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ORIGINAL ARTICLE

The circle criterion for a class of sector-bounded dynamic nonlinearities

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Abstract
We present a circle criterion which is necessary and sufficient for absolute stability with respect to a natural class of sector-bounded nonlinear causal operators. This generalized circle criterion contains the classical result as a special case. Furthermore, we develop a version of the generalized criterion which guarantees input-to-state stability.

Keywords Absolute stability · Circle criterion · Global asymptotic stability · Input-to-state stability · Lur'e systems · Positive realness · Sector-bounded nonlinear operators

Mathematics Subject Classification 34A12 · 34K05 · 34K20 · 93C10 · 93C23 · 93C35 · 93C80 · 93D05 · 93D09 · 93D10 · 93D20 · 93D25

1 Introduction

The stability and convergence properties of Lur'e systems, a common and important class of nonlinear feedback systems, are a much researched area. Absolute stability theory seeks to conclude stability of the feedback system shown in Fig. 1 via the interplay of frequency-domain properties of the linear component, given in state-space by the matrix triple (A, B, C), and sector properties of the *static* nonlinearity Φ. The so-called circle criterion (a natural generalization of the sufficiency part of the Nyquist criterion in the single-input single-output setting) is one of the best-known and most

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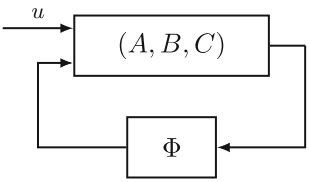

Fig. 1 Forced Lur'e system

used sufficient conditions for absolute stability. It is well known that the circle criterion is not necessary for absolute stability with respect to real static nonlinearities.

Lyapunov approaches have been employed to deduce global asymptotic stability of unforced (that is, $u = 0$) Lur'e systems (see, for example, [2, 11, 13, 14, 16, 17, 19, 28, 29]), whilst input–output methods, pioneered by Sandberg and Zames in the 1960s, have been used to infer L^2 and L^∞ stability (see, for example, [5, 28]). More recently, forced Lur'e systems have been analysed in the context of input-to-state stability (ISS) theory, with attention focussed on the extent to which results from classical absolute stability theory can be generalized to ensure certain ISS properties [1, 6–8, 10, 15, 16, 22–24]. Originating in the paper [25], ISS and its variants, such as integral input-to-state stability, are properties of general controlled nonlinear systems and, roughly, ensure a natural boundedness property of the state, in terms of initial conditions and inputs, see also the survey papers [4, 26].

In the current paper, we consider the situation wherein the nonlinearity Φ of the system shown in Fig. 1 belongs to a natural class of nonlinear causal operators which are sector bounded in an L^2-sense and satisfy a weak local Lipschitz-type condition. In particular, the class generalizes the classical set-up and is sufficiently wide to account for operators with unbounded memory (described by nonlinear integral equations, for example) and for input–output operators of certain dynamical processes. We develop a generalized multivariable circle criterion which is necessary and sufficient for global asymptotic L^2-stability (L^2-GAS) of the closed-loop system shown in Fig. 1 for all operators Φ in the class under consideration. The L^2-GAS property implies in particular, that, for every L^2-input u and all initial conditions, the solution of the closed-loop system converges to 0 as time goes to ∞, see Sect. 2 for details on the concept of L^2-GAS. Furthermore, we derive an ISS version of this result, namely a circle criterion which is necessary and sufficient for ISS of the system in Fig. 1 for all nonlinear operators Φ satisfying an exponentially weighted L^2-sector condition. By applying the sufficiency part of the generalized circle criterion to the case wherein Φ is the Nemytskii operator induced by a static nonlinearity, the classical circle criterion is easily recovered.

We emphasize that the nonlinearities considered are *real* in the sense that real input signals are mapped into real output signals. The key condition of the circle criterion is the positive realness of a certain rational matrix depending on $\mathbf{G}(s) := C(sI - A)^{-1}B$, the transfer function of the linear system given by (A, B, C), and the (possibly dynamic) sector data. As has already been indicated above, the main contribution of the paper is the proof of the equivalence of the positive-real condition in the circle criterion and absolute stability with respect to all real nonlinear causal operators satisfying a suitable L^2-sector condition, a result, which, in a sense, mirrors a well-known theorem from stability radius theory: namely the identity $r_{\mathbb{C}}(A, B, C) = r_{\mathbb{R},\mathrm{d}}(A, B, C)$, where $r_{\mathbb{C}}(A, B, C)$ is the stability radius with respect to complex static linear perturbations and $r_{\mathbb{R},\mathrm{d}}(A, B, C)$ is the stability radius with respect to real nonlinear causal L^2-bounded perturbation operators, where it is assumed that A is asymptotically stable (see [13, Proposition 4.4] and [14, Theorem 5.6.20]). Furthermore, we remark that, in the complex case, the sufficiency of the circle criterion is trivial (in the sense that the proof carries over from the real case without change) and the necessity has been established in [9, Theorems 6.8 and 6.11], where it is shown, in a general infinite-dimensional systems setting, that stability with respect to all *complex linear static* feedbacks satisfying a sector condition determined by two matrices K_1 and K_2 implies the positive realness of $(I - K_1\mathbf{G})(I - K_2\mathbf{G})^{-1}$.

The layout of the paper is as follows. In Sect. 2, we discuss some preliminaries, present a number of auxiliary results and introduce the class of Lur'e systems which will be considered in the rest of the paper. Section 3 is devoted to small gain conditions for L^2-GAS and ISS for Lur'e systems with nonlinear causal L^2-bounded operators in the feedback loop. Contact will be made with the Aizerman conjecture and the work by Hinrichsen and Pritchard [13, 14]. In Sect. 4, it is established that a natural generalization of the positive-real condition familiar from the circle criterion is sufficient for L^2-GAS (ISS) for all nonlinear causal operators satisfying a suitable (exponentially weighted) L^2-sector condition, whilst necessity of the positive-real condition for absolute stability is proved in Sect. 5. Finally, a proof of an auxiliary result from Sect. 2 is presented in Sect. 6.

2 Notation, terminology and auxiliary results

In this section, we present and discuss a number of preliminaries required for the development of the main results of the paper.

Notation

The fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. We set $\mathbb{R}_+ := [0, \infty)$ and, for $\alpha \in \mathbb{R}$, $\mathbb{C}_\alpha := \{s \in \mathbb{C} : \operatorname{Re} s > \alpha\}$. Let $\overline{\mathbb{C}}_\alpha$ denote the closure of $\mathbb{C}_\alpha$, that is, $\overline{\mathbb{C}}_\alpha = \{s \in \mathbb{C} : \operatorname{Re} s \geq \alpha\}$. Throughout, let $\mathbb{F}$ be the field of real or complex numbers, $\mathbb{R}$ or $\mathbb{C}$, respectively. The matrix space $\mathbb{F}^{m\times p}$ is endowed with the operator norm induced by the 2-norm. For $M \in \mathbb{C}^{m\times p}$, let M^T and M^* denote the transposition and Hermitian transposition of M, respectively, and, if $m = p$, we set

$$\operatorname{Re} M := \frac{1}{2}(M + M^*).$$

We say that the matrix $M \in \mathbb{C}^{m\times m}$ is *Hurwitz* if all its eigenvalues have negative real parts. For $M \in \mathbb{C}^{m\times p}$ and $N \in \mathbb{C}^{q\times p}$, we write

$$\operatorname{col}(M, N) := \begin{pmatrix} M \\ N \end{pmatrix} \in \mathbb{C}^{(m+q)\times p}.$$

We say that $\varphi : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^m$ is a (real) *locally Lipschitz Carathéodory function* if the function $t \mapsto \varphi(t, z)$ is Lebesgue measurable for every $z \in \mathbb{R}^p$ and $z \mapsto \varphi(t, z)$ is locally Lipschitz, uniformly in t on compact intervals. The set of all locally Lipschitz Carathéodory functions $\mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^m$ is denoted by $\mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^p, \mathbb{R}^m)$.

We will make use of the Hardy spaces $H^\infty_{p\times m}$ and $H^2_{p\times m}$ of holomorphic functions $\mathbb{C}_0 \to \mathbb{C}^{p\times m}$ with respective norms given by

$$\|\mathbf{H}\|_{H^\infty} := \sup_{s\in\mathbb{C}_0} \|\mathbf{H}(s)\| \quad \text{and} \quad \|\mathbf{H}\|_{H^2} := \sup_{\sigma>0} \left(\int_{-\infty}^{\infty} \|\mathbf{H}(\sigma + i\omega)\|^2 \mathrm{d}\omega\right)^{1/2},$$

where the norm on the RHS is the operator norm induced by the 2-norm. We recall that a holomorphic function is in $H^2_{p\times m}$ if, and only if, it is the Laplace transform of a square-integrable function.

Stability and stability radii in the frequency domain

Let $\mathbf{H}$ be a rational matrix (the coefficients of the entries are not required to be real) of format $p \times m$, frequently interpreted as the transfer function matrix of a finite-dimensional linear time-invariant control system. If $\mathbf{H} \in H^\infty_{p\times m}$, that is, $\mathbf{H}$ does not have any poles in $\overline{\mathbb{C}}_0 \cup \{\infty\}$, then we say that $\mathbf{H}$ is *stable*. Let $R_\mathbb{F} H^\infty_{p\times m}$ be the set of all rational matrices which are in $H^\infty_{p\times m}$ and the entries of which have coefficients in $\mathbb{F}$. Equivalently, $R_\mathbb{F} H^\infty_{p\times m}$ is the set of stable rational matrices of format $p\times m$ which have entries with coefficients in $\mathbb{F}$. For ease of notation, we set $H^\infty := H^\infty_{1\times 1}$, $H^2 := H^2_{1\times 1}$ and $R_\mathbb{F} H^\infty := R_\mathbb{F} H^\infty_{1\times 1}$.

It is said that $\mathbf{K} \in R_\mathbb{C} H^\infty_{m\times p}$ *stabilizes* $\mathbf{H}$ if $\mathbf{H}^\mathbf{K} := \mathbf{H}(I - \mathbf{K}\mathbf{H})^{-1}$ is stable. It is clear that $\mathbf{H}^\mathbf{K}$ is the transfer function of the feedback system with $\mathbf{H}$ and $\mathbf{K}$ in the forward and feedback loops, respectively. We note that $\mathbf{H}^{\mathbf{K}+\mathbf{L}} = (\mathbf{H}^\mathbf{K})^\mathbf{L} = (\mathbf{H}^\mathbf{L})^\mathbf{K}$ for all $\mathbf{K}, \mathbf{L} \in R_\mathbb{C} H^\infty_{m\times p}$. For the special case wherein $\mathbf{K}(s)$ is constant, $\mathbf{K}(s) \equiv K$, we introduce some convenient notation:

$$\mathbb{S}_\mathbb{F}(\mathbf{H}) := \left\{K \in \mathbb{F}^{m\times p} : K \text{ stabilizes } \mathbf{H}\right\}.$$

If the rational matrix $\mathbf{H}$ is stable, we define the $\mathbb{F}$-*stability radius* of $\mathbf{H}$ by

$$r_\mathbb{F}(\mathbf{H}) := \inf\left\{\|K\| : K \in \mathbb{F}^{m\times p} \text{ and } \mathbf{H}^K \text{ is not stable}\right\}.$$

We shall refer to the $\mathbb{F}$-stability radius as the real or complex stability radius, depending on whether $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, respectively. If $\mathbf{H}$ is stable and (A, B, C, D) is a stabilizable and detectable realization of $\mathbf{H}$, then $r_\mathbb{F}(\mathbf{H})$ coincides with the stability radius $r_\mathbb{F}(A; B, C, D; \mathbb{C}_0)$ of A with respect to the weighting (B, C, D) and the stability region $\mathbb{C}_0$ as defined in [14, Sect. 5.3]. It is clear that $r_\mathbb{R}(\mathbf{H}) \geq r_\mathbb{C}(\mathbf{H})$ and if $\mathbf{H}(s) \not\equiv 0$, then there exists a destabilizing feedback $K \in \mathbb{F}^{m\times p}$ with $\|K\| = r_\mathbb{F}(\mathbf{H})$. We note that $r_\mathbb{F}(\mathbf{H})$ can also be expressed in the following form

$$r_\mathbb{F}(\mathbf{H}) = \sup\left\{\rho > 0 : K \in \mathbb{S}_\mathbb{F}(\mathbf{H}) \text{ for all } K \in \mathbb{F}^{m\times p} \text{s.t. } \|K\| < \rho\right\}.$$

Furthermore, it is well known from [12, Proposition 2.1] or [14, Theorem 5.3.9] that

$$r_{\mathbb{C}}(\mathbf{H}) = 1/\|\mathbf{H}\|_{H^\infty}.$$

The above identity implies that, in the context of linear output feedback with complex gains, the small-gain condition is sharp in the sense that there exists a destabilizing output feedback $K \in \mathbb{C}^{m\times p}$ such that $\|\mathbf{H}\|_{H^\infty}\|K\| = 1$ (provided that $\mathbf{H}(s) \not\equiv 0$).

The following result shows that the *complex* stability radius plays a key role in the context of stabilization and destabilization by *real* dynamic linear feedback.

Proposition 2.1 *If* $\mathbf{H}$ *is a real-rational matrix of format* $p \times m$ *and* $\mathbf{K} \in R_{\mathbb{R}}H^\infty_{m\times p}$ *stabilizes* $\mathbf{H}$*, then*

$$\sup\left\{\rho : \mathbf{H}(I-\mathbf{F}\mathbf{H})^{-1} \in H^\infty_{p\times m} \text{ for all } \mathbf{F} \in R_{\mathbb{R}}H^\infty_{m\times p} \text{ s.t. } \|\mathbf{F}-\mathbf{K}\|_{H^\infty} < \rho\right\} = r_{\mathbb{C}}(\mathbf{H}^{\mathbf{K}}). \tag{2.1}$$

Furthermore, assuming that $\mathbf{H}(s) \not\equiv 0$*, there exists* $\mathbf{F} \in R_{\mathbb{R}}H^\infty_{m\times p}$ *such that* $\|\mathbf{F} - \mathbf{K}\|_{H^\infty} = 1/\|\mathbf{H}^{\mathbf{K}}\|_{H^\infty} = r_{\mathbb{C}}(\mathbf{H}^{\mathbf{K}})$ *and* $\mathbf{H}(I-\mathbf{F}\mathbf{H})^{-1}$ *is not stable.*

The key part of the proof of the above result relies on a construction from [27, Proof of Theorem 4, Sect. 7.4]. For completeness, we have included a proof of Proposition 2.1 in Appendix.

Proposition 2.1 shows that, in the context of real dynamic linear output feedback, the small-gain condition is sharp in the sense that there exists a destabilizing real feedback $\mathbf{K} \in R_{\mathbb{R}}H^\infty_{m\times p}$ such that $\|\mathbf{H}\|_{H^\infty}\|\mathbf{K}\|_{H^\infty} = 1$ (provided that $\mathbf{H}(s) \not\equiv 0$).

We recall that a square rational matrix $\mathbf{H}$ is *positive real* if $\operatorname{Re}\mathbf{H}(s)$ is positive semi-definite for all $s \in \mathbb{C}_0$ which are not poles of $\mathbf{H}$. It is well known that if $\mathbf{H}$ is positive real, then $\mathbf{H}$ is holomorphic in $\mathbb{C}_0$. Furthermore, $\mathbf{H}$ is said to be *strictly positive real* (*strongly positive real*) if there exists $\varepsilon > 0$ such that $s \mapsto \mathbf{H}(s-\varepsilon)$ is positive real ($\mathbf{H} - \varepsilon I$ is positive real). If $\mathbf{H}$ is strongly positive real and does not have any poles on the imaginary axis, then $\mathbf{H}$ is strictly positive real.

The following characterization of positive-real properties in terms of norm conditions will be used later on.

Lemma 2.2 *Let* $\mathbf{H}$ *be a square rational matrix.*

(1) $\mathbf{H}$ *is positive real if, and only if,* $\det(I+\mathbf{H}(s)) \not\equiv 0$ *and* $\|(I-\mathbf{H})(I+\mathbf{H})^{-1}\|_{H^\infty} \le 1$.

