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SAMPLED-DATA INTEGRAL CONTROL OF MULTIVARIABLE LINEAR INFINITE-DIMENSIONAL SYSTEMS WITH INPUT NONLINEARITIES

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Abstract. A low-gain integral controller with anti-windup component is presented for exponentially stable, linear, discrete-time, infinite-dimensional control systems subject to input nonlinearities and external disturbances. We derive a disturbance-to-state stability result which, in particular, guarantees that the tracking error converges to zero in the absence of disturbances. The discrete-time result is then used in the context of sampled-data low-gain integral control of stable well-posed linear infinite-dimensional systems with input nonlinearities. The sampled-date control scheme is applied to two examples (including sampled-data control of a heat equation on a square) which are discussed in some detail.

1. Introduction.

Low-gain integral control is a well-studied technique which guarantees that the output of a stable linear system converges to a prescribed constant reference signal. It is well-known (see, for example, [7, 22]) that, for a stable finite-dimensional linear system, by applying an integrator with sufficiently small gain parameter and closing the feedback loop, the corresponding output converges to any prescribed reference value, so long as the eigenvalues of the steady-state gain matrix have positive real part. Much work has been done on the extension of low-gain integral control to more general systems: we refer the reader to works such as [5, 6, 9, 10, 11, 15, 18, 19, 20, 21, 23, 24] for integral control in infinite-dimensional settings, in the presence of input nonlinearities and in sampled-data contexts.

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In much of the literature which addresses the low-gain integral control problem in the presence of nonlinearities (see, for example, [5, 6, 9, 10, 11, 19]), attention is restricted to single-input single-output systems. To the best of our knowledge, the only exceptions are [15, 16] which develop low-gain integral control schemes for multi-input multi-output linear finite-dimensional systems with input nonlinearities. In contrast to [5, 6, 9, 10, 11, 19], the low-gain integral controllers presented in [15, 16] include anti-windup components. We remark that integrator windup can cause performance and/or stability degradation in integral control systems subject to input saturation. Indeed, as [15, Example 10] shows, in the absence of any anti-windup component, the integrator state for the multivariable low-gain integral control scheme considered in [15, 16] may be unbounded. The theory of anti-windup control seeks to avoid or reduce integrator windup and is a much researched area, we only mention the references [1, 2, 12, 28] which provide a small sample of the available literature.

In the current paper, we will extend the main result in [15] to the context of sampled-data integral control of multivariable linear well-posed infinite-dimensional systems with input nonlinearities. Briefly, in sampled-data control, a continuous-time system is controlled by a discrete-time controller, via the use of sample and hold operations. This discrete-time controller may be thought of, for example, as a processor of a digital computer. The underlying class of linear well-posed infinite-dimensional systems is well-developed, see, for example, [26, 27, 29, 30]. Systems in this class allow for considerable unboundedness of the control and observation operators and they encompass many of the most commonly studied partial differential equations (PDEs) with boundary control and observation, and a large class of functional differential equations of retarded and neutral type with delays in the inputs and outputs.

The main result of this paper, Theorem 3.1, provides a stability criterion for the sampled-data feedback interconnection of an exponentially stable well-posed linear system with input nonlinearities and a discrete-time low-gain integral controller combined with an anti-windup component. The sampled-data control scheme under consideration includes continuous as well as discrete-time external disturbances. When these disturbances are small, Theorem 3.1 guarantees good approximate asymptotic tracking, and the asymptotic tracking error goes to zero as the size of the disturbances goes to zero.

To prove Theorem 3.1, we first extend the finite-dimensional continuous-time results of [15] to infinite-dimensional discrete-time systems: under assumptions analogous to those in [15, Theorem 4], we derive a stability estimate of the difference between the output and reference signals which, in the absence of disturbances, guarantees exponentially fast asymptotic tracking. The discrete-time result plays a pivotal role in the main contribution of this paper, namely the development of a sampled-data low-gain integral control scheme with anti-windup component which applies to multivariable well-posed linear infinite-dimensional systems with input nonlinearities.

