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High-gain adaptive stabilization of multivariable linear systems – revisited

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Abstract: We consider the class of multi-input multi-output, finite dimensional, state space systems of the form $\dot{x} = Ax + Bu$, $y = Cx$, where the state dimension is unknown, the system is minimum phase, and it is known that $\det(CB) \neq 0$. For this class a universal adaptive high gain controller – not based on identification or estimation algorithms – is presented which ensures exponential decay of the motion of the closed-loop system. It is shown that the controller is robust with respect to certain nonlinear perturbations.

Keywords: Adaptive stabilization; adaptive control; linear time-invariant systems; nonlinear control.

Nomenclature

$\underline{N} := \{1, \dots, N\}$.
$\| x \|_P = \sqrt{\langle x, Px \rangle}$ for $x \in \mathbb{R}^n$, $P = P^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ positive definite.
$\mathbb{R}_+$ ($\mathbb{R}_-$) the set of non-negative (non-positive) real numbers.
$\mathbb{C}_+$ ($\mathbb{C}_-$) open right- (left-) half complex plane.
$\sigma(A)$ the spectrum of the matrix $A \in \mathbb{C}^{n \times n}$.
$L_p(J)$ vector space of measurable functions $f : J \to \mathbb{R}^n$, $J \subset \mathbb{R}$ some interval, such that $\| f(\cdot) \|_{L_p(J)} < \infty$, where

$$\| f(\cdot) \|_{L_p(J)} := \begin{cases} \left[\int_J \| f(s) \|^p \, \mathrm{d}s \right]^{1/p} & \text{for } p \in [1, \infty). \\ \operatorname*{ess\,sup}_{s \in J} \| f(s) \| & \text{for } p = \infty. \end{cases}$$

1. Introduction

Let Σ denote the class of multi-input multi-output systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) \in \mathbb{R}^n, \\ y(t) &= Cx(t), \end{aligned} \tag{1.1}$$

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$(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$, which satisfy the *minimum phase* condition

$$\det\begin{bmatrix} sI-A & B \\ C & 0 \end{bmatrix} \neq 0 \quad \text{for all } s \in \overline{\mathbb{C}}_+ \tag{1.2}$$

and have invertible high-frequency gain, i.e.

$$\det(CB) \neq 0. \tag{1.3}$$

Notice that the number m of inputs and outputs is fixed while the state dimension is not. The problem of adaptively stabilizing the class Σ, i.e. constructing a (nonlinear) control law which stabilizes *all* systems in Σ, has been studied by Byrnes and Willems [1] and Mårtensson [5,6] using high-gain ideas. The stabilizing control laws obtained in [1] and [5,6] are both based on static output feedback of the form

$$u(t) = k(t)K(k(t))y(t),$$

where the gain parameter k is adaptively increased according to $\dot{k} = \|y\|^2$ in [1] and $\dot{k} = \|y\|^2 + \|u\|^2$ in [5,6] and the gain matrix K is adjusted by a switching strategy which is driven by k and which rotates K among (at most) countably many different matrices K_i. [1] Despite the similarity of the control laws, the proofs in [1] and [5,6] are quite different and more important, they are not convincing because they contain gaps; see Remark 3.8 for details.

Mårtensson [4] has shown that the order of any linear time-invariant stabilizing feedback controller is a sufficient information about a minimal linear time-invariant system in order to apply an adaptive stabilizer. Therefore it is known that adaptive stabilization is feasible for the class Σ considered in the present paper. However, it is interesting to see how the additional asssumptions (1.2) and (1.3) can be used to construct a less complicated controller compared to that in [4].

In this paper we present an adaptive control law which contains the controller proposed in [1] as a special case. By bringing in an additional design parameter we modify the control law given in [1] in such a way that the state trajectory of the closed-loop system is exponentially decaying. In particular, we close the gap contained in the proof of [1]. Moreover we allow gain adaptations of the form $\dot{k}(t) = \|y(t)\|^p$ for $p \geq 1$, and show that closed-loop stability is retained under perturbations of (1.1) given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B[u(t) + h(t, x(t))], \quad x(0) \in \mathbb{R}^n, \\ y(t) &= Cx(t), \end{aligned} \tag{1.4}$$

where the function $h: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable in t, locally Lipschitz in x and of finite gain. The latter condition means that there exists an $\hat{h} > 0$ such that

$$\|h(t, x)\| \leqslant \hat{h}\|x\| \quad \text{for all } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{1.5}$$

The reasons to consider gain adaptation $\dot{k} = \|y\|^p$ for $p \geq 1$ instead of the usual $p = 2$ are: (i) to show that L_p techniques are appropriate in this set-up, (ii) numerical simulations have shown that the transient behaviour is better for p large since the system reacts faster and so the stabilizing gain does not become unnecessarily large. The presentation for $p \geq 1$ is more complicated than for $p = 2$, the reader can set $p = 2$ wherever he likes.

