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A MIXED PASSIVITY/SMALL-GAIN THEOREM FOR SOBOLEV INPUT-OUTPUT STABILITY*

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Abstract. A stability theorem for the feedback connection of two (possibly infinite-dimensional) time-invariant linear systems is presented. The theorem is formulated in the frequency domain and is in the spirit of combined passivity/small-gain results. It places a mixture of positive realness and small-gain assumptions on the two transfer functions to ensure a certain notion of input-output stability, called *Sobolev* stability (which includes the classical L^2-stability concept as a special case). The result is more general than the classical passivity and small-gain theorems; strong positive realness of either the plant or controller is not required, and the small-gain condition only needs to hold on a suitable subset of the open right-half plane. We show that the "mixed" stability theorem is applicable in settings where L^2-stability of the feedback connection is not possible, such as output regulation and disturbance rejection of certain periodic signals by so-called repetitive control.

Key words. feedback control, output regulation, passivity theorem, positive realness, small-gain theorem, Sobolev stability

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1. Introduction.

For time-invariant linear control systems, positive realness is the frequency domain characterization of the time-domain property of passivity. Positive realness in a circuit theory context appears to date back to the 1931/32 papers [7, 9, 10] and is nowadays a key concept in mathematical systems and control theory. Indeed, on the one hand, it is fundamental for the analysis and synthesis of electrical networks [4, 27]; on the other hand, it appears as a natural condition in the study of the stability of certain nonlinear control systems, so-called absolute stability, via the Kalman–Yakubovich–Popov (or positive real) lemma [21, 23]. The upshot is that positive realness is a much-studied property over a vast array of literature. For example, positive realness plays a central role in the recent monograph [6] on dissipative systems. We refer the reader to [4, 6, 19, 26] for more background on the positive real property and note that some authors use the term positive, rather than positive real, such as in [42]. In the time-invariant linear case, the passivity theorem (see [19, Theorem 6.16] for a version that captures a large class of infinite-dimensional systems) states that the feedback connection of an L^2-input–output stable and strongly positive real plant and positive real controller is itself positive real and L^2-input-output stable. This result traces its roots back to the work of Zames [40]. Similar to passivity notions, small loop-gain ideas have been around in control theory for a long time; the first formal statements of the small-gain theorem seem to have appeared in [33, 40] (see also [13, Chapters III and V]). It is well known from [18] that

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the conditions of the passivity and small-gain theorems may be "shared" or "mixed" across the plant and controller and still guarantee stability. We refer the reader to the recent paper [11], and the references therein, for more background on mixed stability results and their generalizations.

The stability criteria (passivity, small gain, and mixed) referred to above are formulated within the framework of L^p-input–output stability (where, usually, $p = 2$ or $p = \infty$). Although this setting is sufficient for many purposes, there are situations in the control of PDEs and repetitive control in which L^2- or L^∞-stability is impossible to achieve, and therefore, more refined stability concepts are required. This is addressed by the state-space concept of polynomial stability of operator semigroups (see, for example, [1, 5, 32]) and the related P-stability notion in the frequency domain [25, 29]. A new stability concept called *Sobolev input-output stability* has been recently introduced in [20], which contains L^2- and P-stability as special cases. This concept is applicable to a rather general class of causal translation-invariant linear input-output operators, the domain and codomain of which are spaces of vector-valued distributions: For real numbers α and β, Sobolev (α, β)-stability of an input-output operator H simply refers to the property that H maps the Sobolev space $H^\alpha(\mathbb{R}, U)$ continuously into the Sobolev space $H^\beta(\mathbb{R}, Y)$, where U and Y are Hilbert spaces. The familiar notion of L^2-stability corresponds to the case wherein $\alpha = \beta = 0$. In the frequency domain, Sobolev (α, β)-stability can be conveniently characterized by the condition that the function $s \mapsto (1 + s)^{\beta-\alpha}\mathbf{H}(s)$ is holomorphic and bounded on the open right-half plane, where $\mathbf{H}$ denotes the transfer function of H; see section 3. Obviously, if $\alpha > \beta$ ($\alpha < \beta$), then application of the input-output operator reduces (increases) the regularity of the input.

In the current paper, we initiate the study of so-called Sobolev stabilizing feedbacks, that is, controllers, which ensure that the closed-loop system is Sobolev stable. The main result, a general mixed passivity/small-gain theorem, is a frequency-domain criterion for the Sobolev stability of the feedback connection of two (possibly infinite-dimensional) time-invariant linear systems; see Theorem 4.2. Loosely speaking, the theorem states that a suitable mixture of positive–realness- and small–gain-type conditions holding on certain subsets of the open right-half plane ensures that the feedback system is Sobolev stable. The passivity and small-gain theorems for linear systems in an L^2-stability setting are contained in Theorem 4.2 as special cases, as is [42, Theorem 4.2]; see section 4 for details.

We apply our mixed passivity/small-gain theorem in the context of a general version of the output-regulation and disturbance-rejection problem (also referred to as the servo problem). Inspired by the frequency-domain theory of the internal model principle [24, 25, 37], a sufficient condition for a Sobolev stabilizing controller to solve the servo problem is given in Theorem 5.1. This result is then applied to the so-called repetitive control problem (see, for example, [38]), for which it is known that L^2-stability of the closed loop is not possible for plant transfer functions, which tend to 0 at high frequencies. However, as we demonstrate, these feedback connections are Sobolev stable, and Corollary 5.4 provides a sufficient condition for a Sobolev stabilizing controller to be a solution to the servo problem in repetitive control.

The paper is organized as follows. Section 2 contains preliminaries on notation and certain spaces of functions and distributions. In section 3, we recall a number of results on Sobolev stability from [20] and introduce the concept of Sobolev stabilizing feedback operators. Section 4 contains the main result, a general mixed passivity/small-gain theorem for Sobolev stability. As has been mentioned already, Sobolev stabilizing feedbacks are used in the context of output regulation and disturbance rejection in section 5. Six examples are presented in section 6, and summarizing comments appear in section 7. Some technical material relating to Example 6.7 is relegated to the Appendix.

2. Preliminaries. We gather some preliminary material required for the statement and proofs of the results in sections 3–6.

2.1. Notation. Let $\mathbb{Z}$ and $\mathbb{N}$ denote the integers and the positive integers, respectively, and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. As usual, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the fields of rational, real, and complex numbers, respectively. For $\mu \in \mathbb{R}$, we let $\mathbb{C}_\mu$ denote the right-half complex plane of complex numbers with real part greater than μ.

For complex Banach spaces X and Y, we let $\mathcal{B}(X,Y)$ denote the vector space of bounded linear operators $X \to Y$, which is a Banach space when equipped with the uniform topology, and set $\mathcal{B}(X) := \mathcal{B}(X,X)$. If X is a Hilbert space and $T \in \mathcal{B}(X)$, then we define the real part $\operatorname{Re} T$ of the operator T by

$$\operatorname{Re} T := \frac{1}{2}(T + T^*),$$

where T^* denotes the adjoint of T. Furthermore, for two operators $S, T \in \mathcal{B}(X)$, X being a complex Hilbert space, we write $T \geq S$ if $T - S$ is positive semidefinite.

2.2. Spaces of holomorphic functions. Here and throughout, U and Y denote complex Hilbert spaces. For nonempty, open $\Omega \subset \mathbb{C}$, we define $\mathcal{H}(\Omega, \mathcal{B}(U,Y))$ as the vector space of holomorphic functions $\Omega \to \mathcal{B}(U,Y)$ and $\mathcal{H}^*(\Omega, \mathcal{B}(U,Y))$ as the space of all $\mathcal{B}(U,Y)$-valued functions that are holomorphic on Ω with the exception of isolated points, namely, poles and essential singularities, understanding that removable singularities have been removed by holomorphic extension. Consequently, if $\mathbf{H} \in \mathcal{H}^*(\Omega, \mathcal{B}(U,Y))$, then $\mathbf{H} \in \mathcal{H}(\Omega \backslash \Sigma_{\mathbf{H}}, \mathcal{B}(U,Y))$, where $\Sigma_{\mathbf{H}}$ denotes the set of singularities (poles and essential singularities) of $\mathbf{H}$ in Ω. For $\mu \in \mathbb{R}$, we write

$$\mathcal{H}_\mu(\mathcal{B}(U,Y)) := \mathcal{H}(\mathbb{C}_\mu, \mathcal{B}(U,Y)) \quad \text{and} \quad \mathcal{H}_\mu^*(\mathcal{B}(U,Y)) := \mathcal{H}^*(\mathbb{C}_\mu, \mathcal{B}(U,Y)).$$

A function $\mathbf{H} \in \mathcal{H}_0^*(\mathcal{B}(U))$ is said to be positive real if $\operatorname{Re} \mathbf{H}(s) \geq 0$ for all $s \in \mathbb{C}_0 \backslash \Sigma_{\mathbf{H}}$ and strongly positive real if there exists $\delta > 0$ such that $\operatorname{Re} \mathbf{H}(s) \geq \delta I$ for all $s \in \mathbb{C}_0 \backslash \Sigma_{\mathbf{H}}$. It is well known (see, for example, [19]) that, if a function $\mathbf{H} \in \mathcal{H}_0^*(\mathcal{B}(U))$ is positive real, then it cannot have any singularities in $\mathbb{C}_0$.

Let $\mathcal{H}^\infty(\Omega, \mathcal{B}(U,Y))$ denote the space of all bounded holomorphic functions $\Omega \to \mathcal{L}(U,Y)$, and set $\mathcal{H}_\mu^\infty(\mathcal{B}(U,Y)) := \mathcal{H}^\infty(\mathbb{C}_\mu, \mathcal{B}(U,Y))$. Endowed with the norm

$$\|\mathbf{H}\|_{\mathcal{H}_\mu^\infty} := \sup_{s \in \mathbb{C}_\mu} \|\mathbf{H}(s)\|,$$

$\mathcal{H}_\mu^\infty(\mathcal{B}(U,Y))$ is a Banach space. For brevity, we abbreviate this to $\mathcal{H}_\mu^\infty$ when $U = Y = \mathbb{C}$. For further background on vector-valued holomorphic and meromorphic functions, we refer the reader to, for example, [14, Chapter 9] or [31, Chapter 4].

2.3. Spaces of function and distributions and integral transforms. Let X denote a complex Banach space. The space of m-times continuously differentiable functions from J to X, $J \subset \mathbb{R}$ being an interval, is denoted by $C^m(J,X)$, while $C_c^\infty(\mathbb{R}, X)$ stands for the space of infinitely differentiable functions $\mathbb{R} \to X$ with compact support. We let $\mathcal{S}$ and $\mathcal{D}$ denote the Schwartz space of rapidly decreasing C^∞-functions $\mathbb{R} \to \mathbb{C}$ and the space of compactly supported C^∞-functions $\mathbb{R} \to \mathbb{C}$ endowed with their usual topologies, respectively. The spaces of all continuous linear maps $\mathcal{D} \to X$ and $\mathcal{S} \to X$ are denoted by $\mathcal{D}'(X)$ and $\mathcal{S}'(X)$, respectively. We have that $\mathcal{S}'(X) \subset \mathcal{D}'(X)$, and the elements in $\mathcal{D}'(X)$ are called X-valued distributions. A distribution in $\mathcal{S}'(X)$ is said to be tempered (or slowly growing). The subspace of distributions in $\mathcal{D}'(X)$ with support bounded on the left is denoted by $\mathcal{D}'_\ell(X)$, and similarly, $\mathcal{S}'_\ell(X)$ stands for the space of tempered distributions having support bounded on the left. For more details on vector-valued distributions, we refer the reader to, for example, [2, Chapter III: sections 4.1 and 4.2], [3, Chapter VII], [12, Chapter XVI: section 2], [16, Chapter 8], and [41, Chapters 3, 5, and 6].

The Fourier transform of a function $f \in L^1(\mathbb{R}, X)$ is defined by

$$(\mathcal{F}f)(y) := \int_{-\infty}^{\infty} \mathrm{e}^{-iyt} f(t)\,\mathrm{d}t \quad \forall\, y \in \mathbb{R}\,.$$

Because $\mathcal{F}$ is an automorphism on $\mathcal{S}$, the definition of the Fourier transform extends to $\mathcal{S}'(X)$ via

$$(\mathcal{F}u)(\phi) := u(\mathcal{F}\phi) \quad \forall\, \phi \in \mathcal{S}, \quad \text{where } u \in \mathcal{S}'(X).$$

It is well known that the Fourier transform $\mathcal{F}$ is an automorphism on $\mathcal{S}'(X)$ with $\mathcal{F}$ and $\mathcal{F}^{-1}$ being sequentially continuous. If $X = U$ is a complex Hilbert space, then the restriction of $\mathcal{F} : \mathcal{S}'(U) \to \mathcal{S}'(U)$ to $L^2(\mathbb{R}, U)$ is an automorphism on $L^2(\mathbb{R}, U)$; in fact, $(1/\sqrt{2\pi})\mathcal{F}$ is a unitary operator on $L^2(\mathbb{R}, U)$, and so, $\|\mathcal{F}u\|_{L^2(\mathbb{R})} = \sqrt{2\pi}\|u\|_{L^2(\mathbb{R})}$ for every $u \in L^2(\mathbb{R}, U)$.

Let U be a complex Hilbert space and $J \subset \mathbb{R}$ an interval. We set $W^{0,2}(J, U) := L^2(J, U)$, and, for $m \in \mathbb{N}$, we let $W^{m,2}(J, U)$ be the space of all $u \in C^{m-1}(J, U)$ such that $u^{(m-1)}$ is (locally) absolutely continuous and $u^{(k)} \in L^2(J, U)$ for $k = 0, 1, \ldots, m$, endowed with the norm

$$\|u\|_{W^{m,2}} := \left(\sum_{k=0}^{m} \int_J \|u^{(k)}(t)\|^2 \mathrm{d}t \right)^{1/2}. \tag{2.1}$$

For $\theta \in \mathbb{R}$ and U a complex Hilbert space, we define the Sobolev space (sometimes also called the Bessel potential space)

$$H^\theta(\mathbb{R}, U) := \left\{ u \in \mathcal{S}'(U) : \left(y \mapsto (1 + y^2)^{\theta/2} (\mathcal{F}u)(y) \right) \in L^2(\mathbb{R}, U) \right\}$$

with inner product and associated norm given by

$$\langle u, v \rangle := \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 + y^2)^\theta \langle (\mathcal{F}u)(y), (\mathcal{F}v)(y) \rangle \,\mathrm{d}y \quad \forall\, u, v \in H^\theta(\mathbb{R}, U)$$

and

$$\|u\|_{H^\theta} := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} (1 + y^2)^\theta \|(\mathcal{F}u)(y)\|^2 \,\mathrm{d}y \right)^{1/2} \quad \forall\, u \in H^\theta(\mathbb{R}, U), \tag{2.2}$$

respectively. The space $H^\theta(\mathbb{R}, U)$ is complete and hence a Hilbert space. We note that $H^0(\mathbb{R}, U) = L^2(\mathbb{R}, U)$ and $\|u\|_{H^0} = \|u\|_{L^2}$ for all $u \in L^2(\mathbb{R}, U)$. If $\theta \geq 0$, then $H^\theta(\mathbb{R}, U) \subset L^2(\mathbb{R}, U)$, while $H^\theta(\mathbb{R}, U)$ contains nonregular distributions when $\theta < 0$. We remark that $H^m(\mathbb{R}, U) = W^{m,2}(\mathbb{R}, U)$ for all $m \in \mathbb{N}_0$ and the norms (2.1) and (2.2) are equivalent. Therefore, it makes sense (and simplifies notation) to set

$$H^m(J, U) := W^{m,2}(J, U) \quad \text{for all } m \in \mathbb{N}_0 \text{ and all intervals } J \subset \mathbb{R}\,. \tag{2.3}$$

Moreover, for arbitrary $\theta \in \mathbb{R}$, let $H^\theta_\ell(\mathbb{R},U)$ denote the subspace of all distributions $H^\theta(\mathbb{R},U)$ with support bounded on the left, while $H^\theta_+(\mathbb{R},U)$ consists of all $u \in H^\theta(\mathbb{R},U)$ such that $\operatorname{supp} u \subset [0,\infty)$. For $m \in \mathbb{N}_0$ and $J \subset \mathbb{R}$ being an interval, the localized version of $H^m(J,U)$ is denoted by $H^m_{\rm loc}(J,U)$, and $H^m_{{\rm loc},\ell}(\mathbb{R},U)$ stands for the subspace of all $u \in H^m_{\rm loc}(\mathbb{R},U)$ such that $\operatorname{supp} u$ is bounded on the left. Furthermore, for arbitrary $\theta \in \mathbb{R}$ and $J \subset \mathbb{R}$ being an open interval, we set

$$H^\theta(J,U) := \big\{u \in \mathcal{D}'(J,U) : \text{there exists } v \in H^\theta(\mathbb{R},U) \text{ such that } u = v|_J\big\}, \tag{2.4}$$

where $\mathcal{D}'(J,U)$ is the space of continuous linear U-valued maps defined on $\mathcal{D}(J) := \{\phi \in \mathcal{D} : \operatorname{supp}\phi \subset J\}$. When $\theta = m \in \mathbb{N}_0$, the definitions (2.4) and (2.3) coincide. In the case of scalar-value Sobolev spaces (that is, if $U = \mathbb{C}$), we write $H^\theta(\mathbb{R})$ for $H^\theta(\mathbb{R},\mathbb{C})$, $H^\theta_\ell(\mathbb{R})$ for $H^\theta_\ell(\mathbb{R},\mathbb{C})$, $H^\theta_+(\mathbb{R})$ for $H^\theta_+(\mathbb{R},\mathbb{C})$, etc.

The Laplace transform $\mathcal{L}u$ of a distribution $u \in \mathcal{D}'(X)$ such that $\operatorname{supp} u \subset [\tau,\infty)$ and $e^{-\mu\,\cdot}u \in \mathcal{S}'(X)$ for some $\tau,\mu \in \mathbb{R}$ is defined by

$$(\mathcal{L}u)(s) := \big(\mathrm{e}^{-\mu\,\cdot}u\big)\big(\eta\mathrm{e}^{-(s-\mu)\,\cdot}\big) \quad \forall\, s \in \mathbb{C}_\mu,$$

where $\eta \in C^\infty(\mathbb{R},\mathbb{C})$ is an arbitrary function such that there exist $t_1 < t_0 < \tau$ such that $\eta(t) = 0$ for all $t < t_1$ and $\eta(t) = 1$ for all $t > t_0$. It is straightforward to show that the definition does not depend on the choice of η and extends the classical Laplace transform. For $u \in \mathcal{D}'_\ell(X)$, the *abscissa of convergence* $\sigma(u)$ is defined as the infimum of all $\mu \in \mathbb{R}$ such that $e^{-\mu\,\cdot}u \in \mathcal{S}'(X)$. If no such μ exists, then we set $\sigma(u) = \infty$. If $\sigma(u) < \infty$, then the Laplace transform of u exists and is holomorphic on $\mathbb{C}_{\sigma(u)}$, and u is said to be *Laplace transformable*.

3. Sobolev input-output stability of feedback systems.

In this section, we recall the Sobolev input-output stability concept, review some relevant results from [20, section 5], and introduce and discuss Sobolev stabilizing compensators.

3.1. Sobolev input-output stability.

The class of linear, translation-invariant, and causal input-output operators to which the Sobolev input-output stability framework applies is described in terms of convolution operators with operator-valued distributional kernels and is reasonably general. In particular, it includes the input-output operators of well-posed linear systems (in the sense of [36]). We refer to [20, Appendix 1] for relevant background on convolutions of vector-valued distributions. Here, we only mention that if $g \in \mathcal{D}'_\ell(\mathcal{B}(U,Y))$, then the convolution product $g \star u$ is a well-defined distribution in $\mathcal{D}'_\ell(Y)$ for all $u \in \mathcal{D}'_\ell(U)$.