(2) $\mathbf{H}$ *is strongly positive real and stable if, and only if,* $\det(I+\mathbf{H}(s)) \not\equiv 0$ *and the strict inequality* $\|(I-\mathbf{H})(I+\mathbf{H})^{-1}\|_{H^\infty} < 1$ *holds.*

Statements (1) and (2) of the above lemma are special cases of [9, Corollaries 3.6 and 4.3], respectively.

We say that $\mathbf{H} \in R_{\mathbb{R}}H^\infty_{p\times m}$ has the *real supremum-value property* if there exists $s^\dagger \in \overline{\mathbb{C}}_0 \cup \{\infty\}$ such that $\mathbf{H}(s^\dagger) \in \mathbb{R}^{p\times m}$ and $\|\mathbf{H}(s^\dagger)\| = \|\mathbf{H}\|_{H^\infty}$. As the function $s \mapsto \|\mathbf{H}(s)\|$ is subharmonic, the maximum principle for subharmonic functions implies that if $\mathbf{H}$ has the real supremum-value property, then, without loss of generality, we may assume that there exists $\omega^\dagger \in \mathbb{R} \cup \{\infty\}$ such that $\mathbf{H}(i\omega^\dagger) \in \mathbb{R}^{p\times m}$ and $\|\mathbf{H}(i\omega^\dagger)\| = \|\mathbf{H}\|_{H^\infty}$.

Case 1 of the proof of Proposition 2.1 (see Appendix) shows that the following lemma holds.

Lemma 2.3 *If $\mathbf{H} \in R_{\mathbb{R}}H^{\infty}_{p\times m}$ has the real supremum-value property, then $r_{\mathbb{R}}(\mathbf{H}) = r_{\mathbb{C}}(\mathbf{H})$.*

We describe a number of scenarios in which the real supremum-value property holds.

Example 2.4 **(a)** Let $\mathbf{H}$ be a stable real-rational transfer function matrix of format $p\times m$, and let (A, B, C, D) be a realization of $\mathbf{H}$, where A is Hurwitz. If the impulse response of $\mathbf{H}$ is nonnegative, that is, $Ce^{At}B \in \mathbb{R}^{p\times m}_{+}$ for all $t \geq 0$ and $D \in \mathbb{R}^{p\times m}_{+}$, then, by [21, Proposition 3], $\|\mathbf{H}\|_{H^\infty} = \|\mathbf{H}(0)\|$, and thus, $\mathbf{H}$ has the real supremum-value property.

(b) Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz, $B \in \mathbb{R}^{n\times m}_{+}$ and $C \in \mathbb{R}^{p\times n}_{+}$. Assume that A is a Metzler matrix (that is, all off-diagonal entries of A are nonnegative); that is, (A, B, C) is a positive and stable continuous-time controlled and observed linear system. Then, $Ce^{At}B \in \mathbb{R}^{p\times m}_{+}$ for all $t \geq 0$, and setting, $\mathbf{H}(s) := C(sI - A)^{-1}B$, it follows from part (a) that $\|\mathbf{H}\|_{H^\infty} = \|\mathbf{H}(0)\|$, showing that $\mathbf{H}$ has the real supremum-value property.

(c) Let $A \in \mathbb{R}^{n\times n}$ be symmetric and Hurwitz, $b \in \mathbb{R}^{n}_{+}$ and consider the single-input single-output symmetric system (A, b, b^T) with transfer function $\mathbf{H}$ given by $\mathbf{H}(s) := b^T(sI - A)^{-1}b$. It follows from the symmetry of A that e^{At} is symmetric and positive definite for all $t \in \mathbb{R}$, and so $b^Te^{At}b \geq 0$ for all $t \in \mathbb{R}$. Consequently, by part (a), $\|\mathbf{H}\|_{H^\infty} = |\mathbf{H}(0)|$. In particular, $\mathbf{H}$ has the real supremum-value property.

(d) Let $\mathbf{H}$ be a proper real-rational matrix of format $m\times m$ such that $\mathbf{H}$ has precisely one pole in $\overline{\mathbb{C}}_0$, namely a simple pole at 0, and the residue matrix $H_0 := \lim_{s\to 0} s\mathbf{H}(s)$ is symmetric and positive definite. Defining $\mathbf{L}(s) := \mathbf{H}(s) - s^{-1}H_0$, it is clear that $\mathbf{L} \in R_{\mathbb{R}}H^{\infty}_{m\times m}$, and so

$$\kappa := \sup\left\{k \geq 0 : \ I + 2k\mathbf{L} \text{ is positive real}\right\} > 0.$$

The positive realness of the function $s \mapsto s^{-1}H_0$ implies that $I + 2k\mathbf{H}$ is also positive real for all $k \in (0, \kappa)$, and thus, it follows from [20, Lemma 3.10] that $\|\mathbf{H}^{-kI}\|_{H^\infty} = 1/k$ for all $k \in (0, \kappa)$. As $\mathbf{H}^{-kI}(0) = (1/k)I$, we see that $\mathbf{H}^{-kI}$ has the real supremum-value property for every $k \in (0, \kappa)$.

(e) In the previous examples, the supremum is achieved at $s = 0$. But there are many examples of transfer functions having the real supremum-value property for which the supremum is achieved at $s = i\omega_0$ for some $\omega_0 \in (0, \infty)$ and not at $s = 0$. Here, we provide one such example. Let $a, b > 0$ and consider the strictly proper stable rational function $\mathbf{H}$ given by

$$\mathbf{H}(s) = \frac{s}{(s+a)(s+b)}.$$

Routine calculations yield

$$\begin{aligned}\|\mathbf{H}\|_{H^\infty} &= \max_{\omega\in\mathbb{R}} |\mathbf{H}(i\omega)| = \mathbf{H}(\pm i\sqrt{ab}) = \frac{1}{a+b} \quad \text{and} \\ |\mathbf{H}(i\omega)| &< \frac{1}{a+b} \;\; \forall\, \omega \in \mathbb{R}\backslash\{\pm\sqrt{ab}\},\end{aligned}$$

showing that $\mathbf{H}$ has the real supremum-value property with the supremum achieved at precisely two points in $\overline{\mathbb{C}}_0$, namely $\pm i\sqrt{ab}$. ◊

Nonlinear operators

For $q \in [1, \infty]$ and $J \subset \mathbb{R}$ an interval, let $L^q(J, \mathbb{R}^n)$ denote the usual Lebesgue space of functions defined on J with values in $\mathbb{R}^n$. The local version of $L^q(J, \mathbb{R}^n)$ is denoted by $L^q_{\text{loc}}(J, \mathbb{R}^n)$. For $0 < \tau \leq \infty$ and $t \in [0, \tau)$, let $\pi_t : L^q_{\text{loc}}([0, \tau), \mathbb{R}^n) \to L^q(\mathbb{R}_+, \mathbb{R}^n)$ be the truncation operator; that is, for $w \in L^q_{\text{loc}}([0, \tau), \mathbb{R}^n)$, $\pi_t w$ is the function in $L^q(\mathbb{R}_+, \mathbb{R}^n)$ which is equal to w on $[0, t]$ and equal to 0 on (t, ∞). We recall that an operator Φ defined on $L^2(\mathbb{R}_+, \mathbb{R}^m)$ or $L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ and mapping into $L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^p)$ is said to be causal if $\pi_t\Phi = \pi_t\Phi\pi_t$ for all $t \geq 0$. If an operator Φ defined on $L^2(\mathbb{R}_+, \mathbb{R}^m)$ is causal, then it naturally extends to a causal operator on $L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ via

$$(\Phi(w))(s) := (\Phi(\pi_t w))(s) \qquad \text{for every } w \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m) \text{ and all } (s, t) \text{ such that } 0 \leq s \leq t < \infty.$$

The causality of Φ guarantees that $\Phi(w)$ is a well-defined function in $L^p_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^p)$. Furthermore, we note that a causal operator $\Phi : L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m) \to L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^p)$ can be "localized" as follows: for every $\tau \in (0, \infty)$ and every $w \in L^2_{\text{loc}}([0, \tau), \mathbb{R}^m)$, we define $\Phi(w) \in L^2_{\text{loc}}([0, \tau), \mathbb{R}^p)$ by setting

$$(\Phi(w))(s) := (\Phi(\pi_t w))(s) \qquad \text{for all } (s, t) \text{ such that } 0 \leq s \leq t < \tau.$$

Again, this definition is meaningful by the causality of Φ.

For $\tau \geq 0$, the shift (or delay) operator $\mathcal{S}_\tau : L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m) \to L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ is defined by

$$(\mathcal{S}_\tau w)(t) = \begin{cases} 0, & 0 \leq t < \tau \\ w(t-\tau), & t \geq \tau. \end{cases} \tag{2.2}$$

A linear operator $H : L^2(\mathbb{R}_+, \mathbb{R}^m) \to L^2(\mathbb{R}_+, \mathbb{R}^p)$ is *shift-invariant* if $\mathcal{S}_\tau H = H\mathcal{S}_\tau$ for all $\tau \geq 0$. We recall that linear shift-invariant operators are causal. A bounded linear shift-invariant operator $H : L^2(\mathbb{R}_+, \mathbb{R}^m) \to L^2(\mathbb{R}_+, \mathbb{R}^p)$ has a transfer function $\mathbf{H} \in H^\infty_{p\times m}$ in the sense that, for every $w \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, the Laplace transform of Hw is given by $\mathbf{Hw}$, where $\mathbf{w}$ denotes the Laplace transform of w. Conversely, every $\mathbf{H} \in H^\infty_{p\times m}$ such that $\mathbf{H}(s)$ is real for all $s \in (0, \infty)$ is the transfer function of a bounded linear shift-invariant operator $H : L^2(\mathbb{R}_+, \mathbb{R}^m) \to L^2(\mathbb{R}_+, \mathbb{R}^p)$. We recall that the L^2-

induced operator norm of a bounded linear shift-invariant operator equals the H^∞-norm of its transfer function.

We say that a causal operator $\Phi : L^2_{\rm loc}(\mathbb{R}_+, \mathbb{R}^m) \to L^2_{\rm loc}(\mathbb{R}_+, \mathbb{R}^p)$ is *weakly Lipschitz* if, for every $t_0 \geq 0$, $z \in \mathbb{R}^p$ and $w \in L^2([0, t_0], \mathbb{R}^m)$, there exist $l \geq 0$, $r > 0$ and $t_1 > t_0$ such that, for all $w_1, w_2 \in L^2_{\rm loc}(\mathbb{R}_+, \mathbb{R}^p)$ satisfying $w_1 = w_2 = w$ on $[0, t_0]$ and $\|w_i(t) - z\| \leq r$ for almost every $t \in [t_0, t_1]$, we have

$$\|\Phi(w_1) - \Phi(w_2)\|_{L^2(t_0,t_1)} \leq l\|w_1 - w_2\|_{L^\infty(t_0,t_1)}.$$

We set

$$\|\Phi\| := \sup_{w\in L^2,\, w\neq 0} \frac{\|\Phi(w)\|_{L^2}}{\|w\|_{L^2}} \leq \infty$$

and say that Φ is *linearly bounded* if $\|\Phi\| < \infty$. Furthermore, if Φ is causal, weakly Lipschitz and $\|\Phi\| < \infty$, then $\Phi(0) = 0$.

The time-domain equivalent of $R_{\mathbb{R}} H^\infty_{p\times m}$ is denoted by $F_{\mathbb{R}} L_{p\times m}$; that is, $F_{\mathbb{R}} L_{p\times m}$ is the space of all real shift-invariant linear operators with transfer function in $R_{\mathbb{R}} H^\infty_{p\times m}$. The operators in $F_{\mathbb{R}} L_{p\times m}$ are precisely the input–output operators of asymptotically stable finite-dimensional linear time-invariant real state-space systems with m inputs and p outputs. Trivially, every operator in $F_{\mathbb{R}} L_{m\times p}$ is causal and weakly Lipschitz.

In the following, let $\mathcal{N}(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R}^p)$ be the set of static nonlinearities $\varphi \in C(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R}^p)$ which are uniformly linearly bounded, that is,

$$\|\varphi\| := \sup_{t\geq 0,\, z\neq 0} \frac{\|\varphi(t, z)\|}{\|z\|} < \infty.$$

The symbol $\|\varphi\|$ should not be confused with the function $(t, z) \mapsto \|\varphi(t, z)\|$.

We provide two classes of examples of operators which are causal and weakly Lipschitz.

Example 2.5 **(a)** *Nemytskii operators*. For $\varphi \in \mathcal{N}(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R}^p)$, let N_φ denote the corresponding Nemytskii operator acting on $L^2_{\rm loc}(\mathbb{R}_+, \mathbb{R}^m)$ via $(N_\varphi(w))(t) = \varphi(t, w(t))$. It is routine to show that $N_\varphi(w)$ is measurable for measurable w, and

$$\|N_\varphi\| = \sup_{w\in L^2,\, w\neq 0} \frac{\|N_\varphi\|_{L^2}}{\|w\|_{L^2}} \leq \|\varphi\| < \infty,$$

implying that N_φ maps $L^2(\mathbb{R}_+, \mathbb{R}^m)$ to $L^2(\mathbb{R}_+, \mathbb{R}^p)$ and $L^2_{\rm loc}(\mathbb{R}_+, \mathbb{R}^m)$ to $L^2_{\rm loc}(\mathbb{R}_+, \mathbb{R}^p)$. Furthermore, N_φ is causal and weakly Lipschitz. It is not difficult to show that $\|N_\varphi\| = \|\varphi\|$ if, for each $z \in \mathbb{R}^p$, the function $t \mapsto \|\varphi(t, z)\|$ is lower semi-continuous. Simple examples show that, in the absence of this lower semi-continuity condition, it is possible that $\|N_\varphi\| < \|\varphi\|$. Finally, it is well known that N_φ is continuous as a map from $L^2(\mathbb{R}_+, \mathbb{R}^m)$ to $L^2(\mathbb{R}_+, \mathbb{R}^p)$, see, for example, [18, Theorem 2.14].

(b) For $k : \{(t,\theta) : t \geq \theta \geq 0\} \to \mathbb{R}^{p\times p}$ and $k_0 \in \mathbb{R}^{p\times p}$ consider the corresponding integral operator

$$(J_k(w))(t) = \int_0^t k(t,\theta)w(\theta)\mathrm{d}\theta + k_0 w(t), \quad \forall\, t \geq 0.$$

Assume that $\int_0^\infty \int_0^t \|k(t,\theta)\|^2 \mathrm{d}\theta \mathrm{d}t < \infty$ or $k(t,\theta) = l(t-\theta)$, where $l \in L^1(\mathbb{R}_+, \mathbb{F}^{p\times p})$, in which case J_k is a causal bounded operator from $L^2(\mathbb{R}_+, \mathbb{R}^p)$ into itself. For $\varphi \in \mathcal{N}(\mathbb{R}_+ \times \mathbb{R}^m, \mathbb{R}^p)$, the operator $J_k \circ N_\varphi$ maps $L^2(\mathbb{R}_+, \mathbb{R}^m)$ to $L^2(\mathbb{R}_+, \mathbb{R}^p)$. If k maps into $\mathbb{R}^{m\times m}$ and $k_0 \in \mathbb{R}^{m\times m}$, then the composition $N_\varphi \circ J_k$ is an operator which maps $L^2(\mathbb{R}_+, \mathbb{R}^m)$ to $L^2(\mathbb{R}_+, \mathbb{R}^p)$. Both of these operators are causal, weakly Lipschitz and linearly bounded. Operators of the form $\mathcal{K} \circ N_\varphi$ and $N_\varphi \circ \mathcal{H}$, where $\mathcal{K} \in F_{\mathbb{R}} L_{p\times p}$ and $\mathcal{H} \in F_{\mathbb{R}} L_{m\times m}$, are special instances of these types of operators. $\diamondsuit$

Lur'e systems with nonlinear causal operators in the feedback loop

Throughout the paper, let $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$ and let $\mathbf{G}$ be the transfer function of the linear system given by (A, B, C), that is, $\mathbf{G}(s) = C(sI - A)^{-1}B$ (a strictly proper real-rational matrix). Furthermore, let $\Phi : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^p) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ be causal and weakly Lipschitz, $(t_0, x^0, v, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times L^2([0, t_0], \mathbb{R}^p) \times L^2_{\mathrm{loc}}([t_0, \infty), \mathbb{R}^n)$ and consider the following initial-value problem

$$\dot{x}(t) = Ax(t) + B\big(\Phi([Cx]_v)\big)(t) + u(t), \quad t \geq t_0, \quad x(t_0) = x^0, \tag{2.3}$$

where the function $[Cx]_v$ is defined as follows

$$[Cx]_v(t) := v(t) \text{ for } t \in [0, t_0] \quad \text{and} \quad [Cx]_v(t) := Cx(t) \text{ for } t > t_0.$$

Note that the differential equation in (2.3) is a forced Lur'e system given by the linear system (A, B, C), the nonlinearity Φ and the forcing (or input) function u, see Fig. 1. Considering (2.3) with $\Phi = N_\varphi$, where $\varphi \in \mathcal{N}(\mathbb{R}_+ \times \mathbb{R}^p, \mathbb{R}^m)$, we obtain the standard ODE initial-value problem

$$\dot{x}(t) = Ax(t) + B\varphi(t, Cx(t)) + u(t), \quad t \geq t_0, \quad x(t_0) = x^0, \tag{2.4}$$

as a special case of (2.3). Obviously, as φ is memoryless, specification of the initial segment v is redundant.