The layout of the paper is as follows. Section 2 is devoted to an infinite-dimensional discrete-time generalisation of the low-gain integral control result in [15]. In Section 3, we use the discrete-time result to prove stability and tracking properties of the feedback interconnection of linear infinite-dimensional systems subject to input nonlinearities with a natural low-gain sampled-data integral controller with anti-windup component. Finally, in Section 4, we provide detailed discussions of two examples. One of which considers the application of the proposed control scheme in the presence of plant output quantization and the other illustrates the developed sampled-data theory in the context of a heat equation on a square.

Notation. We denote the field of complex numbers by $\mathbb{C}$, the field of real numbers by $\mathbb{R}$ and the set of positive integers by $\mathbb{N}$, and define

$$\mathbb{R}_+ := \big\{r \in \mathbb{R} : r \geq 0\big\} \quad \text{and} \quad \mathbb{Z}_+ := \mathbb{N} \cup \{0\}\,.$$

We set $\mathbb{C}_\alpha := \{s \in \mathbb{C} \ : \ \mathrm{Re}\,(s) > \alpha\}$ and $\mathbb{E}_\alpha := \{\xi \in \mathbb{C} : |\xi| > \alpha\}$ for $\alpha \in \mathbb{R}$ and $\alpha > 0$, respectively. For $z \in \mathbb{C}$ and $r > 0$, we let $\mathbb{D}(z, r)$ denote the open disc in $\mathbb{C}$ of radius r centred at z. For ease of notation, we write $\mathbb{E} := \mathbb{E}_1$ and $\mathbb{D} = \mathbb{D}(0, 1)$.

For Banach spaces V, V_1 and V_2, we denote by $\mathcal{L}(V_1, V_2)$ the set of all bounded linear operators from V_1 to V_2 and, as usual, set $\mathcal{L}(V) := \mathcal{L}(V, V)$. We denote the spectrum of $L \in \mathcal{L}(V)$ by $\sigma(L)$ and recall that an operator $L \in \mathcal{L}(V)$ is discrete-time exponentially (or power) stable if $\rho(L) < 1$, where $\rho(L)$ denotes the spectral radius of L. When the context is clear, we will omit the words 'discrete-time' from exponentially stable. The space of functions defined on $\mathbb{Z}_+$ and taking values in V is denoted by $V^{\mathbb{Z}_+}$ and the set of continuous functions $\mathbb{R}_+ \to V$ is denoted by $C(\mathbb{R}_+, V)$. Frequently, we will associate with an element $v \in V$ the constant function in either $V^{\mathbb{Z}_+}$ or $C(\mathbb{R}_+, V)$ with value v, and we will not notationally distinguish between the element and the associated constant function.

For $v \in V^{\mathbb{Z}_+}$ and $t \in \mathbb{Z}_+$, we set

$$(\pi_t v)(s) := \begin{cases} v(s), & \text{if } s \in \{0, 1 \ldots, t\} \\ 0, & \text{otherwise.} \end{cases}$$

For Hilbert spaces V and W, the product space $V \times W$ is itself a Hilbert space when equipped with the inner product

$$\left\langle \begin{pmatrix} \eta_1 \\ \xi_1 \end{pmatrix}, \begin{pmatrix} \eta_2 \\ \xi_2 \end{pmatrix} \right\rangle_{V \times W} := \langle \eta_1, \eta_2 \rangle_V + \langle \xi_1, \xi_2 \rangle_W \quad \forall\, \eta_i \in V,\ \forall\, \xi_i \in W,\ i = 1, 2.$$

When V or W are Banach spaces (and not Hilbert spaces), then we equip the product space $V \times W$ with the norm

$$\left\| \begin{pmatrix} v \\ w \end{pmatrix} \right\|_{V \times W} = \|v\|_V + \|w\|_W \quad \forall\, v \in V,\ w \in W\,,$$

although any equivalent norm on $V \times W$ could be chosen. For ease of presentation, we will typically suppress the space where the norms are taken for clarity.