Exponential stabilization by high-gain adaptive controllers has previously been studied by Logemann [3] and Ilchmann and Owens [2]. In [3] an exponentially stabilizing controller is given for a class of single-input single-output retarded systems which are *strongly* minimum phase, i.e. which have no zeros in $\mathrm{Re}(s) \geq -\varepsilon$ for some $\varepsilon > 0$. It is shown that the controller is robust with respect to various nonlinear perturbations in the state and certain sector-bounded actuator and sensor nonlinearities. In [2] a controller is constructed which produces an exponentially decaying closed-loop trajectory for any square finite dimensional plant, provided it is minimum phase and $\sigma(CB) \subset \mathbb{C}_-$.

The paper is organized as follows. In Section 2 we present some basic properties of the class Σ. The main result is proved in Section 3.

[1] In fact it is shown in [5,6] that *finitely* many gain matrices $K_1, \ldots, K_N$ exist, where N depends on m.

2. Properties of the class Σ

In this section we derive some basic properties of the class Σ which will be used in the following.

2.1. Remark. A system of the form (1.1) is minimum phase if and only if (A, B) is stabilizable, (A, C) is detectable, and the transfer function matrix $C(sI_n - A)^{-1}B \in \mathbb{R}(s)^{m\times m}$ has no zeros inside $\overline{\mathbb{C}}_+$.

In order to include the possibility of exponential stabilization we introduce an exponentially weighted gain adaptation. Logemann [3] has proved that $\dot{k}(t) = e^{\omega t}y(t)^2$, $u(t) = -k(t)y(t)$ leads, in the single-input single-output case and $CB > 0$, to an exponentially decaying output if it is known that

$$\det\begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} \neq 0 \quad \text{for all } s\in\mathbb{C} \text{ with } \operatorname{Re} s \geq -\omega,\ \omega > 0.$$

Ilchmann and Owens [2] do not use the additional assumption but adjust $\omega(\cdot)$ adaptively. Therefore we introduce the following notation.

2.2. Definition Let $\Omega(t_0, t')$, $0 \leq t_0 < t' \leq \infty$, be the class of continuously differentiable functions $\omega(\cdot):[t_0, t') \to \mathbb{R}_+$ which satisfy

$$\omega(t) \text{ is non-increasing on } [t_0, t'), \tag{2.1}$$

$$\omega(t) > 0 \quad \text{for all } t\in[t_0, t') \text{ if } \omega(\cdot) \not\equiv 0, \tag{2.2}$$

$$\lim_{t\to t'} \omega(t) = 0. \tag{2.3}$$

Let $\omega(\cdot):[t_0, t') \to \mathbb{R}_+$ be continuously differentiable and $v(\cdot):[t_0, t') \to \mathbb{R}^r$, $r\in\mathbb{N}$, be a vector-valued function. Then $v_w(\cdot):[t_0, t') \to \mathbb{R}^r$ is defined by

$$v_w(t) := e^{\omega(t)t}v(t). \tag{2.4}$$

For the sake of simplicity the reader may set $\omega \equiv 0$ in the following and he will get the result for asymptotic (not exponential) stabilization.

Since $\det(CB) \neq 0$ the state space can be composed into the direct sum $\mathbb{R}^n = \operatorname{im} B \oplus \ker C$ which leads to the following convenient decomposition of the system (1.4).

2.3. Proposition. *Suppose $\omega(\cdot)\in\Omega(0, t')$ and $(A, B, C)\in\Sigma$. If $V\in\mathbb{R}^{n\times(n-m)}$ denotes a basis matrix of* ker C, *then $U := [B(CB)^{-1}, V]$ is invertible, and under the state space transformation $[y^T, z^T]^T = U^{-1}x$ and new coordinates y_ω, z_ω the equations in* (1.4) *are equivalent to*

$$\dot{y}_\omega(t) = [A_1 + (\omega(t) + \dot{\omega}(t)t)I_m]y_\omega(t) + A_2 z_\omega(t) + CBu_\omega(t) + CB\tilde{h}(t, y_\omega(t), z_\omega(t)), \tag{2.5a}$$

$$\dot{z}_\omega(t) = A_3 y_\omega(t) + [A_4 + (\omega(t) + \dot{\omega}(t)t)I_{n-m}]z_\omega(t), \tag{2.5b}$$

where

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = U^{-1}AU$$

and

$$\tilde{h}(t, \eta, \xi) := e^{\omega(t)t}h\left(t, e^{-\omega(t)t}U[\eta^T, \xi^T]^T\right) \quad \textit{for all } (t, \eta, \xi)\in\mathbb{R}_+\times\mathbb{R}^m\times\mathbb{R}^{n-m}.$$