An absolutely continuous function $x : [t_0, \tau) \to \mathbb{R}^n$, where $t_0 < \tau \leq \infty$, is said to be a solution of (2.3) on $[t_0, \tau)$ if $x(t_0) = x^0$ and the differential equation in (2.3) is satisfied for almost every $t \in [t_0, \tau)$.

By a suitable modification of the arguments used in [13] (where the uncontrolled case $u = 0$ is treated), it can be shown that, for every $(t_0, x^0, v, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times L^2([0, t_0], \mathbb{R}^p) \times L^2_{\mathrm{loc}}([t_0, \infty), \mathbb{R}^n)$, there exists $t_0 < \tau \leq \infty$ such that (2.3) has a solution on $[t_0, \tau)$ and that, for given $t_0 < \tau \leq \infty$, there exists at most one solution of (2.3) on the interval $[t_0, \tau)$. Furthermore, if $\|\Phi\| < \infty$, then (2.3) has a unique solution defined on $[t_0, \infty)$.

Remark 2.6 The case wherein $t_0 = 0$ has a special feature which we wish to highlight. In this case, the initial segment v is simply a point in $\mathbb{R}^p$. Assume that x is a solution of (2.3) with $t_0 = 0$ and let $\hat{v} \in \mathbb{R}^p$, $\hat{v} \neq v$. Then, the functions $[Cx]_v$ and $[Cx]_{\hat{v}}$ coincide on the open interval $(0, \tau)$, and so, they are equal almost everywhere in $[0, \tau)$. Hence, $\Phi([Cx]_v)$ and $\Phi([Cx]_{\hat{v}})$ are also equal almost everywhere in $[0, \tau)$ and x solves (2.3) with v replaced by $\hat{v}$. Consequently, if $t_0 = 0$, then the initial segment v is irrelevant and, without loss of generality, we may assume that $v = Cx^0$, in which case $[Cx]_v = Cx$. ◊

We say that (2.3) is *globally asymptotically L^2-stable (L^2-GAS)*, if, for every $(t_0, x^0, v, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times L^2([0, t_0], \mathbb{R}^p) \times L^2_{\text{loc}}([t_0, \infty), \mathbb{R}^n)$, there exists a solution $x = x(\cdot\,; t_0, x^0, v, u)$ of (2.3) defined on $[t_0, \infty)$ and the following two conditions are satisfied:

(i) the origin is *L^2 -stable in the large*; that is, there exists $\kappa \geq 0$ such that, for all $(t_0, x^0, v, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times L^2([0, t_0], \mathbb{R}^p) \times L^2([t_0, \infty), \mathbb{R}^n)$,

$$\begin{aligned} &\|x(\cdot\,; t_0, x^0, v, u)\|_{L^2(t_0,t)} + \|x(t; t_0, x^0, v, u)\| \\ &\quad \leq \kappa\big(\|x^0\| + \|v\|_{L^2(0,t_0)} + \|u\|_{L^2(t_0,t)}\big) \quad \forall\, t \geq t_0; \end{aligned}$$

(ii) the origin is *globally L^2-attractive*, that is, $x(t; t_0, x^0, v, u) \to 0$ as $t \to \infty$ for all $(t_0, x^0, v, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times L^2([0, t_0], \mathbb{R}^p) \times L^2([t_0, \infty), \mathbb{R}^n)$.

When the system (2.3) is considered without forcing ($u = 0$), then the origin is said to be *globally asymptotically stable (GAS)* if there exists $\kappa \geq 0$ such that, for all $(t_0, x^0, v) \in \mathbb{R}_+ \times \mathbb{R}^n \times L^2([0, t_0], \mathbb{R}^p)$,

$$\|x(t; t_0, x^0, v, 0)\| \leq \kappa\big(\|x^0\| + \|v\|_{L^2(0,t_0)}\big) \quad \forall\, t \geq t_0,$$

and (ii) holds with $u = 0$. If property (ii) is satisfied with $u = 0$ and for some (fixed) $t_0 \geq 0$, then the origin is said to be *globally attractive at time t_0*.

Not surprisingly, if, in (2.3), $\Phi \in F_{\mathbb{R}}L_{m\times p}$, then the above stability and attractivity concepts are closely related to well-known frequency-domain properties.

Lemma 2.7 *Let $\mathcal{F} \in F_{\mathbb{R}}L_{m\times p}$, with transfer function denoted by $\mathbf{F}$, and consider (2.3) with $\Phi = \mathcal{F}$.*

(1) *If the origin of (2.3) is globally attractive at time t_0 for some $t_0 \geq 0$, then $\mathbf{G}^{\mathbf{F}} \in H^\infty_{p\times m}$.*

(2) *If (A, B, C) is stabilizable and detectable and $\mathbf{G}^{\mathbf{F}} \in H^\infty_{p\times m}$, then (2.3) is L^2-GAS.*

(3) *If (A, B, C) is stabilizable and detectable and the origin of (2.3) is globally attractive at time t_0 for some $t_0 \geq 0$, then (2.3) is L^2-GAS.*

We remark that if $\mathbf{F}$ is not stable, then statement (2) is in general not true.

Proof of Lemma 2.7. Let $(A_{\mathcal{F}}, B_{\mathcal{F}}, C_{\mathcal{F}}, D_{\mathcal{F}})$ be a minimal realization of $\mathcal{F}$. We note that $A_{\mathcal{F}}$ is Hurwitz and

$$(\mathcal{F}w)(t) = \int_0^t C_{\mathcal{F}} e^{A_{\mathcal{F}}(t-\theta)} B_{\mathcal{F}} w(\theta)\mathrm{d}\theta + D_{\mathcal{F}} w(t) \quad \forall\, t \geq 0,\ \forall\, w \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^p).$$

For arbitrary $t_0 \geq 0$, $v \in L^2([0, t_0], \mathbb{R}^p)$, $u \in L^2([t_0, \infty), \mathbb{R}^n)$ and $x^0 \in \mathbb{R}^n$, let $x = x(\,\cdot\,; t_0, x^0, v, u)$ be the solution of (2.3). Setting

$$z(t) := \int_0^t e^{A_{\mathcal{F}}(t-\theta)} B_{\mathcal{F}} [Cx]_v(\theta)\mathrm{d}\theta \quad \forall\, t \geq 0, \tag{2.5}$$

we have that, on $[t_0, \infty)$, the function $\mathcal{F}([Cx]_v)$ can be written as

$$(\mathcal{F}[Cx]_v)(t) = C_{\mathcal{F}} z(t) + D_{\mathcal{F}} Cx(t) \quad \forall\, t \geq t_0.$$

We also note that there exists $c \geq 0$ (not depending on t_0, x^0, v or u) such that

$$\|z(t_0)\| \leq c\|v\|_{L^2(0,t_0)}. \tag{2.6}$$

Consequently, on $[t_0, \infty)$,

$$\dot{x} = Ax + B(C_{\mathcal{F}} z + D_{\mathcal{F}} Cx) + u \quad \text{and} \quad \dot{z} = A_{\mathcal{F}} z + B_{\mathcal{F}} Cx.$$

Setting $x_*(t) = x(t + t_0)$, $z_*(t) = z(t + t_0)$ and $u_*(t) = u(t + t_0)$ for all $t \geq 0$, we have that on $[0, \infty)$,

$$\begin{aligned} \dot{x}_* &= Ax_* + B(C_{\mathcal{F}} z_* + D_{\mathcal{F}} Cx_*) + u_*, \quad x_*(0) = x^0, \\ \dot{z}_* &= A_{\mathcal{F}} z_* + B_{\mathcal{F}} Cx_*, \quad z_*(0) = z(t_0). \end{aligned}$$

Using Laplace transformation, a routine calculation shows that

$$C\mathbf{x}_*(s) = (I - \mathbf{G}(s)\mathbf{F}(s))^{-1}\big[C(sI - A)^{-1}(x^0 + \mathbf{u}_*(s)) + \mathbf{G}(s)C_{\mathcal{F}}(sI - A_{\mathcal{F}})^{-1} z(t_0)\big], \tag{2.7}$$

where $\mathbf{x}_*$ and $\mathbf{u}_*$ denote the Laplace transforms of x_* and u_*, respectively.

(1) Let $v = 0$ (and so, $z(t_0) = 0$), $u = 0$ and let $t_0 \geq 0$ be fixed, but arbitrary. By (2.7)

$$C\mathbf{x}_*(s) = (I - \mathbf{G}(s)\mathbf{F}(s))^{-1} C(sI - A)^{-1} x^0. \tag{2.8}$$

The components of $C\mathbf{x}_*$ are strictly proper rational functions, and so since $Cx_*(t) \to 0$ as $t \to \infty$ (because, by hypothesis, $x(t) \to 0$ as $t \to \infty$), we conclude that $Cx_*(t)$ converges to 0 exponentially fast as $t \to \infty$. This in turn implies via (2.8) that, for every $x^0 \in \mathbb{R}^n$, the function $(I - \mathbf{G}(s)\mathbf{F}(s))^{-1} C(sI - A)^{-1} x^0$ does not have any poles in $\overline{\mathbb{C}}_0$. Therefore, $\mathbf{G}^{\mathbf{F}} = (I - \mathbf{G}\mathbf{F})^{-1}\mathbf{G}$ does not have any poles in $\overline{\mathbb{C}}_0$, showing that $\mathbf{G}^{\mathbf{F}} \in H^\infty_{p\times m}$.

(2) Assume that (A, B, C) is stabilizable and detectable and $\mathbf{G}^{\mathbf{F}} \in H^\infty_{p\times m}$. Rearranging (2.7) leads to

$$C\mathbf{x}_*(s) = \mathbf{G}^{\mathbf{F}}(s)C_{\mathcal{F}}(sI - A_{\mathcal{F}})^{-1}z(t_0) + (I - \mathbf{G}(s)\mathbf{F}(s))^{-1}C(sI - A)^{-1}(x^0 + \mathbf{u}_*(s)). \tag{2.9}$$

We proceed in two steps.

Step 1. Define a rational matrix $\mathbf{H}$ by $\mathbf{H}(s) := (I - \mathbf{G}(s)\mathbf{F}(s))^{-1}C(sI - A)^{-1}$. We claim that $\mathbf{H} \in H^\infty_{p\times n} \cap H^2_{p\times n}$. To this end, set $\mathbf{T}(s) := A + B\mathbf{F}(s)C$ and note that

$$\begin{aligned}\mathbf{H}(s) &= C(sI - A)^{-1}\big(I - B\mathbf{F}(s)C(sI - A)^{-1}\big)^{-1}\\ &= C(sI - \mathbf{T}(s))^{-1} \quad \text{and} \quad \mathbf{G}^{\mathbf{F}}(s) = C(sI - \mathbf{T}(s))^{-1}B.\end{aligned}$$

By stabilizability, there exists $K \in \mathbb{R}^{m\times n}$ such that $A + BK$ is Hurwitz. Trivially, $sI - \mathbf{T}(s) - B(K - \mathbf{F}(s)C = sI - A - BK$ and so,

$$(sI - \mathbf{T}(s))(sI - A - BK)^{-1} - B(K - \mathbf{F}(s)C)(sI - A - BK)^{-1} \equiv I.$$

Multiplying from the left by $C(sI - \mathbf{T}(s))^{-1}$ gives

$$C(sI - A - BK)^{-1} - \mathbf{G}^{\mathbf{F}}(s)(K - \mathbf{F}(s)C)(sI - A - BK)^{-1} = C(sI - \mathbf{T}(s)^{-1},$$

showing that $C(sI - \mathbf{T}(s))^{-1} = \mathbf{H}(s)$ is stable, that is, $\mathbf{H} \in H^\infty_{p\times n}$. As $\mathbf{H}(s) = \mathrm{O}(1/s)$ as $|s| \to \infty$, it is clear that $\mathbf{H}$ is also in $H^2_{p\times n}$.

Step 2. Using that $\mathbf{G}^{\mathbf{F}}$ is stable, the Hurwitz property of $A_{\mathcal{F}}$ and Step 1, it follows from (2.6) and (2.9) that there exists a constant $b \geq 0$ (not depending on t_0, x_0, v or u) such that

$$\|C\mathbf{x}_*\|_{H^2} \leq b\big(\|x^0\| + \|v\|_{L^2(0,t_0)} + \|\mathbf{u}_*\|_{H^2}\big),$$

and thus

$$\|Cx\|_{L^2(t_0,\infty)} \leq b\big(\|x^0\| + \|v\|_{L^2(0,t_0)} + \|u\|_{L^2(t_0,\infty)}\big).$$

By detectability, there exists $H \in \mathbb{R}^{n\times p}$ such that $A + HC$ is Hurwitz. As x satisfies

$$\dot{x}(t) = (A + HC)x(t) + B(\mathcal{F}[Cx]_v)(t) - HCx(t) + u(t) \quad \forall t \geq t_0$$

and

$$\|B\mathcal{F}[Cx]_v - HCx + u\|_{L^2(t_0,\infty)} \leq a\big(\|x^0\| + \|v\|_{L^2(0,t_0)} + \|u\|_{L^2(t_0,\infty)}\big)$$

for suitable $a \geq 0$ (not depending t_0, x_0, v or u), we conclude that (2.3) with $\Phi = \mathcal{F}$ is L^2-GAS.

(3) This is an immediate consequence of statements (1) and (2). □

3 Small gain conditions for L^2-GAS and ISS

The first result in this section is a small-gain theorem for L^2-GAS.

Theorem 3.1 *Consider (2.3) and assume that (A, B, C) is stabilizable and detectable. Let $\Phi : L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^p) \to L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ be causal and weakly Lipschitz, and let $\mathcal{K} \in F_{\mathbb{R}}L_{m\times p}$ have transfer function $\mathbf{K}$. If $\mathbf{G}^{\mathbf{K}} \in H^\infty_{p\times m}$ and*

$$\|\Phi - \mathcal{K}\| = \sup_{w\in L^2,\, w\neq 0} \frac{\|\Phi(w) - \mathcal{K}w\|_{L^2}}{\|w\|_{L^2}} < \frac{1}{\|\mathbf{G}^{\mathbf{K}}\|_{H^\infty}} = r_{\mathbb{C}}(\mathbf{G}^{\mathbf{K}}), \qquad (3.1)$$

then (2.3) is L^2-GAS.

We note that (3.1) is equivalent to the small-gain condition $\|\mathbf{G}^{\mathbf{K}}\|_{H^\infty}\|\Phi - \mathcal{K}\| < 1$, where we adopt the common engineering jargon wherein the norms of operators or transfer functions are referred to as gains of the corresponding systems.

Proof of Theorem 3.1 Let Φ satisfy (3.1). We proceed in two steps.