In the following, let V be a Banach space. The Hardy space given by

$$H^\infty(V) := \big\{h : \mathbb{E} \to V \ : \ h \text{ is holomorphic and bounded}\big\},$$

is a Banach space when endowed with the norm

$$\|h\|_{H^\infty} := \sup_{\xi \in \mathbb{E}} \|h(\xi)\|_V \quad \forall\, h \in H^\infty(V)\,.$$

For $\alpha \in \mathbb{R}$, we define the exponentially weighted L^2-space $L^2_\alpha(\mathbb{R}_+, V)$ by

$$L^2_\alpha(\mathbb{R}_+, V) := \big\{w \in L^2_{\mathrm{loc}}(\mathbb{R}_+, V) : e_{-\alpha} w \in L^2(\mathbb{R}_+, V)\big\},$$

with norm $\|w\|_{L^2_\alpha} := \|e_{-\alpha} w\|_{L^2}$ and where, for $\beta \in \mathbb{R}$, $e_\beta(t) := e^{\beta t}$, for all $t \in \mathbb{R}_+$. For $\alpha \in \mathbb{R}$, we define the Hardy space

$$H^2(\mathbb{C}_\alpha, V) := \Big\{ h : \mathbb{C}_\alpha \to V \ : \ h \text{ is holomorphic and} \\ \sup_{x>\alpha} \Big(\int_{-\infty}^{\infty} \|h(x+iy)\|_V^2 dy \Big) < \infty \Big\}.$$

For $u : T \to V$ and $v \in V$, where $T = \mathbb{Z}_+$ or $T = \mathbb{R}_+$, we say that $u(t)$ converges exponentially to v as $t \to \infty$ in T if there exists $\delta > 0$ such that $\limsup_{t\to\infty,\, t\in T} \big(e^{\delta t} \|u(t) - v\|_V \big) < \infty$.

Finally, if $u \in L^1_{\rm loc}(\mathbb{R}_+, V)$ (which means in particular that u is an equivalence class of functions, the difference of any two of which is equal to zero almost everywhere) and $v \in V$, then we say that $u(t)$ converges to v as $t \to \infty$ if, for every $\varepsilon > 0$, there exists $t_\varepsilon \geq 0$ such that

$$\text{meas}\,\{t \geq t_\varepsilon : \|u(t) - v\| > \varepsilon\} = 0, \quad \text{where meas} = \text{Lebesgue measure.}$$

It is easy to see that this is equivalent to saying that there exists a representative $\widetilde{u}$ of the equivalence class u such that $\widetilde{u}(t) \to v$ as $t \to \infty$ in the usual sense.

2. **Low-gain integral control in discrete-time.** In this section we derive a low-gain integral control result for infinite-dimensional, discrete-time linear systems with input nonlinearities and external disturbances. We proceed to outline the mathematical formulation. Let X and V be complex Banach spaces and let Y and U be complex Hilbert spaces. We consider the discrete-time system of difference equations given by

$$\left.\begin{aligned} x^+ &= Ax + B\phi(u) + B_{\rm e} v, \quad x(0) = x^0 \in X, \\ y &= Cx + D\phi(u) + D_{\rm e} v\,. \end{aligned}\right\} \tag{2.1}$$

Here x^+ denotes the image of x under the left shift operator, that is, $x^+(t) = x(t+1)$ for all $t \in \mathbb{Z}_+$. Further,

$$(A, B, B_{\rm e}, C, D, D_{\rm e}) \in \mathbb{L} := \mathcal{L}(X) \times \mathcal{L}(U, X) \times \mathcal{L}(V, X) \times \mathcal{L}(X, Y) \times \mathcal{L}(U, Y) \times \mathcal{L}(V, Y)\,,$$

and $u \in U^{\mathbb{Z}_+}$, $v \in V^{\mathbb{Z}_+}$, $\phi : U \to U$ and A is assumed to be exponentially stable. As usual, we shall denote the transfer function of (2.1) (from $\phi(u)$ to y) by $\mathbf{G}$, that is, $\mathbf{G}(z) = C(zI - A)^{-1}B + D$. For ease of notation, we set $\Sigma := (A, B, B_{\rm e}, C, D, D_{\rm e}) \in \mathbb{L}$.