It is known (see, for example, [20, Proposition 5.2], which, in turn, is based on results of [41, Chapter 5]) that, if $G : \operatorname{dom} G \subset \mathcal{D}'(U) \to \mathcal{D}'(Y)$ is a continuous, causal, and translation-invariant linear operator such that $C^\infty_{\rm c}(\mathbb{R},U) \subset \operatorname{dom} G$, then there exists a unique $g \in \mathcal{D}'(\mathcal{B}(U,Y))$ such that $\operatorname{supp} g \subset [0,\infty)$ and $Gu = g \star u$ for all $u \in \mathcal{D}'_\ell(U) \cap \operatorname{dom} G$. Conversely, if there exists $g \in \mathcal{D}'(\mathcal{B}(U,Y))$ such that $\operatorname{supp} g \subset [0,\infty)$ and $Gu = g \star u$ for all $u \in C^\infty_{\rm c}(\mathbb{R},U)$, then G is continuous, causal, and translation invariant. The distribution g is called the kernel or impulse response of the operator G. If $\sigma(g) < \infty$ (finite abscissa of convergence), then $\mathbf{G}(s) := (\mathcal{L}g)(s)$ exists for all $s \in \mathbb{C}_{\sigma(g)}$, and the function $\mathbf{G}$, a $\mathcal{B}(U,Y)$-valued holomorphic function defined on $\mathbb{C}_{\sigma(g)}$, is referred to as the transfer function of G. If $u \in \mathcal{D}'_\ell(U)$ is such that $\sigma(u) < \infty$, then $g \star u$ is Laplace transformable and

$$(\mathcal{L}Gu)(s) = \mathbf{G}(s)(\mathcal{L}u)(s) \quad \forall\, s \in \mathbb{C}_\mu,$$

where $\mu := \max\big(\sigma(g),\sigma(u)\big)$.

Let $\alpha, \beta \in \mathbb{R}$. A linear operator $G : \operatorname{dom} G \subset \mathcal{D}'(U) \to \mathcal{D}'(Y)$ is said to be *Sobolev (α,β)-stable* if $C_c^\infty(\mathbb{R}, U) \subset \operatorname{dom} G$, $G(C_c^\infty(\mathbb{R},U)) \subset H^\beta(\mathbb{R}, Y)$, and there exists $\gamma > 0$ such that

$$\|Gu\|_{H^\beta} \leq \gamma \|u\|_{H^\alpha} \quad \forall\, u \in C_c^\infty(\mathbb{R}, U).$$

Throughout, we shall use the function

$$\mathbf{r}_\alpha(s) := (1+s)^{-\alpha} \quad \forall\, s \in \mathbb{C}_{-1}, \quad \text{where } \alpha \in \mathbb{R}.$$

On the right-hand side, we identify the complex power function with its principal branch on the domain $\mathbb{C}\backslash(-\infty, 0]$, and thus, $\mathbf{r}_\alpha(s) \in (0,\infty)$ for all $s \in (-1,\infty)$.

The next theorem provides several characterizations of Sobolev (α,β)-stability in terms of transfer functions and is a combination of results in [20, sections 3 and 5].

Theorem 3.1 (see [20, Theorems 3.1 and 5.4]). *Let $G : \operatorname{dom} G \subset \mathcal{D}'(U) \to \mathcal{D}'(Y)$ be a causal translation-invariant continuous linear operator such that $C_c^\infty(\mathbb{R}, U) \subset \operatorname{dom} G$, and let $g \in \mathcal{D}'(\mathcal{B}(U,Y))$ be the kernel of G. For arbitrary $\alpha, \beta \in \mathbb{R}$, the following statements are equivalent.*

(1) *G is Sobolev (α,β)-stable.*

(2) *There exists a unique causal and translation-invariant operator $G^{\mathrm{e}} \in \mathcal{B}(H^\alpha(\mathbb{R}, U), H^\beta(\mathbb{R}, Y))$ such that $G^{\mathrm{e}} u = Gu$ for all $u \in H^\alpha(\mathbb{R}, U) \cap \operatorname{dom} G$.*

(3) *g is Laplace transformable, $\sigma(g) \leq 0$, and $\mathbf{r}_{\alpha-\beta}\mathbf{G} \in \mathcal{H}_0^\infty(\mathcal{B}(U,Y))$, where $\mathbf{G}$ is the transfer function of G.*

(4) *g is Laplace transformable, and there exist $\mu > \max\{0, \sigma(g)\}$ and a holomorphic $\mathbf{G}^{\mathrm{e}} : \mathbb{C}_0 \to \mathcal{B}(U,Y)$ that coincides with the transfer function $\mathbf{G}$ of G on $\mathbb{C}_{\max\{0,\sigma(g)\}}$ and such that*

$$\sup_{0 < \operatorname{Re} s < \mu} \|\mathbf{r}_{\alpha-\beta}(s)\mathbf{G}^{\mathrm{e}}(s)\| < \infty.$$

(5) *g is Laplace transformable, and there exists a holomorphic $\mathbf{G}^{\mathrm{e}} : \mathbb{C}_0 \to \mathcal{B}(U,Y)$ that coincides with the transfer function $\mathbf{G}$ of G on $\mathbb{C}_{\max\{0,\sigma(g)\}}$ and such that $\mathbf{r}_{\alpha-\beta}\mathbf{G}^{\mathrm{e}} \in \mathcal{H}_0^\infty(\mathcal{B}(U,Y))$.*

If one of the above statements holds, then

$$\|G^{\mathrm{e}}\|_{\mathcal{B}(H^\alpha, H^\beta)} = \sup_{u \in C_c^\infty,\, u \neq 0} \frac{\|Gu\|_{H^\beta}}{\|u\|_{H^\alpha}} = \|\mathbf{r}_{\alpha-\beta}\mathbf{G}\|_{\mathcal{H}_0^\infty} = \|\mathbf{r}_{\alpha-\beta}\mathbf{G}^{\mathrm{e}}\|_{\mathcal{H}_0^\infty}.$$

As an immediate consequence of the above theorem, we note that Sobolev (α,β)-stability implies Sobolev $(\alpha+\theta, \beta+\theta)$-stability for all $\theta \in \mathbb{R}$. We emphasize that the classical input-output notion of L^2-stability is contained in the above concept as the special case of Sobolev $(0,0)$-stability.

The following proposition shows that, under suitable assumptions, Sobolev (α,β)-stability follows if the transfer function satisfies a natural boundedness condition on the imaginary axis.

Proposition 3.2 (see [20, Corollary 5.6]). *Let $G : \operatorname{dom} G \subset \mathcal{D}'(U) \to \mathcal{D}'(Y)$ be a causal translation-invariant continuous linear operator such that $C_c^\infty(\mathbb{R}, U) \subset \operatorname{dom} G$, let $g \in \mathcal{D}'(\mathcal{B}(U,Y))$ be the kernel of G, and let $\alpha, \beta \in \mathbb{R}$. Assume that g is Laplace transformable and that there exists a holomorphic $\mathbf{G}^{\mathrm{e}} : \mathbb{C}_0 \to \mathcal{B}(U,Y)$ that coincides with the transfer function $\mathbf{G}$ of G on $\mathbb{C}_{\max\{0,\sigma(g)\}}$. Furthermore, assume that U and Y are separable and $\mathbf{G}^{\mathrm{e}}$ is polynomially bounded on the strip $0 < \operatorname{Re} s < \mu$ for some*

$\mu > \sigma(g)$. *Under these conditions, the limit* $\mathbf{G}^{\mathrm{e}}_0(y) = \lim_{x\downarrow 0} \mathbf{G}^{\mathrm{e}}(x+iy)$ *exists in the strong operator topology for almost every* $y \in \mathbb{R}$, *and, if*

$$\operatorname*{ess\,sup}_{y\in\mathbb{R}} \|\mathbf{r}_{\alpha-\beta}(iy)\mathbf{G}^{\mathrm{e}}_0(y)\| < \infty\,,$$

then G *is Sobolev* (α,β)*-stable, in which case*

$$\|\mathbf{r}_{\alpha-\beta}\mathbf{G}^{\mathrm{e}}\|_{\mathcal{H}^\infty_0} = \operatorname*{ess\,sup}_{y\in\mathbb{R}} \|\mathbf{r}_{\alpha-\beta}(iy)\mathbf{G}^{\mathrm{e}}_0(y)\|.$$

3.2. Sobolev stabilizing feedback controllers. Following [19], we say that $\mathbf{K} \in \mathcal{H}^*_\mu(\mathcal{B}(Y,U))$ is an *admissible feedback* for $\mathbf{P} \in \mathcal{H}^*_\mu(\mathcal{B}(U,Y))$, where $\mu \in \mathbb{R}$, if

$$\mathbf{S} := \begin{pmatrix} I & \mathbf{K} \\ -\mathbf{P} & I \end{pmatrix} \tag{3.1}$$

has an inverse that belongs to $\mathcal{H}^*_\nu(\mathcal{B}(U\times Y))$ for some $\nu \geq \mu$, or, equivalently, the set

$$\Xi_{\mathbf{P},\mathbf{K}} := \big\{ s \in \mathbb{C}_\mu \backslash \big(\Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}}\big) : \mathbf{S}(s) \text{ is not invertible}\big\}$$

does not have any accumulation points in $\mathbb{C}_\nu$; in particular, the inverse

$$\mathbf{S}^{-1}(s) := \big(\mathbf{S}(s)\big)^{-1} = \begin{pmatrix} I & \mathbf{K}(s) \\ -\mathbf{P}(s) & I \end{pmatrix}^{-1} \tag{3.2}$$

exists for all $s \in \mathbb{C}_\nu$ such that $s \notin \Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}} \cup \Xi_{\mathbf{P},\mathbf{K}}$. This definition coincides with [19, Definition 6.15] up to a sign change in $\mathbf{K}$. A necessary and sufficient condition for $\mathbf{K}$ to be an admissible feedback for $\mathbf{P}$ is that $I + \mathbf{KP}$ has an inverse in $\mathcal{H}^*_\nu(\mathcal{B}(U))$ for some $\nu \geq \mu$, or, equivalently, $I + \mathbf{PK}$ has an inverse in $\mathcal{H}^*_\nu(\mathcal{B}(Y))$, in which case

$$\mathbf{S}^{-1} = \begin{pmatrix} I & \mathbf{K} \\ -\mathbf{P} & I \end{pmatrix}^{-1} = \begin{pmatrix} (I+\mathbf{KP})^{-1} & -\mathbf{K}(I+\mathbf{PK})^{-1} \\ \mathbf{P}(I+\mathbf{KP})^{-1} & (I+\mathbf{PK})^{-1} \end{pmatrix} \quad \text{on } \mathbb{C}_\nu. \tag{3.3}$$

Recall that the feedback connection of $\mathbf{P}$ and admissible $\mathbf{K}$ is called *well posed* if there exists $\omega \geq \mu$ such that $\mathbf{S}^{-1} \in \mathcal{H}^\infty_\omega(\mathcal{B}(U\times Y))$.

The following lemma shows that, under certain conditions, admissibility of the feedback $\mathbf{K}$ is guaranteed provided that $\mathbf{S}(s)$ is invertible at one point $s = s_0$ in $\mathbb{C}_\mu$.

Lemma 3.3. *Let* $\mathbf{P} \in \mathcal{H}^*_\mu(\mathcal{B}(U,Y))$ *and* $\mathbf{K} \in \mathcal{H}^*_\mu(\mathcal{B}(Y,U))$, *where* $\mu \in \mathbb{R}$, *and let* $s_0 \in \mathbb{C}_\mu\backslash\big(\Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}}\big)$. *Assume that at least one of the operators* $\mathbf{P}(s)$ *and* $\mathbf{K}(s)$ *is compact for every* $s \in \mathbb{C}_\mu\backslash\big(\Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}}\big)$ *and that* $\mathbf{P}$ *and* $\mathbf{K}$ *are holomorphic on* $\mathbb{C}_\nu$ *for some* $\nu \geq \mu$. *If the inverse* $\mathbf{S}^{-1}(s)$ *in* (3.2) *exists for* $s = s_0$, *then the set* $\Xi_{\mathbf{P},\mathbf{K}}$ *does not have any accumulation points in* $\mathbb{C}_\mu\backslash\big(\Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}}\big)$. *Furthermore,* $\mathbf{S}^{-1}$ *is meromorphic on* $\mathbb{C}_\nu$; *in particular,* $\mathbf{K}$ *is an admissible feedback for* $\mathbf{P}$.

We remark that, without the compactness assumption, Lemma 3.3 is not true; see, for example, [26, Example 4.2]. Trivially, the compactness hypothesis is satisfied whenever U or Y is finite dimensional.

Proof of Lemma 3.3. For $s \in \mathbb{C}_\mu\backslash\big(\Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}}\big)$, the operator $\mathbf{S}(s)$ is invertible if and only if $I + \mathbf{K}(s)\mathbf{P}(s)$ is invertible (and $\mathbf{S}^{-1}(s)$ is given by (3.3)). Therefore,

$$\Xi_{\mathbf{P},\mathbf{K}} = \{ s \in \mathbb{C}_\mu\backslash\big(\Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}}\big) : I + \mathbf{K}(s)\mathbf{P}(s) \text{ is not invertible}\}.$$

Because $\mathbf{K}(s)\mathbf{P}(s)$ is compact for all $s \in \mathbb{C}_\mu\backslash\big(\Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}}\big)$, it follows from [19, Lemma 5.8] that $\Xi_{\mathbf{P},\mathbf{K}}$ does not have any accumulation points in $\mathbb{C}_\mu\backslash\big(\Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}}\big)$ and $(I + \mathbf{KP})^{-1}$

is meromorphic on $\mathbb{C}_\mu \backslash \big(\Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}}\big)$. Because $\mathbf{P}$ and $\mathbf{K}$ are holomorphic on $\mathbb{C}_\nu$, we have that $\mathbb{C}_\nu \subset \mathbb{C}_\mu \backslash \big(\Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}}\big)$, and we conclude that $(I + \mathbf{KP})^{-1}$, and hence $\mathbf{S}^{-1}$, are meromorphic on $\mathbb{C}_\nu$. ☐

Let $\mathbf{K} \in \mathcal{H}^*_\mu(\mathcal{B}(Y,U))$ be an admissible feedback for $\mathbf{P} \in \mathcal{H}^*_\mu(\mathcal{B}(U,Y))$. Then, there exists a $\nu \geq \mu$ such that $\Xi_{\mathbf{P},\mathbf{K}}$ does not have any accumulation points in $\mathbb{C}_\nu$ and $\mathbf{S}^{-1} \in \mathcal{H}^*_\nu(\mathcal{B}(U \times Y))$. Setting

$$\xi = \xi_{\mathbf{P},\mathbf{K}} := \inf\{\omega \leq \nu : \text{there exists } \mathbf{E} \in \mathcal{H}^*_\omega(\mathcal{B}(U \times Y)) \text{ extending } \mathbf{S}^{-1}\},$$

we define $\mathbf{F}_{\mathbf{P},\mathbf{K}}$ to be the uniquely determined function in $\mathcal{H}^*_\xi(\mathcal{B}(U \times Y))$ such that

$$\mathbf{F}_{\mathbf{P},\mathbf{K}}(s) = \mathbf{S}^{-1}(s) = \begin{pmatrix} I & \mathbf{K}(s) \\ -\mathbf{P}(s) & I \end{pmatrix}^{-1} \quad \forall\, s \in \mathbb{C}_\nu \backslash \big(\Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}} \cup \Xi_{\mathbf{P},\mathbf{K}}\big). \tag{3.4}$$

If $\xi = -\infty$, then $\mathcal{H}^*_\xi(\mathcal{B}(U \times Y))$ should be interpreted as $\mathcal{H}^*(\mathbb{C}, \mathcal{B}(U \times Y))$.

Remark 3.4. In (3.4), ν may be replaced by any ω such that $\max\{\mu, \xi\} \leq \omega \leq \nu$. To see this, we observe that, by (3.4),

$$\mathbf{S}(s)\mathbf{F}_{\mathbf{P},\mathbf{K}}(s) = \mathbf{F}_{\mathbf{P},\mathbf{K}}(s)\mathbf{S}(s) = I \quad \forall\, s \in \mathbb{C}_\nu \backslash \big(\Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}} \cup \Xi_{\mathbf{P},\mathbf{K}}\big). \tag{3.5}$$

Since the set $\mathbb{C}_\omega \backslash \big(\Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}} \cup \Xi_{\mathbf{P},\mathbf{K}}\big)$ is connected, we can invoke the identity theorem for holomorphic functions to conclude that (3.5) extends to all $s \in \mathbb{C}_\omega \backslash \big(\Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}} \cup \Xi_{\mathbf{P},\mathbf{K}}\big)$.

DEFINITION 3.5. *Let $\mathbf{P} \in \mathcal{H}^*_\mu(\mathcal{B}(U,Y))$ and $\mathbf{K} \in \mathcal{H}^*_\mu(\mathcal{B}(Y,U))$ for some $\mu \in \mathbb{R}$, and let $\alpha, \beta \in \mathbb{R}$. We say that $\mathbf{K}$ is a* Sobolev (α,β)-stabilizing feedback *for $\mathbf{P}$ if $\mathbf{K}$ is an admissible feedback for $\mathbf{P}$ and $\mathbf{r}_{\alpha-\beta}\mathbf{F}_{\mathbf{P},\mathbf{K}} \in \mathcal{H}^\infty_0(\mathcal{B}(U \times Y))$. In this case, we say that the feedback connection (of $\mathbf{P}$ and $\mathbf{K}$) is Sobolev (α,β)-stable, or just Sobolev stable.*

To explain how the above frequency-domain concept is related to the time-domain notion of Sobolev (α,β)-stability from subsection 3.1, assume that $\mathbf{K}$ is a Sobolev (α,β)-stabilizing feedback for $\mathbf{P}$. Then, trivially, $\mathbf{F}_{\mathbf{P},\mathbf{K}}$ is polynomially bounded on $\mathbb{C}_0$, and [41, Theorem 6.5-1 and Corollary 6.5-1a] guarantee that there exists a causal translation-invariant operator $F : \mathcal{D}'_\ell(U \times Y) \to \mathcal{D}'_\ell(U \times Y)$, the transfer function of which is $\mathbf{F}_{\mathbf{P},\mathbf{K}}$. It follows from Theorem 3.1 that F is Sobolev (α,β)-stable in the sense of subsection 3.1.

We present some immediate consequences of the above definition in the following lemma.

LEMMA 3.6. *Let $\mathbf{P} \in \mathcal{H}^*_\mu(\mathcal{B}(U,Y))$ and $\mathbf{K} \in \mathcal{H}^*_\mu(\mathcal{B}(Y,U))$ for some $\mu \in \mathbb{R}$, and let $\alpha, \beta \in \mathbb{R}$.*

(1) *$\mathbf{K}$ is a Sobolev (α,β)-stabilizing feedback for $\mathbf{P}$ if and only if $\mathbf{P}$ is a Sobolev (α,β)-stabilizing feedback for $\mathbf{K}$.*

(2) *Under the additional assumptions that $Y = U$ and $\mathbf{P}$ and $\mathbf{K}$ are invertible with inverses $\mathbf{P}^{-1}, \mathbf{K}^{-1} \in \mathcal{H}^*_\nu(\mathcal{B}(U))$ for some $\nu \geq \mu$, the following statements hold.*
   - (i) *If $\alpha \geq \beta$, then $\mathbf{K}$ is a Sobolev (α,β)-stabilizing feedback for $\mathbf{P}$ if and only if $\mathbf{K}^{-1}$ is a Sobolev (α,β)-stabilizing feedback for $\mathbf{P}^{-1}$.*
   - (ii) *If $\mathbf{K}$ is a Sobolev (α,β)-stabilizing feedback for $\mathbf{P}$ and $\mathbf{K}^{-1}$ is a Sobolev (α,β)-stabilizing feedback for $\mathbf{P}^{-1}$, then $\alpha \geq \beta$.*

(3) *Assume that $\mathbf{K}$ is holomorphic on $\mathbb{C}_\nu$ for some $\nu \geq \mu$, $\mathbf{K}(s)$ is compact for all $s \in \mathbb{C}_\nu$, and $\mathbf{K}$ is a Sobolev (α,β)-stabilizing feedback for $\mathbf{P}$. Then, $\mathbf{P}$ is meromorphic on $\mathbb{C}_\nu$.*

An important scenario in which the invertibility assumption in statement (2) holds is the following: If $\mathbf{P}$ and $\mathbf{K}$ are positive real (and a fortiori holomorphic on $\mathbb{C}_0$ by [19, Proposition 3.3]) and there exist s_1 and s_2 in $\mathbb{C}_0$ such that $\mathbf{P}(s_1)$ and $\mathbf{K}(s_2)$ are invertible, then $\mathbf{P}(s)$ and $\mathbf{K}(s)$ are invertible for all $s \in \mathbb{C}_0$, and $\mathbf{P}^{-1}$ and $\mathbf{K}^{-1}$ are positive real by [26, Corollary 4.3] and hence holomorphic on $\mathbb{C}_0$.