The minimum phase condition implies $\sigma(A_4)\subset\mathbb{C}_-$ and (1.5) gives

$$\|\tilde{h}(t, \eta, \xi)\| \leq \hat{h}\|U\|\,\|[\eta^T, \xi^T]^T\| \quad \text{for all } (t, \eta, \xi)\in\mathbb{R}_+\times\mathbb{R}^m\times\mathbb{R}^{n-m}. \tag{2.6}$$

The class of systems under consideration satisfies the following fundamental inequality relating the weighted output $y_\omega(t)$ at time t to the weighted input $u_\omega(\cdot)$ and the weighted output $y_\omega(\cdot)$ on the past interval $[0, t)$.

2.4. Proposition. *Consider system* (1.4) *and suppose that* $(A, B, C) \in \Sigma$ *and* $\omega(\cdot) \in \Omega(t_0, t')$, *where* $0 \le t_0 < t' \le \infty$. *Furthermore let* $u(\cdot): [t_0, t') \to \mathbb{R}^m$ *be locally integrable*, $P \in \mathbb{R}^{n \times n}$ *be positive definite, and* $p \ge 1$. *Define*

$$\beta: \mathbb{R}^m \to \mathbb{R}^m, \quad y \mapsto \beta(y) = \begin{cases} \dfrac{y}{\| y \|_P}, & y \neq 0, \\ 0, & y = 0. \end{cases}$$

Then there exists $M = M(x(0)) > 0$ *such that for all* $t \in [t_0, t')$,

$$\frac{1}{p} \| y_\omega(t) \|_P^p \le M + M \int_{t_0}^{t} \| y_\omega(s) \|_P^p \, \mathrm{d}s + \int_{t_0}^{t} \| y_\omega(s) \|_P^{p-1} \langle \beta(y_\omega(s)), PCBu_\omega(s) \rangle \, \mathrm{d}s. \tag{2.7}$$

Proof. Assume (A, B, C) is of the form (2.5). Since A_4 is exponentially stable, $\lim_{t \to t'} \omega(t) = 0$, and $\dot{\omega}(t) \le 0$, the homogeneous part of (2.5b) is an exponentially stable system. Therefore it can be shown (use e.g. a similar argument as in [7], p.258) that there exists an $M_1 > 0$ such that

$$\| z_\omega(\cdot) \|_{L_p(t_0, t)} \le M_1 + M_1 \| y_\omega(\cdot) \|_{L_p(t_0, t)}. \tag{2.8}$$

Let $J_1 \subset [t_0, t')$ be the set of measure zero where $y_\omega(\cdot)$ is not differentiable and

$$J_2 := \{ t \in [t_0, t') \setminus J_1 \mid y_\omega(t) = 0, \ \dot{y}_\omega(t) \neq 0 \}.$$

Now it is easy to see that $\| y_\omega(\cdot) \|_P$ is not differentiable in any point of J_2. However $\| y_\omega(\cdot) \|_P$ is absolutely continuous because $y_\omega(\cdot)$ is and hence J_2 must be of measure zero. It follows that $J := J_1 \cup J_2$ is of measure zero and a routine calculation gives

$$\frac{\mathrm{d}}{\mathrm{d}s} \| y_\omega(s) \|_P = \begin{cases} \dfrac{\langle y_\omega(s), P\dot{y}_\omega(s) \rangle}{\| y_\omega(s) \|_P}, & s \in [t_0, t') \setminus J \text{ and } y_\omega(s) \neq 0, \\ 0, & s \in [t_0, t') \setminus J \text{ and } y_\omega(s) = 0. \end{cases} \tag{2.9}$$

Using (2.5a), (2.6), and (2.9) yields for an appropriate $M_2 > 0$ and all $s \notin J$,

$$\begin{aligned} \frac{1}{p} \frac{\mathrm{d}}{\mathrm{d}s} (\| y_\omega(s) \|_P^p) \le{} & M_2 \| y_\omega(s) \|_P^{p-2} \left[\| y_\omega(s) \|_P^2 + \| y_\omega(s) \| \cdot \| z_\omega(s) \| \right] \\ & + \| y_\omega(s) \|_P^{p-1} \langle \beta(y_\omega(s)), PCBu_\omega(s) \rangle. \end{aligned} \tag{2.10}$$