Step 1. Assume that $\mathbf{G} \in H^\infty_{p\times m}$ and $\mathcal{K} = 0$. Note that, by stabilizability and detectability, the matrix A is Hurwitz. Let $(t_0, x^0, v, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times L^2([0, t_0], \mathbb{R}^p) \times L^2([t_0, \infty), \mathbb{R}^n)$. As in [13, proof of Theorem 3.12], it can be shown that there exists a constant $c \geq 0$ (not depending on (t_0, x^0, v, u)) such that

$$\|Cx(\,\cdot\,; t_0, x^0, v, u)\|_{L^2(t_0,t)} \leq c\big(\|x^0\| + \|v\|_{L^2(0,t_0)} + \|u\|_{L^2(t_0,t)}\big) \quad \forall\, t \geq t_0,$$

and routine arguments using the variation-of-parameters formula, the Hurwitz property of A and the linear boundedness of Φ show that (2.3) is L^2-GAS. We remark that [13, Theorem 3.12] is about GAS in the uncontrolled case $u = 0$ and not about L^2-GAS, but an inspection of the proof shows that obvious modifications of the arguments in [13] establishes L^2-GAS.

Step 2. Let $(A_{\mathcal{K}}, B_{\mathcal{K}}, C_{\mathcal{K}}, D_{\mathcal{K}})$ be a minimal realization of $\mathcal{K}$ (with state dimension $n_{\mathcal{K}}$) and set

$$\hat{A} := \begin{pmatrix} A + BD_{\mathcal{K}}C & BC_{\mathcal{K}} \\ B_{\mathcal{K}}C & A_{\mathcal{K}} \end{pmatrix}.$$

Furthermore, define $\hat{B} := \operatorname{col}(B, 0)$ and $\hat{C} := (C, 0)$ and note that the transfer function of $(\hat{A}, \hat{B}, \hat{C})$ is $\mathbf{G}^{\mathbf{K}}$. As $\mathbf{G}^{\mathbf{K}}$ and $\mathbf{K}$ are stable, it follows that $(I - \mathbf{KG})^{-1}$, $\mathbf{K}(I - \mathbf{GK})^{-1}$ and $(I - \mathbf{GK})^{-1}$ are also stable. Combining this with the stabilizability and detectability of (A, B, C) and $(A_{\mathcal{K}}, B_{\mathcal{K}}, C_{\mathcal{K}}, D_{\mathcal{K}})$ shows that $\hat{A}$ is Hurwitz [27, Lemma 17, Sect. 5.1]. Setting $\hat{n} := n + n_{\mathcal{K}}$ and letting $(t_0, \zeta^0, \hat{v}, \hat{u}) \in$

$\mathbb{R}_+ \times \mathbb{R}^{\hat{n}} \times L^2([0, t_0], \mathbb{R}^p) \times L^2_{\mathrm{loc}}([t_0, \infty), \mathbb{R}^{\hat{n}})$, it now follows from Step 1 that

$$\dot{\zeta}(t) = \hat{A}\zeta(t) + \hat{B}\big((\Phi([\hat{C}\zeta]_{\hat{v}}))(t) - (\mathcal{K}[\hat{C}\zeta]_{\hat{v}})(t)\big) + \hat{u}(t), \quad t \geq t_0, \quad \zeta(t_0) = \zeta^0 \tag{3.2}$$

is L^2-GAS.

Let x be the solution of (2.3). Obviously,

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B(\mathcal{K}[Cx]_v)(t) + B\big((\Phi([Cx]_v))(t) \\ &\quad -(\mathcal{K}[Cx]_v)(t)\big) + u(t) \quad \forall t \geq t_0.\end{aligned} \tag{3.3}$$

Defining z as in (2.5) (with $\mathcal{F}$ replaced by $\mathcal{K}$), we have that

$$\dot{z}(t) = A_{\mathcal{K}}z(t) + B_{\mathcal{K}}Cx(t) \quad \forall t \geq t_0, \tag{3.4}$$

and there exists $c \geq 0$ (not depending on (t_0, x^0, v, u)) such that

$$\|z(t_0)\| \leq c\|v\|_{L^2(0,t_0)}. \tag{3.5}$$

Moreover, $C_{\mathcal{K}}z(t) + D_{\mathcal{K}}Cx(t) = (\mathcal{K}[Cx]_v)(t)$ for all $t \geq t_0$. Combining this with (3.3) shows

$$\begin{aligned}\dot{x}(t) &= (A + BD_{\mathcal{K}}C)x(t) + BC_{\mathcal{K}}z(t) + B\big((\Phi([Cx]_v))(t) \\ &\quad -(\mathcal{K}[Cx]_v)(t)\big) + u(t) \quad \forall t \geq t_0.\end{aligned} \tag{3.6}$$

With $\zeta := \mathrm{col}(x, z)$, we have that $[\hat{C}\zeta]_v = [Cx]_v$, and we conclude from (3.4) and (3.6) that ζ satisfies (3.2) with $\zeta^0 = \mathrm{col}(x^0, z(t_0))$, $\hat{v} = v$ and $\hat{u} = \mathrm{col}\,(u, 0)$. The claim now follows from (3.5) and the L^2-GAS property of (3.2). □

Corollary 3.2 *Let $\mathcal{K} \in F_{\mathbb{R}}L_{m\times p}$ with transfer function $\mathbf{K}$, let $\rho > 0$ and assume that (A, B, C) is stabilizable and detectable. If $\mathbf{G}^{\mathbf{F}} \in H^\infty_{p\times m}$ for all $\mathbf{F} \in R_{\mathbb{R}}H^\infty_{m\times p}$ such that $\|\mathbf{F} - \mathbf{K}\|_{H^\infty} \leq \rho$, then (2.3) is L^2-GAS for all causal and weakly Lipschitz nonlinearities $\Phi : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^p) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ satisfying $\|\Phi - \mathcal{K}\| \leq \rho$.*

Proof By Proposition 2.1, $\rho < r_{\mathbb{C}}(\mathbf{G}^{\mathbf{K}})$, and thus, the claim follows from Theorem 3.1. □

Roughly speaking, the above corollary says that if we consider all real feedback operators in the ball of radius ρ centred at $\mathcal{K}$, then stability for all real linear compensators implies stability for all real nonlinear operators. As such, the corollary is reminiscent of the Aizerman conjecture.

The next result is a straightforward consequence of Proposition 2.1, Lemma 2.7 and Theorem 3.1.

Corollary 3.3 *Let $\mathcal{K} \in F_{\mathbb{R}}L_{m\times p}$ with transfer function $\mathbf{K}$, let $\rho > 0$, and assume that (A, B, C) is stabilizable and detectable. System (2.3) is L^2-GAS for all causal and weakly Lipschitz nonlinearities $\Phi : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^p) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ satisfying $\|\Phi - \mathcal{K}\| \le \rho$ if, and only if, $\rho < r_{\mathbb{C}}(\mathbf{G}^{\mathbf{K}})$.*

Let $\Phi : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^p) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ be causal and weakly Lipschitz, and let $\nu \ge 0$. It is a routine exercise to show that the operator $\Phi^\nu : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^p) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ defined by $\Phi^\nu(w) := e^{\nu\cdot}\Phi(e^{-\nu\cdot}w)$ is also causal and weakly Lipschitz. The operator Φ^ν is called the ν-*exponential weighting* of Φ and will play a key role in the context of exponential input-to-state stability (ISS), a concept which we will now define. We say that (2.3) is *exponentially ISS* if there exist positive constants Γ and γ such that, for all $(t_0, x^0, v, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times L^2([0, t_0], \mathbb{R}^p) \times L^\infty_{\mathrm{loc}}([t_0, \infty), \mathbb{R}^n)$, the solution $x = x(\cdot\,; t_0, x^0, v, u)$ of (2.3) satisfies

$$\|x(t)\| \le \Gamma\big[e^{-\gamma(t-t_0)}\big(\|x^0\| + \|v\|_{L^2(0,t_0)}\big) + \|u\|_{L^\infty(t_0,t)}\big] \quad \forall\, t \ge t_0. \tag{3.7}$$

A sufficient condition for exponential ISS of (2.3) is provided by the next result.

Corollary 3.4 *Let $\Phi : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^p) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ be causal and weakly Lipschitz and $\mathcal{K} \in F_{\mathbb{R}}L_{m\times p}$ with transfer function $\mathbf{K}$. If (A, B, C) is stabilizable and detectable, $\mathbf{G}^{\mathbf{K}} \in H^\infty_{p\times m}$ and there exists $\mu > 0$ such that*

$$\sup_{0\le\nu\le\mu} \|\Phi^\nu - \mathcal{K}\| < \frac{1}{\|\mathbf{G}^{\mathbf{K}}\|_{H^\infty}} = r_{\mathbb{C}}(\mathbf{G}^{\mathbf{K}}), \tag{3.8}$$

then (2.3) is exponentially ISS.

We identify a number of scenarios for which (3.8) is satisfied for sufficiently small $\mu > 0$.

Example 3.5 **(a)** Let $\varphi \in \mathcal{N}(\mathbb{R}_+ \times \mathbb{R}^p, \mathbb{R}^m)$, $K \in \mathbb{R}^{m\times p}$ and $\nu > 0$. Define $\varphi^\nu \in \mathcal{N}(\mathbb{R}_+ \times \mathbb{R}^p, \mathbb{R}^m)$ by $\varphi^\nu(t, z) := e^{\nu t}\varphi(t, e^{-\nu t}z)$ and note that $N^\nu_\varphi = N_{\varphi^\nu}$. As

$$\sup_{t\ge0,\, z\ne0} \frac{\|\varphi^\nu(t, z) - Kz\|}{\|z\|} = \sup_{t\ge0,\, z\ne0} \frac{\|\varphi(t, z) - Kz\|}{\|z\|},$$

we conclude that if

$$\sup_{t\ge0,\, z\ne0} \frac{\|\varphi(t, z) - Kz\|}{\|z\|} < r_{\mathbb{C}}(\mathbf{G}^K),$$

then

$$\|N^\nu_\varphi - N_K\| = \|N_{\varphi^\nu} - N_K\| \le \sup_{t\ge0,\, z\ne0} \frac{\|\varphi^\nu(t, z) - Kz\|}{\|z\|} < r_{\mathbb{C}}(\mathbf{G}^K),$$

and so (3.8) is satisfied for $\Phi = N_\varphi$, $\mathcal{K} = N_K$ and all $\mu > 0$.

Finally, we remark that whilst $\|N_\varphi^\nu(w) - N_\varphi(w)\| = \|N_{\varphi^\nu}(w) - N_\varphi(w)\| \to 0$ as $\nu \to 0$ for every $w \in L^2(\mathbb{R}_+, \mathbb{R}^p)$, in general, $\|N_\varphi^\nu - N_\varphi\|$ does not converge to 0 as $\nu \to 0$. In fact convergence in this sense is very rare: for time-independent φ, it can be shown that $\|N_\varphi^\nu - N_\varphi\| \to 0$ as $\nu \to 0$ if, and only if, φ is positively homogeneous of degree 1.

(b) Let $\mathcal{L}$ be a shift-invariant bounded linear operator from $L^2(\mathbb{R}_+, \mathbb{R}^p)$ into itself such that the transfer function $\mathbf{L}$ of $\mathcal{L}$ is bounded and holomorphic on $\mathbb{C}_{-\varepsilon}$ for some $\varepsilon > 0$. Then, for every $\nu \in (0, \varepsilon)$, $\mathcal{L}^\nu$ is a shift-invariant bounded linear operator from $L^2(\mathbb{R}_+, \mathbb{R}^p)$ into itself with transfer function $\mathbf{L}^\nu(s) = \mathbf{L}(s - \nu)$ and thus $\|\mathcal{L}^\nu\| = \sup_{\operatorname{Re} s > -\nu} \|\mathbf{L}(s)\|$. It follows from [3, Theorem 3.7] that $\|\mathbf{L}^\nu - \mathbf{L}\|_{H^\infty} \to 0$ as $\nu \to 0$, and so, $\|\mathcal{L}^\nu - \mathcal{L}\| \to 0$ as $\nu \to 0$. Let $\varphi \in \mathcal{N}(\mathbb{R}_+ \times \mathbb{R}^p, \mathbb{R}^m)$ and consider the operator $\Phi := N_\varphi \circ \mathcal{L}$. It is clear that $\Phi^\nu = N_{\varphi^\nu} \circ \mathcal{L}^\nu$ and thus

$$\|\Phi^\nu\| = \|(N_\varphi \circ \mathcal{L})^\nu\| \le \|\varphi\| \|\mathcal{L}^\nu\|. \tag{3.9}$$

Consequently, assuming that $\mathbf{G} \in H^\infty_{p\times m}$, the condition $\|\varphi\|\|\mathcal{L}\| < r_{\mathbb{C}}(\mathbf{G})$ is sufficient for (3.8) to hold with $\mathcal{K} = 0$ and for sufficiently small $\mu > 0$.

(c) In most instances, the inequality in (3.9) will of course be strict. Here, we consider a special case of part (b) for which equality holds. Let φ be time-independent and $\mathcal{L} = \mathcal{S}_\tau$, where $\tau > 0$ and $\mathcal{S}_\tau$ is the delay (or shift) operator defined in (2.2). It is clear that $\|\mathcal{S}_\tau^\nu\| = e^{\tau\nu}$ for all $\nu \ge 0$, and it can be proved that

$$\|(N_\varphi \circ \mathcal{S}_\tau)^\nu\| = \|\varphi\| \|\mathcal{S}_\tau^\nu\| = e^{\tau\nu} \|\varphi\|.$$

Hence, assuming that $\mathbf{G} \in H^\infty_{p\times m}$, the condition $\|\varphi\| < r_{\mathbb{C}}(\mathbf{G})$ is necessary and sufficient for (3.8) to hold with $\Phi = N_\varphi \circ \mathcal{S}_\tau$ and $\mathcal{K} = 0$ and for sufficiently small $\mu > 0$. ◊

Proof of Corollary 3.4. Let $\nu > 0$ and, for $(t_0, \zeta^0, \hat{v}, \hat{u}) \in \mathbb{R}_+ \times \mathbb{R}^n \times L^2([0, t_0], \mathbb{R}^p) \times L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, consider the initial-value problem

$$\dot{\zeta}(t) = (A + \nu I)\zeta(t) + B(\Phi^\nu([C\zeta]_{\hat{v}}))(t) + \hat{u}(t), \;\; t \ge t_0, \;\; \zeta(t_0) = \zeta^0. \tag{3.10}$$

The transfer function $\mathbf{G}^\nu$ of the linear system $(A + \nu I, B, C)$ is given by $\mathbf{G}^\nu(s) = \mathbf{G}(s - \nu)$. For sufficiently small $\nu > 0$, the linear system $(A + \nu I, B, C)$ is stabilizable and detectable, $\mathbf{G}^\nu(I - \mathbf{K}\mathbf{G}^\nu)^{-1}$ is stable, and as $\nu \to 0$, we have that $\|\mathbf{G}^\nu(I - \mathbf{K}\mathbf{G}^\nu)^{-1}\|_{H^\infty} \to \|\mathbf{G}^{\mathbf{K}}\|_{H^\infty}$. It follows from (3.8) and Theorem 3.1 that system (3.10) is L^2-GAS for all sufficiently small $\nu > 0$. Let $x = x(\cdot\,; t_0, x^0, v, u)$ be the solution of (2.3) and note that ζ given by $\zeta(t) := e^{\nu t}x(t)$ satisfies (3.10) with $\zeta^0 = e^{\nu t_0}x^0$, $\hat{v}(t) = e^{\nu t}v(t)$ and $\hat{u}(t) = e^{\nu t}u(t)$. Consequently, choosing $\nu > 0$ sufficiently small, there exists $\kappa > 0$ such that, for all $(t_0, x^0, v, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times L^2([0, t_0], \mathbb{R}^p) \times L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$,

$$\|e^{\nu t}x(t)\| \le \kappa\big(\|e^{\nu t_0}x^0\| + \|e^{\nu\,\cdot}v\|_{L^2(0,t_0)} + \|e^{\nu\,\cdot}u\|_{L^2(t_0,t)}\big) \;\; \forall\, t \ge t_0.$$

Now, $\|e^{\nu\,\cdot} v\|_{L^2(0,t_0)} \le e^{\nu t_0}\|v\|_{L^2(0,t_0)}$ and

$$\|e^{\nu\,\cdot} u\|_{L^2(t_0,t)} \le \frac{e^{\nu t}}{\sqrt{2\nu}}\|u\|_{L^\infty(t_0,t)} \quad \forall\, t \ge t_0,$$

and thus

$$\|x(t)\| \le \kappa\big[e^{-\nu(t-t_0)}\big(\|x^0\| + \|v\|_{L^2(0,t_0)}\big) + (2\nu)^{-1/2}\|u\|_{L^\infty(t_0,t)}\big] \quad \forall\, t \ge t_0.$$

Hence, (3.7) holds with $\Gamma = \kappa \max(1, (2\nu)^{-1/2})$ and $\gamma = \nu$, showing that (2.3) is exponentially ISS. □

The following result is an immediate consequence of Corollary 3.4 and part (a) of Example 3.5.