Proof of Lemma 3.6. Statement (1) follows immediately from the equality

$$\begin{pmatrix} I & \mathbf{K} \\ -\mathbf{P} & I \end{pmatrix} = -\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} I & \mathbf{P} \\ -\mathbf{K} & I \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

We proceed to prove part (i) of statement (2). Assume that $\mathbf{K}$ is a Sobolev (α,β)-stabilizing feedback for $\mathbf{P}$. Because $\mathbf{K}$ is an admissible feedback for $\mathbf{P}$, the transfer function $I + \mathbf{PK}$ has an inverse in $\mathcal{H}^*_\omega(\mathcal{B}(U))$ for some $\omega \geq \mu$, and the trivial identity $I + \mathbf{K}^{-1}\mathbf{P}^{-1} = \mathbf{K}^{-1}\mathbf{P}^{-1}(I + \mathbf{PK})$ implies that $\mathbf{K}^{-1}$ is an admissible feedback for $\mathbf{P}^{-1}$. Using (3.3) and

$$(I + \mathbf{K}^{-1}\mathbf{P}^{-1})^{-1} = \mathbf{PK}(I + \mathbf{PK})^{-1} = I - (I + \mathbf{PK})^{-1},$$

we obtain that

$$\begin{aligned} \begin{pmatrix} I & \mathbf{K}^{-1} \\ -\mathbf{P}^{-1} & I \end{pmatrix}^{-1} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} -(I + \mathbf{PK})^{-1} & -\mathbf{P}(I + \mathbf{KP})^{-1} \\ \mathbf{K}(I + \mathbf{PK})^{-1} & -(I + \mathbf{KP})^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & \mathbf{P} \\ -\mathbf{K} & I \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = I - J\mathbf{F}_{\mathbf{K},\mathbf{P}}J, \end{aligned} \tag{3.6}$$

where $J := \operatorname{diag}(I, -I)$. By the hypothesis and statement (1), $\mathbf{P}$ is a Sobolev (α,β)-stabilizing feedback for $\mathbf{K}$, and so, $\mathbf{r}_{\alpha-\beta}\mathbf{F}_{\mathbf{K},\mathbf{P}} \in \mathcal{H}^\infty_0(\mathcal{B}(U \times U))$. Because $\alpha \geq \beta$, we conclude that $\mathbf{r}_{\alpha-\beta}(I - J\mathbf{F}_{\mathbf{K},\mathbf{P}}(s)J)$ is also in $\mathcal{H}^\infty_0(\mathcal{B}(U \times U))$. It now follows from (3.6) that $\mathbf{K}^{-1}$ is a Sobolev (α,β)-stabilizing feedback for $\mathbf{P}^{-1}$. The converse claim can be proved by a similar argument.

As for part (ii) of statement (2), we note that the equality (3.6), combined with the hypotheses, yields that the function $\mathbf{r}_{\alpha-\beta}$ is bounded on $\mathbb{C}_0$, implying that $\alpha \geq \beta$.

To establish statement (3), we observe that, by the hypothesis of $\mathbf{K}$ being Sobolev stabilizing, there exists holomorphic $\mathbf{Q} : \mathbb{C}_0 \to \mathcal{B}(U,Y))$ such that $\mathbf{Q} = \mathbf{P}(I + \mathbf{KP})^{-1}$ on $\mathbb{C}_\omega$ for some $\omega \geq \mu$. Because $I - \mathbf{KQ} = (I + \mathbf{KP})^{-1}$, we conclude that $-\mathbf{K}$ is an admissible feedback for $\mathbf{Q}$. Because $\mathbf{KQ}$ is holomorphic on $\mathbb{C}_\nu$ and $\mathbf{K}(s)\mathbf{Q}(s)$ is compact for all $s \in \mathbb{C}_\nu$, [19, Lemma 5.8] then shows that $(I - \mathbf{KQ})^{-1}$ is meromorphic on $\mathbb{C}_\nu$. The claim now follows since $\mathbf{Q}(I - \mathbf{KQ})^{-1} = \mathbf{P}$ on $\mathbb{C}_\nu$. ◻

4. A mixed passivity/small-gain condition for Sobolev input-output stability.

Here, we state our main result—a mixed passivity/small-gain theorem that ensures that the feedback connection of $\mathbf{P}$ and $\mathbf{K}$ is Sobolev stable. By Theorem 3.1, it follows that it is the difference $\beta - \alpha$, rather than α, β, that is crucial in determining Sobolev stability. Because the feedback connections to be considered do not have any smoothing properties (frequently, they are not even L^2-stable), we shall focus on Sobolev $(\theta, 0)$-stabilizing feedbacks with $\theta \geq 0$. To simplify terminology, we refer to these feedbacks as Sobolev θ-stabilizing and shall say that the corresponding feedback system is Sobolev θ-stable.

The following subsets of the complex plane shall play a key role in our main result.

DEFINITION 4.1. *Let* $\mathbf{P}, \mathbf{K} \in \mathcal{H}^*_\lambda(\mathcal{B}(U))$, *where* $\lambda \in \mathbb{R}$, *and let* $\Omega \subset \mathbb{C}_\lambda \backslash (\Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}})$. *For* $\theta \geq 0$, $\mu, \varepsilon, \nu > 0$, *and* $\gamma \in (0,1)$, *set* $\Pi_\theta(\mathbf{P},\mathbf{K};\Omega,\mu,\varepsilon) := \{s \in \Omega :$

$\operatorname{Re}\mathbf{P}(s) \geq \varepsilon|\mathbf{r}_\theta(s)|I,\ \|\mathbf{P}(s)\| \leq \mu$ *and* $\operatorname{Re}\mathbf{K}(s) \geq 0\big\}$ *and* $\Gamma_\theta(\mathbf{P},\mathbf{K};\Omega,\nu,\gamma) := \big\{s \in \Omega : \max\big\{\|\mathbf{K}(s)\mathbf{P}(s)\|, \|\mathbf{P}(s)\mathbf{K}(s)\|\big\} \leq \gamma$ *and* $\|\mathbf{P}(s)\| + \|\mathbf{K}(s)\| \leq \nu|\mathbf{r}_{-\theta}(s)|\big\}$.

We shall be mainly interested in the cases $\Omega = \mathbb{C}_0$ and $\Omega = i\mathbb{R}$ and comment that there exist $\mu > 0$ and $\varepsilon > 0$ such that $\Pi_0(\mathbf{P},\mathbf{K};\mathbb{C}_0,\mu,\varepsilon) = \mathbb{C}_0$ if and only if $\mathbf{P}$ is L^2-stable and strongly positive real and $\mathbf{K}$ is positive real. Roughly speaking, the sets $\Pi_\theta(\mathbf{P},\mathbf{K};\mathbb{C}_0,\mu,\varepsilon)$ are subsets of $\mathbb{C}_0$ where $\mathbf{P}$ and $\mathbf{K}$ have desirable properties from the perspective of the passivity theorem—stability (that is, boundedness) and a positive-realness property that is "between" positive real and strongly positive real in the case of $\mathbf{P}$ and positive realness in the case of $\mathbf{K}$. Similarly, on the set $\Gamma_\theta(\mathbf{P},\mathbf{K};\mathbb{C}_0,\nu,\gamma)$, the functions $\mathbf{P}$ and $\mathbf{K}$ are jointly well behaved on $\mathbb{C}_0$ from a small-gain perspective.

The following theorem, the main result of the paper, provides sufficient conditions for the feedback connection of positive real functions to be well posed or Sobolev stable in terms of inclusion conditions involving certain unions of the Π_θ and Γ_θ sets. For the presentation of the theorem, it is convenient to define

$$\Phi(x_1,\ldots,x_6) := 2\max\left\{1+\frac{x_1}{x_2}, 1+\frac{x_3}{x_4}, \frac{x_1^2}{x_2}, \frac{x_3^2}{x_4}, \frac{1}{x_2}, \frac{1}{x_4}, \frac{x_5}{1-x_6}, \frac{1}{1-x_6}\right\},$$
$$x_1,\ldots,x_5 > 0,\ \ x_6 \in (0,1).$$

Theorem 4.2. *Let $\mu \leq 0$, $\eta \geq 0$, $\theta \geq 0$, and let $\mathbf{P},\mathbf{K} \in \mathcal{H}^*_\mu(\mathcal{B}(U))$ be such that $\mathbf{P}$ and $\mathbf{K}$ are holomorphic on $\mathbb{C}_0$.*

(1) *Assume that there exist $\mu_\mathbf{P}, \varepsilon_\mathbf{P}, \mu_\mathbf{K}, \varepsilon_\mathbf{K}, \nu > 0$ and $\gamma \in (0,1)$ such that*

$$\mathbb{C}_\eta \subseteq \Pi_\theta(\mathbf{P},\mathbf{K};\mathbb{C}_0,\mu_\mathbf{P},\varepsilon_\mathbf{P}) \cup \Pi_\theta(\mathbf{K},\mathbf{P};\mathbb{C}_0,\mu_\mathbf{K},\varepsilon_\mathbf{K}) \cup \Gamma_\theta(\mathbf{P},\mathbf{K};\mathbb{C}_0,\nu,\gamma). \tag{4.1}$$

The following statements hold.

(i) *The transfer function $I+\mathbf{KP}$ has an inverse in $\mathcal{H}^*_\eta(\mathcal{B}(U))$; in particular, $\mathbf{F}_{\mathbf{P},\mathbf{K}} \in \mathcal{H}^*_\eta(\mathcal{B}(U\times U))$ and $\mathbf{K}$ is an admissible feedback for $\mathbf{P}$.*

(ii) *If $\theta = 0$, then $\mathbf{F}_{\mathbf{P},\mathbf{K}} \in \mathcal{H}^\infty_\eta(\mathcal{B}(U\times U))$.*

(iii) *If $\eta = 0$, then $\mathbf{K}$ is a Sobolev θ-stabilizing feedback for $\mathbf{P}$, and furthermore,*

$$\|\mathbf{r}_\theta \mathbf{F}_{\mathbf{P},\mathbf{K}}\|_{\mathcal{H}^\infty_0} \leq \Phi(\mu_\mathbf{P},\varepsilon_\mathbf{P},\mu_\mathbf{K},\varepsilon_\mathbf{K},\nu,\gamma). \tag{4.2}$$

(2) *Set $\Sigma := \Sigma_\mathbf{P} \cup \Sigma_\mathbf{K}$, and assume that U is separable, $\mu < 0$, $I+\mathbf{KP}$ has an inverse in $\mathcal{H}^*_0(\mathcal{B}(U))$, and $\mathbf{F}_{\mathbf{P},\mathbf{K}}$ is polynomially bounded on $\mathbb{C}_0$. If there exist $\mu_\mathbf{P}, \varepsilon_\mathbf{P}, \mu_\mathbf{K}, \varepsilon_\mathbf{K}, \nu > 0$, $\gamma \in (0,1)$, and a null set $E \subset \mathbb{R}$ such that*

$$i(\mathbb{R}\backslash E) \subset \Pi_\theta(\mathbf{P},\mathbf{K};i\mathbb{R}\backslash\Sigma,\mu_\mathbf{P},\varepsilon_\mathbf{P}) \cup \Pi_\theta(\mathbf{K},\mathbf{P};i\mathbb{R}\backslash\Sigma,\mu_\mathbf{K},\varepsilon_\mathbf{K}) \cup \Gamma_\theta(\mathbf{P},\mathbf{K};i\mathbb{R}\backslash\Sigma,\nu,\gamma), \tag{4.3}$$

then $\mathbf{K}$ is a Sobolev θ-stabilizing feedback for $\mathbf{P}$ and (4.2) holds.

Before we come to the proof of Theorem 4.2 (given below, towards the end of this section), we provide some commentary and draw some conclusions.

Condition (4.1) involves a "mix" of positive real and small-gain properties (specified by Π_θ and Γ_θ sets, respectively). If $\theta = \eta = 0$, then the inclusion in (4.1) becomes an equality, and the conclusion is that the feedback connection of $\mathbf{P}$ and $\mathbf{K}$ is L^2-stable.

DEFINITION 4.3.

(i) *We say that* $\mathbf{P} \in \mathcal{H}^*_0(\mathcal{B}(U))$ *is* Sobolev positive real with exponent $\theta \geq 0$ *if there exists* $\varepsilon > 0$ *such that*

$$\operatorname{Re} \mathbf{P}(s) \geq \varepsilon |\mathbf{r}_\theta(s)| I = \varepsilon |1+s|^{-\theta} I \quad \forall\, s \in \mathbb{C}_0 \backslash \Sigma_{\mathbf{P}}.$$

(ii) $\mathbf{P}$ *and* $\mathbf{K}$ *in* $\mathcal{H}^*_0(\mathcal{B}(U))$ *are said to satisfy a* Sobolev small-gain condition with exponent $\theta \geq 0$ *if there exist* $\nu > 0$ *and* $\gamma \in (0,1)$ *such that* $\Gamma_\theta(\mathbf{P},\mathbf{K};\mathbb{C}_0,\nu,\gamma) = \mathbb{C}_0$.

Trivially, Sobolev positive realness implies positive realness, and Sobolev positive realness with exponent 0 is the same as strong positive realness.

The following immediate corollaries of Theorem 4.2 are generalizations of the passivity and small-gain theorems, respectively.

COROLLARY 4.4. *Let* $\mathbf{P} \in \mathcal{H}^\infty_0(\mathcal{B}(U))$ *be Sobolev positive real with exponent* $\theta \geq 0$*. Then, every positive real* $\mathbf{K} \in \mathcal{H}^*_0(\mathcal{B}(U))$ *is a Sobolev* θ*-stabilizing feedback for* $\mathbf{P}$.

COROLLARY 4.5. *Let* $\mathbf{P} \in \mathcal{H}^*_0(\mathcal{B}(U))$*. If* $\mathbf{K} \in \mathcal{H}^*_0(\mathcal{B}(U))$ *is such that* $\mathbf{P}$ *and* $\mathbf{K}$ *satisfy a Sobolev small-gain condition with exponent* θ *for some* $\theta \geq 0$*, then* $\mathbf{K}$ *is a Sobolev* θ*-stabilizing feedback for* $\mathbf{P}$.

In the following, for notational convenience, we shall suppress the dependence of the sets Π_θ and Γ_θ on μ,ε,ν, and γ when the values of these constants involved are unimportant. We remark that the sets $\Pi_\theta(\mathbf{P},\mathbf{K};\Omega)$ and $\Gamma_\theta(\mathbf{P},\mathbf{K};\Omega)$ have certain obvious monotonicity properties; for example,

$$\begin{aligned}&\Pi_{\theta_1}(\mathbf{P},\mathbf{K};\Omega,\mu,\varepsilon) \subset \Pi_{\theta_2}(\mathbf{P},\mathbf{K};\Omega,\mu,\varepsilon) \quad \text{and}\\ &\Gamma_{\theta_1}(\mathbf{P},\mathbf{K};\Omega,\nu,\gamma) \subset \Gamma_{\theta_2}(\mathbf{P},\mathbf{K};\Omega,\nu,\gamma) \quad \text{for } 0 \leq \theta_1 \leq \theta_2\,.\end{aligned}$$

Consequently, if, for some $\theta_1,\theta_2,\theta_3 \geq 0$,

$$\mathbb{C}_\eta \subset \Pi_{\theta_1}(\mathbf{P},\mathbf{K};\mathbb{C}_0,\mu_{\mathbf{P}},\varepsilon_{\mathbf{P}}) \cup \Pi_{\theta_2}(\mathbf{K},\mathbf{P};\mathbb{C}_0,\mu_{\mathbf{K}},\varepsilon_{\mathbf{K}}) \cup \Gamma_{\theta_3}(\mathbf{P},\mathbf{K};\mathbb{C}_0,\nu,\gamma), \tag{4.4}$$

then (4.1) holds with $\theta := \max\{\theta_1,\theta_2,\theta_3\}$. Moreover, an inspection of the proof Theorem 4.2 shows that there is nothing to be gained (such as, for example, Sobolev θ-stability for a smaller value of θ) by using (4.4) instead of (4.1).

Statement (2) of Theorem 4.2 replaces (4.1) by an imaginary axis condition (which is typically simpler to check) at the expense of imposing an additional polynomial-boundedness hypothesis on the feedback connection. The hypothesis that $\mathbf{P},\mathbf{K} \in \mathcal{H}^*_\mu(\mathcal{B}(U))$ for $\mu < 0$ gives that $\mathbf{P}$ and $\mathbf{K}$ are defined on $i\mathbb{R}$ with the exception of possible imaginary axis singularities in $\Sigma_{\mathbf{P}}$ and $\Sigma_{\mathbf{K}}$ so that the right-hand side of (4.3) is meaningful. In fact, if condition (4.1) holds with $\eta = 0$ and for some $\theta = \theta_0 \geq 0$, then the hypothesis in statement (2) that $\mathbf{F}_{\mathbf{P},\mathbf{K}}$ is polynomially bounded on $\mathbb{C}_0$ is satisfied. Therefore, the feedback connection is Sobolev θ_1-stable if the imaginary axis condition (4.3) holds with $\theta = \theta_1$, which may yield a smaller (and hence "improved") $\theta_1 < \theta_0$.

Before we come to the proof of Theorem 4.2, we provide a comparison between Theorem 4.2 and other results available in the literature. In particular, [28, section 3] contains several results that are of a similar nature to Theorem 4.2 in that a frequency-dependent lower bound on the real part of a transfer function is considered and some type of stability is concluded. However, a detailed comparison is difficult because [28] develops a state-space theory based on the concept of a regular infinite-dimensional linear system (see, for example, [36]). We note that the class of transfer functions considered in Theorem 4.2 is very general and contains transfer functions that do not admit regular (or even well-posed) state-space realizations. Furthermore, we point out that [42, Theorem 4.2] is essentially a special case of Theorem 4.2. Indeed, the hypotheses of [42, Theorem 4.2] ensure the existence of $\{\omega_1,\dots,\omega_n\}\subset\mathbb{R}$ and $r>0$ such that

$$\bigcap_{j=1}^{n}\left\{s\in\mathbb{C}_0 : |s-i\omega_j|\geq r\right\}\subset\Pi_0(\mathbf{K},\mathbf{P};\mathbb{C}_0)\quad\text{and}$$
$$\bigcup_{j=1}^{n}\left\{s\in\mathbb{C}_0 : |s-i\omega_j|\leq r\right\}\subset\Gamma_0(\mathbf{P},\mathbf{K};\mathbb{C}_0),$$

and thus, an application of part of statement (1)(iii) of Theorem 4.2 (with $\theta=0$) shows that $\mathbf{K}$ stabilizes $\mathbf{P}$ in the sense of L^2-stability. Finally, we mention the papers [11, 18], which derive mixed small-gain and passivity theorems guaranteeing L^2-stability (in incremental form in the case of [11]) for certain classes of nonlinear feedback systems.

The remainder of the section is dedicated to proving Theorem 4.2. We start with three technical results that will facilitate the proof of Theorem 4.2. The following lemma 4.6 is an immediate consequence of [28, Lemma A.1].

LEMMA 4.6. *Let $A,B\in\mathcal{B}(U)$ and $\delta_A,\delta_B>0$. The following statements hold.*

(1) *If $\operatorname{Re}A\geq 0$ and $\operatorname{Re}B\geq\delta_B I$, then $I+AB$ is invertible and*

$$\|B(I+AB)^{-1}\|\leq\frac{\|B\|^2}{\delta_B}. \tag{4.5}$$

(2) *If $\operatorname{Re}A\geq\delta_A I$ and $\operatorname{Re}B\geq 0$, then $I+AB$ is invertible and*

$$\|B(I+AB)^{-1}\|\leq\frac{1}{\delta_A}. \tag{4.6}$$

From Lemma 4.6, we obtain the following corollary.

COROLLARY 4.7. *Let $S,T\in\mathcal{B}(U)$, and let $\delta_S,\delta_T>0$. The following statements hold.*

(1) *If $\operatorname{Re}T\geq\delta_T I$, then*

$$\left\|\begin{pmatrix} I & S\\ -T & I\end{pmatrix}^{-1}\right\|\leq 2\max\left\{1+\frac{\|T\|}{\delta_T},\frac{\|T\|^2}{\delta_T},\frac{1}{\delta_T}\right\}$$

whenever $\operatorname{Re}S\geq 0$.

(2) *If $\operatorname{Re}S\geq\delta_S I$, then*

$$\left\|\begin{pmatrix} I & S\\ -T & I\end{pmatrix}^{-1}\right\|\leq 2\max\left\{1+\frac{\|S\|}{\delta_S},\frac{\|S\|^2}{\delta_S},\frac{1}{\delta_S}\right\}$$

whenever $\operatorname{Re}T\geq 0$.

Note that the right-hand sides of the inequalities in statements (1) and (2) are independent of S and T, respectively.