Integrating (2.10) and using the inequality

$$\int_{t_0}^{t} \| y_\omega(s) \|^{p-1} \| z_\omega(s) \| \, \mathrm{d}s \le \| y_\omega(\cdot) \|_{L_p(t_0, t)}^{p-1} \cdot \| z_\omega(\cdot) \|_{L_p(t_0, t)}$$

together with (2.8) gives the result. □

3. Main result

The switching strategy is based on the following result from linear algebra.

3.1. Lemma. *There exists a finite set* $\{K_1, \ldots, K_N\} \subset \mathrm{GL}_m(\mathbb{R})$ *so that, for any* $M \in GL_m$ *there exists* $i \in \underline{N}$ *such that* $\sigma(MK_i) \subset \mathbb{C}_-$.

Proof. See Mårtensson [6], Section 8, or Mårtensson [5], p.81. □

The set given in Lemma 3.1 is often called the *unmixing set*. Unfortunately, the cardinality of the unmixing sets constructed by Mårtensson [5,6] is far too large than would be convenient for applications. Hardly anything is known on the minimum cardinality of unmixing sets. However, for $m = 1$ the set $\{1, -1\}$ is obviously unmixing, while for $m = 2$ there exists an unmixing set of cardinality 6 (see [6]). It has been shown by Zhu [9] that $\mathrm{GL}_3(\mathbb{R})$ can be unmixed by a set having cardinality 32.

In order to extend the result of single-output systems with unknown high frequency gain, see Willems and Byrnes [8], we introduce the following concept of switching sequences and functions.

3.2. Definition. A strictly increasing sequence $0 < \tau_1 < \tau_2 < \cdots$ is called a *switching sequence* if it satisfies the following growth condition

$$\lim_{i \to \infty} \frac{\tau_{i-1}}{\tau_i} = 0. \tag{3.1}$$

A function $S(\cdot): \mathbb{R} \to \underline{N}$, $N \in \mathbb{N}$, is called a *switching function* if there exists a switching sequence $\{\tau_i\}_{i \in \mathbb{N}}$ such that

$$S(k) = \begin{cases} 1 & \text{if } k \in (-\infty, \tau_1), \\ i & \text{if } k \in [\tau_{lN+i}, \tau_{lN+i+1}) \text{ for some } l \in \mathbb{N}_0,\ i \in \underline{N}. \end{cases} \tag{3.2}$$

Note that a switching sequence $\{\tau_i\}_{i \in \mathbb{N}}$ necessarily satisfies

$$\lim_{i \to \infty} \tau_i = \infty. \tag{3.3}$$

3.3. Examples. (i) It is obvious that the sequence $\tau_i := \tau_{i-1}^2$ for $i \in \mathbb{N}$, $\tau_0 > 1$ is a switching sequence.

(ii) The sequence $\tau_i := i^2$, $i \in \mathbb{N}$, has been used by Willems and Byrnes [8] in the single-input single-output case. However, it is not a switching sequence in the sense of Definition 3.2.

(iii) The sequence $\tau_{i+1} := \tau_i + \mathrm{e}^{(i^2)}$, $i \in \mathbb{N}$, has been suggested by Byrnes and Willems [1]. It is in fact a switching sequence since

$$\tau_i/\tau_{i+1} = \left[1 + \frac{\mathrm{e}^{(i^2)}}{\tau_i}\right]^{-1} \quad \text{and} \quad \frac{\mathrm{e}^{(i^2)}}{\tau_i} \geq \frac{\mathrm{e}^{1+(2i-1)}}{\tau_0\, \mathrm{e}^{-(i-1)^2} + \mathrm{e}^{-(i-1)^2} i\, \mathrm{e}^{(i-1)^2}} = \frac{\mathrm{e}^{1+(2i-1)}}{\tau_0\, \mathrm{e}^{-(i-1)^2} + i}.$$

The right hand side tends to $+\infty$ as i goes to $+\infty$.

In order to prove the main result the following lemma is needed.