Corollary 3.6 *Let $\varphi \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^p, \mathbb{R}^m)$ and $K \in \mathbb{R}^{m\times p}$. If (A, B, C) is stabilizable and detectable, $\mathbf{G}^K \in H^\infty_{p\times m}$ and*

$$\sup_{t\ge 0,\, z\ne 0} \frac{\|\varphi(t,z) - Kz\|}{\|z\|} < \frac{1}{\|\mathbf{G}^K\|_{H^\infty}} = r_{\mathbb{C}}(\mathbf{G}^K),$$

then (2.4) is exponentially ISS; that is, there exist positive constants Γ and γ such that, for all $(t_0, x^0, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times L^\infty_{\mathrm{loc}}([t_0,\infty), \mathbb{R}^n)$, the solution $x = x(\,\cdot\,; t_0, x^0, u)$ of (2.4) satisfies

$$\|x(t)\| \le \Gamma\big(e^{-\gamma(t-t_0)}\|x^0\| + \|u\|_{L^\infty(t_0,t)}\big) \quad \forall\, t \ge t_0.$$

4 The circle criterion

In this section, we formulate a number of circle criteria for Lur'e systems of the form (2.3), including an ISS version. The textbook form of the circle criterion is contained in our considerations as a special case.

Throughout, let $\mathcal{K}_1, \mathcal{K}_2 \in F_{\mathbb{R}}L_{m\times p}$ be given and let $\Phi : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^p) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ be causal and weakly Lipschitz. If

$$\langle \pi_t(\Phi(w) - \mathcal{K}_1 w), \Phi(w) - \mathcal{K}_2 w\rangle_{L^2} \le 0 \quad \forall\, w \in L^2(\mathbb{R}_+, \mathbb{R}^p)\ \forall\, t \ge 0, \qquad (4.1)$$

then Φ is said to satisfy a *sector condition* (determined by the sector data $\mathcal{K}_1$ and $\mathcal{K}_2$). Similarly, we say that Φ satisfies a *strict sector condition* if

$$\sup_{w\in L^2,\, \pi_t w\ne 0,\, t\ge 0} \frac{\langle \pi_t(\Phi(w) - \mathcal{K}_1 w), \Phi(w) - \mathcal{K}_2 w\rangle_{L^2}}{\|\pi_t w\|^2_{L^2}} < 0. \qquad (4.2)$$

The next lemma relates the above sector conditions to L^2 norm conditions and facilitates the proofs of the main results of this section

Lemma 4.1 *Let $\mathcal{K}_1, \mathcal{K}_2 \in F_{\mathbb{R}} L_{m\times p}$ and set*

$$\mathcal{L} := \frac{1}{2}(\mathcal{K}_2 - \mathcal{K}_1) \quad \textit{and} \quad \mathcal{M} := \frac{1}{2}(\mathcal{K}_1 + \mathcal{K}_2). \tag{4.3}$$

(1) *Assume that* $\ker \mathcal{L} = \{0\}$. *The operator* Φ *satisfies (4.1) if, and only if,*

$$\sup_{w\in L^2,\, w\neq 0} \frac{\|\Phi(w) - \mathcal{M}w\|_{L^2}}{\|\mathcal{L}w\|_{L^2}} \le 1.$$

In particular, if (4.1) holds, then Φ *is linearly bounded.*

(2) *Assume that* $\ker \mathcal{L} = \{0\}$. *If*

$$\sup_{w\in L^2,\, \pi_t w\neq 0,\, t\ge 0} \frac{\langle \pi_t(\Phi(w) - \mathcal{K}_1 w), \Phi(w) - \mathcal{K}_2 w\rangle_{L^2}}{\|\pi_t w\|_{L^2}^2} \le -\varepsilon \tag{4.4}$$

for some $\varepsilon > 0$, *then*

$$\sup_{w\in L^2,\, w\neq 0} \frac{\|\Phi(w) - \mathcal{M}w\|_{L^2}}{\|\mathcal{L}w\|_{L^2}} \le 1 - \frac{\varepsilon}{\|\mathcal{L}\|^2}. \tag{4.5}$$

(3) *Assume that* $c := \inf_{\|w\|_{L^2}=1} \|\mathcal{L}w\|_{L^2} > 0$. *If*

$$\sup_{w\in L^2,\, w\neq 0} \frac{\|\Phi(w) - \mathcal{M}w\|_{L^2}}{\|\mathcal{L}w\|_{L^2}} \le \theta \tag{4.6}$$

for some $\theta \in (0, 1)$, *then*

$$\sup_{w\in L^2,\, w\neq 0} \frac{\langle \Phi(w) - \mathcal{K}_1 w, \Phi(w) - \mathcal{K}_2 w\rangle_{L^2}}{\|w\|_{L^2}^2} \le -c^2(1-\theta). \tag{4.7}$$

Proof Noting that $\mathcal{K}_1 = \mathcal{M} - \mathcal{L}$ and $\mathcal{K}_2 = \mathcal{M} + \mathcal{L}$, a straightforward calculation shows that

$$\begin{aligned} &\langle \Phi(w) - \mathcal{K}_1 w, \Phi(w) - \mathcal{K}_2 w\rangle_{L^2(0,t)} \\ &\quad = \|\Phi(w) - \mathcal{M}w\|_{L^2(0,t)}^2 - \|\mathcal{L}w\|_{L^2(0,t)}^2 \quad \forall\, w \in L^2(\mathbb{R}_+, \mathbb{R}^p), \quad \forall\, t \ge 0. \end{aligned} \tag{4.8}$$

Statement (1) is an immediate consequence of this identity. Furthermore, if (4.1) holds, then $\|\Phi(w)\|_{L^2} \le (\|\mathcal{M}\| + \|\mathcal{L}\|)\|w\|_{L^2}$ for all $w \in L^2(\mathbb{R}_+, \mathbb{R}^p)$.

To prove statement (2), assume that (4.4) is satisfied. It then follows from (4.8) that

$$\|\Phi(w) - \mathcal{M}w\|_{L^2}^2 - \|\mathcal{L}w\|_{L^2}^2 \le -\varepsilon\|w\|_{L^2}^2 \quad \forall\, w \in L^2(\mathbb{R}_+, \mathbb{R}^p),$$

implying that

$$\frac{\|\Phi(w)-\mathcal{M}w\|_{L^2}^2}{\|\mathcal{L}w\|_{L^2}^2} \le 1-\frac{\varepsilon}{\|\mathcal{L}\|^2} \quad \forall\, w \in L^2(\mathbb{R}_+,\mathbb{R}^p),\ \ w\neq 0,$$

establishing (4.5).

We proceed to prove statement (3). Assuming that (4.6) is satisfied, (4.8) yields

$$\begin{aligned}&\frac{\langle(\Phi(w)-\mathcal{K}_1 w),\Phi(w)-\mathcal{K}_2 w\rangle_{L^2}}{\|w\|_{L^2}^2}\\ &\quad\le -(1-\theta)\frac{\|\mathcal{L}w\|_{L^2}^2}{\|w\|_{L^2}^2} \le -c^2(1-\theta) \quad \forall\, w\in L^2(\mathbb{R}_+,\mathbb{R}^p),\ \ w\neq 0,\end{aligned}$$

showing that (4.7) holds. □

It is convenient to define the following sets of sector-bounded operators:

$S[\mathcal{K}_1,\mathcal{K}_2]$:= set of all causal weakly Lipschitz operators $L^2_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^p) \to L^2_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^m)$ satisfying the sector condition (4.1),

$S(\mathcal{K}_1,\mathcal{K}_2)$:= set of all causal weakly Lipschitz operators $L^2_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^p) \to L^2_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^m)$ satisfying the strict sector condition (4.2).

Obviously, $S(\mathcal{K}_1,\mathcal{K}_2)\subset S[\mathcal{K}_1,\mathcal{K}_2]$.

The following corollary is a straightforward consequence of Lemma 4.1.

Corollary 4.2 *The following statements hold.*

(1) *If* $\ker(\mathcal{K}_2-\mathcal{K}_1)=\{0\}$*, then* $\Phi\in S[\mathcal{K}_1,\mathcal{K}_2]$ *if, and only if,* $\Phi(L^2(\mathbb{R}_+,\mathbb{R}^p))\subset L^2(\mathbb{R}_+,\mathbb{R}^m)$ *and* $\langle\Phi(w)-\mathcal{K}_1 w,\Phi(w)-\mathcal{K}_2 w\rangle_{L^2}\le 0$ *for all* $w\in L^2(\mathbb{R}_+,\mathbb{R}^p)$.

(2) *If* $\inf_{\|w\|_{L^2}=1}\|(\mathcal{K}_2-\mathcal{K}_1)w\|_{L^2} > 0$*, then* $\Phi\in S(\mathcal{K}_1,\mathcal{K}_2)$ *if, and only if,* $\Phi(L^2(\mathbb{R}_+,\mathbb{R}^p))\subset L^2(\mathbb{R}_+,\mathbb{R}^m)$ *and*

$$\sup_{w\in L^2,\, w\neq 0}\frac{\langle\Phi(w)-\mathcal{K}_1 w,\Phi(w)-\mathcal{K}_2 w\rangle_{L^2}}{\|w\|_{L^2}^2}<0.$$

Let $\varphi\in\mathcal{C}(\mathbb{R}_+\times\mathbb{R}^p,\mathbb{R}^m)$ and $K_1,K_2\in\mathbb{R}^{m\times p}$ and note that

- if
$$\langle\varphi(t,z)-K_1 z,\varphi(t,z)-K_2 z\rangle\le 0 \quad \forall\, t\ge 0,\ \forall\, z\in\mathbb{R}^p, \tag{4.9}$$
then $N_\varphi\in S[N_{K_1},N_{K_2}]$;
- if φ satisfies
$$\sup_{t\ge 0,\, z\neq 0}\frac{\langle\varphi(t,z)-K_1 z,\varphi(t,z)-K_2 z\rangle}{\|z\|^2}<0, \tag{4.10}$$
then $N_\varphi\in S(N_{K_1},N_{K_2})$.

Throughout the rest of the section, let $\mathbf{K}_1$ and $\mathbf{K}_2$ be the transfer functions of the operators $\mathcal{K}_1 \in F_{\mathbb{R}}L_{m\times p}$ and $\mathcal{K}_2 \in F_{\mathbb{R}}L_{m\times p}$, respectively.

To formulate the main result of this section, we introduce the following assumption.

(A) There exists an operator $\mathcal{K}^\# \in F_{\mathbb{R}}L_{p\times m}$ such that $\mathcal{K}^\#(\mathcal{K}_2 - \mathcal{K}_1) = I$ and $\|(\mathcal{K}_2 - \mathcal{K}_1)\mathcal{K}^\#\| \leq 1$.

Assumption (A) is equivalent to the existence of a rational matrix $\mathbf{K}^\# \in R_{\mathbb{R}}H^\infty_{p\times m}$ such that $\mathbf{K}^\#(\mathbf{K}_2 - \mathbf{K}_1) = I$ and $\|(\mathbf{K}_2 - \mathbf{K}_1)\mathbf{K}^\#\|_{H^\infty} \leq 1$. If assumption (A) holds, then, for all $y \in \operatorname{im}(\mathcal{K}_2 - \mathcal{K}_1)$, we have that $(\mathcal{K}_2 - \mathcal{K}_1)\mathcal{K}^\# y = y$, showing that $\|(\mathcal{K}_2 - \mathcal{K}_1)\mathcal{K}^\#\| = 1$, and consequently, $\|(\mathbf{K}_2 - \mathbf{K}_1)\mathbf{K}^\#\|_{H^\infty} = 1$.

Below we describe some situations in which assumption (A) is satisfied.

Example 4.3 **(a)** Assume that the sector data $\mathcal{K}_1$ and $\mathcal{K}_2$ are static, that is, $\mathcal{K}_1 = N_{K_1}$ and $\mathcal{K}_2 = N_{K_2}$ for some $K_1, K_2 \in \mathbb{R}^{m\times p}$. If $\ker(K_2 - K_1) = \{0\}$, then, setting $K := K_2 - K_1$, the matrix $K^T K$ is invertible and $K^\# := (K^T K)^{-1} K^T$ is a left-inverse of K. Moreover, $KK^\#$ is the orthogonal projection onto $\operatorname{im} K$ along $\ker K^\# = \ker K^T = (\operatorname{im} K)^\perp$, and thus, $\|KK^\#\| = 1$. It follows that assumption (A) holds for N_{K_1} and N_{K_2}, provided that $\ker(K_2 - K_1) = \{0\}$.

(b) Assume that $m = p$. Invertibility of $\mathbf{K}_2 - \mathbf{K}_1$ in $R_{\mathbb{R}}H^\infty_{m\times m}$ is equivalent to

$$\inf_{s\in\mathbb{C}_0} |\det(\mathbf{K}_2(s) - \mathbf{K}_1(s))| > 0. \tag{4.11}$$

Consequently, assumption (A) is satisfied if, and only if, (4.11) holds. In other words, assumption (A) is equivalent to $\det(\mathbf{K}_2 - \mathbf{K}_1)$ being a unit in the ring $R_{\mathbb{R}}H^\infty$.

(c) Assume that there exist $K_1, K_2 \in \mathbb{R}^{m\times p}$ and $\mathcal{K} \in F_{\mathbb{R}}L_{p\times p}$ such that $\mathcal{K}_1 = N_{K_1} \circ \mathcal{K}$ and $\mathcal{K}_2 = N_{K_2} \circ \mathcal{K}$. Assumption (A) holds if $\ker(K_2 - K_1) = \{0\}$ and $\inf_{s\in\mathbb{C}_0} |\det \mathbf{K}(s)| > 0$, where $\mathbf{K}$ is the transfer function of $\mathcal{K}$.

(d) Assume that there exist $K \in \mathbb{R}^{m\times p}$ and $\mathcal{L}_1, \mathcal{L}_2 \in F_{\mathbb{R}}L_{p\times p}$ such that $\mathcal{K}_1 = N_K \circ \mathcal{L}_1$ and $\mathcal{K}_2 = N_K \circ \mathcal{L}_2$. Assumption (A) holds if $\ker K = \{0\}$ and $\inf_{s\in\mathbb{C}_0} |\det(\mathbf{L}_2(s) - \mathbf{L}_1(s))| > 0$, where $\mathbf{L}_1$ and $\mathbf{L}_2$ are the transfer functions of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. $\diamond$

We are now in the position to state and prove the following circle criterion for Lur'e systems of the form (2.3).

Theorem 4.4 (Circle criterion: L^2- GAS version) *Assume that* (A, B, C) *is stabilizable and detectable and that assumption* (A) *holds. The following statements hold.*

(1) *If* $(I - \mathbf{K}_1\mathbf{G})(I - \mathbf{K}_2\mathbf{G})^{-1}$ *is positive real, then (2.3) is* L^2*-GAS for all* $\Phi \in S(\mathcal{K}_1, \mathcal{K}_2)$.

(2) *If* $(I - \mathbf{K}_1\mathbf{G})(I - \mathbf{K}_2\mathbf{G})^{-1}$ *is strictly positive real, then (2.3) is* L^2*-GAS for all* $\Phi \in S[\mathcal{K}_1, \mathcal{K}_2]$.