Proof. To prove statement (1), we note that, by Lemma 4.6, $I+ST$ is invertible. Therefore, $I+TS$ is also invertible, and so,

$$\begin{pmatrix} I & S\\ -T & I\end{pmatrix}\quad\text{is invertible and}\quad\begin{pmatrix} I & S\\ -T & I\end{pmatrix}^{-1}=\begin{pmatrix}(I+ST)^{-1} & -S(I+TS)^{-1}\\ T(I+ST)^{-1} & (I+TS)^{-1}\end{pmatrix}. \tag{4.7}$$

Using (4.5) with $A=S$ and $B=T$ yields that

$$\|T(I+ST)^{-1}\| \leq \frac{\|T\|^2}{\delta_T}. \tag{4.8}$$

From (4.6) with $A=T$ and $B=S$, we have that

$$\|S(I+TS)^{-1}\| \leq \frac{1}{\delta_T}. \tag{4.9}$$

Because $(I+ST)^{-1} = I - S(I+TS)^{-1}T$ and $(I+TS)^{-1} = I - TS(I+TS)^{-1}$, it follows from (4.9) that

$$\|(I+ST)^{-1}\| \leq 1 + \frac{\|T\|}{\delta_T} \quad \text{and} \quad \|(I+TS)^{-1}\| \leq 1 + \frac{\|T\|}{\delta_T}. \tag{4.10}$$

Setting

$$c := \max\left\{1 + \frac{\|T\|}{\delta_T}, \frac{\|T\|^2}{\delta_T}, \frac{1}{\delta_T}\right\} < \infty,$$

we apply (4.7)–(4.10) to obtain that, for $u_1, u_2 \in U$,

$$\begin{aligned}\left\|\begin{pmatrix} I & S \\ -T & I\end{pmatrix}^{-1}\begin{pmatrix} u_1 \\ u_2\end{pmatrix}\right\|^2 &= \left\|(I+ST)^{-1}u_1 - S(I+TS)^{-1}u_2\right\|^2 \\ &\quad + \left\|T(I+ST)^{-1}u_1 + (I+TS)^{-1}u_2\right\|^2 \\ &\leq \left(\left\|(I+ST)^{-1}\right\|\|u_1\| + \left\|S(I+TS)^{-1}\right\|\|u_2\|\right)^2 \\ &\quad + \left(\left\|T(I+ST)^{-1}\right\|\|u_1\| + \left\|(I+TS)^{-1}\right\|\|u_2\|\right)^2 \\ &\leq 4c^2(\|u_1\|^2 + \|u_2\|^2)\end{aligned}$$

since $(\|u_1\| + \|u_2\|)^2 \leq 2(\|u_1\|^2 + \|u_2\|^2)$. Hence,

$$\left\|\begin{pmatrix} I & S \\ -T & I\end{pmatrix}^{-1}\right\| = \sup_{\substack{u_1,u_2\in U \\ \|u_1\|^2+\|u_2\|^2=1}} \left\|\begin{pmatrix} I & S \\ -T & I\end{pmatrix}^{-1}\begin{pmatrix} u_1 \\ u_2\end{pmatrix}\right\| \leq 2c,$$

as required.

The second statement follows from the identity

$$\begin{pmatrix} I & S \\ -T & I\end{pmatrix}^{-1} = \begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}\begin{pmatrix} I & T \\ -S & I\end{pmatrix}^{-1}\begin{pmatrix} 0 & -I \\ I & 0\end{pmatrix},$$

combined with an application of statement (1) (with the roles of S and T interchanged) to the second operator matrix on the right-hand side. ∎

LEMMA 4.8. *Let $S, T \in \mathcal{B}(U)$. If $\max\{\|ST\|, \|TS\|\} \leq \rho < 1$, then $I+ST$ and $I+TS$ are invertible,*

$$\begin{pmatrix} I & S \\ -T & I\end{pmatrix} \text{ is invertible, and } \left\|\begin{pmatrix} I & S \\ -T & I\end{pmatrix}^{-1}\right\| \leq \frac{2}{1-\rho}\max\{1, \|S\|, \|T\|\}. \tag{4.11}$$

Proof. It follows from the hypotheses that $I+ST$ and $I+TS$ are invertible, with each inverse given in terms of a Neumann series, and

$$\max\left\{\|(I+ST)^{-1}\|, \|(I+TS)^{-1}\|\right\} \leq 1/(1-\rho).$$

The inequality in (4.11) now follows from the identity in (4.7). ∎

Proof of Theorem 4.2. For notational convenience, we set

$$\Pi_\theta(\mathbf{P},\mathbf{K}) := \Pi_\theta(\mathbf{P},\mathbf{K};\mathbb{C}_0,\mu_{\mathbf{P}},\varepsilon_{\mathbf{P}}), \quad \Pi_\theta(\mathbf{K},\mathbf{P}) := \Pi_\theta(\mathbf{K},\mathbf{P};\mathbb{C}_0,\mu_{\mathbf{K}},\varepsilon_{\mathbf{K}}),$$
$$\Gamma_\theta(\mathbf{P},\mathbf{K}) := \Gamma_\theta(\mathbf{P},\mathbf{K};\mathbb{C}_0,\nu,\gamma).$$

(1) To prove part (i), let $\theta,\eta \geq 0$ be such that (4.1) holds. Because $\operatorname{Re}\mathbf{K}(s) \geq 0$ and $\operatorname{Re}\mathbf{P}(s) \geq \varepsilon_{\mathbf{P}}|\mathbf{r}_\theta(s)|$ for $s \in \Pi_\theta(\mathbf{P},\mathbf{K})$, it follows from Lemma 4.6 that $I+\mathbf{K}(s)\mathbf{P}(s)$ is invertible for all $s \in \Pi_\theta(\mathbf{P},\mathbf{K})$. Similarly, $I+\mathbf{K}(s)\mathbf{P}(s)$ is invertible for all $s \in \Pi_\theta(\mathbf{K},\mathbf{P})$. An application of Lemma 4.8 with $S=\mathbf{K}(s)$ and $T=\mathbf{P}(s)$ yields that $I+\mathbf{K}(s)\mathbf{P}(s)$ is invertible for all $s \in \Gamma_\theta(\mathbf{P},\mathbf{K})$. Invoking (4.1), we conclude that $I+\mathbf{K}(s)\mathbf{P}(s)$ is invertible for all $s \in \mathbb{C}_\eta$. Therefore, $(I+\mathbf{KP})^{-1} \in \mathcal{H}^*_\eta(\mathcal{B}(U))$, showing that $\mathbf{K}$ is an admissible feedback for $\mathbf{P}$.

To prove parts (ii) and (iii), assume that (4.1) holds with $\theta,\eta \geq 0$. Part (i) gives that $\mathbf{K}$ is an admissible feedback for $\mathbf{P}$ and $\mathbf{F}_{\mathbf{P},\mathbf{K}} \in \mathcal{H}^*_\eta(\mathcal{B}(U\times U))$. Invoking statement (1) of Corollary 4.7 with $S=\mathbf{K}(s)$ and $T=\mathbf{P}(s)$, it follows that

$$\begin{aligned}\big\|\mathbf{r}_\theta(s)\mathbf{F}_{\mathbf{P},\mathbf{K}}(s)\big\| &\leq 2\max\left\{|\mathbf{r}_\theta(s)| + \frac{\mu_{\mathbf{P}}}{\varepsilon_{\mathbf{P}}}, \frac{\mu_{\mathbf{P}}^2}{\varepsilon_{\mathbf{P}}}, \frac{1}{\varepsilon_{\mathbf{P}}}\right\}\\ &\leq \Phi(\mu_{\mathbf{P}},\varepsilon_{\mathbf{P}},\mu_{\mathbf{K}},\varepsilon_{\mathbf{K}},\nu,\gamma) \quad \forall\, s \in \Pi_0(\mathbf{P},\mathbf{K}).\end{aligned}$$

Similarly, applying statement (2) of Corollary 4.7 with $S=\mathbf{P}(s)$ and $T=\mathbf{K}(s)$, we obtain that

$$\begin{aligned}\big\|\mathbf{r}_\theta(s)\mathbf{F}_{\mathbf{P},\mathbf{K}}(s)\big\| &\leq 2\max\left\{|\mathbf{r}_\theta(s)| + \frac{\mu_{\mathbf{K}}}{\varepsilon_{\mathbf{K}}}, \frac{\mu_{\mathbf{K}}^2}{\varepsilon_{\mathbf{K}}}, \frac{1}{\varepsilon_{\mathbf{K}}}\right\}\\ &\leq \Phi(\mu_{\mathbf{P}},\varepsilon_{\mathbf{P}},\mu_{\mathbf{K}},\varepsilon_{\mathbf{K}},\nu,\gamma) \quad \forall\, s \in \Pi_0(\mathbf{K},\mathbf{P}).\end{aligned}$$

Furthermore, an application of Lemma 4.8 with $S=\mathbf{K}(s)$ and $T=\mathbf{P}(s)$ shows that

$$\big\|\mathbf{r}_\theta(s)\mathbf{F}_{\mathbf{P},\mathbf{K}}(s)\big\| \leq \frac{2}{1-\gamma}\max\{|\mathbf{r}_\theta(s)|,\nu\} \leq \Phi(\mu_{\mathbf{P}},\varepsilon_{\mathbf{P}},\mu_{\mathbf{K}},\varepsilon_{\mathbf{K}},\nu,\gamma) \quad \forall\, s \in \Gamma_0(\mathbf{P},\mathbf{K}).$$

The last three estimates together with (4.1) yield that

$$\big\|\mathbf{r}_\theta(s)\mathbf{F}_{\mathbf{P},\mathbf{K}}(s)\big\| \leq \Phi(\mu_{\mathbf{P}},\varepsilon_{\mathbf{P}},\mu_{\mathbf{K}},\varepsilon_{\mathbf{K}},\nu,\gamma) \quad \forall\, s \in \mathbb{C}_\eta.$$

Parts (ii) and (iii) now follow by considering the special cases $\theta=0$ and $\eta=0$.

(2) Assume that U is separable and $\mu<0$. Because $\mathbf{P},\mathbf{K} \in \mathcal{H}^*_\mu(\mathcal{B}(U))$ and both functions are holomorphic on $\mathbb{C}_0$, it follows that $\mathbb{C}_0 \cap \Sigma = \emptyset$, and there exists an open set $\Omega \subset \mathbb{C}$ such that $(\overline{\mathbb{C}}_0\backslash\Sigma) \subset \Omega \subset \mathbb{C}_\mu$ and $\mathbf{S}$ is holomorphic on Ω, where $\mathbf{S}$ is given by (3.1). By definition, $\mathbf{F} := \mathbf{F}_{\mathbf{P},\mathbf{K}}$ is an extension of $\mathbf{S}^{-1}$. Because $(I+\mathbf{KP})^{-1} \in \mathcal{H}^*_0(\mathcal{B}(U))$ by hypothesis, we have that $\mathbf{S}^{-1} \in \mathcal{H}^*_0(\mathcal{B}(U\times U))$, and thus, $\mathbf{F} \in \mathcal{H}^*_{-\delta}(\mathcal{B}(U\times U))$ for some $\delta \geq 0$ (but, of course, there is no guarantee that $\delta>0$). Polynomial boundedness of $\mathbf{F}$ on $\mathbb{C}_0$ implies that $\mathbf{F}$ is holomorphic on $\mathbb{C}_0$; in particular, $\mathbf{S}(s)$ is invertible for every $s \in \mathbb{C}_0$. Furthermore, it follows from an application of Proposition 3.2 (with $\mathbf{G}^{\mathrm{e}}=\mathbf{F}$) that there exists a set $B \subset \mathbb{R}$ such that $\mathbb{R}\backslash B$ is a null set and the limit $\mathbf{F}_0(y) := \lim_{x\downarrow 0}\mathbf{F}(x+iy)$ exists in the strong operator topology for every $y \in B$. Let $a>0$, and note that, by polynomial boundedness on $\mathbb{C}_0$, $f_y := \sup_{x\in(0,a)}\|\mathbf{F}(x+iy)\| < \infty$ for every $y \in \mathbb{R}$. Setting $B' := \{y \in B : iy \notin \Sigma\}$, we observe that

$$\begin{aligned}\mathbf{F}(x+iy)\mathbf{S}(iy)u = u + \mathbf{F}(x+iy)\big(&\mathbf{S}(iy)u\\ &- \mathbf{S}(x+iy)u\big) \quad \forall\, u \in U,\ \forall\, x \in (0,a), \text{ and } \forall\, y \in B'.\end{aligned}$$

Consequently, for all $u \in U$ and all $y \in B'$,

$$\|\mathbf{F}(x+iy)\mathbf{S}(iy)u - u\| \le f_y \|\mathbf{S}(iy)u - \mathbf{S}(x+iy)u\| \to 0 \quad \text{as } x \downarrow 0,$$

showing that $\mathbf{F}_0(y)$ is a left inverse of $\mathbf{S}(iy)$ for every $y \in B'$. A similar argument shows that, for each $y \in B'$, $\mathbf{F}_0(y)$ is also a right inverse of $\mathbf{S}(iy)$. Therefore,

$$\mathbf{F}_0(y) = (\mathbf{S}(iy))^{-1} = \begin{pmatrix} I & \mathbf{K}(iy) \\ -\mathbf{P}(iy) & I \end{pmatrix}^{-1} \quad \forall\, y \in B'. \tag{4.12}$$

Defining $B'' := B' \backslash E$, we have that, if $y \in B''$, then iy is contained in the right-hand side of the set inclusion (4.3). Together with (4.12), this means that the estimates in the proof of parts (ii) and (iii) of statement (1) can be used to show that

$$\|\mathbf{r}_\theta(iy)\mathbf{F}_0(y)\| \le \Phi(\mu_{\mathbf{P}}, \varepsilon_{\mathbf{P}}, \mu_{\mathbf{K}}, \varepsilon_{\mathbf{K}}, \nu, \gamma) \quad \forall\, y \in B''.$$

Because $\mathbb{R}\backslash B''$ is a null set, we conclude that

$$\operatorname*{ess\,sup}_{y\in\mathbb{R}} \|\mathbf{r}_\theta(iy)\mathbf{F}_0(y)\| \le \Phi(\mu_{\mathbf{P}}, \varepsilon_{\mathbf{P}}, \mu_{\mathbf{K}}, \varepsilon_{\mathbf{K}}, \nu, \gamma) < \infty\,. \tag{4.13}$$

Proposition 3.2, together with Theorem 3.1, yields that $\mathbf{r}_\theta \mathbf{F} \in \mathcal{H}_0^\infty(\mathcal{B}(U \times U))$, showing that $\mathbf{K}$ is a Sobolev θ-stabilizing feedback for $\mathbf{P}$. Furthermore,

$$\|\mathbf{r}_\theta \mathbf{F}\|_{\mathcal{H}_0^\infty} = \sup_{s\in\mathbb{C}_0} \|\mathbf{r}_\theta(s)\mathbf{F}(s)\| = \operatorname*{ess\,sup}_{y\in\mathbb{R}} \|\mathbf{r}_\theta(iy)\mathbf{F}_0(y)\|,$$

and it follows from (4.13) that (4.2) holds. ∎

Remark 4.9.

(i) An inspection of the above proof shows that statement (1) of Theorem 4.2 remains valid without the a priori assumption of holomorphicity of $\mathbf{P}$ and $\mathbf{K}$ on $\mathbb{C}_0$ provided that (4.1) is replaced by

$$\begin{aligned} \mathbb{C}_\eta \backslash \Sigma \subseteq \Pi_\theta(\mathbf{P}, \mathbf{K}; \mathbb{C}_0 \backslash \Sigma, \mu_{\mathbf{P}}, \varepsilon_{\mathbf{P}}) \cup \Pi_\theta(\mathbf{K}, \mathbf{P}; \mathbb{C}_0 \backslash \Sigma, \mu_{\mathbf{K}}, \varepsilon_{\mathbf{K}}) \\ \cup\, \Gamma_\theta(\mathbf{P}, \mathbf{K}; \mathbb{C}_0 \backslash \Sigma, \nu, \gamma), \end{aligned} \tag{4.14}$$

where $\Sigma := \Sigma_{\mathbf{P}} \cup \Sigma_{\mathbf{K}}$. However, this does not enlarge the class of transfer functions that can be handled because it can be proved (by invoking arguments similar to those used in [19, Proof of Proposition 3.3]) that, if (4.14) holds, then $\Sigma \cap \mathbb{C}_0 = \emptyset$.

(ii) The above proof suggests that Theorem 4.2 may be viewed as a special case of a broader principle. Roughly speaking, the principle is that, if $\mathbb{C}_0$ can be divided into sets on which positive–real- or small–gain-type conditions hold for $\mathbf{P}$ and $\mathbf{K}$, in the sense of (4.1) with $(\eta = 0)$, then some resulting stability property of the feedback connection of $\mathbf{P}$ and $\mathbf{K}$ may be inferred. Indeed, routine modifications to Lemma 4.8 and the proof of Theorem 4.2 yield that the conclusions of Theorem 4.2 remain true if the set $\Gamma_\theta(\mathbf{P}, \mathbf{K}; \nu, \gamma)$ in (4.1) is replaced by

$$\begin{aligned} \Gamma'_\theta(\mathbf{P}, \mathbf{K}; \Omega, \nu, \gamma) := \big\{ s \in \Omega \,:\, \|\mathbf{K}(s)\mathbf{P}(s)\| \le \gamma,\ \|\mathbf{K}(s)\| \le \nu, \\ \|\mathbf{P}(s)\| \le \nu |\mathbf{r}_{-\theta}(s)| \big\}\,. \end{aligned}$$

We expect that other suitable modifications of the Π_θ and Γ_θ sets are possible. For brevity, we do not give formal statements.

5. Sobolev stability for output regulation and disturbance rejection. In this section, we invoke the concept of Sobolev stability in the context of output-regulation and disturbance-rejection problems. For this purpose, consider the following feedback system expressed in the frequency domain:

$$\left.\begin{aligned} \hat{y} &= \mathbf{P}\hat{u}, \qquad \hat{z} = \mathbf{K}\hat{e}\,, \\ u &= d - z, \qquad e = y - r\,, \end{aligned}\right\} \tag{5.1}$$

where y, z, e, u, r, and d denote the plant output, the controller output, the tracking error, the plant input, the reference, and the disturbance, respectively, which we also refer to as signals. The hat $\hat{}$-notation in (5.1) denotes the bilateral Laplace transform, and the signals are all assumed to be functions on the real axis $\mathbb{R}$ (or distributions) with support bounded to the left (where we assume that input-output operators associated with $\mathbf{P}$, $\mathbf{K}$ and the feedback connection of $\mathbf{P}$ and $\mathbf{K}$ are causal).

Given a plant $\mathbf{P}$, the objective is to design a controller $\mathbf{K}$ such that the error $e = y - r$ is in $L^2(\mathbb{R})$ (or preferably better—in $H^1(\mathbb{R})$, for example) for a class of persistent reference and disturbance signals. This is a fundamental control problem (sometimes also referred to as the servo problem) and is consequently well studied, with numerous papers on the subject including, but by no means limited to, [22, 25, 28, 38, 39]. The well-known internal model principle (see, for example, [17], [24], [25], or [37, section 7.5]) plays a key role in the servo problem. In the internal model principle, it is assumed that the signals r and d are generated by two signal generators (or exo-systems) (see the configuration shown in Figure 5.1), and the principle says, roughly speaking, that $\mathbf{K}$ is a robust solution to the servo problem if and only if $\mathbf{K}$ is stabilizing and "contains" so-called internal models of $\mathbf{D}$ and $\mathbf{R}$ (essentially meaning that all unstable poles of $\mathbf{D}$ and $\mathbf{R}$ are also poles of $\mathbf{K}$).

Inspired by the frequency-domain theory of the internal model principle [24, 25, 37], a sufficient condition for a Sobolev stabilizing controller to solve the servo problem is given in Theorem 5.1.

In the following result, we shall assume that the transfer functions $\mathbf{D} \in \mathcal{H}^*_\mu((\mathcal{B}(V, U))$ and $\mathbf{R} \in \mathcal{H}^*(\mathcal{B}(W, Y))$ of the signal generators are polynomially bounded on some right-half plane, where V and W are complex Hilbert spaces, the spaces in which the functions g_d and g_r take their values, respectively. Then, by [41, Theorem 6.5-1 and Corollary 6.5-1a], $\mathbf{D}$ and $\mathbf{R}$ are the transfer functions of causal translation-invariant operators $D : \mathcal{D}'_\ell(V) \to \mathcal{D}'_\ell(U)$ and $R : \mathcal{D}'_\ell(W) \to \mathcal{D}'_\ell(Y)$, respectively.