We highlight that the formulation (2.1) encompasses systems of the form

$$\begin{aligned} x^+ &= Ax + B\phi(u) + v_1, \quad x(0) = x^0 \in X, \\ y &= Cx + D\phi(u) + v_2, \end{aligned}$$

where $v_1 \in X^{\mathbb{Z}_+}$ and $v_2 \in Y^{\mathbb{Z}_+}$. Indeed, the above is a special case of (2.1) with $V = X \times Y$,

$$B_{\rm e} = (I, 0), \ \ D_{\rm e} = (0, I) \ \text{ and } \ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

As usual, the variables v, u, x and y in (2.1) denote a disturbance signal, a control signal, the state and output, respectively, and the function ϕ represents an input nonlinearity. The control goal is that the output y asymptotically tracks a constant reference vector $r \in Y$. We shall propose a low-gain integral controller to determine u in (2.1) so that the closed-loop system is disturbance-to-error stable. In particular, when $v = 0$, the low-gain control law will guarantee that the error $y(t) - r$ converges to 0 exponentially as $t \to \infty$.

Given $\Sigma \in \mathbb{L}$, we say that $r \in Y$ is feasible (with respect to (2.1)) if the set

$$U^r := \left\{ w \in U : \mathbf{G}(1)\phi(w) = r \right\},$$

is non-empty, and we say that a subset $R \subseteq Y$ is feasible (with respect to (2.1)) if every $r \in R$ is feasible. For $\Sigma \in \mathbb{L}$, feasible $R \subseteq Y$, $r \in R$ and $u^r \in U^r$, consider the control law

$$u^+ = u + gK(r - y) - g\Gamma(u - \phi(u) - u^r + \phi(u^r)) + w, \quad u(0) = u^0 \in U, \tag{2.2}$$

where $g > 0$, $K \in \mathcal{L}(Y, U)$ and $\Gamma \in \mathcal{L}(U)$ are design parameters and $w \in U^{\mathbb{Z}_+}$ is a disturbance. The signal w could model a measurement disturbance d, that is, instead of y, the disturbed output $y + d$ is fed into the control law, in which case w is of the form $w = -gKd$. Further, w could also be used to model errors which occur when the control law is implemented digitally. We highlight that if $\Gamma = 0$ or $\phi = \mathrm{id}$ is the identity function, then (2.2) becomes 'standard' integral control (subject to the disturbance w). The term $g\Gamma(u - \phi(u) - u^r + \phi(u^r))$, the so-called *anti-windup* component, seeks to mitigate against integrator windup [1] when the input nonlinearity ϕ is a saturation function, in which case ϕ usually has a 'linear regime', meaning that there exists a neighbourhood U_0 of zero such that $\phi|_{U_0} = \mathrm{id}$. In this scenario, if $u^r \in U_0 \cap U^r$, then $\phi(u^r) = u^r$ (and $U_0 \cap U^r = \{\mathbf{G}(1)^{-1}r\}$, provided that $\mathbf{G}(1)$ is invertible), and so, the difference $-u^r + \phi(u^r)$ is zero, implying that knowledge of u^r is not required to implement the control law (2.2). Furthermore, if $u^r \in U_0 \cap U^r$, then the contribution of the anti-windup component is zero as long as u remains in U_0 and, when u is outside of U_0, the anti-windup action helps to drive u towards U_0. As mentioned in the introduction, integrator windup often leads to undesirable outcomes in the context of PI control, including destabilisation: for example, in a finite-dimensional continuous-time context, [15, Example 10] shows that, in the absence of any anti-windup action, the integrator state of a multivariable low-gain integral control scheme with input saturation may diverge to infinity. We refer the reader to [12] and [28] for tutorials concerning anti-windup control.