3.4. Lemma. *Suppose* $S(\cdot): \mathbb{R} \to \underline{N}$, $N \in \mathbb{N}$, *is a switching function associated with a switching sequence* $\{\tau_i\}_{i \in \mathbb{N}}$. *If we define for arbitrary* $\alpha > 0$ *and every* $i \in \underline{N}$,

$$f_i^\alpha(x) := \begin{cases} 1 & \text{if } S(x) = i, \\ -\alpha & \text{if } S(x) \neq i, \end{cases} \tag{3.4}$$

then it follows that

$$\sup_{k > 0} \frac{1}{k} \int_0^k x \cdot f_i^\alpha(x)\, \mathrm{d}x = +\infty \tag{3.5}$$

Proof. Without restriction of generality we only consider the function $f_N^\alpha(\cdot)$, i.e. $i = N$. It follows from the definition of $S(\cdot)$ that

$$\int_{\tau_{jN+1}}^{\tau_{(j+1)N+1}} x \cdot f_N^\alpha(x)\,\mathrm{d}x = -\int_{\tau_{jN+1}}^{\tau_{jN+N}} \alpha \cdot x\,\mathrm{d}x + \int_{\tau_{jN+N}}^{\tau_{(j+1)N+1}} x\,\mathrm{d}x = \tfrac{1}{2}\alpha\tau_{jN+1}^2 - \tfrac{1}{2}(\alpha+1)\tau_{jN+N}^2 + \tfrac{1}{2}\tau_{(j+1)N+1}^2.$$

Therefore

$$\begin{aligned}\frac{1}{\tau_{(l+1)N+1}-\tau_1}\int_{\tau_1}^{\tau_{(l+1)N+1}} x\cdot f_N^\alpha(x)\,\mathrm{d}x &= \frac{1}{\tau_{(l+1)N+1}-\tau_1}\sum_{j=0}^{l}\int_{\tau_{jN+1}}^{\tau_{(j+1)N+1}} x\cdot f_N^\alpha(x)\,\mathrm{d}x\\ &= \frac{1}{2}\frac{1}{\tau_{(l+1)N+1}-\tau_1}\sum_{j=0}^{l-1}\alpha\tau_{jN+1}^2 - [\alpha+1]\tau_{(j+1)N}^2 + \tau_{(j+1)N+1}^2\\ &\quad+\frac{1}{2}\frac{1}{\tau_{(l+1)N+1}-\tau_1}\left[\alpha\tau_{lN+1}^2 - (\alpha+1)\tau_{(l+1)N}^2 + \tau_{(l+1)N+1}^2\right].\end{aligned} \tag{3.6}$$

Since

$$\alpha\tau_{jN+1}^2 - [\alpha+1]\tau_{(j+1)N}^2 + \tau_{(j+1)N+1}^2 \geq \tau_{(j+1)N+1}^2\left[-(\alpha+1)\frac{\tau_{(j+1)N}^2}{\tau_{(j+1)N+1}^2} + 1\right]$$

it follows from condition (3.1) that the first term in (3.6) is bounded from below by some $L \in \mathbb{R}$ (independent of l). Therefore

$$\frac{1}{\tau_{(l+1)N+1}-\tau_1}\int_{\tau_1}^{\tau_{(l+1)N+1}} x\cdot f_N^\alpha(x)\,\mathrm{d}x \geq L + \frac{1}{2}\frac{\tau_{(l+1)N+1}^2}{\tau_{(l+1)N+1}-\tau_1}\left[1-(\alpha+1)\left(\frac{\tau_{(l+1)N}}{\tau_{(l+1)N+1}}\right)^2\right] \tag{3.7}$$

and the second summand in (3.7) goes to $+\infty$ as l tends to ∞. This completes the proof. □

3.5. Remark. It is easy to see that condition (3.5) is satisfied if and only if

$$\sup_{k>k_0}\frac{1}{k-k_0}\int_{k_0}^{k} x\cdot f_i^\alpha(x)\,\mathrm{d}x = +\infty \quad \text{for all } k_0 \in \mathbb{R}. \tag{3.8}$$

The switching function will be build into the feedback via

$$u(t) = k(t)\cdot K_{S(k(t))}\cdot y(t)$$

where the gain $k(\cdot)$ increases monotonically and different K_i's (given by Lemma 3.1) are picked. The intuition behind the above control law is as follows: If the 'correct' K_i is hit, the gain $k(\cdot)$ is large enough, and the time interval until the next possible switch is long enough (which is ensured by the condition (3.1)), then the system settles down and no more switchings occur. However this does not guarantee that the terminal gain $\lim_{t\to\infty}k(t)K_{S(k(t))}$ is a stabilizing output feedback gain. It is only ensured that the trajectory of the closed-loop system corresponding to a particular initial value is forced to zero.