THEOREM 5.1. *Consider the feedback system* (5.1) *for given* $\mathbf{P} \in \mathcal{H}^*_\mu(\mathcal{B}(U, Y))$ *and* $\mathbf{K} \in \mathcal{H}^*_\mu(\mathcal{B}(Y, U))$, *where* $\mu \le 0$, *and let* $\mathbf{D} \in \mathcal{H}^*_\mu(\mathcal{B}(V, U))$ *and* $\mathbf{R} \in \mathcal{H}^*_\mu(\mathcal{B}(W, Y))$ *be polynomially bounded on some right-half plane. Assume that there exist a function* $\mathbf{q} \in \mathcal{H}_0(\mathbb{C})$, $\mathbf{q}(s) \not\equiv 0$, *and* $\rho \in \mathbb{R}$ *such that* $\mathbf{r}_\rho \mathbf{q}\mathbf{D} \in \mathcal{H}^\infty_0(\mathcal{B}(V, U))$ *and* $\mathbf{r}_\rho \mathbf{q}\mathbf{R} \in \mathcal{H}^\infty_0(\mathcal{B}(W, Y))$. *Furthermore, assume that there exist an open set* $\Omega \subset \mathbb{C}_0$ *and* $\sigma \in \mathbb{R}$

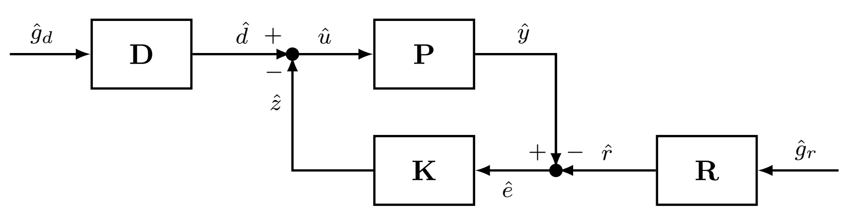

FIG. 5.1. *Feedback connection of plant* $\mathbf{P}$ *and controller* $\mathbf{K}$ *with signal generators* $\mathbf{D}$ *and* $\mathbf{R}$.

such that $\mathbf{r}_\sigma/\mathbf{q}$ *is bounded on* $\mathbb{C}_0\backslash\Omega$. *Let* $\kappa\in\mathbb{R}$ *and* $\theta\geq 0$. *If* $\mathbf{K}$ *satisfies the two conditions*

(a) $\mathbf{K}$ *is a Sobolev* θ*-stabilizing feedback for* $\mathbf{P}$ *and*

(b) $\mathbf{K}$ *has a left-inverse* $\mathbf{K}^\#\in\mathcal{H}^*(\Omega,\mathcal{B}(U,Y))$ *such that* $(\mathbf{r}_\sigma/\mathbf{q})\mathbf{K}^\#\in\mathcal{H}^\infty(\Omega,\mathcal{B}(U,Y))$,

then $e\in H_\ell^{\kappa-\alpha}(\mathbb{R},Y)$ *for all* $d\in\{Dg : g\in H_\ell^\kappa(\mathbb{R},V)\}+H_\ell^\kappa(\mathbb{R},U)$ *and* $r\in\{Rg : g\in H_\ell^\kappa(\mathbb{R},W)\}+H_\ell^\kappa(\mathbb{R},Y)$, *where* $\alpha:=\max\{\theta,\rho+\sigma+\theta\}$. *Furthermore,* $e\in H_\ell^{\kappa-(\rho+\sigma+\theta)}(\mathbb{R},Y)$ *whenever* $d\in\{Dg:g\in H_\ell^\kappa(\mathbb{R},V)\}$ *and* $r\in\{Rg:g\in H_\ell^\kappa(\mathbb{R},W)\}$.

In the case wherein $U=Y$ and $\mathbf{P}$ and $\mathbf{K}$ are holomorphic on $\mathbb{C}_0$, Theorem 4.2 provides a sufficient condition for condition (a) to hold.

Remark 5.2.

(i) The above result guarantees that $e\in L^2(\mathbb{R},U)$ whenever $\kappa\geq\alpha$, implying tracking in measure in the sense that $\lim_{T\to\infty}\operatorname{meas}\big(\{t\geq T:\|e(t)\|\geq\varepsilon\}\big)=0$ for every $\varepsilon>0$, where "meas" denotes the Lebesgue measure. Furthermore, $e(t)\to 0$ as $t\to\infty$ (asymptotic tracking) whenever $\kappa\geq\alpha+1$.

(ii) Set $Z_\mathbf{q}:=\{z\in\overline{\mathbb{C}}_0:\liminf_{s\to z,s\in\mathbb{C}_0}\mathbf{q}(s)=0\}$. Clearly, $Z_\mathbf{q}\cap\mathbb{C}_0=\{z\in\mathbb{C}_0:\mathbf{q}(z)=0\}$. Moreover, if $\mathbf{q}$ has a holomorphic extension (also denoted by $\mathbf{q}$) to an open set containing $\overline{\mathbb{C}}_0$, then $Z_\mathbf{q}=\{z\in\overline{\mathbb{C}}_0:\mathbf{q}(z)=0\}$. The boundedness of $\mathbf{r}_\sigma/\mathbf{q}$ on $\mathbb{C}_0\backslash\Omega$ implies that $\operatorname{dist}(z,\mathbb{C}_0\backslash\Omega)>0$ for every $z\in Z_\mathbf{q}$. Thus, $Z_\mathbf{q}\subset\overline{\Omega}$, $Z_\mathbf{q}\cap\mathbb{C}_0\subset\Omega$, and, for every $z\in Z_\mathbf{q}$, there exists $\varepsilon_z>0$ such that $\{s\in\mathbb{C}_0:|s-z|<\varepsilon_z\}\subset\Omega$.

(iii) Let $z\in\overline{\mathbb{C}}_0$ be such that $\limsup_{s\to z,s\in\mathbb{C}_0}\big(\|\mathbf{D}(s)\|+\|\mathbf{R}(s)\|\big)=\infty$ (z is a pole of $\mathbf{D}$ or $\mathbf{R}$ if $z\in\mathbb{C}_0$). Since $\mathbf{r}_\rho\mathbf{q}\mathbf{D}\in\mathcal{H}_0^\infty(\mathcal{B}(V,U))$ and $\mathbf{r}_\rho\mathbf{q}\mathbf{R}\in\mathcal{H}_0^\infty(\mathcal{B}(W,Y))$, we see that $z\in Z_\mathbf{q}$, and condition (b) implies that $\limsup_{s\to z,s\in\mathbb{C}_0}\|\mathbf{K}(s)\|=\infty$. Furthermore, if $\mu<0$, then it follows that every pole of $\mathbf{D}$ or $\mathbf{R}$ in $\overline{\mathbb{C}}_0$ is also a pole of the controller $\mathbf{K}$. Consequently, condition (b) can be viewed as a version of the internal model principle.

(iv) Assume that $U=Y$ and there exists $\delta>0$ such that

$$|\mathbf{r}_{-\sigma}(s)\mathbf{q}(s)|\|\mathbf{K}(s)v\|\geq\delta\|v\|\quad\text{and}$$
$$|\mathbf{r}_{-\sigma}(s)\mathbf{q}(s)|\|\mathbf{K}^*(s)v\|\geq\delta\|v\|\quad\forall\, s\in\big(\mathbb{C}_0\cap\Omega\big)\backslash\Sigma_\mathbf{K}\quad\text{and}\ \forall\, v\in U.$$

Then it follows from [30, Proposition 3.2.6] that $\mathbf{K}(s)$ is invertible for all $s\in\big(\mathbb{C}_0\cap\Omega\big)\backslash\Sigma_\mathbf{K}$, and the first inequality guarantees that $(\mathbf{r}_\sigma/\mathbf{q})\mathbf{K}^{-1}\in\mathcal{H}^\infty(\Omega,\mathcal{B}(U))$. Consequently, condition (b) holds in this case with $\mathbf{K}^\#=\mathbf{K}^{-1}$.

Proof of Theorem 5.1. By hypothesis, $\mathbf{K}$ is a Sobolev θ-stabilizing feedback for $\mathbf{P}$ and $\mu\leq 0$. Hence, $\Xi_{\mathbf{P},\mathbf{K}}\cap\mathbb{C}_0=\emptyset$, and it follows from Remark 3.4 that

$$\mathbf{F}_{\mathbf{P},\mathbf{K}}(s)=\mathbf{S}^{-1}(s)=\begin{pmatrix} I & \mathbf{K}(s)\\ -\mathbf{P}(s) & I\end{pmatrix}^{-1}\quad\forall\, s\in\mathbb{C}_0\backslash\big(\Sigma_\mathbf{P}\cup\Sigma_\mathbf{K}\big). \tag{5.2}$$

Let $d=Dg_d+h_d$ and $r=Rg_r+h_r$, where $g_d\in H_\ell^\kappa(\mathbb{R},V)$, $g_r\in H_\ell^\kappa(\mathbb{R},W)$, $h_d\in H_\ell^\kappa(\mathbb{R},U)$, and $h_r\in H_\ell^\kappa(\mathbb{R},Y)$. System (5.1) can be expressed as

$$\begin{pmatrix} I & \mathbf{K}\\ -\mathbf{P} & I\end{pmatrix}\begin{pmatrix}\hat{u}\\ \hat{e}\end{pmatrix}=\begin{pmatrix}\hat{d}\\ -\hat{r}\end{pmatrix}=\begin{pmatrix}\mathbf{D} & 0\\ 0 & \mathbf{R}\end{pmatrix}\begin{pmatrix}\hat{g}_d\\ -\hat{g}_r\end{pmatrix}+\begin{pmatrix}\hat{h}_d\\ -\hat{h}_r\end{pmatrix}.$$

Routine calculations invoking (3.3) and (5.2) yield that

$$\hat{e} = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} I & \mathbf{K} \\ -\mathbf{P} & I \end{pmatrix}^{-1} \left(\begin{pmatrix} \mathbf{D} & 0 \\ 0 & \mathbf{R} \end{pmatrix} \begin{pmatrix} \hat{g}_d \\ -\hat{g}_r \end{pmatrix} + \begin{pmatrix} \hat{h}_d \\ -\hat{h}_r \end{pmatrix} \right)$$
$$= \big(\mathbf{P}(I+\mathbf{KP})^{-1}\mathbf{D}, (I+\mathbf{PK})^{-1}\mathbf{R}\big) \begin{pmatrix} \hat{g}_d \\ -\hat{g}_r \end{pmatrix} + (0,I)\mathbf{F}_{\mathbf{P},\mathbf{K}} \begin{pmatrix} \hat{h}_d \\ -\hat{h}_r \end{pmatrix}.$$

The hypothesis on $\mathbf{D}$ and $\mathbf{R}$ implies that there exist $\mathbf{H_D} \in \mathcal{H}_0(\mathcal{B}(V,U))$ and $\mathbf{H_R} \in \mathcal{H}_0(\mathcal{B}(W,Y))$ such that $\mathbf{D} = \mathbf{H_D}/\mathbf{q}$, $\mathbf{R} = \mathbf{H_R}/\mathbf{q}$, $\mathbf{r}_\rho \mathbf{H_D} \in \mathcal{H}_0^\infty(\mathcal{B}(V,U))$, and $\mathbf{r}_\rho \mathbf{H_R} \in \mathcal{H}_0^\infty(\mathcal{B}(W,Y))$. Defining $\mathbf{F}_1 := \mathbf{P}(I+\mathbf{KP})^{-1}\mathbf{D}$ and $\mathbf{F}_2 := \mathbf{P}(I+\mathbf{PK})^{-1}\mathbf{R}$, we have that

$$\begin{cases} \mathbf{F}_1 = (1/\mathbf{q})\mathbf{P}(I+\mathbf{KP})^{-1}\mathbf{H_D}, \quad \mathbf{F}_2 = (1/\mathbf{q})(I+\mathbf{PK})^{-1}\mathbf{H_R} \quad \text{on } \mathbb{C}_0 \supset \mathbb{C}_0 \backslash \Omega, \\ \mathbf{F}_1 = (\mathbf{K}^\#/\mathbf{q})\big(I - (I+\mathbf{KP})^{-1}\big)\mathbf{H_D}, \quad \mathbf{F}_2 = (\mathbf{K}^\#/\mathbf{q})(I+\mathbf{KP})^{-1}\mathbf{K}\mathbf{H_R} \quad \text{on } \Omega. \end{cases} \tag{5.3}$$

The function $\hat{e}$ can be expressed in the form

$$\hat{e} = \big(\mathbf{F}_1, \mathbf{F}_2\big) \begin{pmatrix} \hat{g}_d \\ -\hat{g}_r \end{pmatrix} + (0,I)\mathbf{F}_{\mathbf{P},\mathbf{K}} \begin{pmatrix} \hat{h}_d \\ -\hat{h}_r \end{pmatrix}. \tag{5.4}$$

Set $\beta := \rho + \sigma + \theta$ so that $\alpha = \max\{\theta, \beta\}$. Because $\mathbf{K}$ is Sobolev θ-stabilizing for $\mathbf{P}$, $\mathbf{r}_\rho \mathbf{H_D} \in \mathcal{H}_0^\infty(\mathcal{B}(V,U))$, $\mathbf{r}_\rho \mathbf{H_R} \in \mathcal{H}_0^\infty(\mathcal{B}(W,Y))$, $\mathbf{r}_\sigma \mathbf{K}^\#/\mathbf{q} \in \mathcal{H}^\infty(\Omega, \mathcal{B}(U,Y))$, and $\mathbf{r}_\sigma/\mathbf{q}$ is bounded on $\mathbb{C}_0 \backslash \Omega$, it follows from (5.3) that $\mathbf{r}_\beta(\mathbf{F}_1, \mathbf{F}_2) \in \mathcal{H}_0^\infty(\mathcal{B}(V \times W, Y))$. Furthermore, $\mathbf{r}_\theta(0,I)\mathbf{F}_{\mathbf{P},\mathbf{K}} \in \mathcal{H}_0^\infty(\mathcal{B}(U \times Y, Y))$. Appealing to Theorem 3.1, we conclude that the causal translation-invariant operators that have $(\mathbf{F}_1, \mathbf{F}_2)$ and $(0,I)\mathbf{F}_{\mathbf{P},\mathbf{K}}$ as transfer functions are Sobolev β-stable and Sobolev θ-stable, respectively, in the sense of subsection 3.1. Hence, $e \in H_\ell^{\kappa-\alpha}(\mathbb{R}, Y)$. Finally, if $h_r = h_d = 0$, then, invoking (5.4), it follows that e is in the image of $H_\ell^\kappa(\mathbb{R}, V \times W)$ under a Sobolev β-stable operator, whence $e \in H_\ell^{\kappa-\beta}(\mathbb{R}, Y)$, completing the proof. ∎

Next, we apply Theorem 5.1 in the context of repetitive control, in which the controller

$$\mathbf{K}_\tau(s) := \frac{1}{1 - \mathrm{e}^{-\tau s}} I \quad \forall\, s \in \mathbb{C}_0, \quad \text{where } \tau > 0, \tag{5.5}$$

plays a key role. Since, for $x > 0$ and $y \in \mathbb{R}$,

$$\operatorname{Re} \frac{1}{1 - \mathrm{e}^{-\tau(x+iy)}} = \frac{1 - \mathrm{e}^{-\tau x}\cos(\tau y)}{1 + \mathrm{e}^{-2\tau x} - 2\mathrm{e}^{-\tau x}\cos(\tau y)} \geq \frac{1 - \mathrm{e}^{-\tau x}\cos(\tau y)}{2(1 - \mathrm{e}^{-\tau x}\cos(\tau y))} = \frac{1}{2}, \tag{5.6}$$

it follows that $\mathbf{K}_\tau$ is strongly positive real.

We remark that $\mathbf{K}_\tau$ qualifies as a so-called *repetitive controller* in the sense of [22]. Repetitive controllers have been considered across numerous papers on output regulation, and we refer the reader to, for example, [38] for more information. It was noted in [22, Proposition 2] that L^2-input–output stability of the feedback system (5.1) is impossible when $\mathbf{P}(s) \to 0$ as $|s| \to \infty$ in $\mathbb{C}_0$ and $\mathbf{K}$ has infinitely many imaginary axis poles that accumulate at ∞ (pole-zero cancelation at ∞). However, using Theorem 4.2, we show, for several examples in section 6, that these feedback systems may well be Sobolev stable.

The lemma below establishes that, in a certain sense, τ-periodic functions are the images of suitable compactly supported functions under the input-output operator of $\mathbf{K}_\tau$.

LEMMA 5.3. *Let $f \in H^k_{\rm loc}(\mathbb{R}_+, U)$ be τ-periodic, where $k \in \mathbb{N}_0$. Then, there exist $w_f \in H^k_{{\rm loc},\ell}(\mathbb{R}, U)$ and a compactly supported $g \in H^k(\mathbb{R}, U)$ such that $\mathbf{K}_\tau \hat{g} = \hat{w}_f$ and $w_f = f$ on $\mathbb{R}_+$.*

For the definition of the Sobolev spaces $H^k(\mathbb{R}, U)$, $H^k_{\rm loc}(\mathbb{R}_+, U)$, and $H^k_{{\rm loc},\ell}(\mathbb{R}, U)$, see subsection 2.3.

Proof of Lemma 5.3. Fix $\lambda \in (0, \tau)$, and let $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that $\psi(t) = 0$ for all $t \le -\lambda/2$ and $\psi(t) = 1$ for all $t \ge -\lambda/4$. Setting $\phi(t) := \psi(t) f(t + \tau)$ for all $t \in [-\lambda, 0]$, we have that

$$\phi^{(m)}(-\lambda) = 0 \quad \text{and} \quad \phi^{(m)}(0) = f^{(m)}(\tau) = f^{(m)}(0) \quad \forall\, m \in \{0, 1, \ldots, k-1\}\,. \tag{5.7}$$

Define

$$g(t) := \begin{cases} 0, & t \le -\tau - \lambda, \\ \phi(t+\tau), & t \in [-\tau - \lambda, -\tau], \\ f(t+\tau), & t \in [-\tau, -\lambda], \\ f(t+\tau) - \phi(t), & t \in [-\lambda, 0], \\ 0, & t \ge 0 \end{cases} \qquad \text{and}$$

$$w_f(t) := \begin{cases} 0, & t \le -\tau - \lambda, \\ \phi(t+\tau), & t \in [-\tau - \lambda, -\tau], \\ f(t+\tau), & t \ge -\tau\,. \end{cases}$$

Clearly, g is compactly supported, w_f has support bounded to the left, and $w_f = f$ on $\mathbb{R}_+$ because f is τ-periodic. Invoking (5.7), it is routine to also verify that $g \in H^k(\mathbb{R}, U)$, $w_f \in H^k_{{\rm loc},\ell}(\mathbb{R}, U)$ and that

$$w_f(t) - w_f(t - \tau) = g(t) \quad \forall\, t \in \mathbb{R}\,. \tag{5.8}$$

Taking the (bilateral) Laplace transform of both sides of (5.8) shows that $\mathbf{K}_\tau \hat{g} = \hat{w}_f$. ☐

We comment that a simpler construction is available if either $k = 0$ or $f^{(j)}(0) = 0$ for all $j \in \{0, \ldots, k-1\}$ when $k \ge 1$. Indeed, in this case, the function $g : \mathbb{R} \to U$ defined by

$$g(t) = f(t) \text{ if } t \in [0, \tau] \quad \text{and} \quad g(t) = 0 \text{ if } t \notin [0, \tau]$$

satisfies $g \in L^2(\mathbb{R}, U)$ (if $k = 0$) and $g \in H^k(\mathbb{R}, U)$ (if $k \ge 1$), is compactly supported, and $\mathbf{K}_\tau \hat{g} = \hat{f}$, showing that Lemma 5.3 holds with w_f given by $w_f(t) = 0$ for $t < 0$ and $w_f(t) = f(t)$ for $t \ge 0$.

In the following corollary, we consider the feedback system (5.1), where we assume that $U = Y$ and the signals r and d are in

$$\mathcal{P}^k_\tau(U) := \big\{ h_1 + h_2 : h_1 \in H^k_{{\rm loc},\ell}(\mathbb{R}, U) \text{ such that } h_1 \text{ is } \tau\text{-periodic on } \mathbb{R}_+ \text{ and } h_2 \in H^k_\ell(\mathbb{R}, U) \big\} \tag{5.9}$$

for some $k \in \mathbb{N}_0$.