The feedback interconnection of (2.1) and (2.2) yields the closed-loop system

$$\left.\begin{aligned} x^+ &= Ax + B\phi(u) + B_{\mathrm{e}}v, \quad x(0) = x^0 \in X, \\ y &= Cx + D\phi(u) + D_{\mathrm{e}}v, \\ u^+ &= u + gK(r - y) - g\Gamma(u - \phi(u) - u^r + \phi(u^r)) + w, \quad u(0) = u^0 \in U. \end{aligned}\right\} \tag{2.3}$$

We shall show in Theorem 2.1 that the above feedback interconnection is well-posed, meaning here that for all $(x^0, u^0) \in X \times U$, and all $(v, w) \in V^{\mathbb{Z}_+} \times U^{\mathbb{Z}_+}$, there is a unique solution of (2.3), that is, a pair $(u, x) \in U^{\mathbb{Z}_+} \times X^{\mathbb{Z}_+}$ satisfying (2.3).

The following theorem is the main result of this section and provides a disturbance-to-state estimate for the low-gain integral control feedback system (2.3).

Theorem 2.1. *Let $\Sigma \in \mathbb{L}$, $K \in \mathcal{L}(Y, U)$, $\Gamma \in \mathcal{L}(U)$ and $R \subseteq Y$, where A and $I - \Gamma$ are exponentially stable and R is feasible. Assume that there exists $L > 0$ such that*

$$\|\phi(\xi + \zeta) - \phi(\zeta)\|_U \le L\|\xi\|_U \quad \forall\, \xi \in U, \quad \forall\, \zeta \in \bigcup_{r \in R} U^r, \tag{2.4}$$

and

$$\sup_{\xi \in \mathbb{E}} \|(\xi I - (I - \Gamma))^{-1}\| \cdot \|\Gamma - K\mathbf{G}(1)\| < 1/L. \tag{2.5}$$

The following statements hold.

(1) *For all $g > 0$, all $(x^0, u^0) \in X \times U$, and all $(v, w) \in V^{\mathbb{Z}_+} \times U^{\mathbb{Z}_+}$, there exists a unique solution $(u, x) \in U^{\mathbb{Z}_+} \times X^{\mathbb{Z}_+}$ of (2.3).*

(2) *There exists $g^* \in (0, 1]$ such that for all $g \in (0, g^*)$, there exist constants $c, d > 0$ and $\theta \in (0, 1)$ such that for all $r \in R$, $u^r \in U^r$ and all $(x^0, u^0) \in X \times U$, and all $(v, w) \in V^{\mathbb{Z}_+} \times U^{\mathbb{Z}_+}$, the solution $(u, x) \in U^{\mathbb{Z}_+} \times X^{\mathbb{Z}_+}$ of (2.3) satisfies, for all $t \in \mathbb{N}$,*

$$\left\| \begin{pmatrix} x(t) - (I - A)^{-1} B\phi(u^r) \\ u(t) - u^r \end{pmatrix} \right\| \leq c \left(\theta^t \left\| \begin{pmatrix} x^0 - (I - A)^{-1} B\phi(u^r) \\ u^0 - u^r \end{pmatrix} \right\| + \left\| \pi_{t-1} \begin{pmatrix} v \\ w \end{pmatrix} \right\|_{\ell^\infty} \right) \tag{2.6}$$

and the corresponding output y of (2.3) satisfies, for all $t \in \mathbb{Z}_+$,

$$\|y(t) - r\| \leq d \left(\theta^t \left\| \begin{pmatrix} x^0 - (I - A)^{-1} B\phi(u^r) \\ u^0 - u^r \end{pmatrix} \right\| + \left\| \pi_t \begin{pmatrix} v \\ w \end{pmatrix} \right\|_{\ell^\infty} \right). \tag{2.7}$$

Before proving Theorem 2.1, we provide some commentary. The reader is referred to [15, Remark 7] for a related discussion in the finite-dimensional, continuous-time setting.