Proof Let $\mathcal{L}$ and $\mathcal{M}$ be as in (4.3). By assumption (A), the operator $\mathcal{L}^\# := 2\mathcal{K}^\# \in F_{\mathbb{R}}L_{p\times m}$ is a left-inverse of $\mathcal{L}$ and such that $\|\mathcal{L}\mathcal{L}^\#\| \leq 1$. Setting $\mathbf{H} := (I - \mathbf{K}_1\mathbf{G})(I - \mathbf{K}_2\mathbf{G})^{-1}$, then by positive realness of $\mathbf{H}$, the rational matrix $\mathbf{H} + I$ is invertible (see statement (1) of Lemma 2.2), and a routine calculation shows that

$$(\mathbf{H} - I)(\mathbf{H} + I)^{-1} = \mathbf{L}\mathbf{G}(I - \mathbf{M}\mathbf{G})^{-1} = \mathbf{L}\mathbf{G}^{\mathbf{M}}, \tag{4.12}$$

where $\mathbf{L}$ and $\mathbf{M}$ are the transfer functions of $\mathcal{L}$ and $\mathcal{M}$, respectively.

(1) Let $\Phi \in S(\mathcal{K}_1, \mathcal{K}_2)$. An application of statement (2) of Lemma 4.1 shows that there exists $\theta \in (0, 1)$ such that

$$\|\Phi(y) - \mathcal{M}y\|_{L^2} \leq \theta \|\mathcal{L}y\|_{L^2} \quad \forall\, y \in L^2(\mathbb{R}_+, \mathbb{R}^p).$$

Defining a causal weakly Lipschitz operator $\Psi : L^2(\mathbb{R}_+, \mathbb{R}^m) \to L^2(\mathbb{R}_+, \mathbb{R}^m)$ by

$$\Psi(w) := \Phi(\mathcal{L}^\# w) - \mathcal{M}\mathcal{L}^\# w \quad \forall\, w \in L^2(\mathbb{R}_+, \mathbb{R}^m), \tag{4.13}$$

we obtain that

$$\|\Psi(w)\|_{L^2} \leq \theta \|\mathcal{L}\mathcal{L}^\# w\|_{L^2} \leq \theta \|w\|_{L^2} \quad \forall\, w \in L^2(\mathbb{R}_+, \mathbb{R}^m),$$

where we have used that $\|\mathcal{L}\mathcal{L}^\#\| \leq 1$. By hypothesis, $\mathbf{H}$ is positive real and thus it follows from (4.12) that $\|\mathbf{L}\mathbf{G}^{\mathbf{M}}\|_{H^\infty} \leq 1$ (see statement (1) of Lemma 2.2). We may now conclude that

$$\|\Psi\| \leq \frac{\theta}{\|\mathbf{L}\mathbf{G}^{\mathbf{M}}\|_{H^\infty}} < \frac{1}{\|\mathbf{L}\mathbf{G}^{\mathbf{M}}\|_{H^\infty}} = r_{\mathbb{C}}(\mathbf{L}\mathbf{G}^{\mathbf{M}}). \tag{4.14}$$

Let $(A_\mathcal{L}, B_\mathcal{L}, C_\mathcal{L}, D_\mathcal{L})$ and $(A_\mathcal{M}, B_\mathcal{M}, C_\mathcal{M}, D_\mathcal{M})$ be minimal realizations of $\mathbf{L}$ and $\mathbf{M}$, respectively, with state dimensions denoted by $n_\mathcal{L}$ and $n_\mathcal{M}$. Set $\tilde{n} := n + n_\mathcal{L} + n_\mathcal{M}$ and define $\tilde{A} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$, $\tilde{B} \in \mathbb{R}^{\tilde{n}\times m}$ and $\tilde{C} \in \mathbb{R}^{p\times\tilde{n}}$ by

$$\tilde{A} := \begin{pmatrix} A + BD_\mathcal{M}C & BC_\mathcal{M} & 0 \\ B_\mathcal{M}C & A_\mathcal{M} & 0 \\ B_\mathcal{L}C & 0 & A_\mathcal{L} \end{pmatrix}, \quad \tilde{B} := \begin{pmatrix} B \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{C} := (D_\mathcal{L}C, 0, C_\mathcal{L}). \tag{4.15}$$

For $(t_0, \zeta^0, \tilde{v}, \tilde{u}) \in \mathbb{R}_+ \times \mathbb{R}^{\tilde{n}} \times L^2([0, t_0], \mathbb{R}^p) \times L^2([t_0, \infty), \mathbb{R}^{\tilde{n}})$, consider the initial-value problem

$$\dot{\zeta}(t) = \tilde{A}\zeta(t) + \tilde{B}(\Psi([\tilde{C}]_{\tilde{v}}))(t) + \tilde{u}(t), \quad t \geq t_0, \quad \zeta(t_0) = \zeta^0. \tag{4.16}$$

We now proceed in two steps.

Step 1: (4.16) is L^2-GAS. As $\mathbf{L}\mathbf{G}^{\mathbf{M}}$ is the transfer function of $(\tilde{A}, \tilde{B}, \tilde{C})$, it follows from (4.14) and Theorem 3.1 that (4.16) is L^2-GAS (where the roles of $\mathbf{G}$ and $\mathbf{K}$ in Theorem 3.1 are played by $\mathbf{L}\mathbf{G}^{\mathbf{M}}$ and 0, respectively), provided that $(\tilde{A}, \tilde{B}, \tilde{C})$ is stabilizable and detectable. We will show that $\tilde{A}$ is Hurwitz, which trivially implies stabilizability and detectability of $(\tilde{A}, \tilde{B}, \tilde{C})$. To this end, define

$$\hat{A} := \begin{pmatrix} A + BD_\mathcal{M}C & BC_\mathcal{M} \\ B_\mathcal{M}C & A_\mathcal{M} \end{pmatrix}, \quad \hat{B} := \begin{pmatrix} B \\ 0 \end{pmatrix}, \quad \hat{C} := (C, 0)$$

and note that $\mathbf{G}^{\mathbf{M}}$ is the transfer function of $(\hat{A}, \hat{B}, \hat{C})$. Now, $\mathbf{L}\mathbf{G}^{\mathbf{M}}$ is stable and so $\mathbf{G}^{\mathbf{M}}$ is stable because the transfer function of $\mathcal{L}^{\#}$ is a stable left-inverse of $\mathbf{L}$. It now follows as in Step 2 of the proof of Theorem 3.1 that $\hat{A}$ is Hurwitz which in turn implies that $\tilde{A}$ is also Hurwitz.

Step 2: (2.3) *is L^2-GAS.* For $(x^0, v, u) \in \mathbb{R}^n \times L^2([0, t_0], \mathbb{R}^p) \times L^2_{\mathrm{loc}}([t_0, \infty), \mathbb{R}^n)$, let $x = x(\cdot\,; x^0, v, u)$ be the solution of (2.3). Defining

$$\begin{aligned} z_{\mathcal{M}}(t) &:= \int_0^t e^{A_{\mathcal{M}}(t-\theta)} B_{\mathcal{M}}[Cx]_v(\theta)\mathrm{d}\theta \quad \text{and} \\ z_{\mathcal{L}}(t) &:= \int_0^t e^{A_{\mathcal{L}}(t-\theta)} B_{\mathcal{L}}[Cx]_v(\theta)\mathrm{d}\theta; \quad \forall\, t \geq 0, \end{aligned}$$

we have that

$$\dot{z}_{\mathcal{M}}(t) = A_{\mathcal{M}} z_{\mathcal{M}}(t) + B_{\mathcal{M}} Cx(t) \quad \text{and} \quad \dot{z}_{\mathcal{L}}(t) = A_{\mathcal{L}} z_{\mathcal{L}}(t) + B_{\mathcal{L}} Cx(t); \quad \forall\, t \geq t_0. \tag{4.17}$$

Setting $\zeta := \mathrm{col}(x, z_{\mathcal{M}}, z_{\mathcal{L}})$, we conclude that

$$\begin{aligned} (\mathcal{M}[Cx]_v)(t) = C_{\mathcal{M}} z_{\mathcal{M}}(t) + D_{\mathcal{M}} Cx(t), \quad &(\mathcal{L}[Cx]_v)(t) \\ = C_{\mathcal{L}} z_{\mathcal{L}}(t) + D_{\mathcal{L}} Cx(t) &= \tilde{C}\zeta(t); \quad \forall\, t \geq t_0. \end{aligned} \tag{4.18}$$

Furthermore, there exists $c \geq 0$ (not depending on (t_0, x^0, v, u)) such that

$$\|z_{\mathcal{M}}(t_0)\| + \|z_{\mathcal{L}}(t_0)\| \leq c\|v\|_{L^2(0,t_0)}. \tag{4.19}$$

It follows from (4.18) that $[\tilde{C}\zeta]_{\mathcal{L}v} = \mathcal{L}[Cx]_v$. Consequently,

$$\Psi([\tilde{C}\zeta]_{\mathcal{L}v}) = \Phi(\mathcal{L}^{\#}[\tilde{C}\zeta]_{\mathcal{L}v}) - \mathcal{M}\mathcal{L}^{\#}[\tilde{C}\zeta]_{\mathcal{L}v} = \Phi([Cx]_v) - \mathcal{M}[Cx]_v,$$

implying that

$$(\Psi([\tilde{C}\zeta]_{\mathcal{L}v}))(t) = (\Phi([Cx]_v))(t) - C_{\mathcal{M}} z_{\mathcal{M}}(t) - D_{\mathcal{M}} Cx(t), \quad \forall\, t \geq t_0.$$

Therefore, on $[t_0, \infty)$,

$$\dot{x} = Ax + B\Phi[Cx]_v + u = Ax + BD_{\mathcal{M}} Cx + BC_{\mathcal{M}} z_{\mathcal{M}} + B\Psi[\tilde{C}\zeta]_{\mathcal{L}v} + u,$$

and combining this with (4.17) shows that ζ satisfies (4.16) with $\zeta^0 = \mathrm{col}\,(x^0, z_{\mathcal{M}}(t_0), z_{\mathcal{L}}(t_0))$, $\tilde{v} = \mathcal{L}v$ and $\tilde{u} = \mathrm{col}\,(u, 0, 0)$. Using (4.19) and the fact that (4.16) is L^2-GAS shows that (2.3) is L^2-GAS.

(2) Set

$$\mathcal{K} := 2\mathcal{L} = \mathcal{K}_2 - \mathcal{K}_1, \tag{4.20}$$

denote the transfer function of $\mathcal{K}$ by $\mathbf{K}$ and, for $\rho \geq 0$, define

$$\mathbf{H}_\rho := (I - (\mathbf{K}_1 - \rho\mathbf{K})\mathbf{G})(I - (\mathbf{K}_2 + \rho\mathbf{K})\mathbf{G})^{-1}.$$

The idea is to prove that, for sufficiently small $\rho > 0$, the conditions of statement (1) hold with $\mathcal{K}_1$ and $\mathcal{K}_2$ replaced by $\mathcal{K}_1 - \rho\mathcal{K}$ and $\mathcal{K}_2 + \rho\mathcal{K}$, respectively. A routine calculation shows that

$$(\mathbf{H}_\rho - I)(\mathbf{H}_\rho + I)^{-1} = (\mathbf{K}_2 - \mathbf{K}_1 + 2\rho\mathbf{K})\mathbf{G}(2I - (\mathbf{K}_1 + \mathbf{K}_2)\mathbf{G})^{-1} = (1 + 2\rho)\mathbf{L}\mathbf{G}^{\mathbf{M}}.$$

By hypothesis, $\mathbf{H}_0 = \mathbf{H}$ is strictly positive real and hence does not have any poles in $\overline{\mathbb{C}}_0$. As $\mathbf{H}_0(\infty) = I$, $\mathbf{H}_0$ is also holomorphic at ∞ and we conclude that $\mathbf{H}_0$ is stable. It follows from [9, Corollary 4.5] that $\mathbf{H}_0$ is strongly positive real, and an application of statement (2) of Lemma 2.2 shows that

$$\|(\mathbf{H}_0 - I)(\mathbf{H}_0 + I)^{-1}\|_{H^\infty} = \|\mathbf{L}\mathbf{G}^{\mathbf{M}}\|_{H^\infty} < 1.$$

Consequently,

$$\|(\mathbf{H}_\rho - I)(\mathbf{H}_\rho + I)^{-1}\|_{H^\infty} \leq 1 \quad \forall\, \rho \in [0, \rho^*], \quad \text{where } \ \rho^* := \frac{1}{2}\big(\|\mathbf{L}\mathbf{G}^{\mathbf{M}}\|_{H^\infty}^{-1} - 1\big),$$

and thus, $\mathbf{H}_\rho$ is positive real for every $\rho \in [0, \rho^*]$ by statement (1) of Lemma 2.2.

Furthermore, a straightforward calculation shows that, for arbitrary $\rho \geq 0$ and for all $w \in L^2(\mathbb{R}_+, \mathbb{R}^p)$,

$$\begin{aligned}&\langle \Phi(w) - (\mathcal{K}_1 - \rho\mathcal{K})w, \Phi(w) - (\mathcal{K}_2 + \rho\mathcal{K})w\rangle_{L^2}\\ &\quad = \langle \Phi(w) - \mathcal{K}_1 w, \Phi(w) - \mathcal{K}_2 w\rangle_{L^2} - \rho(1+\rho)\|\mathcal{K}w\|_{L^2}^2.\end{aligned}$$

By assumption (A), there exists $c > 0$ such that $\|\mathcal{K}w\|_{L^2} \geq c\|w\|_{L^2}$ for all $w \in L^2(\mathbb{R}_+, \mathbb{R}^p)$, and thus,

$$\begin{aligned}\langle \Phi(w) - (\mathcal{K}_1 - \rho\mathcal{K})w, \Phi(w) - (\mathcal{K}_2 + \rho\mathcal{K})w\rangle_{L^2} \ &\leq -c^2\rho(1+\rho)\|w\|_{L^2}^2\\ &\qquad \forall\, w \in L^2(\mathbb{R}_+, \mathbb{R}^p),\end{aligned}$$

showing that $\Phi \in S(\mathcal{K}_1 - \rho\mathcal{K}, \mathcal{K}_2 + \rho\mathcal{K})$ for every $\rho > 0$. An application of statement (1) (with K_1 and K_2 replaced by $K_1 - \rho K$ and $K_2 + \rho K$, respectively, $0 < \rho \leq \rho^*$) shows that (2.3) is L^2-GAS. □

Let $S^{\mathrm{e}}[\mathcal{K}_1, \mathcal{K}_2]$ be the subset of all operators Φ in $S[\mathcal{K}_1, \mathcal{K}_2]$ for which there exists $\mu = \mu(\Phi) > 0$ such that $\Phi^\nu \in S[\mathcal{K}_1, \mathcal{K}_2]$ for all $\nu \in [0, \mu]$. Furthermore, $S^{\mathrm{e}}(\mathcal{K}_1, \mathcal{K}_2)$ is the set of all operators $\Phi \in S(\mathcal{K}_1, \mathcal{K}_2)$ for which there exists $\mu = \mu(\Phi) > 0$ such that $\Phi^\nu \in S(\mathcal{K}_1, \mathcal{K}_2)$ for all $\nu \in [0, \mu]$ and

$$\sup_{w \in L^2,\, w \neq 0,\, \nu \in [0,\mu]} \frac{\langle \Phi^\nu(w) - \mathcal{K}_1 w, \Phi^\nu(w) - \mathcal{K}_2 w\rangle_{L^2}}{\|w\|_{L^2}^2} < 0.$$

We now turn our attention to exponential ISS. Recall that (2.3) is said to be exponentially ISS if there exist positive constants Γ and γ such that (3.7) holds for all $(t_0, x^0, v, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times L^2([0, t_0], \mathbb{R}^p) \times L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^n)$.

Corollary 4.5 (Circle criterion: ISS version) *Assume that* (A, B, C) *is stabilizable and detectable and that assumption* (A) *holds. The following statements hold.*

(1) *If* $(I - \mathbf{K}_1\mathbf{G})(I - \mathbf{K}_2\mathbf{G})^{-1}$ *is positive real, then (2.3) is exponentially ISS for all* $\Phi \in S^{\text{e}}(\mathcal{K}_1, \mathcal{K}_2)$.

(2) *If* $(I - \mathbf{K}_1\mathbf{G})(I - \mathbf{K}_2\mathbf{G})^{-1}$ *is strictly positive real, then (2.3) is exponentially ISS for all* $\Phi \in S^{\text{e}}[\mathcal{K}_1, \mathcal{K}_2]$.