COROLLARY 5.4. *Consider the feedback system* (5.1) *for $\mathbf{P} \in \mathcal{H}^*_\mu(\mathcal{B}(U))$ and $\mathbf{K} = \mathbf{K}_\tau \mathbf{H}_1 + \mathbf{H}_2$ with $\mathbf{K}_\tau$ given by* (5.5) *and $\mathbf{H}_1, \mathbf{H}_2 \in \mathcal{H}^*_\mu(\mathcal{B}(U))$, where $\mu \le 0$ and $\tau > 0$. Let open $\Omega \subset \mathbb{C}_0$ and $\sigma \in \mathbb{R}$ be such that $\mathbf{r}_\sigma \mathbf{K}_\tau$ is bounded on $\mathbb{C}_0 \backslash \Omega$, and furthermore, let $k \in \mathbb{N}_0$ and $\theta \ge 0$. If $\mathbf{K}$ is a Sobolev θ-stabilizing feedback for $\mathbf{P}$,*

$\mathbf{H}_1 + \mathbf{K}_\tau^{-1}\mathbf{H}_2$ *is invertible in* $\mathcal{H}^*(\Omega, \mathcal{B}(U))$, *and* $\mathbf{r}_\sigma(\mathbf{H}_1 + \mathbf{K}_\tau^{-1}\mathbf{H}_2)^{-1} \in \mathcal{H}^\infty(\Omega, \mathcal{B}(U))$, *then* $e \in H_\ell^{k-\alpha}(\mathbb{R}, U)$ *for all* $d, r \in \mathcal{P}_\tau^k(U)$, *where* $\alpha := \max\{\theta, \sigma + \theta\}$.

Because $\mathbf{r}_\sigma \mathbf{K}_\tau$ is bounded on $\mathbb{C}_0 \backslash \Omega$, we see that the poles $p_k := (i2\pi k)/\tau$ of $\mathbf{K}_\tau$ satisfy $p_k \in \overline{\Omega}$ for every $k \in \mathbb{Z}$. The repetitive controller model originally considered in [22] is of the form $\mathbf{K}(s) = \mathrm{e}^{-\tau s}\mathbf{K}_\tau(s) + \mathbf{A}(s)$ with $\mathbf{A} \in \mathcal{H}_0^\infty(\mathcal{B}(U))$ (that is, in the notation of the above corollary, $\mathbf{H}_1(s) = \mathrm{e}^{-\tau s} I$ and $\mathbf{H}_2 = \mathbf{A}$); it is straightforward to show that the condition on $\mathbf{H}_1 + \mathbf{K}_\tau^{-1}\mathbf{H}_2 = \mathrm{e}^{-\tau\,\cdot} I + \mathbf{K}_\tau^{-1}\mathbf{A}$ is satisfied with $\sigma = 0$ and $\Omega = \{s \in \mathbb{C}_0 : \mathrm{Re}\, s < a\}$ provided that $a > 0$ is sufficiently small and $\|\mathbf{A}\|_{\mathcal{H}_0^\infty} < 1/2$.

Proof of Corollary 5.4. The idea is to apply Theorem 5.1 with $U = V = W = Y$, $\mathbf{D} = \mathbf{R} = \mathbf{K}_\tau$, $\mathbf{q}(s) = 1 - \mathrm{e}^{-\tau s}$, and $\rho = 0$. To this end, we note that $\mathbf{q}\mathbf{K}_\tau = I \in \mathcal{H}_0^\infty(\mathcal{B}(U))$ and that $\mathbf{r}_\sigma/\mathbf{q}$ is bounded on $\mathbb{C}_0 \backslash \Omega$ since $(\mathbf{r}_\sigma/\mathbf{q})I = \mathbf{r}_\sigma \mathbf{K}_\tau$. Furthermore, $\mathbf{K} = (\mathbf{H}_1 + \mathbf{q}\mathbf{H}_2)/\mathbf{q}$, and thus,

$$\frac{\mathbf{r}_\sigma}{\mathbf{q}}\mathbf{K}^{-1} = \mathbf{r}_\sigma\big(\mathbf{H}_1 + \mathbf{q}\mathbf{H}_2\big)^{-1} = \mathbf{r}_\sigma\big(\mathbf{H}_1 + \mathbf{K}_\tau^{-1}\mathbf{H}_2\big)^{-1} \in \mathcal{H}^\infty(\Omega, \mathcal{B}(U)),$$

showing that condition (b) of Theorem 5.1 holds. Because $\mathbf{K}$ is a Sobolev θ-stabilizing feedback for $\mathbf{P}$, condition (a) of Theorem 5.1 is also satisfied. Consequently, by Theorem 5.1, it is now sufficient to show that

$$\mathcal{P}_\tau^k(U) \subset \{K_\tau g + h : g, h \in H_\ell^k(\mathbb{R}, U)\}, \tag{5.10}$$

where K_τ denotes the causal translation-invariant operator with transfer function $\mathbf{K}_\tau$. To establish (5.10), let $f \in \mathcal{P}_\tau^k(U)$. Then, $f = f_1 + f_2$ with $f_1 \in H_{\mathrm{loc},\ell}^k(\mathbb{R}, U)$ such that f_1 is τ-periodic on $\mathbb{R}_+$ and $f_2 \in H_\ell^k(\mathbb{R}, U)$. By Lemma 5.3, there exists (compactly supported) $g \in H_\ell^k(\mathbb{R}, U)$ such that $K_\tau g \in H_{\mathrm{loc},\ell}^k(\mathbb{R}, U)$ and $(K_\tau g)|_{\mathbb{R}_+} = f_1|_{\mathbb{R}_+}$. Consequently, setting $h := f_1 + f_2 - K_\tau g$, we have that $h \in H_\ell^k(\mathbb{R}, U)$ and $f = K_\tau g + h$, showing that (5.10) holds. ∎

Remark 5.5. In this remark on Corollary 5.4, we assume, for simplicity, that $d = 0$; that is, we focus on the tracking problem. Corollary 5.4 guarantees that, for any given τ-periodic function $\tilde{r} \in H_{\mathrm{loc},\ell}^k(\mathbb{R}_+, U)$ on the half-line, the output y of the feedback system (5.1), driven by any reference signal $r \in H_{\mathrm{loc},\ell}^k(\mathbb{R}, U)$ such that $r = \tilde{r}$ on $\mathbb{R}_+$, satisfies $(y - \tilde{r})|_{(0,\infty)} \in H^{k-\alpha}\big((0,\infty), U\big)$; in particular, if $k \geq \alpha + 1$, then $\|y(t) - \tilde{r}(t)\| \to 0$ as $t \to \infty$ for any such r. Furthermore, in the case wherein $\tilde{r}(0) \neq 0$, $\alpha > 0$, and r is defined by $r(t) = 0$ for $t < 0$ and $r(t) = \tilde{r}(t)$ for $t \geq 0$ ($r \in \mathcal{P}_\tau^0(U)$ but $r \notin \mathcal{P}_\tau^1(U)$), then $e \in H_\ell^{-\alpha}(\mathbb{R}, U)$, and so, e may not be square integrable.

We conclude this section by discussing the relation between the results in this subsection to those in [25] and [28]. Both of these papers consider the output-regulation problem when a polynomial stability condition is imposed on the closed-loop system, referred to as P-stability. Roughly speaking, adopting an algebraic approach inspired by the factorization approach to control system synthesis [37], the paper [25] establishes, in frequency-domain terms, an internal model principle that guarantees that the Laplace transform of the error is bounded on $\mathbb{C}_\delta$ for every $\delta > 0$ and polynomially bounded on the imaginary axis. We note that P-stability imposes a growth bound equal to zero (that is, $\mathbf{F}_{\mathbf{P},\mathbf{K}}$ is bounded on $\mathbb{C}_\delta$ for every $\delta > 0$), which is not a requirement of Sobolev stability. For a comparison of Sobolev stability and P-stability, we refer to [20, Proposition 5.8]. Positive realness does not play a role in [25]. The paper [28] invokes the state-space theory of regular infinite-dimensional linear systems to analyze the robust output-regulation problem for passive systems with reference and disturbance signals being contained in various classes of trigonometric polynomials. In particular, closed-loop exponential stability is ensured under an assumption on the plant that resembles strong positive realness [28, Theorem 5.2]. In the absence of this property, only strong stability of the closed-loop system is possible in general (see [28, Theorem 5.11]), and the error is shown to satisfy a certain polynomial convergence rate [28, Theorem 5.4]. As already noted, the class of transfer functions considered in the current paper is very general and contains transfer functions that do not admit regular (or even well-posed) state-space realizations. Finally, a direct comparison between our results and those in [25] or [28] is difficult owing to the different approaches adopted; while there is some overlap, the theories developed in [25] and [28] are not suitable to derive Theorem 5.1 and/or Corollary 5.4.

6. Examples.

We illustrate the results in sections 4 and 5 with six examples. In Examples 6.2 and 6.5, the following simple lemma will be used.

LEMMA 6.1. *Consider the transfer functions* $\mathbf{J}(s) := 1/s$ *and* $\mathbf{T}_\omega(s) := \omega/(s^2+\omega^2)$, *where* $\omega > 0$, *and denote the corresponding causal translation-invariant operators by* J *and* T_ω, *respectively. Let* $k \in \mathbb{N}_0$.

(1) *If* $\zeta \in H^k(\mathbb{R})$ *has compact support in* $(-\infty, 0)$ *and* $\int_{-\infty}^0 \zeta(t)\mathrm{d}t = 1$, *then* $J\zeta \in H^k_\ell(\mathbb{R})$ *and* $(J\zeta)(t) = 1$ *for all* $t \geq 0$.

(2) *If* $\phi, \psi \in H^k(\mathbb{R})$ *have compact support in* $(-\infty, 0)$ *and satisfy*

$$\begin{aligned} &\int_{-\infty}^0 \cos(\omega t)\phi(t)\mathrm{d}t = 1, \quad \int_{-\infty}^0 \sin(\omega t)\phi(t)\mathrm{d}t = \int_{-\infty}^0 \cos(\omega t)\psi(t)\mathrm{d}t = 0, \\ &\int_{-\infty}^0 \sin(\omega t)\psi(t)\mathrm{d}t = -1, \end{aligned} \tag{6.1}$$

then $T_\omega\phi, T_\omega\psi \in H^k_\ell(\mathbb{R})$ *and* $(T_\omega\phi)(t) = \sin(\omega t)$ *and* $(T_\omega\psi)(t) = \cos(\omega t)$ *for all* $t \geq 0$.

While the proof of the above lemma is elementary, we have, for completeness, included it in the appendix.

Example 6.2. We consider as plant the following controlled and observed heat equation on the unit spatial domain $(0, 1)$:

$$w_t = w_{\xi\xi}, \quad w(0,t) = 0, \quad w_\xi(1,t) = u(t), \quad y(t) = w(1,t) + \kappa w_\xi(1,t),$$

where $\kappa \geq 0$ and u and y denote the input and output, respectively. The transfer function $\mathbf{P}$ is given by $\mathbf{P}(s) = \kappa + \tanh(\sqrt{s})/\sqrt{s}$, which belongs to $\mathcal{H}^\infty_0$ and is positive real.

If $\kappa > 0$, then $\mathbf{P}$ is strongly positive real so that $\Pi_0(\mathbf{P}, \mathbf{K}; \mathbb{C}_0) = \mathbb{C}_0$ whenever $\mathbf{K}$ is positive real. Consequently, Theorem 4.2 shows that any positive real $\mathbf{K}$ is a Sobolev 0-stabilizing feedback for $\mathbf{P}$ (in this scenario, Theorem 4.2 reduces to the passivity theorem for L^2-stability).

Let us now assume that $\kappa = 0$. Then, $\mathbf{P}$ is positive real, but not strongly positive real. However, it is straightforward to establish that $\mathbf{P}$ is Sobolev positive real with exponent $1/2$. Consequently, because $\mathbf{P}$ is also in $\mathcal{H}^\infty_0$, Corollary 4.4 ensures that every positive real $\mathbf{K}$ is a Sobolev $(1/2)$-stabilizing feedback for $\mathbf{P}$. We shall consider the positive real transfer function $\mathbf{K}$ given by

$$\mathbf{K}(s) := \frac{1}{s} + \frac{s}{s^2+\omega_1^2} + \frac{s}{s^2+\omega_2^2}, \quad \text{where } \omega_1, \omega_2 > 0, \omega_1 \neq \omega_2. \tag{6.2}$$

An application of Theorem 5.1 with $U=V=Y=\mathbb{C}$, $W=\mathbb{C}^2$, $\mathbf{D}=\mathbf{J}$, $\mathbf{R}=(\mathbf{T}_{\omega_1},\mathbf{T}_{\omega_2})$, $\rho=\sigma=0$, $\theta=1/2$, $\mathbf{q}(s)=s(s^2+\omega_1^2)(s^2+\omega_2^2)/(s+1)^4$, and

$$\Omega=\bigcup_{\omega\in\{0,\pm\omega_1,\pm\omega_2\}}\{s\in\mathbb{C}_0:|s-i\omega|<a\},\quad \text{where } a>0 \text{ is sufficiently small,}$$

shows that the error e of the feedback system (5.1) satisfies

$$e\in H_\ell^{\kappa-1/2}(\mathbb{R})\quad \forall\, d\in\{Jg:g\in H_\ell^\kappa(\mathbb{R})\}\ \text{ and } \forall\, r\in\{T_{\omega_1}g_1+T_{\omega_2}g_2:g_1,g_2\in H_\ell^\kappa(\mathbb{R})\}. \tag{6.3}$$

Finally, let $\phi_0,\phi_1,\phi_2,\psi_1,\psi_2\in H^2(\mathbb{R})$ be compactly supported in $(-\infty,0)$ and such that $\int_{-\infty}^0\phi_0(t)\mathrm{d}t=1$ and (6.1) holds for $\omega=\omega_j$, $\phi=\phi_j$, and $\psi=\psi_j$ for $j=1,2$. Let $a_0,a_1,a_2,b_1,b_2\in\mathbb{R}$, and consider $d:=a_0J\phi_0$ and $r=T_{\omega_1}(a_1\phi_1+b_1\psi_1)+T_{\omega_2}(a_2\phi_2+b_2\psi_2)$. Appealing to Lemma 6.1, we see that $d(t)=a_0$ and $r(t)=a_1\sin(\omega_1 t)+b_1\cos(\omega_1 t)+a_2\sin(\omega_2 t)+b_2\cos(\omega_2 t)$ for all $t\geq 0$. It follows from (6.3) that $e\in H_\ell^{3/2}(\mathbb{R})$. In particular, $e\in H_\ell^1(\mathbb{R})$, and thus,

$$\begin{aligned} e(t)&=y(t)-r(t)\\ &=y(t)-\big(a_1\sin(\omega_1 t)+b_1\cos(\omega_1 t)+a_2\sin(\omega_2 t)+b_2\cos(\omega_2 t)\big)\to 0 \text{ as } \ t\to\infty\end{aligned}$$

in the presence of constant disturbances. By increasing the regularity of the input functions of the signal generators, the error e becomes more regular. If, for example, $\phi_0,\phi_1,\phi_2,\psi_1,\psi_2\in H^3(\mathbb{R})$, then $e\in H_\ell^{5/2}(\mathbb{R})$, and we have that $e(t)\to 0$ and $\dot e(t)\to 0$ as $t\to\infty$.

Example 6.3. To illustrate Corollary 5.4, consider again $\mathbf{P}(s)=\kappa+\tanh(\sqrt{s})/\sqrt{s}$, $\kappa\geq 0$, the transfer function of the heat equation from Example 6.2. As was noted in Example 6.2, $\mathbf{P}\in\mathcal{H}_0^\infty$ and $\mathbf{P}$ is Sobolev positive real with exponent 1/2 (strongly positive real if $\kappa>0$). Let $\mathbf{K}_\tau$, $\tau>0$ be the repetitive controller given by (5.5), and recall that $\mathbf{K}_\tau$ is (strongly) positive real.

If $\kappa>0$, it follows from Corollary 4.4 that $\mathbf{K}_\tau$ is a Sobolev 0-stabilizing (equivalently, L^2-stabilizing) feedback for $\mathbf{P}$, and thus, invoking Corollary 5.4 (with $U=\mathbb{C}$, $\mathbf{K}=\mathbf{K}_\tau$, and $\sigma=0$), we conclude that the feedback system (5.1) with $\mathbf{K}=\mathbf{K}_\tau$ satisfies $e\in H_\ell^k(\mathbb{R})$ whenever $d,r\in\mathcal{P}_\tau^k(\mathbb{C})$, $k\in\mathbb{N}_0$, where the space $\mathcal{P}_\tau^k(\mathbb{C})$ is defined in (5.9).

Let us assume now that $\kappa=0$. Applying Corollary 4.4 once more, we obtain that $\mathbf{K}_\tau$ is a Sobolev (1/2)-stabilizing feedback for $\mathbf{P}$, and Corollary 5.4 (with $U=\mathbb{C}$, $\mathbf{K}=\mathbf{K}_\tau$ and $\sigma=0$) guarantees that $e\in H_\ell^{k-1/2}(\mathbb{R})$ whenever $d,r\in\mathcal{P}_\tau^k(\mathbb{C})$, $k\in\mathbb{N}_0$. For a numerical example, we consider the case where, in Corollary 5.4, $U=\mathbb{C}$, $\mathbf{K}=\mathbf{K}_\tau$, $\sigma=0$, $d(t)\equiv 0$, and r is given by

$$r(t)=\sum_{j=0}^\infty r_0(t-j\tau),\qquad \text{where}\qquad r_0(t):=\begin{cases} at, & t\in[0,\tau/2],\\ a(\tau-t), & t\in(\tau/2,\tau],\\ 0, & t\in\mathbb{R}\backslash[0,\tau],\end{cases} \tag{6.4}$$

and $a>0$ is an amplitude parameter. We see that $r(t)=0$ for all $t\leq 0$ and, on $\mathbb{R}_+$, the function r is a τ-periodic nonnegative triangular wave obtained by periodic extension of r_0; see Figure 6.1(a). Obviously, $r\in\mathcal{P}_\tau^1(\mathbb{C})$ (but $r\notin\mathcal{P}_\tau^k(\mathbb{C})$ for $k\geq 2$), whence $e\in H_\ell^{1/2}(\mathbb{R})$. A routine calculation shows that

$$\hat r_0(s)=\int_0^\tau \mathrm{e}^{-st}r_0(t)\mathrm{d}t=\frac{a}{s^2}(1-\mathrm{e}^{-\tau s/2})^2\quad \forall\, s\in\mathbb{C},$$

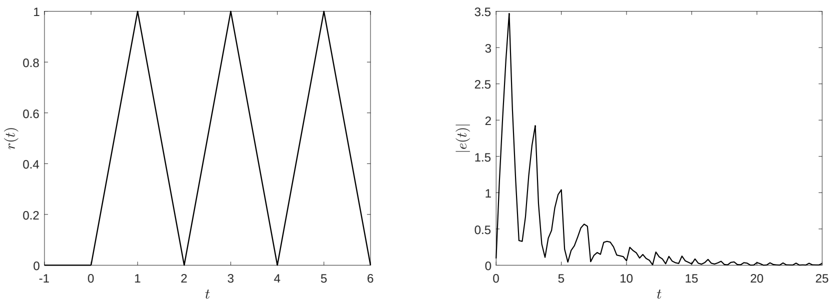

FIG. 6.1. *Graphical illustration and numerical computation for Example* 6.3 *with parameter values $a=1$ and $\tau=2$.* (a) *Graph of $r(t)$ given by* (6.4) *against t.* (b) *Graph of error $e(t)$ against t.*

where we note that $\hat{r}_0$ has a removable singularity at $s=0$. Consequently,

$$\hat{r}(s)=\int_0^\infty \mathrm{e}^{-st}r(t)\mathrm{d}t=\sum_{j=0}^{\infty}\mathrm{e}^{-j\tau s}\hat{r}_0(s)=\frac{\hat{r}_0(s)}{1-\mathrm{e}^{-\tau s}}=\mathbf{K}_\tau(s)\hat{r}_0(s)\quad \forall\, s\in\mathbb{C}_0.$$

The tracking error $e=y-r$ satisfies

$$\hat{e}(s)=\hat{y}(s)-\hat{r}(s)=-\frac{\mathbf{K}_\tau(s)\hat{r}_0(s)}{1+\mathbf{P}(s)\mathbf{K}_\tau(s)}=-\frac{a(1-\mathrm{e}^{-\tau s/2})^2}{s^2\big[1-\mathrm{e}^{-\tau s}+\tanh(\sqrt{s})/\sqrt{s}\big]}\quad \forall\, s\in\mathbb{C}_0.$$

Computing the inverse Laplace transform of $\hat{e}$ above analytically seems intractable. We numerically compute e by using a standard result from the L^2-theory of the Fourier transform, according to which

$$\frac{1}{2\pi}\int_{-\omega}^{\omega}\mathrm{e}^{iy\,\cdot}\,\hat{e}(iy)\,\mathrm{d}y=\frac{1}{2\pi}\int_{-\omega}^{\omega}\mathrm{e}^{iy\,\cdot}\,(\mathcal{F}e)(y)\,\mathrm{d}y\to e\quad\text{in } L^2(\mathbb{R})\text{ as }\omega\to\infty, \tag{6.5}$$

where $(\mathcal{F}e)(y):=\int_{-\infty}^{\infty}\mathrm{e}^{-iyt}e(t)\,\mathrm{d}t$ is the Fourier transform of e. For $\tau=2$ and $a=1$, Figure 6.1(b) shows a plot of the error $e(t)$ against t, where convergence of the error to zero over time is observed. The integral in (6.5) was computed in MATLAB using the numerical integration command `quadgk`. The optional error bound output of `quadgk` returned values of orders between 10^{-8} and 10^{-12}, varying over $t\geq 0$.