Remark 2.2. (i) We note that Theorem 2.1 guarantees that, under zero forcing, $u(t) \to u^r$, $x(t) \to (I - A)^{-1} B\phi(u^r)$ and $y(t) \to r$ as $t \to \infty$, and, in each case, the convergence is exponential.

(ii) Assumption (2.4) is evidently satisfied if ϕ is globally Lipschitz with Lipschitz constant L. Moreover, if there exist distinct $u_1, u_2 \in U$ such that $\phi(u_1) = u_1$ and $\phi(u_2) = u_2$, then an immediate consequence of (2.4) is that $L \geq 1$.

(iii) The assumptions of Theorem 2.1 imply that any $u^r \in U^r$ which satisfies $u^r = \phi(u^r)$ is in fact unique. To see this, suppose that $w^r \in U^r$ also satisfies $w^r = \phi(w^r)$. From (2.4) we see that

$$\|u^r - w^r\|_U = \|\phi(u^r) - \phi(w^r)\|_U \leq L \|u^r - w^r\|_U .$$

We conclude that either $u^r = w^r$, or that $L \geq 1$. In the latter case, (2.5) yields that

$$\|\Gamma - K\mathbf{G}(1)\| < \frac{1}{\sup_{\xi \in \mathbb{E}} \|(\xi I - (I - \Gamma))^{-1}\|} .$$

Invoking [13, Lemma 2.1] and the exponential stability of $I - \Gamma$, it follows that

$$\sigma(I - K\mathbf{G}(1)) = \sigma\left((I - \Gamma) + (\Gamma - K\mathbf{G}(1))\right) \subseteq \mathbb{D} ,$$

which in turn implies that $K\mathbf{G}(1)$ is invertible. Finally, by definition of $u^r, w^r \in U^r$,

$$K\mathbf{G}(1)u^r = K\mathbf{G}(1)\phi(u^r) = r = K\mathbf{G}(1)\phi(w^r) = K\mathbf{G}(1)w^r ,$$

and so $u^r = w^r$.

(iv) Note that, for $\lambda \in \mathbb{C}$,

$$\lambda I - K\mathbf{G}(1) = -\big((1 - \lambda)I - (I - K\mathbf{G}(1))\big) ,$$

meaning $\lambda \in \sigma(K\mathbf{G}(1))$ if, and only if, $1 - \lambda \in \sigma(I - K\mathbf{G}(1))$. In the case that $L \geq 1$, we argued in item (iii) that the condition (2.5) implies that $\sigma(I - K\mathbf{G}(1)) \subseteq \mathbb{D}$. Hence, the conditions $L \geq 1$ and (2.5) together imply that $\sigma(K\mathbf{G}(1)) \subseteq \mathbb{C}_0$, the usual spectral condition for low-gain integral control.

(v) Theorem 2.1 remains true if the spaces V, U, X, and Y are real, provided that the estimate (2.5) holds for the canonical extensions of Γ and $K\mathbf{G}(1)$ to the complexifications U^{c} and Y^{c} of U and Y, respectively. In the situation wherein $U = \mathbb{R}^m$ and $Y = \mathbb{R}^p$ for some $m, p \in \mathbb{N}$, if $L \geq 1$, then part (iii) shows that $\operatorname{rk} \mathbf{G}(1) = m$ is a necessary condition for the assumptions of Theorem 2.1 to hold.

(vi) The condition (2.5) is trivially satisfied for any $L > 0$ if we choose K as a left inverse of $\mathbf{G}(1)$ and $\Gamma = I$. Such a choice naturally requires knowledge of $\mathbf{G}(1)$ to be implemented, although the condition (2.5) carries robustness with respect to uncertainty in $\mathbf{G}(1)$. We note that with the choice $\Gamma = I$, the supremum on the left-hand side of (2.5) equals 1 and thus, (2.5) holds, provided that steady-state gain information is available which is sufficient for the design of an integrator gain K satisfying $\|I - K\mathbf{G}(1)\| < 1/L$.

