this with the power stability of $e^{A\tau}+HC$ and the fact that $(y(t_j))_{j\in\mathbb{N}_0}\in\ell^1(\mathbb{N}_0,\mathbb{R}^p)$ (guaranteed by Remark 2.1), we conclude that $(x(t_j;x^0))_{j\in\mathbb{N}_0}\in\ell^1(\mathbb{N}_0,\mathbb{R}^n)$. This implies in particular that

$$\lim_{j\to\infty} x(t_j;x^0) = 0. \tag{4.11}$$

Setting

$$\bar\tau := \sup_{j\in\mathbb{N}_0}\tau_j < \infty \quad\text{and}\quad M := \sup_{\theta\in[0,\bar\tau]}\left\|e^{A\theta} + \int_0^\theta e^{As}\mathrm{d}s\,BFC\right\|,$$

we obtain from (4.10) that

$$\|x(t_j+\theta;x^0)\| \leq M\|x(t_j;x^0)\|, \quad \forall\theta\in[0,\tau_j],\ \forall j\in\mathbb{N}_0.$$

Consequently, by (4.11),

$$\lim_{t\to\infty} x(t;x^0) = 0.$$

Finally,

$$\begin{aligned}\int_0^\infty \|x(t)\|\mathrm{d}t &= \sum_{j=0}^\infty \int_{t_j}^{t_{j+1}} \|x(t;x^0)\|\mathrm{d}t\\ &\leq M\bar\tau\sum_{j=0}^\infty \|x(t_j;x^0)\| < \infty,\end{aligned}$$

showing that $x\in L^1(\mathbb{R}_+,\mathbb{R}^n)$ and completing the proof of statement (ii). □

Proof of Lemma 2.6: By assumption, (C, A) is continuous-time detectable and δ_l is non-pathological relative to A for all $l\in\mathbb{N}_0$. Therefore, by a standard result (Francis and Georgiou (1988), Lemma 8), the pair $(C, e^{A\delta_l})$ is discrete-time detectable for all $l\in\mathbb{N}_0$, showing that hypothesis (D) holds. □

Proof of Corollary 3.2: Let $x^0\in\mathbb{R}^{n_\mathrm{p}+n_\mathrm{c}}$ be fixed, but arbitrary. Moreover, let the matrices B, C and F be defined as in (3.4). Invoking the variation-of-parameters formula, we conclude that

$$\begin{pmatrix} x_\mathrm{p}(t_j+\theta;x^0)\\ x_\mathrm{c}^\mathrm{d}(j+1;x^0)\end{pmatrix} = \left[\begin{pmatrix} e^{A_\mathrm{p}\theta} & 0\\ 0 & e^{A_\mathrm{c}\tau_j}\end{pmatrix} + \begin{pmatrix}\int_0^\theta e^{A_\mathrm{p}s}\mathrm{d}s & 0\\ 0 & \int_0^{\tau_j} e^{A_\mathrm{c}s}\mathrm{d}s\end{pmatrix}BFC\right] \times\begin{pmatrix} x_\mathrm{p}(t_j;x^0)\\ x_\mathrm{c}^\mathrm{d}(j;x^0)\end{pmatrix}, \quad \forall\theta\in[0,\tau_j),\ \forall j\in\mathbb{N}_0. \tag{4.12}$$

Since, by continuity of $x_\mathrm{p}(\cdot;x^0)$, $x_\mathrm{p}(t_j+\theta;x^0)\to x_\mathrm{p}(t_{j+1};x^0)$ as $\theta\uparrow\tau_j$, we obtain the following from (4.12), as $\theta\uparrow\tau_j$,

$$\begin{pmatrix} x_\mathrm{p}(t_{j+1};x^0)\\ x_\mathrm{c}^\mathrm{d}(j+1;x^0)\end{pmatrix} = \Delta_j\begin{pmatrix} x_\mathrm{p}(t_j;x^0)\\ x_\mathrm{c}^\mathrm{d}(j;x^0)\end{pmatrix}, \quad \forall j\in\mathbb{N}_0;\qquad \begin{pmatrix} x_\mathrm{p}(0;x^0)\\ x_\mathrm{c}^\mathrm{d}(0;x^0)\end{pmatrix} = x^0, \tag{4.13}$$

where $\Delta_j := e^{A\tau_j} + \int_0^{\tau_j} e^{As}\mathrm{d}s\,BFC$ with A, B, C and F given by (3.4). Now consider the adaptive sampled-data system defined by (2.1) and (2.2), where again A, B, C and F are given by (3.4) and, furthermore, $n = n_\mathrm{p}+n_\mathrm{c}$. Denoting its solution by $x(\cdot;x^0)$, it follows that

$$x(t_{j+1};x^0) = \Delta_j x(t_j;x^0), \quad \forall j\in\mathbb{N}_0;\quad x(0;x^0) = x^0.$$