3.6. Theorem. *Suppose $p \geq 1$, $\omega(\cdot) \in \Omega(0,\infty)$, $S:\mathbb{R}\to \underline{N}$ is a switching function, and $(A, B, C) \in \Sigma$. Let $K_1, \ldots, K_N$ be as in Lemma 3.1 and apply the feedback law*

$$u(t) = k(t)\cdot K_{(S\circ k)(t)}\cdot y(t), \tag{3.9}$$

$$\dot{k}(t) = \|\mathrm{e}^{(\omega\circ k)(t)t}y(t)\|^p, \quad k(0)\in\mathbb{R}_+, \tag{3.10}$$

to the system (1.4). *Then the resulting closed-loop system*

$$\dot{x}(t) = \left[A + k(t)BK_{(S\circ k)(t)}C\right]x(t) + Bh(t, x(t)), \quad x(0) \in \mathbb{R}^n,$$
$$\dot{k}(t) = \|\mathrm{e}^{(\omega\circ k)(t)t}y(t)\|^p, \quad k(0) \in \mathbb{R}_+, \tag{3.11}$$

has the following properties:

(i) *The solution of* (3.11), *i.e. an absolutely continuous function satisfying* (3.11) *a.e. on* $\mathbb{R}_+$, *exists.*
(ii) $\lim_{t\to\infty} k(t) = k_\infty < \infty$, *and hence* $K_{S(k(t))} = K_i$ *for all* $t \geq t^*$ *for some* $i \in \underline{N}$ *and some* $t^* > 0$.
(iii) $\lim_{t\to\infty}(\omega\circ k)(t) = \omega_\infty > 0$ *if* $\omega(\cdot) \not\equiv 0$.
(iv) *If* $\omega(\cdot) \not\equiv 0$ *then there exist* $M = M(x(0), k(0)) > 0$, *and* $\lambda = \lambda(x(0), k(0)) > 0$ *such that*

$$\|x(t)\| \leq M\,\mathrm{e}^{-\lambda t} \quad \textit{for all } t \geq 0,$$

while if $\omega(\cdot) \equiv 0$ *then* $x(\cdot) \in L_p(0, \infty) \cap L_\infty(0, \infty)$ *and*

$$\lim_{t\to\infty} x(t) = 0.$$

Due to the discontinuity in k of the right hand side of (3.11) the proof of Theorem 3.6 requires a certain amount of technicalities. The idea is as follows. Existence and uniqueness of the solution (x, k) is ensured on an interval where $S\circ k$ is constant. Since $k(\cdot)$ is monotonically increasing the solution can be stuck together as long as $k(t)$ remains bounded. We will show that $k(t)$ cannot escape to infinity, either in finite or infinite time, since the length of the intervals where $(S\circ k)(t)$ is kept constant is increasing, by virtue of (3.1), and finally if the 'correct' gain is hit the system settles down so that no more switchings occur. Now the properties in (iv) follow by standard arguments.

Proof. The right hand side of (3.11) is discontinuous in k, discontinuities occur for $k = \tau_i$, where $\{\tau_i\}_{i\in\mathbb{N}}$ is the switching sequence associated with $S(\cdot)$. Set $\tau_0 := \min\{k(0), \tau_1\}$. It is clear that there exists an $j \in \mathbb{N}$ such that

$$\tau_{j-1} \leq k(0) < \tau_j.$$

(a) First we investigate the solution of (3.11) under the constraint that $k(t) < \tau_j$, that means no switching occurs. It follows from the assumptions that the right hand side of (3.11) is locally integrable in t and locally Lipschitz in (x, k) for $(x, k) \in \mathbb{R}^n \times [\tau_{j-1}, \tau_j)$. Therefore there exists a unique solution of (3.11) on $[0, T_j)$ for some $T_j > 0$. Let $[0, T_j)$ be the largest interval on which (3.11) has unique solution $(x(\cdot), k(\cdot))$ with $k(t) < \tau_j$ for all $t \in [0, T_j)$.

(b) Since $k(t)$ is bounded on $[0, T_j)$ and $h(t, x)$ is linearly bounded in x (uniformly in t), it follows from the first equation of (3.11) and the theory of ordinary differential equations that $x(\cdot) \in L_\infty(0, T_j)$, too.