The next example involves an operator-valued transfer function.

Example 6.4. Consider the following heat equation on the square $(0,1)\times(0,1)$:

$$w_t=w_{\xi_1\xi_1}+w_{\xi_2\xi_2},\quad w(0,\xi_2,t)=w(1,\xi_2,t)=w_{\xi_2}(\xi_1,0,t)=0,$$
$$w_{\xi_2}(\xi_1,1,t)=u(\xi_1,t),\quad y(\xi_1,t)=w(\xi_1,1,t).$$

Choosing $U:=L^2(0,1)$, we have that $t\mapsto u(\,\cdot\,,t)$ and $t\mapsto y(\,\cdot\,,t)$ are U-valued functions. It is shown in [19, Example 7.14] that the transfer function $\mathbf{P}$ of the above system, given by

$$\mathbf{P}(s)v=\sum_{k=1}^{\infty}\frac{\sqrt{2}\gamma_k(v)\sin(k\pi\,\cdot)}{\sqrt{s+k^2\pi^2}\tanh(\sqrt{s+k^2\pi^2})}\quad\forall\, v\in U,\quad\text{where}$$
$$\gamma_k(v):=\sqrt{2}\int_0^1 v(\xi)\sin(k\pi\xi)\mathrm{d}\xi,$$

is positive real. Furthermore, $\mathbf{P} \in \mathcal{H}_0^\infty(\mathcal{B}(U))$ and $\lim_{|s|\to\infty,\, s\in\mathbb{C}_0} \|\mathbf{P}(s)\| = 0$. If $\mathbf{K} \in \mathcal{H}_0^\infty(\mathcal{B}(U))$ has the property that, for every bounded set $B \subset \mathbb{C}_0$, there exists $\varepsilon > 0$ such that $\operatorname{Re}\mathbf{K}(s) \geq \varepsilon I$ for all $s \in B$ (for example, if $\mathbf{K}(s) = \tilde{\mathbf{K}}(s) + 1/(s+1)$, where $\tilde{\mathbf{K}} \in \mathcal{H}_0^\infty(\mathcal{B}(U))$ is positive real), then, for given $\gamma \in (0,1)$, there exist $\varepsilon > 0$ and $r > 0$ such that

$$\{s \in \mathbb{C}_0 : |s| < r\} \subset \Pi_0(\mathbf{K}, \mathbf{P}; \mathbb{C}_0, \mu, \varepsilon) \quad \text{and} \quad \{s \in \mathbb{C}_0 : |s| \geq r\} \subset \Gamma_0(\mathbf{P}, \mathbf{K}; \mathbb{C}_0, \nu, \gamma),$$

where $\mu := \|\mathbf{K}\|_{H^\infty}$ and $\nu := \|\mathbf{P}\|_{H^\infty} + \|\mathbf{K}\|_{H^\infty}$. It follows from Theorem 4.2 that $\mathbf{K}$ is a L^2-stabilizing feedback for $\mathbf{P}$.

In the following example, the plant is a fractional derivative and hence not well posed (in the sense of [36]).

Example 6.5. We consider the feedback system (5.1) with $\mathbf{P}$ and $\mathbf{K}$ given by

$$\mathbf{P}(s) = s^\delta \quad \text{and} \quad \mathbf{K}(s) = \frac{s}{s^2 + \omega^2}, \quad \text{where } 0 < \delta < 1 \text{ and } \omega > 0.$$

We claim that $\mathbf{K}$ is a Sobolev δ-stabilizing feedback for $\mathbf{P}$. To this end, set

$$A_\eta := \{s \in \mathbb{C}_0 : |s| < \eta\} \cup \{s \in \mathbb{C}_0 : |s| > 1/\eta\}, \quad \text{where } \eta \in (0,1).$$

It is clear that $A_\eta \subset \Gamma_\delta(\mathbf{P}, \mathbf{K}; \mathbb{C}_0, 2, 1/2)$ for sufficiently small $\eta \in (0,1)$. Moreover,

$$\begin{aligned} &\operatorname{Re}\mathbf{P}(s) \geq \eta^\delta \cos(\pi\delta/2) =: \varepsilon > 0, \quad \|\mathbf{P}(s)\| \leq 1/\eta^\delta \quad \text{and} \\ &\operatorname{Re}\mathbf{K}(s) \geq 0 \quad \forall\, s \in \mathbb{C}_0 \backslash A_\eta, \end{aligned}$$

showing that $\mathbb{C}_0 \backslash A_\eta \subset \Pi_0(\mathbf{P}, \mathbf{K}; \mathbb{C}_0, 1/\eta^\delta, \varepsilon)$. Consequently,

$$\mathbb{C}_0 \subset \Pi_\delta(\mathbf{P}, \mathbf{K}; \mathbb{C}_0, 1/\eta^\delta, \varepsilon) \cup \Gamma_\delta(\mathbf{P}, \mathbf{K}; \mathbb{C}_0, 2, 1/2),$$

and Sobolev δ-stability follows from Theorem 4.2. It is easy to see that this stability result is optimal in the sense that $\mathbf{K}$ is not Sobolev θ-stabilizing for $\mathbf{P}$ for any $\theta < \delta$.

An application of Theorem 5.1 with $U = V = W = Y = \mathbb{C}$, $\mathbf{D} = \mathbf{R} = \mathbf{T}_\omega$, $\rho = \sigma = 0$, $\theta = \delta$, $\mathbf{q}(s) = (s^2 + \omega^2)/(s+1)^2$, and

$$\Omega = \{s \in \mathbb{C}_0 : |s - i\omega| < a\} \cup \{s \in \mathbb{C}_0 : |s + i\omega| < a\}, \quad \text{where } a > 0 \text{ is sufficiently small},$$

shows that, for all $d, r \in \{T_\omega g : g \in H_\ell^\kappa(\mathbb{R})\} + H_\ell^\kappa(\mathbb{R})$, the error e of the feedback system (5.1) satisfies $e \in H_\ell^{\kappa-\delta}(\mathbb{R})$. Now, let $d, r \in H^2_{\mathrm{loc},\ell}(\mathbb{R})$ be such that

$$d(t) = a_d \sin(\omega t) + b_d \cos(\omega t) \quad \text{and} \quad r(t) = a_r \sin(\omega t) + b_r \cos(\omega t) \quad \forall\, t \geq 0$$

for given constants $a_d, b_d, a_r, b_r \in \mathbb{R}$. It follows from Lemma 6.1 that there exist $g_d, g_r \in H_\ell^2(\mathbb{R})$ with support in $(-\infty, 0)$ such that the functions $d - T_\omega g_d$ and $r - T_\omega g_r$ are in $H_\ell^2(\mathbb{R})$. Consequently, $e \in H_\ell^{2-\delta}(\mathbb{R})$. In particular, because $2 - \delta > 1$, it follows that

$$y(t) - \big(a_r \sin(\omega t) + b_r \cos(\omega t)\big) = e(t) \to 0 \quad \text{as } t \to \infty$$

in the presence of the persistent oscillating disturbance d.

In our final two examples, we study stability problems involving a controlled and observed wave equation.

Example 6.6. Consider the following wave equation on the spatial domain $(0,1)$

$$w_{tt} = w_{\xi\xi}, \quad w(0,t) = 0, \quad w_t(1,t) = u(t), \quad y(t) = w_\xi(1,t). \tag{6.6}$$

The transfer function $\mathbf{Q}$ is given by

$$\mathbf{Q}(s) = \frac{1}{\tanh(s)}, \tag{6.7}$$

and we note that $\mathbf{Q}$ is positive real. Let $\mathbf{P}(s) := \kappa + \tanh(\sqrt{s})/\sqrt{s}$, $\kappa \geq 0$; that is, $\mathbf{P}(s)$ is the transfer function of the heat equation of Example 6.2. As pointed out in Example 6.2, the function $\mathbf{P}$ is Sobolev positive real with exponent $1/2$ and in $\mathcal{H}_0^\infty$, and thus, an application of Corollary 4.4 shows that $\mathbf{Q}$ is a Sobolev $(1/2)$-stabilizing feedback for $\mathbf{P}$ when $\kappa = 0$. Furthermore, if $\kappa > 0$, then $\mathbf{Q}$ is a Sobolev 0-stabilizing feedback.

Next, we consider repetitive control of the wave equation from Example 6.6.

Example 6.7. We consider again the wave equation (6.6) and its positive real transfer function $\mathbf{Q}$ given by (6.7). For $\tau > 0$, consider the repetitive controller $\mathbf{K}_\tau$ given by (5.5). The objective is to investigate the stability properties of the feedback connection of $\mathbf{Q}$ and $\mathbf{K}_\tau$. As we shall see, this will crucially depend on the value of τ. It is more convenient to apply our results to $1/\mathbf{Q}$ and $1/\mathbf{K}_\tau$ and invoke part (i) of statement (2) of Lemma 3.6 to infer the stability properties of the original feedback connection. Accordingly, we define

$$\mathbf{G}(s) := \frac{1}{\mathbf{Q}(s)} = \tanh(s) = \frac{\sinh(s)}{\cosh(s)}, \qquad \mathbf{H}_\tau(s) := \frac{1}{\mathbf{K}_\tau(s)} = 1 - \mathrm{e}^{-\tau s}.$$

The poles p_k of $\mathbf{G}$ and the zeros z_k of $\mathbf{H}_\tau$ are given by

$$p_k := i\frac{\pi}{2}(1+2k) \quad \text{and} \quad z_k = z_k(\tau) := i\frac{2k\pi}{\tau} \quad \forall\, k \in \mathbb{Z},$$

and we set $P := \{p_k : k \in \mathbb{Z}\}$ and $Z_\tau := \{z_k(\tau) : k \in \mathbb{Z}\}$. To outline what follows, the feedback connection of $\mathbf{G}$ and $\mathbf{H}_\tau$ exhibits a range of stability properties (or lack thereof), depending crucially on whether $\operatorname{dist}(Z_\tau, P) = 0$ or $\operatorname{dist}(Z_\tau, P) > 0$ and so ultimately on the value of τ. More precisely, the following three cases are possible: $Z_\tau \cap P \neq \emptyset$, $\operatorname{dist}(Z_\tau, P) > 0$, or finally, $\operatorname{dist}(Z_\tau, P) = 0$ while $Z_\tau \cap P = \emptyset$.

CASE 1: $Z_\tau \cap P \neq \emptyset$.

There exist $k, l \in \mathbb{Z}$ such that $p_k = z_l(\tau)$, trivially implying that τ is rational. Because $\mathbf{G}$ and $\mathbf{H}_\tau$ have a pole-zero cancellation at p_k, it follows that $\mathbf{G}/(1+\mathbf{H}_\tau\mathbf{G})$ has a pole at p_k. This, in turn, shows that there does not exist $\theta \geq 0$ such that $(1+s)^{-\theta}\mathbf{F}_{\mathbf{G},\mathbf{H}_\tau}$ is bounded on $\mathbb{C}_0$. We conclude that $\mathbf{H}_\tau$ is not a θ-stabilizing feedback for $\mathbf{G}$ no matter what the value of θ is.

CASE 2: $\operatorname{dist}(Z_\tau, P) > 0$.

In this case, there exists $\rho > 0$ such that the disc of radius ρ centered at $z_k(\tau)$ intersected with $\mathbb{C}_0$ is a subset of $\Gamma_0(\mathbf{G}, \mathbf{H}_\tau; \mathbb{C}_0)$ for every $k \in \mathbb{Z}$. The complement of the union of these semidiscs with respect to $\mathbb{C}_0$ is included in $\Pi_0(\mathbf{H}_\tau, \mathbf{G}; \mathbb{C}_0)$. Hence, $\mathbb{C}_0 = \Pi_0(\mathbf{H}_\tau, \mathbf{G}; \mathbb{C}_0) \cup \Gamma_0(\mathbf{G}, \mathbf{H}_\tau; \mathbb{C}_0)$, and it follows from Theorem 4.2 that $\mathbf{H}_\tau$ is a Sobolev 0-stabilizing feedback for $\mathbf{G}$; that is, the feedback system is L^2-stable.

CASE 3: $Z_\tau \cap P = \emptyset$ and $\operatorname{dist}(Z_\tau, P) = 0$.

The above conditions can be expressed in the following equivalent form

$$\operatorname{dist}(z_k(\tau), P) > 0 \ \forall\, k \in \mathbb{Z} \quad \text{and} \quad \liminf_{|k|\to\infty} \operatorname{dist}(z_k(\tau), P) = 0. \tag{6.8}$$

It is straightforward to show that, in this case, τ must be irrational. For given $\theta \geq 0$, there are two possible scenarios:

$$\liminf_{|k|\to\infty} |k|^\theta \operatorname{dist}(z_k(\tau), P) > 0 \quad \text{(S1)} \qquad \text{and} \qquad \liminf_{|k|\to\infty} |k|^\theta \operatorname{dist}(z_k(\tau), P) = 0 \quad \text{(S2)}.$$

It is shown in the appendix that, if (S1) holds, then $\mathbf{H}_\tau$ is a Sobolev (2θ)-stabilizing feedback for $\mathbf{G}$, and if (S2) is satisfied, then $\mathbf{H}_\tau$ is not a Sobolev θ-stabilizing feedback.

DISCUSSION. We provide some comments and observations relating to the above three cases.

• As already mentioned, Case 3 requires τ to be irrational. We claim that, conversely, if τ is irrational, then (6.8) holds. To see this, assume that τ is irrational. Then, trivially, $\operatorname{dist}(z_k(\tau), P) > 0$ for all $k \in \mathbb{Z}$. Furthermore, by [35], there exist infinitely many $k \in \mathbb{Z}$ and $l \in \mathbb{N}_0$ such that

$$\left|\frac{\tau}{2} - \frac{2k}{1+2l}\right| < \frac{1}{(1+2l)^2}.$$

Together with

$$|z_k(\tau) - p_l| = \frac{\pi}{\tau}|1+2l|\left|\frac{\tau}{2} - \frac{2k}{1+2l}\right| \quad \forall\, k,l \in \mathbb{Z}, \tag{6.9}$$

this shows that $\liminf_{|k|\to\infty} \operatorname{dist}(z_k(\tau), P) = 0$. Consequently, we have the following equivalences:

$$\tau \in \mathbb{Q} \Leftrightarrow \big(Z_\tau \cap P \neq \emptyset \;\vee\; \operatorname{dist}(Z_\tau, P) > 0\big) \quad \text{and}$$
$$\tau \in \mathbb{R}\backslash\mathbb{Q} \Leftrightarrow \big(Z_\tau \cap P = \emptyset \;\wedge\; \operatorname{dist}(Z_\tau, P) = 0\big).$$

• The conditions $Z_\tau \cap P \neq \emptyset$ and $\operatorname{dist}(Z_\tau, P) > 0$ are equivalent to

$$\tau \in \mathcal{I} := \left\{\frac{4k}{1+2l} : k,l \in \mathbb{N}\right\} \subset \mathbb{Q} \cap (0,\infty) \quad \text{and} \quad \tau \in \mathcal{I}^{\mathrm{c}} := \big(\mathbb{Q} \cap (0,\infty)\big)\backslash\mathcal{I},$$

respectively. Because $\mathcal{I}$ and $\mathcal{I}^{\mathrm{c}}$ are dense in $(0,\infty)$, we see that, in every neighborhood of a given $\tau > 0$, there exist $\tau_1 \in \mathcal{I}$ and $\tau_2 \in \mathcal{I}^{\mathrm{c}}$ such that $\mathbf{H}_{\tau_1}$ is not Sobolev θ-stabilizing for any $\theta \geq 0$ while $\mathbf{H}_{\tau_2}$ is a Sobolev 0-stabilizing feedback for $\mathbf{G}$ (that is, the closed-loop system is L^2-stable) and hence Sobolev θ-stabilizing for every $\theta \geq 0$. We conclude that the stability properties of the feedback connection of $\mathbf{G}$ and $\mathbf{H}_\tau$ are extremely sensitive to variations in τ.

• To connect the scenarios (S1) and (S2) in Case 3 more directly to τ, we recall that the *irrationality exponent* $\mu(\omega)$ of a real number ω (also known as the *irrationality measure*) is the supremum of all $\nu > 0$ such that the inequality

$$0 < \left|\omega - \frac{m}{q}\right| < \frac{1}{q^\nu}$$

has infinitely many solutions in integers $m \in \mathbb{Z}$ and $q \in \mathbb{N}$; see, for example, [8, Appendix E]. Equivalently, $\mu(\omega)$ is the infimum of all $\nu > 0$ for which there exists $q_\nu \in \mathbb{N}$ such that

$$\left|\omega - \frac{m}{q}\right| \geq \frac{1}{q^\nu}$$

for all integers m and $q \geq q_\nu$. It is well known that $\mu(\omega) = 1$ for rational ω, $\mu(\omega) \geq 2$ for irrational ω, and $\mu(\omega) = 2$ for almost every $\omega \in \mathbb{R}$ (in the sense of Lebesgue measure); see [8, Theorems E.1 and E.2]. Furthermore, Roth's theorem, a deep result from Diophantine approximation theory, guarantees that $\mu(\omega) = 2$ for every irrational algebraic number ω (see [8, Theorem E.7] for a statement of Roth's theorem and

[34, Chapter V] for a detailed treatment). There exist $\omega \in \mathbb{R}$, the so-called Liouville numbers, such that $\mu(\omega) = \infty$. The even/odd irrationality exponent $\mu_{\rm eo}(\omega)$ of a real number ω is defined in the same way but restricting m and q to be even and odd, respectively. Clearly, $\mu_{\rm eo}(\omega) \leq \mu(\omega)$ for all $\omega \in \mathbb{R}$. If $\mu(\omega) = 2$, then $\mu_{\rm eo}(\omega) = 2$, as follows from [35]. In particular, $\mu_{\rm eo}(\omega) = 2$ for almost every $\omega \in \mathbb{R}$.

We claim that

$$\big(1+\theta > \mu_{\rm eo}(\tau/2)\big) \Rightarrow \text{(S1)} \qquad \text{and} \qquad \big(1+\theta < \mu_{\rm eo}(\tau/2)\big) \Rightarrow \text{(S2)}. \tag{6.10}$$

To show this, choose, for each $k \in \mathbb{Z}$, a number $l(k) \in \mathbb{Z}$ such that

$$|z_k(\tau) - p_{l(k)}| = \operatorname{dist}(z_k(\tau), P) > 0.$$

There may be two choices for $l(k)$, but it is irrelevant which one we make. It is clear that

$$|z_k(\tau) - p_{l(k)}| \leq \pi/2 \quad \forall\, k \in \mathbb{Z}.$$

Combining this with (6.9), we conclude that

$$0 < \inf_{k\in\mathbb{Z},\, k\neq 0} \frac{|k|}{|1+2l(k)|} \leq \sup_{k\in\mathbb{Z},\, k\neq 0} \frac{|k|}{|1+2l(k)|} < \infty\,.$$

Consequently, invoking (6.9) once more, there exists a constant $c > 0$ such that

$$|k|^\theta |z_k(\tau) - p_{l(k)}| \geq c|1+2l(k)|^{1+\theta} \left|\frac{\tau}{2} - \frac{2k}{1+2l(k)}\right| \quad \forall\, k \in \mathbb{Z},\ k \neq 0. \tag{6.11}$$

To establish the first implication in (6.10), assume that $1+\theta > \mu_{\rm eo}(\tau/2)$. Then, for all sufficiently large $|k|$,

$$\left|\frac{\tau}{2} - \frac{2k}{1+2l(k)}\right| \geq \frac{1}{(1+2l(k))^{1+\theta}}.$$

It follows from (6.11) that

$$\liminf_{|k|\to\infty} |k|^\theta\, |z_k(\tau) - p_{l(k)}| > 0,$$

showing that (S1) holds.