Combining this with (4.13) shows that

$$x(t_j;x^0) = \begin{pmatrix} x_\mathrm{p}(t_j;x^0)\\ x_\mathrm{c}^\mathrm{d}(j;x^0)\end{pmatrix}, \quad \forall j\in\mathbb{N}_0.$$

An application of Corollary 2.7 to the sampled-data system defined by (2.1) and (2.2), with A, B, C and F given by (3.4), then shows that $(\tau_j)_{j\in\mathbb{N}_0}$ is ultimately constant and the sequence $(x(t_j;x^0))_{j\in\mathbb{N}_0}$ is in $\ell^1(\mathbb{N}_0,\mathbb{R}^n)$. In particular,

$$\lim_{j\to\infty} x(t_j;x^0) = \lim_{j\to\infty}\begin{pmatrix} x_{\rm p}(t_j;x^0) \\ x_{\rm c}^{\rm d}(j;x^0)\end{pmatrix} = 0. \tag{4.14}$$

Finally, we note that by using (4.12) and (4.14) in combination with an argument similar to that adopted at the end of the proof of Theorem 2.2 (after Equation (4.11)), it follows that $\lim_{t\to\infty} x_{\rm p}(t;x^0)=0$ and $x_{\rm p}(\cdot\,;x^0)\in L^1(\mathbb{R}_+,\mathbb{R}^{n_{\rm p}})$, completing the proof. □

5. Conclusions

We have proved that if the controlled continuous-time system $\dot{x}=Ax+Bu$ with output $y=Cx$ is exponentially stabilised by the static output feedback $u=Fy$ and if hypothesis (D) or hypothesis (D′) holds, then the corresponding indirect sampled-data control together with the adaptive strategy (2.2) leads to a stable sampled-data system in the sense that, for all initial states, the adaptation of the sampling period terminates after finitely many time steps and the state is integrable and converges to zero as time goes to infinity. Furthermore, we have shown how this result can be generalised to dynamic output feedback.

Acknowledgements

This research was supported in part by the UK Engineering & Physical Sciences Research Council (Grant Ref: GR/S94582/01) and the FRG Deutsche Forschungsgemeinschaft (Grant Ref: IL25/4).

References

Chen, T., and Francis, B.A. (1991), 'Input–Output Stability of Sampled-data Systems', *IEEE Transactions on Automatic Control*, 36, 50–58.

Dragan, V. (1990), 'Discrete Implementation of Stabilising Linear Controls', *Revue Roumaine des Sciences Techniques, Série: Électrotechnique et Énergétique*, 35, 389–396.

Francis, B.A., and Georgiou, T.T. (1988), 'Stability Theory for Linear Time-invariant Plants with Periodic Digital Controllers', *IEEE Transactions on Automatic Control*, 33, 820–832.

Ilchmann, A., and Townley, S. (1999), 'Adaptive Sampling Control of High-gain Stabilisable Systems', *IEEE Transactions on Automatic Control*, 44, 1961–1966.

Ke, Z. (2008), 'Sampled-data Control: Stabilisation, Tracking and Disturbance Rejection', Ph.D. thesis, University of Bath, 2008, http://www.maths.bath.ac.uk/~hl/THESES/ke_thesis.pdf

Logemann, H., Rebarber, R., and Townley, S. (2003), 'Stability of Infinite-dimensional Sampled-data Systems', *Transactions of the American Mathematical Society*, 355, 3301–3328.

Owens, D.H. (1996), 'Adaptive Stabilisation using a Variable Sampling Rate', *International Journal of Control*, 63, 107–119.

Özdemir, N., and Townley, S. (2003), 'Integral Control by Variable Sampling Based on Steady-state Data', *Automatica*, 39, 135–140.

Sontag, E.D. (1998), *Mathematical Control Theory* (2nd ed.), New York: Springer.

Tokarzewski, J., and Olbrot, A.W. (1995), 'Sufficient Stability Condition for a Sampled-data System with Digital Controller', *IEEE Transactions on Automatic Control*, 40, 1241–1243.

Townley, S. (1996), 'Topological Aspects of Universal Adaptive Stabilisation', *SIAM Journal of Control and Optimization*, 34, 1044–1070.

Townley, S. (1999), 'An Example of a Globally Stabilising Adaptive Controller with a Generically Destabilising Parameter Estimate', *IEEE Transactions on Automatic Control*, 44, 2238–2241.