Proof **(1)** Let $\Phi \in S^{\text{e}}(\mathcal{K}_1, \mathcal{K}_2)$. Then, there exist $\mu > 0$ and $\varepsilon > 0$ such that $\Phi^\nu \in S(\mathcal{K}_1, \mathcal{K}_2)$ for all $\nu \in [0, \mu]$ and

$$\sup_{w \in L^2,\, w \neq 0,\, \nu \in [0,\mu]} \frac{\langle \Phi^\nu(w) - \mathcal{K}_1 w, \Phi^\nu(w) - \mathcal{K}_2 w \rangle_{L^2}}{\|w\|^2_{L^2}} \leq -\varepsilon.$$

Defining $\mathcal{L}$ and $\mathcal{M}$ as in (4.3), an application of statement (2) of Lemma 4.1 shows that there exists $\theta \in (0, 1)$ such that

$$\|\Phi^\nu(y) - \mathcal{M}y\|_{L^2} \leq \theta \|\mathcal{L}y\|_{L^2} \quad \forall\, y \in L^2(\mathbb{R}_+, \mathbb{R}^p), \quad \forall\, \nu \in [0, \mu].$$

By assumption (A), there exists $\mathcal{L}^\# \in F_{\mathbb{R}}L_{p\times m}$ such that $\mathcal{L}^\#\mathcal{L} = I$ and $\|\mathcal{L}\mathcal{L}^\#\| \leq 1$. Defining Ψ as in (4.13), it is clear that $\Psi^\nu(w) = \Phi^\nu((\mathcal{L}^\#)^\nu w) - \mathcal{M}^\nu(\mathcal{L}^\#)^\nu w$ for all $w \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ and $\nu \in [0, \mu]$, and, furthermore,

$$\begin{aligned} \|\Psi^\nu(w)\| &\leq \theta \|\mathcal{L}(\mathcal{L}^\#)^\nu w\| + \|\mathcal{M}^\nu - \mathcal{M}\| \|(\mathcal{L}^\#)^\nu\| \|w\| \\ &\qquad \forall\, w \in L^2(\mathbb{R}_+, \mathbb{R}^m), \quad \forall\, \nu \in [0, \mu]. \end{aligned}$$

Since $\theta \in (0, 1)$, $\|\mathcal{L}\mathcal{L}^\#\| \leq 1$ and $\|\mathcal{M}^\nu - \mathcal{M}\| + \|(\mathcal{L}^\#)^\nu - \mathcal{L}^\#\| \to 0$ as $\nu \to 0$, we conclude that there exist $\nu^\dagger \in (0, \mu]$ and $\theta^\dagger \in (\theta, 1)$ such that

$$\|\Psi^\nu(w)\| \leq \theta^\dagger \|w\| \quad \forall\, w \in L^2(\mathbb{R}_+, \mathbb{R}^m), \quad \forall\, \nu \in [0, \nu^\dagger].$$

As in the proof of statement (1) of Theorem 4.4, we have that $\|\mathbf{L}\mathbf{G}^{\mathbf{M}}\|_{H^\infty} \leq 1$ (where $\mathbf{L}$ and $\mathbf{M}$ are the transfer functions of $\mathcal{L}$ and $\mathcal{M}$, respectively) and thus

$$\sup_{0 \leq \nu \leq \nu^\dagger} \|\Psi^\nu\| \leq \frac{\theta^\dagger}{\|\mathbf{L}\mathbf{G}^{\mathbf{M}}\|_{H^\infty}} < \frac{1}{\|\mathbf{L}\mathbf{G}^{\mathbf{M}}\|_{H^\infty}} = r_{\mathbb{C}}(\mathbf{L}\mathbf{G}^{\mathbf{M}}).$$

An application of Corollary 3.4 to the system (4.16) (where the roles of Φ, $\mathbf{G}$ and $\mathbf{K}$ in Corollary 3.4 are played by Ψ, $\mathbf{L}\mathbf{G}^{\mathbf{M}}$ and 0, respectively) yields that (4.16) is exponentially ISS. The exponential ISS property of (2.3) follows in the same way as L^2-GAS of (2.3) was derived in the proof of statement (1) of Theorem 4.4.

(2) Defining $\mathcal{K}$ as in (4.20), denoting the transfer function of $\mathcal{K}$ by $\mathbf{K}$ and using the arguments from the proof of statement (2) of Theorem 4.4, it can be shown that, for sufficiently small $\rho > 0$, the rational matrix $(I - (\mathbf{K}_1 - \rho\mathbf{K})\mathbf{G})(I - (\mathbf{K}_2 + \rho\mathbf{K})\mathbf{G})^{-1}$ is positive real and $\Phi \in S^{\mathrm{e}}(\mathcal{K}_1 - \rho\mathcal{K}, \mathcal{K}_2 + \rho\mathcal{K})$. It now follows from statement (1) that (2.3) is exponentially ISS. □

We note that if $\varphi \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^p, \mathbb{R}^m)$ and $\nu \geq 0$, then φ^ν defined by $\varphi^\nu(t, z) := e^{\nu t}\varphi(t, e^{-\nu t}z)$ is in $\mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^p, \mathbb{R}^m)$. If φ satisfies (4.9), then, trivially, for all $\nu \geq 0$,

$$\langle \varphi^\nu(t, z) - K_1 z, \varphi^\nu(t, z) - K_2 z\rangle \leq 0 \quad \forall\, t \geq 0,\ \forall\, z \in \mathbb{R}^p.$$

Similarly, if (4.10) holds, then, for all $\nu \geq 0$,

$$\begin{aligned}&\sup_{t\geq 0,\, z\neq 0} \frac{\langle \varphi^\nu(t, z) - K_1 z, \varphi^\nu(t, z) - K_2 z\rangle}{\|z\|^2}\\ &= \sup_{t\geq 0,\, z\neq 0} \frac{\langle \varphi(t, z) - K_1 z, \varphi(t, z) - K_2 z\rangle}{\|z\|^2} < 0.\end{aligned}$$

As $N_\varphi^\nu = N_{\varphi^\nu}$, it follows that $N_\varphi \in S^{\mathrm{e}}[N_{K_1}, N_{K_2}]$ or $N_\varphi \in S^{\mathrm{e}}(N_{K_1}, N_{K_2})$, whenever φ satisfies (4.9) or (4.10), respectively. Combining this with Corollary 4.5 leads to the following result.

Corollary 4.6 (Circle criterion: ISS version for static nonlinearities) *Let $K_1, K_2 \in \mathbb{R}^{m\times p}$. Assume that (A, B, C) is stabilizable and detectable and that* $\ker(K_2 - K_1) = \{0\}$. *The following statements hold.*

(1) *If $(I - K_1\mathbf{G})(I - K_2\mathbf{G})^{-1}$ is positive real, then (2.4) is exponentially ISS for every $\varphi \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^p, \mathbb{R}^m)$ satisfying (4.10).*

(2) *If $(I - K_1\mathbf{G})(I - K_2\mathbf{G})^{-1}$ is strictly positive real, then (2.4) is exponentially ISS for every $\varphi \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^p, \mathbb{R}^m)$ satisfying (4.9).*

The textbook version of the circle criterion given in [2, 11, 17, 28] is essentially statement (2) of Corollary 4.6 in the absence of forcing ($u = 0$); that is, the stability conclusion in the textbook version is global exponential stability for the unforced version of (2.4). The proofs in [2, 11, 17, 28] are based on the positive-real lemma and Lyapunov theory, whilst Corollaries 4.5 and 4.6 have been derived by a small-gain argument.

If, in statement (1), the sector condition (4.10) is replaced by the weaker sector condition

$$\sup_{t\geq 0}\langle \varphi(t, z) - K_1 z, \varphi(t, z) - K_2 z\rangle \leq -\alpha(\|z\|)\|z\| \quad \forall\, z \in \mathbb{R}^p, \tag{4.21}$$

where $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is an arbitrary $\mathcal{K}_\infty$ function (that is, α is continuous, strictly increasing and surjective), then, in general, system (2.4) is not exponentially ISS, but it is still ISS in the usual sense (see, for example, [4, 25, 26]) for all φ satisfying (4.21) [22, Corollary 3.10]. Furthermore, if in statement (1) of Corollary 4.5 the sector condition (4.10) is further weakened to

$$\sup_{t\geq 0}\langle \varphi(t, z) - K_1 z, \varphi(t, z) - K_2 z\rangle < 0 \quad \forall\, z \in \mathbb{R}^p,\ z \neq 0, \tag{4.22}$$

then it can be shown (by using Lyapunov methods) that (2.4) with $u = 0$ is GAS for all φ satisfying (4.22) [23, Corollary 11]. (We remark that [23] considers discrete-time systems, but it is easy to see that the results carry over to the continuous-time case.)

5 Necessity of the circle criterion

Here, we will investigate scenarios in which the (strict) positive realness of $(I - \mathbf{K}_1\mathbf{G})(I - \mathbf{K}_2\mathbf{G})^{-1}$ is necessary for absolute stability. In particular, we will show that the positive real conditions in Theorem 4.4 are necessary for absolute stability with respect to all real nonlinear causal operators satisfying the relevant L^2-sector condition. Throughout this section, let $\mathcal{K}_1, \mathcal{K}_2 \in F_{\mathbb{R}}L_{m\times p}$ and let $\mathbf{K}_1$ and $\mathbf{K}_2$ denote the transfer functions of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively

Proposition 5.1 *Let (A, B, C) be stabilizable and detectable.*

(1) *Assume that* $\inf_{\|w\|_{L^2}=1} \|(\mathcal{K}_2 - \mathcal{K}_1)w\|_{L^2} > 0$. *If the origin of (2.3) is globally attractive at time* 0 *for all* $\Phi \in F_{\mathbb{R}}L_{m\times p} \cap S(\mathcal{K}_1, \mathcal{K}_2)$, *then* $(I - \mathbf{K}_1\mathbf{G})(I - \mathbf{K}_2\mathbf{G})^{-1}$ *is positive real.*

(2) *Assume that* $\ker(\mathcal{K}_2 - \mathcal{K}_1) = \{0\}$. *If the origin of (2.3) is globally attractive at time* 0 *for all* $\Phi \in F_{\mathbb{R}}L_{m\times p} \cap S[\mathcal{K}_1, \mathcal{K}_2]$, *then* $(I - \mathbf{K}_1\mathbf{G})(I - \mathbf{K}_2\mathbf{G})^{-1}$ *is strictly and strongly positive real.*

Proof Define the linear operators $\mathcal{L}$ and $\mathcal{M}$ as in (4.3) with corresponding minimal state-space realizations $(A_\mathcal{L}, B_\mathcal{L}, C_\mathcal{L}, D_\mathcal{L})$ and $(A_\mathcal{M}, B_\mathcal{M}, C_\mathcal{M}, D_\mathcal{M})$ and transfer functions $\mathbf{L}$ and $\mathbf{M}$, respectively. Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be defined as in (4.15) and consider the initial-value problem

$$\dot{\zeta} = \tilde{A}\zeta(t) + \tilde{B}(\Psi(\tilde{C}\zeta))(t), \quad t \geq 0, \quad \zeta(0) = \zeta^0 \tag{5.1}$$

with $\Psi \in F_{\mathbb{R}}L_{m\times m}$. Denoting the transfer function of $(\tilde{A}, \tilde{B}, \tilde{C})$ by $\tilde{\mathbf{G}}$, we have that $\tilde{\mathbf{G}} = \mathbf{L}\mathbf{G}^\mathbf{M}$. Setting $\mathbf{H} := (I - \mathbf{K}_1\mathbf{G})(I - \mathbf{K}_2\mathbf{G})^{-1}$, (4.12) shows that $(\mathbf{H} - I)(\mathbf{H} + I)^{-1} = \mathbf{L}\mathbf{G}^\mathbf{M} = \tilde{\mathbf{G}}$. Define $\Phi \in F_{\mathbb{R}}L_{m\times p}$ by $\Phi(w) := \Psi(\mathcal{L}w) + \mathcal{M}w$.

Let ζ be the solution of (5.1). Partitioning $\zeta = \mathrm{col}\,(x, z_\mathcal{M}, z_\mathcal{L})$ and $\zeta^0 = \mathrm{col}\,(x^0, z^0_\mathcal{M}, z^0_\mathcal{L})$, we have

$$\tilde{C}\zeta = C_\mathcal{L}z_\mathcal{L} + D_\mathcal{L}Cx = \mathcal{L}Cx + y_\mathcal{L}, \quad \text{where } y_\mathcal{L}(t) := C_\mathcal{L}e^{A_\mathcal{L}t}z^0_\mathcal{L} \quad \forall\, t \geq 0,$$
$$C_\mathcal{M}z_\mathcal{M} + D_\mathcal{M}Cx = \mathcal{M}Cx + y_\mathcal{M}, \quad \text{where } y_\mathcal{M}(t) := C_\mathcal{M}e^{A_\mathcal{M}t}z^0_\mathcal{M} \quad \forall\, t \geq 0,$$

and thus,

$$\begin{aligned}\dot{x} &= Ax + B\big(\Psi(\mathcal{L}(Cx) + y_\mathcal{L}) + \mathcal{M}(Cx) + y_\mathcal{M}\big)\\ &= Ax + B\big(\Phi(Cx) + \Psi(y_\mathcal{L}) + y_\mathcal{M}\big).\end{aligned}$$

Setting $f := B(\Psi(y_\mathcal{L}) + y_\mathcal{M})$, we arrive at

$$\dot{x}(t) = Ax(t) + B(\Phi(Cx))(t) + f(t), \quad t \geq 0, \quad x(0) = x^0, \tag{5.2}$$

which is (2.3) with $t_0 = 0$ and $u = f \in L^2(\mathbb{R}_+, \mathbb{R}^n)$.

(1) Assume that $\|\Psi\| < 1$. Then,

$$\sup_{w \in L^2,\, w \neq 0} \frac{\|\Phi(w) - \mathcal{M}w\|_{L^2}}{\|\mathcal{L}w\|_{L^2}} < 1,$$

and statement (3) of Lemma 4.1 yields that $\Phi \in S(\mathcal{K}_1, \mathcal{K}_2)$, whence $\Phi \in F_{\mathbb{R}}L_{m\times p} \cap S(\mathcal{K}_1, \mathcal{K}_2)$. By hypothesis, the origin of (2.3) is globally attractive at time 0, and hence, (2.3) is also L^2-GAS by Lemma 2.7. Since x satisfies (5.2), we conclude that $x(t) \to 0$ as $t \to \infty$, which in turn implies that $\zeta(t) \to 0$ as $t \to \infty$. This shows that the origin of (5.1) is globally attractive at time 0 for every $\Psi \in F_{\mathbb{R}}L_{m\times m}$ such that $\|\Psi\| < 1$, and so, in particular, $\tilde{A}$ is Hurwitz and $\tilde{\mathbf{G}} \in H^\infty_{m\times m}$. An application of statement (1) of Lemma 2.7 to (5.1) shows that $\tilde{\mathbf{G}}^{\mathbf{F}} \in H^\infty_{m\times m}$ for all $\mathbf{F} \in R_{\mathbb{R}}H^\infty_{m\times m}$ with $\|\mathbf{F}\|_{H^\infty} < 1$. Appealing to Proposition 2.1, we arrive at $1 \le r_{\mathbb{C}}(\tilde{\mathbf{G}}) = 1/\|\tilde{\mathbf{G}}\|_{H^\infty}$. Therefore, $\|(\mathbf{H} - I)(\mathbf{H} + I)^{-1}\|_{H^\infty} = \|\tilde{\mathbf{G}}\|_{H^\infty} \le 1$ which in conjunction with statement (1) of Lemma 2.2 establishes that $\mathbf{H}$ is positive real.

(2) Assume that $\|\Psi\| \le 1$. Proceeding as in the proof of statement (1), with statement (3) of Lemma 4.1 replaced by statement (1) of Lemma 4.1, shows that $\tilde{\mathbf{G}}^{\mathbf{F}} \in H^\infty_{m\times m}$ for all $\mathbf{F} \in R_{\mathbb{R}}H^\infty_{m\times m}$ with $\|\mathbf{F}\|_{H^\infty} \le 1$. Another application of Proposition 2.1 yields that $1 < r_{\mathbb{C}}(\tilde{\mathbf{G}}) = 1/\|\tilde{\mathbf{G}}\|_{H^\infty}$, whence $\|(\mathbf{H} - I)(\mathbf{H} + I)^{-1}\|_{H^\infty} = \|\tilde{\mathbf{G}}\|_{H^\infty} < 1$. It follows from statement (2) of Lemma 2.2 that $\mathbf{H}$ is strongly positive real and $\mathbf{H} \in H^\infty_{p\times m}$ and therefore $\mathbf{H}$ is also strictly positive real. □

The following corollary shows that stability for all (real) nonlinear causal operators in a sector and the corresponding positive realness property are equivalent.