(c) If $T_j = \infty$ then statements (i)–(iii) follow. Suppose $T_j < \infty$. We claim that the solution of (3.11) can be extended beyond T_j. To this end notice that $\lim_{t\to T_j} k(t) = \tau_j$ by (b) and the definition of T_j. Furthermore we obtain from (b) and the first equation in (3.11) that $\dot{x} \in L_\infty(0, T_j) \subset L_1(0, T_j)$. Using the fact that

$$x(t) = x(0) + \int_0^t \dot{x}(s)\,\mathrm{d}s \quad \text{for all } t \in [0, T_j)$$

it follows that $\bar{x}_j := \lim_{t\to T_j} x(t)$ exists. Thus there exists a maximal $T_{j+1} > T_j$ such that on $[T_j, T_{j+1})$ the initial value problem

$$\dot{x}(t) = \left[A + k(t)BK_{S(k(t))}C\right]x(t) + Bh(t, x(t)), \quad x(T_j) = \bar{x}_j,$$
$$\dot{k}(t) = \|\mathrm{e}^{(\omega\circ k)(t)t}y(t)\|^p, \quad k(T_j) = \tau_j,$$

has a unique solution $(x(\cdot), k(\cdot))$ which satisfies $k(t) < \tau_{j+1}$ for $t \in [T_j, T_{j+1})$ and which extends the solution of (3.11) to $[0, T_{j+1})$.

(d) If there exists a $j_0 \in \mathbb{N}$ such that $T_{j_0} = \infty$ then (i)–(iii) follow. Assume the contrary, i.e. there are infinitely many switching times

$$T_1 < T_2 < \cdots < \infty \quad \text{with} \quad \lim_{j\to\infty} T_j =: T \in \mathbb{R}_+ \cup \{\infty\}.$$

By (a)–(c) the solution (x, k) exists on $[0, T)$. If $k(\cdot)$ were bounded on $[0, T)$ then at most finitely many switches would occur during the time interval $[0, T)$. This would contradict the assumption, therefore $k(\cdot) \notin L_\infty(0, T)$. We will show that this leads to a contradiction. By Lemma 3.1 there exists $i \in \underline{N}$ such that $\sigma(CBK_i) \subset \mathbb{C}_-$. Let $P = P^{\mathrm{T}} \in \mathbb{R}^{m\times m}$ be the unique positive-definite solution of

$$K_i^{\mathrm{T}}(CB)^{\mathrm{T}}P + PCBK_i = -I_m.$$

Choose $\alpha > 0$ so that

$$K_l^{\mathrm{T}}(CB)^{\mathrm{T}}P + PCBK_l \le \alpha I_m \quad \text{for all } l \in \underline{N}. \tag{3.12}$$

It then follows from the definition of $f_i^\alpha(\cdot)$, (3.9), and from (3.12) that

$$\begin{aligned} 2\langle y_{\omega\circ k}(t), PCBu_{\omega\circ k}(t)\rangle &= 2k(t)\langle y_{\omega\circ k}(t), PCBK_{(S\circ k)(t)}y_{\omega\circ k}(t)\rangle \\ &\le -k(t)f_i^\alpha(k(t))\,\| y_{\omega\circ k}(t)\|^2. \end{aligned} \tag{3.13}$$

Since $k(\cdot)$ is unbounded on $[0,T)$, we have that $\lim_{t\to T}(\omega\circ k)(t) = 0$. Therefore we can apply Proposition 2.4, and inserting (3.13) into the inequality yields for all $t \in [0, T)$,

$$\begin{aligned} \frac{1}{p}\| y_{\omega\circ k}(t)\|_P^p &\le M + M\int_0^t \| y_{\omega\circ k}(s)\|_P^p \,\mathrm{d}s \\ &\quad - \tfrac{1}{2}\int_0^t f_i^\alpha(k(s))k(s)\,\|\beta(y_{\omega\circ k}(s))\|^2\cdot \| y_{\omega\circ k}(s)\|_P^{p-2}\,\mathrm{d}s. \end{aligned} \tag{3.14}$$

Let M_1 be a positive number such that for all $y_{\omega\circ k}(s) \ne 0$ we have

$$-\| y_{\omega\circ k}(s)\|^2\cdot \| y_{\omega\circ k}(s)\|_P^{p-2} \le -M_1\,\| y_{\omega\circ k}(s)\|^p = -M_1\cdot \dot{k}(s).$$

Inserting this inequality into (3.14) and changing variables gives for suitable $M_2, M_3 > 0$,

$$\begin{aligned} \frac{1}{p}\| y_{\omega\circ k}(t)\|_P^p &\le M + M_2[k(t)-k(0)] - M_3\int_{k(0)}^{k(t)} f_i^\alpha(\mu)\mu\,\mathrm{d}\mu \\ &= M + M_2[k(t)-k(0)]\left[1 - \frac{M_3}{M_2}\frac{1}{k(t)-k(0)}\int_{k(0)}^{k(t)} f_i^\alpha(\mu)\mu\,\mathrm{d}\mu\right]. \end{aligned} \tag{3.15}$$

Since $k(t) \to \infty$ as $t \to T$ it follows from Lemma 3.4 that the right hand side of (3.15) becomes negative, which is impossible. This contradiction proves that there exists a $j_0 \in \mathbb{N}$ such that $T_{j_0} = \infty$. It remains to prove (iv).