To prove the second implication in (6.10), assume that $1+\theta < \mu_{\rm eo}(\tau/2)$. Letting ν be such that $1+\theta < \nu < \mu_{\rm eo}(\tau/2)$, there exist $k_j, l_j \in \mathbb{N}$, $j \in \mathbb{N}$ such that $l_j \to \infty$ and

$$\left|\frac{\tau}{2} - \frac{2k_j}{1+2l_j}\right| \leq \frac{1}{|1+2l_j|^\nu} \quad \forall\, j \in \mathbb{N}. \tag{6.12}$$

It follows that there exists $c > 0$ such that $|k_j| \leq c|1+2l_j|$ for all $j \in \mathbb{N}$. Hence, by (6.9) and (6.12),

$$|k_j|^\theta\, |z_{k_j}(\tau) - p_{l(k_j)}| \leq |k_j|^\theta\, |z_{k_j}(\tau) - p_{l_j}| = \frac{\pi}{\tau}\frac{|k_j|^\theta}{|1+2l_j|^{\nu-1}} \leq \frac{c^\theta \pi}{\tau}|1+2l_j|^{1+\theta-\nu} \to 0$$
as $j \to \infty$,

showing that (S2) holds.

• We close our discussion with the consideration of two specific irrational values for τ, namely, (i) $\tau = \sqrt{2}$ (or, more generally, $\tau =$ arbitrary irrational algebraic number) and (ii) $\tau = 4\sum_{n=1}^\infty 3^{-n!}$.

(i) When τ is an irrational algebraic number, then so is $\tau/2$, and, by the above commentary, $\mu(\tau/2) = 2 = \mu_{\rm eo}(\tau/2)$. Consequently, $\mathbf{H}_\tau$ is a Sobolev θ-stabilizing feedback for $\mathbf{G}$ whenever $\theta > 2$.

(ii) Let $\tau = 4\sum_{n=1}^{\infty} 3^{-n!}$. Defining even and odd numbers $m_j := 2\sum_{n=1}^{j} 3^{j!-n!}$ and $q_j := 3^{j!}$, respectively, we have that

$$\left|\frac{\tau}{2} - \frac{m_j}{q_j}\right| = 2\sum_{n=j+1}^{\infty}\frac{1}{3^{n!}} \leq \frac{2}{3^{(j+1)!}}\sum_{n=0}^{\infty}\frac{1}{3^n} = \frac{3}{q_j^{j+1}} \quad \forall\, j\in\mathbb{N},$$

implying that $\mu_{\rm eo}(\tau/2) = \infty$. We conclude that there does not exist any $\theta \geq 0$ such that $\mathbf{H}_\tau$ is a Sobolev θ-stabilizing feedback for $\mathbf{G}$.

SUMMARY. The stability properties of the feedback connection of $\mathbf{G}$ and $\mathbf{H}_\tau$ are extremely sensitive to variations in τ, the sensitivity being caused by both the plant and the controller having infinitely many poles/zeros on the imaginary axis (compare this to the feedback system considered in Example 6.3, which is Sobolev stable for all $\tau > 0$). If $\operatorname{dist}(Z_\tau, P) > 0$, then the feedback $\mathbf{H}_\tau$ is Sobolev 0-stabilizing, and if $\operatorname{dist}(Z_\tau, P) = 0$ without $\mathbf{G}$ and $\mathbf{H}_\tau$ having any pole-zero cancelations, then $\mathbf{H}_\tau$ is Sobolev θ-stabilizing provided that $\theta > 2(\mu_{\rm eo}(\tau/2) - 1)$. In particular, as mentioned above, $\mu_{\rm eo}(\tau/2) = 2$ for almost every $\tau \in (0,\infty)$, and thus, the feedback $\mathbf{H}_\tau$ is Sobolev θ-stabilizing for $\mathbf{G}$ for almost every $\tau \in (0,\infty)$ whenever $\theta > 2$. While the example is purely mathematical, without much relevance in a control engineering context, it demonstrates that Theorem 4.2 can be successfully applied in a situation that does not lack subtlety.

7. Conclusions. The concept of Sobolev stabilizing feedback compensators has been introduced and studied, based on the Sobolev input-output stability concept from [20]. A mixed passivity/small-gain condition has been presented and shown to be sufficient for the feedback connection of two time-invariant linear (possibly infinite-dimensional) systems to be Sobolev input-output stable. The result contains the well-known passivity and small-gain theorems for L^2-stability as special cases. We have considered scenarios in which it is impossible to achieve L^2-stability (for example, if there is a "pole-zero cancelation" at ∞ due to an infinite number of poles in the controller and vanishing gain of the plant at high frequencies), but the mixed passivity/small-gain theorem may be applied to establish Sobolev stability. It has been demonstrated how Sobolev stabilizing feedback compensators can be used in the context of a general version of the servo problem. We have shown that, if a controller $\mathbf{K}$ is a Sobolev θ-stabilizing controller, then the tracking error is in $H_\ell^\beta(\mathbb{R}, U)$ provided that $\mathbf{K}$ satisfies a condition that is reminiscent of the internal model principle, where β depends on θ, the regularity of the reference and disturbance signals r and d, and properties of the stable parts of the signal generators. In particular, smaller θ and higher regularity of r and d give a larger β. The servo result has been applied to a repetitive control problem that features a controller with infinitely many poles on the imaginary axis.

Appendix A. In this appendix, we give a proof of Lemma 6.1 and provide details for Example 6.7.

A.1. Proof of Lemma 6.1. The proof of statement (1) is trivial. To prove statement (2), we note that the existence of $\phi, \psi \in H^k(\mathbb{R},\mathbb{C})$ having compact support in $(-\infty, 0)$ and satisfying (6.1) follows from an inspection of the graphs of $t \mapsto \sin(\omega t)$ and $t \mapsto \cos(\omega t)$ (a formal proof is a routine exercise, which we leave to the interested reader). For $g \in H_\ell^k(\mathbb{R},\mathbb{C})$, we have that

$$
\begin{aligned}
(T_\omega g)(t) &= \int_{-\infty}^{t} \sin\big(\omega(t-\tau)\big)g(\tau)\mathrm{d}\tau \\
&= \sin(\omega t)\int_{-\infty}^{t}\cos(\omega\tau)g(\tau)\mathrm{d}\tau - \cos(\omega t)\int_{-\infty}^{t}\sin(\omega\tau)g(\tau)\mathrm{d}\tau \quad \forall\, t\in\mathbb{R}.
\end{aligned}
$$

Let $\phi,\psi\in H^k(\mathbb{R},\mathbb{C})$ have compact support in $(-\infty,0)$ and such that (6.1) is satisfied. The claim now follows by invoking the above identity with $g=\phi$ and $g=\psi$.

A.2. Details for Example 6.7. Here, we prove the following claim made in Case 3.

CLAIM A.1. *Assume that* $Z_\tau\cap P=\emptyset$ *and* $\operatorname{dist}(Z_\tau,P)=0$.
(a) *If* (S1) *holds, then* $\mathbf{H}_\tau$ *is a Sobolev* (2θ)*-stabilizing feedback for* $\mathbf{G}$.
(b) *If* (S2) *is satisfied, then* $\mathbf{H}_\tau$ *is not a Sobolev* θ*-stabilizing feedback.*

Proof. For ease of notation, we write $z_k:=z_k(\tau)$. For $k\in\mathbb{Z}$, let $l(k)\in\mathbb{Z}$ be such that $|z_k-p_{l(k)}|=\operatorname{dist}(z_k,P)$. Defining

$$K:=\{k\in\mathbb{Z}: |z_k-p_{l(k)}|\le \pi/4\}\subset\mathbb{Z} \quad\text{and}\quad K^{\mathrm{c}}:=\mathbb{Z}\backslash K,$$

it follows from (6.8) that K is infinite.

(a) Assume that (S1) holds. Then, there exists $\eta>0$ such that

$$|k|^\theta \operatorname{dist}(z_k,P)\ge\eta \quad \forall\, k\in\mathbb{Z},\ k\neq 0.$$

The set

$$B:=\bigcup_{k\in K^{\mathrm{c}}}\big(z_k+\{s\in\overline{\mathbb{C}}_0: |\operatorname{Im}s|\le\pi/8\}\big)$$

has the property that $\operatorname{dist}(B,P)\ge\pi/8>0$ and the function $\mathbf{G}$ is bounded on B. Consequently, there exists $\rho\in(0,\pi/8)$ such that

$$|\mathbf{H}_\tau(s)\mathbf{G}(s)|\le 1/2 \quad \forall\, s\in\bigcup_{k\in K^{\mathrm{c}}}\big(z_k+\{s\in\overline{\mathbb{C}}_0: |\operatorname{Im}s|,\operatorname{Re}s\le\rho\}\big). \tag{A.1}$$

Set

$$\kappa:=\min\left\{\rho,\frac{\eta}{2},\frac{1}{2\tau\sqrt{2(1+4/\eta^2)}}\right\}<\frac{\pi}{8}, \tag{A.2}$$

and define, for all $k\in\mathbb{Z}$, rectangles R_k in $\overline{\mathbb{C}}_0$ by

$$R_k:=z_k+\begin{cases}\{s\in\overline{\mathbb{C}}_0: |\operatorname{Im}s|,\operatorname{Re}s\le\kappa(1+|k|)^{-\theta}\}, & k\in K,\\ \{s\in\overline{\mathbb{C}}_0: |\operatorname{Im}s|,\operatorname{Re}s\le\kappa\}, & k\in K^{\mathrm{c}}.\end{cases}$$

Setting $R:=\bigcup_{k\in\mathbb{Z}}R_k$, we claim that

$$\lambda:=\sup_{s\in R}\frac{|\mathbf{G}(s)|}{|1+s|^\theta}<\infty \tag{A.3}$$

and

$$|\mathbf{H}_\tau(s)\mathbf{G}(s)|\le 1/2 \quad \forall\, s\in R, \tag{A.4}$$

which, when combined, yield that

$$R \subseteq \Gamma_\theta(\mathbf{G}, \mathbf{H}_\tau; \mathbb{C}_0, \nu, \gamma) \quad \forall\, k \in \mathbb{Z}, \tag{A.5}$$

where $\nu := \|\mathbf{H}_\tau\|_{H^\infty} + \lambda = 2 + \lambda$ and $\gamma := 1/2$. We additionally claim that

$$\inf_{s \in \mathbb{C}_0 \setminus R} |1+s|^{2\theta} \operatorname{Re} \mathbf{H}_\tau(s) > 0 \tag{A.6}$$

so that there exists $\varepsilon > 0$ such that

$$\mathbb{C}_0 \setminus R \subseteq \Pi_{2\theta}(\mathbf{H}_\tau, \mathbf{G}; \mathbb{C}_0, 2, \varepsilon). \tag{A.7}$$

The conjunction of (A.5) and (A.7) entails condition (4.1) with $\delta = 0$ and θ there replaced by 2θ. An application of part (iii) of statement (1) of Theorem 4.2 yields that $\mathbf{H}$ is a Sobolev (2θ)-stabilizing feedback for $\mathbf{G}$.

It remains to prove (A.3), (A.4), and (A.6). To this end, it is convenient to define the horizontal strip

$$H_k := z_k + \{ s \in \overline{\mathbb{C}}_0 : |\operatorname{Im} s| \le (1 + 1/\tau)\pi/2 \} \quad \forall\, k \in \mathbb{Z}.$$

It is clear that there exists $c > 0$ such that

$$1 + |k| \le c|1+s| \quad \forall\, s \in H_k \ \text{ and } \forall\, k \in \mathbb{Z}. \tag{A.8}$$

As $R_k \subset H_k$, the above bound applies to all $s \in R_k$ for every $k \in \mathbb{Z}$.

Proof of (A.3). As $R_k \subset B$ for all $k \in K^c$, we have that

$$\sup_{k \in K^c} \Big(\sup_{s \in R_k} |\mathbf{G}(s)| \Big) < \infty\,.$$

Thus, it suffices to show that

$$\sup_{k \in K} \left(\sup_{s \in R_k} \frac{|\mathbf{G}(s)|}{|1+s|^\theta} \right) < \infty\,. \tag{A.9}$$

Therefore, let $k \in K$. Invoking the addition formula for the hyperbolic tangent, we obtain, for $\omega \in \mathbb{R}$ and $x \ge 0$, that

$$|\mathbf{G}(p_k + i\omega + x)|^2 = |\tanh(p_k + i\omega + x)|^2 = \left| \frac{\tanh(x) + i \tan(\operatorname{Im} p_k + \omega)}{1 + i \tanh(x) \tan(\operatorname{Im} p_k + \omega)} \right|^2.$$

Bounding the denominator from below by 1 gives

$$|\mathbf{G}(p_k + i\omega + x)|^2 \le \tanh^2(x) + \tan^2(\operatorname{Im} p_k + \omega).$$

Using that $|\tanh(x)| \le 1$ and $\tan^2(\pi/2 + \omega) \le 1/\omega^2$ for $\omega \in (-\pi/2, \pi/2)\setminus\{0\}$ (and periodicity) gives that

$$|\mathbf{G}(p_k + i\omega + x)|^2 \le 1 + 1/\omega^2, \qquad 0 < |\omega| < \pi/2\,. \tag{A.10}$$

Consider

$$\omega_y := \operatorname{Im} z_k - \operatorname{Im} p_{l(k)} + y, \qquad |y| \le \kappa(1 + |k|)^{-\theta},$$

and note that

$$|\omega_y| \geq |z_k - p_{l(k)}| - |y| \geq \frac{\eta - \kappa}{(1+|k|)^\theta} \geq \frac{\eta}{2(1+|k|)^\theta}, \qquad |y| \leq \frac{\kappa}{(1+|k|)^\theta}.$$

Furthermore, because $k \in K$,

$$|\omega_y| \leq |z_k - p_{l(k)}| + |y| \leq \frac{\pi}{4} + \kappa \leq \frac{3}{\pi/8}, \qquad |y| \leq \frac{\kappa}{(1+|k|)^\theta}.$$

Therefore, invoking (A.10), we obtain that

$$|\mathbf{G}(z_k + iy + x)|^2 = |\mathbf{G}(p_{l(k)} + i\omega_y + x)|^2 \leq 1 + \frac{4(1+|k|)^{2\theta}}{\eta^2}, \qquad |y| \leq \frac{\kappa}{(1+|k|)^\theta},$$

and hence,

$$|\mathbf{G}(s)|^2 \leq 1 + \frac{4(1+|k|)^{2\theta}}{\eta^2} \quad \forall\, s \in R_k \text{ and } \forall\, k \in K. \tag{A.11}$$

Invoking (A.8) yields

$$|\mathbf{G}(s)|^2 \leq 1 + \frac{4c^{2\theta}}{\eta^2}|1+s|^{2\theta} \quad \forall\, s \in R_k \text{ and } \forall\, k \in K,$$

showing that (A.9) holds.

Proof of (A.4). By (A.1), $|\mathbf{H}_\tau(s)\mathbf{G}(s)| \leq 1/2$ for all $s \in R_k$ whenever $k \in K^{\mathrm{c}}$. Therefore, it suffices to show that

$$|\mathbf{H}_\tau(s)\mathbf{G}(s)| \leq 1/2 \quad \forall\, s \in R_k \text{ and } \forall\, k \in K. \tag{A.12}$$

For $x \geq 0$ and $y \in \mathbb{R}$, we have that

$$|\mathbf{H}_\tau(z_k + iy + x)|^2 \geq \big(1 - \mathrm{e}^{-\tau x}\big)^2 + 2\mathrm{e}^{-\tau x}\big(1 - \cos(\tau y)\big).$$

Using that $1 - \mathrm{e}^{-\xi} \leq \xi$ for $\xi \geq 0$ and $1 - \cos(\xi) \leq \xi^2/2$ for $\xi \in \mathbb{R}$, we obtain that

$$|\mathbf{H}_\tau(z_k + iy + x)|^2 \leq \tau^2(x^2 + y^2) \quad \forall\, x \geq 0 \ \text{ and } \forall\, y \in \mathbb{R},$$

and thus,

$$|\mathbf{H}_\tau(s)|^2 \leq \frac{2\tau^2\kappa^2}{(1+|k|)^{2\theta}} \quad \forall\, s \in R_k \text{ and } \forall\, k \in K. \tag{A.13}$$

It now follows from (A.2), (A.11), and (A.13) that (A.12) holds.

Proof of (A.6). Let $s \in \mathbb{C}_0 \backslash R$. There exists $k \in \mathbb{Z}$ such that $s = z_k(\tau) + iy + x$, where $x = \operatorname{Re} s$ and $y \in \mathbb{R}$ is such that $|y| \leq \pi/\tau$. We note that

$$\operatorname{Re} \mathbf{H}_\tau(z_k(\tau) + iy + x) = 1 - \mathrm{e}^{-\tau x}\cos(\tau y)\,. \tag{A.14}$$

Then, either (1) $|y| \geq \pi/(2\tau)$ or (2) $|y| \leq \pi/(2\tau)$.

(1) If $|y| \geq \pi/(2\tau)$, then, by (A.14),

$$\operatorname{Re} \mathbf{H}_\tau(s) = \operatorname{Re} \mathbf{H}_\tau(z_k(\tau) + iy + x) \geq 1.$$

(2) If $|y| \leq \pi/(2\tau)$, then we distinguish the subcases (i) $x > \kappa(1+|k|)^{-\theta}$ and (ii) $x \leq \kappa(1+|k|)^{-\theta}$.

(i) If $x > \kappa(1+|k|)^{-\theta}$, then it follows from (A.15) that

$$\text{(A.15)} \qquad \operatorname{Re}\mathbf{H}_\tau(s) = \operatorname{Re}\mathbf{H}_\tau(z_k(\tau)+iy+x) \geq 1 - \mathrm{e}^{-\tau\kappa(1+|k|)^{-\theta}}.$$

For all $0 \leq \xi \leq \xi_0$, $\xi_0 > 0$, we have that $1-\mathrm{e}^{-\xi} \geq \mathrm{e}^{-\xi_0}\xi$, as follows from an application of the mean-value theorem. Using this estimate on the right-hand side of (A.15), we obtain that

$$\operatorname{Re}\mathbf{H}_\tau(s) \geq \tau\kappa\mathrm{e}^{-\tau\kappa}(1+|k|)^{-\theta} \geq \tau\kappa c^{-\theta}\mathrm{e}^{-\tau\kappa}|1+s|^{-\theta},$$

where we have used (A.8).

(ii) If $x \leq \kappa(1+|k|)^{-\theta}$, then, since $s \notin R_k$, it follows that $|y| > \kappa(1+|k|)^{-\theta}$. Because $\cos(\xi) \leq 1-(2/\pi^2)/\xi^2$ for $\xi \in [-\pi,\pi]$, it follows from (A.14) that

$$\operatorname{Re}\mathbf{H}_\tau(s) = \operatorname{Re}\mathbf{H}_\tau(z_k(\tau)+iy+x) \geq \frac{2\tau^2}{\pi^2}y^2 \geq \frac{2\tau^2\kappa^2}{\pi^2}(1+|k|)^{-2\theta} \geq \frac{2\tau^2\kappa^2}{c^{2\theta}\pi^2}|1+s|^{-2\theta},$$

where, once again, we have used (A.8).

The above analysis shows that $\inf_{s\in\mathbb{C}_0\setminus R}|1+s|^{2\theta}\operatorname{Re}\mathbf{H}_\tau(s) > 0$, establishing (A.6).

(b) Assume that (S2) is satisfied. Then, there exist $k_j, l_j \in \mathbb{Z}$, $j \in \mathbb{N}$ such that $|k_j| \to \infty$ and $|l_j| \to \infty$ as $j \to \infty$ and

$$|k_j|^\theta|z_{k_j} - p_{l_j}| \to 0 \quad \text{as } j \to \infty.$$

Since $\tan^2(\pi/2+\xi) \geq 1/(4\xi^2)$ for all $\xi \in (-1,1)$, $\xi \neq 0$, and $\mathbf{G}(iy) = \tanh(iy) = i\tan(y)$ for all $y \in \mathbb{R}$, it follows that

$$\begin{aligned}|\mathbf{G}(z_{k_j}|^2 &= \tan^2(\operatorname{Im} z_{k_j}) = \tan^2(\pi/2 + \operatorname{Im} z_{k_j} - \operatorname{Im} p_{l_j}) \\ &\geq \frac{1}{4|z_{k_j} - p_{l_j}|^2} \quad \text{for all sufficiently large } j.\end{aligned}$$

Consequently,

$$\begin{aligned}|z_{k_j}|^{-\theta}\left|\frac{\mathbf{G}(z_{k_j})}{1+\mathbf{H}_\tau(z_{k_j})\mathbf{G}(z_{k_j})}\right| &= |z_{k_j}|^{-\theta}|\mathbf{G}(z_{k_j}| \\ &\geq \frac{\tau^\theta}{2^{1+\theta}\pi^\theta|k_j|^\theta|z_{k_j}-p_{l_j}|} \to \infty \quad \text{as } j \to \infty,\end{aligned}$$

implying that $\mathbf{H}_\tau$ is not a Sobolev θ-stabilizing feedback for $\mathbf{G}$. ∎

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