Corollary 5.2 *Assume that (A, B, C) is stabilizable and detectable and assumption* (A) *holds. Set* $\mathbf{H} := (I - \mathbf{K}_1\mathbf{G})(I - \mathbf{K}_2\mathbf{G})^{-1}$. *The following statements hold.*

(1) *System (2.3) is L^2-GAS for all $\Phi \in S(\mathcal{K}_1, \mathcal{K}_2)$ if, and only if,* $\mathbf{H}$ *is positive real.*

(2) *System (2.3) is L^2-GAS for all $\Phi \in S[\mathcal{K}_1, \mathcal{K}_2]$ if, and only if,* $\mathbf{H}$ *is strictly positive real.*

(3) *System (2.3) is GAS for all $\Phi \in S(\mathcal{K}_1, \mathcal{K}_2)$ if, and only if,* $\mathbf{H}$ *is positive real.*

(4) *System (2.3) is GAS for all $\Phi \in S[\mathcal{K}_1, \mathcal{K}_2]$ if, and only if,* $\mathbf{H}$ *is strictly positive real.*

(5) *System (2.3) is exponentially ISS for all $\Phi \in S^{\mathrm{e}}(\mathcal{K}_1, \mathcal{K}_2)$ if, and only if,* $\mathbf{H}$ *is positive real.*

Proof Statements (1)–(4) are immediate consequences of Theorem 4.4 and Proposition 5.1, and the sufficiency part of statement (5) follows from Corollary 4.5. To prove the necessity part of statement (5), assume that (2.3) is exponentially ISS for all $\Phi \in S^{\mathrm{e}}(\mathcal{K}_1, \mathcal{K}_2)$. Then, trivially, the origin of (2.3) is globally attractive for all $\Phi \in F_{\mathbb{R}}L_{m\times p} \cap S^{\mathrm{e}}(\mathcal{K}_1, \mathcal{K}_2)$. Since $\|\mathcal{F}^\nu - \mathcal{F}\| \to 0$ as $\nu \to 0$ for every $\mathcal{F} \in F_{\mathbb{R}}L_{m\times p}$, it follows from Lemma 4.1 that $F_{\mathbb{R}}L_{m\times p} \cap S^{\mathrm{e}}(\mathcal{K}_1, \mathcal{K}_2) = F_{\mathbb{R}}L_{m\times p} \cap S(\mathcal{K}_1, \mathcal{K}_2)$. Invoking Proposition 5.1 shows that $\mathbf{H}$ is positive real. □

We remark that statement (1) of Corollary 5.2 has some overlap with [28, Theorem 126, Sect. 6.6] where, in a single-input single-output context, it is shown that stability (in the sense of L^2-input–output theory) for all causal nonlinearities in a L^2-sector (defined by static sector data) implies the positive-real condition of the circle criterion. The proof given in [28] relies on a result on minimal norm destabilization by linear delayed feedback of the form $\rho\mathcal{S}_\tau$, where $\rho \in \mathbb{R}$ (see [28, Lemma 112, Sect. 6.6]) and does not generalize to the general multivariable set-up considered in this paper.

Finally, we look at a scenario wherein, in the context of (real) static nonlinearities, positive realness is necessary for absolute stability.

Proposition 5.3 *Consider (2.4) and assume that (A, B, C) is stabilizable and detectable. Let $K_1, K_2 \in \mathbb{R}^{m\times p}$, define*

$$L := \frac{1}{2}(K_2 - K_1) \quad \text{and} \quad M := \frac{1}{2}(K_1 + K_2),$$

and assume that $\ker L = \ker(K_2 - K_1) = \{0\}$.

(1) *If the origin of (2.4) is globally attractive for all φ given by $\varphi(t, z) = Kz$ with $K \in \mathbb{R}^{m\times p}$ satisfying*

$$\sup_{z\in\mathbb{R}^p,\ \|z\|=1} \langle Kz - K_1 z, Kz - K_2 z\rangle < 0, \tag{5.3}$$

and $L\mathbf{G}^M$ has the real supremum-value property, then $(I - K_1\mathbf{G})(I - K_2\mathbf{G})^{-1}$ is positive real.

(2) *If the origin of (2.4) is globally attractive for all φ given by $\varphi(t, z) = Kz$ with $K \in \mathbb{R}^{m\times p}$ satisfying*

$$\langle Kz - K_1 z, Kz - K_2 z\rangle \le 0 \quad \forall z \in \mathbb{R}^p, \tag{5.4}$$

and $L\mathbf{G}^M$ has the real supremum-value property, then $(I - K_1\mathbf{G})(I - K_2\mathbf{G})^{-1}$ is strictly and strongly positive real.

We remark that in general, on its own, global attractivity for all $K \in \mathbb{R}^{m\times p}$ satisfying (5.3) does not imply positive realness of $(I - K_1\mathbf{G})(I - K_2\mathbf{G})^{-1}$. Similarly, global attractivity for all $K \in \mathbb{R}^{m\times p}$ satisfying (5.4) does not guarantee strict or strong positive realness of $(I - K_1\mathbf{G})(I - K_2\mathbf{G})^{-1}$. The situation changes if complex feedback gains are considered: for example, if global attractivity holds for all φ given by $\varphi(t, z) = Kz$ with complex $K \in \mathbb{C}^{m\times p}$ satisfying

$$\sup_{z\in\mathbb{C}^p,\ \|z\|=1} \operatorname{Re}\langle Kz - K_1 z, Kz - K_2 z\rangle < 0,$$

then $(I - K_1\mathbf{G})(I - K_2\mathbf{G})^{-1}$ is positive real, see [9, Theorem 6.8].

Proof of Proposition 5.3. Note that $\langle Mz - K_1 z, Mz - K_2 z\rangle = -\|Lz\|^2$ for all $z \in \mathbb{R}^p$, and so, as $\ker L = \{0\}$, (5.3) holds with $K = M$. Consequently, under the hypothesis of statements (1) or (2), $A + BMC$ is Hurwitz. This implies that $M \in \mathbb{S}_{\mathbb{R}}(\mathbf{G})$ and thus, $\tilde{\mathbf{G}} := L\mathbf{G}^M \in H^\infty_{m\times m}$.

(1) Let F be an arbitrary element in $\mathbb{R}^{m\times m}$ such that $\|F\| < 1$ and set $K := FL + M \in \mathbb{R}^{m\times p}$. Then,

$$\sup_{z\in\mathbb{R}^p,\, z\neq 0} \frac{\|Kz - Mz\|}{\|Lz\|} < 1,$$

and since $\langle Kz - K_1 z, Kz - K_2 z\rangle = \|Kz - Mz\|^2 - \|Lz\|^2$ for all $z \in \mathbb{R}^p$ and $\ker L = \{0\}$, we see that K satisfies (5.3). Now, the argument from the proof of statement (1) of Proposition 5.1 can be used to conclude that all $F \in \mathbb{R}^{m\times m}$ such that $\|F\| < 1$ are in $\mathbb{S}_{\mathbb{R}}(\tilde{\mathbf{G}})$, whence $1 \leq r_{\mathbb{R}}(\tilde{\mathbf{G}})$. As $\tilde{\mathbf{G}}$ as the real supremum-value property, Lemma 2.3 shows that $r_{\mathbb{R}}(\tilde{\mathbf{G}}) = r_{\mathbb{C}}(\tilde{\mathbf{G}})$, and so, $1 \leq r_{\mathbb{C}}(\tilde{\mathbf{G}}) = 1/\|\tilde{\mathbf{G}}\|_{H^\infty}$. As in the proof of statement (1) of Proposition 5.1, it now follows that $(I - K_1\mathbf{G})(I - K_2\mathbf{G})^{-1}$ is positive real.

(2) Arguing as in the prove of statement (1), we see that all $F \in \mathbb{R}^{m\times m}$ such that $\|F\| \leq 1$ are in $\mathbb{S}_{\mathbb{R}}(\tilde{\mathbf{G}})$, implying that $1 < r_{\mathbb{C}}(\tilde{\mathbf{G}}) = 1/\|\tilde{\mathbf{G}}\|_{H^\infty}$. As in the proof of statement (2) of Proposition 5.1, it can be shown that $(I - K_1\mathbf{G})(I - K_2\mathbf{G})^{-1}$ is strictly and strongly positive real. □

In the remark below, we describe how Proposition 5.3 can be used in the context of positive systems.

Remark 5.4 Let the matrices B and C be nonnegative and assume that there exists $F \in \mathbb{R}^{m\times p}$ such that $A + BFC$ is Hurwitz and Metzler. Let $P \in \mathbb{R}^{m\times m}$ be nonnegative and set $K_1 := F - P$ and $K_2 := F + P$. Then, defining L and M as in (4.3), we see that $L\mathbf{G}^M = P\mathbf{G}^F$ is the transfer function of the stable positive system $(A + BFC, B, PC)$, and so, by part (b) of Example 2.4, has the real supremum-value property. Proposition 5.3 shows that if the origin of (2.4) is globally attractive for all φ given by $\varphi(t, z) = Kz$ with $K \in \mathbb{R}^{m\times p}$ satisfying (5.3), then $(I - K_1\mathbf{G})(I - K_2\mathbf{G})^{-1}$ is positive real (strictly and strongly positive real if global attractivity holds for all K satisfying (5.4)). ♢

6 Appendix

For completeness, we give a proof of Proposition 2.1.

Proof of Proposition 2.1. As (2.1) is trivially true in the case wherein $\mathbf{H}(s) \equiv 0$, we may, without loss of generality, assume that $\mathbf{H}(s) \not\equiv 0$. It is clear that if $\|\mathbf{F} - \mathbf{K}\|_{H^\infty} < r_{\mathbb{C}}(\mathbf{H}^{\mathbf{K}}) = 1/\|\mathbf{H}^{\mathbf{K}}\|_{H^\infty}$, then $\mathbf{H}^{\mathbf{F}} = (\mathbf{H}^{\mathbf{K}})^{\mathbf{F}-\mathbf{K}} \in H^\infty_{p\times m}$. It remains to show that there exists $\mathbf{F} \in R_{\mathbb{R}}H^{\infty}_{m\times p}$ such that $\|\mathbf{F}-\mathbf{K}\|_{H^\infty} = r_{\mathbb{C}}(\mathbf{H}^K)$ and $\mathbf{H}^{\mathbf{F}}$ is not stable. To this end, we note that there exists $0 \le \omega^\dagger \le \infty$ such that $\|\mathbf{H}^{\mathbf{K}}(i\omega^\dagger)\| = \|\mathbf{H}^{\mathbf{K}}\|_{H^\infty} =: \gamma$. Let $v \in \mathbb{C}^m$ be such that $\|v\| = 1$ and $\|\mathbf{H}^{\mathbf{K}}(i\omega^\dagger)v\| = \gamma$. Defining

$$u := \frac{1}{\gamma}\mathbf{H}^{\mathbf{K}}(i\omega^\dagger)v \in \mathbb{C}^p \qquad \text{and} \qquad F := \frac{1}{\gamma}vu^* \in \mathbb{C}^{m\times p},$$

we have that $\|u\| = 1$ and $Fw = (1/\gamma)\langle w, u\rangle v$ for all $w \in \mathbb{C}^p$, and so, $\|F\| = 1/\gamma$. Furthermore,

$$(I - F\mathbf{H}^{\mathbf{K}}(i\omega^\dagger))v = v - \frac{1}{\gamma}\langle \mathbf{H}^{\mathbf{K}}(i\omega^\dagger)v, u\rangle v = v - v = 0,$$

showing that $(I - F\mathbf{H}^{\mathbf{K}})^{-1} = F\mathbf{H}^{\mathbf{K}}(I - F\mathbf{H}^{\mathbf{K}})^{-1} + I$ has a pole at $i\omega^\dagger$. Consequently, $(\mathbf{H}^{\mathbf{K}})^F = \mathbf{H}^{\mathbf{K}}(I - F\mathbf{H}^{\mathbf{K}})^{-1}$ has a pole at $i\omega^\dagger$. Setting $\mathbf{F} := \mathbf{K} + F$, we conclude that $\|\mathbf{F}-\mathbf{K}\|_{H^\infty} = 1/\gamma$ and $\mathbf{H}^{\mathbf{F}} = \mathbf{H}^{\mathbf{K}+F} = (\mathbf{H}^{\mathbf{K}})^F$ has a pole at $i\omega^\dagger$.

We now consider two cases; namely, $\mathbf{H}^{\mathbf{K}}(i\omega^\dagger)$ is real (which happens, for example, if $\omega^\dagger = 0$ or $\omega^\dagger = \infty$) and $\mathbf{H}^{\mathbf{K}}(i\omega^\dagger)$ is not real.

Case 1: $\mathbf{H}^{\mathbf{K}}(i\omega^\dagger)$ *is real*. In this case, v can be chosen to be real, and so u and F are real and the claim follows with $\mathbf{F}(s) = \mathbf{K} + F$.

Case 2: $\mathbf{H}(i\omega^\dagger)$ *is not real*. In this case, $0 < \omega^\dagger < \infty$ and v and u are in general not real. Let v_j and u_j denote the components of v and u, respectively. Using a construction from [27, Proof of Theorem 4, Sect. 7.4], we note that there exist real vectors $(\hat{v}_1, \ldots, \hat{v}_m)^T \in \mathbb{R}^m$ and $(\hat{u}_1, \ldots, \hat{u}_p)^T \in \mathbb{R}^p$ and $\psi_j, \theta_j \in [0, \pi)$ such that

$$v_j = \hat{v}_j e^{\psi_j}, \quad j = 1, \ldots, m \qquad \text{and} \qquad \bar{u}_j = \hat{u}_j e^{\theta_j}, \quad j = 1, \ldots, p.$$

Define

$$\mathbf{v}(s) := \begin{pmatrix} \hat{v}_1(s-a_1)/(s+a_1) \\ \vdots \\ \hat{v}_m(s-a_m)/(s+a_m) \end{pmatrix} \qquad \text{and} \qquad \mathbf{u}(s) := \begin{pmatrix} \hat{u}_1(s-b_1)/(s+b_1) \\ \vdots \\ \hat{u}_p(s-b_p)/(s+b_p) \end{pmatrix},$$

where a_j and b_j are nonnegative real constants such that

$$\frac{i\omega^\dagger - a_j}{i\omega^\dagger + a_j} = e^{i\psi_j}, \quad j = 1, \ldots, m \qquad \text{and} \qquad \frac{i\omega^\dagger - b_j}{i\omega^\dagger + b_j} = e^{i\theta_j}, \quad j = 1, \ldots, p.$$

Setting

$$\mathbf{F}(s) := \mathbf{K}(s) + \frac{1}{\gamma}\mathbf{v}(s)\mathbf{u}(s)^T,$$

it is clear that $\mathbf{F} \in R_{\mathbb{R}} H^{\infty}_{m \times p}$ and $\|\mathbf{F}-\mathbf{K}\|_{H^\infty} \le 1/\gamma$. As $\mathbf{F}(i\omega^\dagger)-\mathbf{K}(i\omega^\dagger) = (1/\gamma) v u^*$, we conclude that $\|\mathbf{F} - \mathbf{K}\|_{H^\infty} = 1/\gamma$. Furthermore,

$$\big(I - (\mathbf{F}(i\omega^\dagger) - \mathbf{K}(i\omega^\dagger))\mathbf{H}^{\mathbf{K}}(i\omega^\dagger)\big)v = 0,$$

showing that $\mathbf{H}^{\mathbf{F}} = \mathbf{H}^{\mathbf{K}}(I - (\mathbf{F} - \mathbf{K})\mathbf{H}^{\mathbf{K}})^{-1}$ has a pole at $i\omega^\dagger$, completing the proof. $\square$

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