(e) We consider the case $\omega \not\equiv 0$, only. The case $\omega \equiv 0$ is simpler and therefore omitted. It follows from (3.10) and statement (ii) that $y_{\omega\circ k}(\cdot) \in L_p(0, \infty)$. Hence $y_\lambda(\cdot) \in L_p(0, \infty)$ for all $\lambda \in [0, \omega_\infty]$, where $\omega_\infty = \lim_{t\to\infty}(\omega\circ k)(t)$. Consider next (2.5b) for λ instead of $\omega(\cdot)$. Then for $\lambda > 0$ sufficiently small $\sigma(A_4 + \lambda I_{n-m}) \subset \mathbb{C}_-$ and it follows from (2.5b) and from $y_\lambda(\cdot) \in L_p(0, \infty)$ that $z_\lambda(\cdot) \in L_p(0, \infty)$. This together with (2.5) yields

$$\left[y_\lambda(\cdot)^{\mathrm{T}}, z_\lambda(\cdot)^{\mathrm{T}}\right]^{\mathrm{T}}, \left[\dot{y}_\lambda(\cdot)^{\mathrm{T}}, \dot{z}_\lambda(\cdot)^{\mathrm{T}}\right]^{\mathrm{T}} \in L_p(0, \infty).$$

Therefore $\lim_{t\to\infty} [y_\lambda(t)^T, z_\lambda(t)^T] = 0$ *and we obtain*

$$\lim_{t\to\infty} x(t)\, e^{\lambda t} = \lim_{t\to\infty} U\big[y(t)^T, z(t)^T\big]^T e^{\lambda t} = 0.$$

Therefore (iv) is proved. □

3.7. Remark. It can be shown that Theorem 3.6 remains true if the system (1.4) is subjected to the following additional disturbances:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + g(t, x(t)) + d(t) + B[u(t) + h(t, x(t))], \\ y(t) &= Cx(t), \end{aligned} \tag{3.16}$$

where

$$g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n, \quad (t, x) \mapsto g(t, x) \quad \text{with } \| g(t, x) \| \le \hat{g} \| x \| \text{ for all } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

is a function which is measurable in t, locally Lipschitz in x, and has sufficiently small gain $\hat{g}$, and $d(\cdot)\, e^{\varepsilon \cdot} \in L_p(0, \infty)$ for some $\varepsilon > 0$.

If we are only interested in asymptotically stability, i.e. $\omega(\cdot) \equiv 0$, then we may allow that $d(\cdot) \in L_p(0, \infty)$. For the sake of brevity we omit the proof here.

3.8. Remark. (i) The proof of Byrnes and Willems [1] is incomplete because their inequality (3.4) is not valid for $\alpha > 1$. However, our approach is in their spirit. It is essentially Lemma 3.4 in the present paper together with a rigorous proof that the closed loop system (3.11) with discontinuous right hand side admits a unique absolutely continuous solution on $\mathbb{R}_+$ which completes the argument in [1].

(ii) The incompleteness in Mårtensson [6], see also [5], is more subtle. The proof of Theorem 9.1 in [6] (Th. 6.14 in [5]) is based on the claim that for the adaptive control system

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t), \quad x(0) \in \mathbb{R}^n, \\ &u(t) = k(t) Q y(t) \\ &\dot{k}(t) = \| y(t) \|^2 + \| u(t) \|^2, \quad k(0) \in \mathbb{R}, \end{aligned}$$

there exists constants c, $T > 0$ such that

$$\int_{t_0}^{\infty} \| y(s) \|^2 + \| u(s) \|^2 \, ds \le c \| x(t_0) \|^2 \quad \text{for all } t_0 \ge T,\ x(0) \in \mathbb{R}^n,\ k(0) \in \mathbb{R},$$

provided that (A, B, C) is minimum phase and $\sigma(CBQ) \subset \mathbb{C}_-$. This property is not shown in [5,6] and we expect that it is not valid in general.

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