(vii) Observe that the choice $\Gamma = 0$ does not satisfy the hypotheses of Theorem 2.1, as $I - \Gamma$ is required to be exponentially stable. When $\Gamma = 0$, then (2.2) for feasible $r \in Y$ reduces to "pure" integral control, leading to the closed-loop system

$$\left.\begin{aligned} x^+ &= Ax + B\phi(u), \quad x(0) = x^0 \in X, \\ y &= Cx + D\phi(u), \\ u^+ &= u + gK(r - y), \quad u(0) = u^0 \in U, \end{aligned}\right\} \tag{2.8}$$

where, for simplicity, we have assumed that $v = 0$ and $w = 0$. It can be shown that, if $\sigma(K\mathbf{G}(1)) \subseteq \mathbb{C}_0$, $K\mathbf{G}(1)$ is self-adjoint, and ϕ satisfies (2.4) with $L = 1$, then there exists $g^* > 0$ such that, for all $(x^0, u^0) \in X \times U$ and all $g \in (0, g^*)$, it follows that the output y of (2.8) satisfies $y(t) \to r$ as $t \to \infty$. However, the assumption that $K\mathbf{G}(1)$ is self-adjoint is highly non-robust to uncertainty, and essentially restricts the result to the single-input single-output case, that is, $Y = U = \mathbb{R}$ or $\mathbb{C}$. If $K\mathbf{G}(1)$ is not self-adjoint, then the anti-windup component in (2.3) is crucial for the stability of the closed-loop system as [15, Example 10] shows. $\diamondsuit$

The proof of Theorem 2.1 requires the following lemma, which is a discrete-time, infinite-dimensional version of [15, Lemma 6], extended to the case of potentially non-zero feedthrough.

Lemma 2.3. *Let $\Sigma \in \mathbb{L}$, $K \in \mathcal{L}(Y, U)$, $\Gamma \in \mathcal{L}(U)$ and assume that A and $I - \Gamma$ are exponentially stable. For $g > 0$, we define*

$$\widetilde{A} := \begin{pmatrix} A & 0 \\ -gKC & I - g\Gamma \end{pmatrix}, \quad \widetilde{B} := \begin{pmatrix} B \\ g(\Gamma - KD) \end{pmatrix}, \quad \widetilde{C} := \begin{pmatrix} 0 & I \end{pmatrix}, \tag{2.9}$$

and let $\widetilde{\mathbf{G}}$ denote the transfer function of $(\widetilde{A}, \widetilde{B}, \widetilde{C})$, that is, $\widetilde{\mathbf{G}}(\xi) = \widetilde{C}(\xi I - \widetilde{A})^{-1}\widetilde{B}$. The following statements hold.

(1) *For each $g \in (0, 1]$, $\widetilde{A}$ is exponentially stable.*

(2) *For all $\varepsilon > 0$, there exists $g^* \in (0, 1]$ such that, for all $g \in (0, g^*)$,*

$$\|\widetilde{\mathbf{G}}\|_{H^\infty} \leq \varepsilon + \sup_{\xi \in \mathbb{E}} \|(\xi I - (I - \Gamma))^{-1}\| \cdot \|\Gamma - K\mathbf{G}(1)\|\,. \tag{2.10}$$

Proof. To prove statement (1), fix $g \in (0, 1]$. From the block triangular structure of $\widetilde{A}$, it follows that $\sigma(\widetilde{A}) \subseteq \sigma(A) \cup \sigma(I - g\Gamma)$. Since $\sigma(A) \subseteq \mathbb{D}$, our aim is thus to prove that $\sigma(I-g\Gamma)\subseteq\mathbb{D}$. For which purpose, first note that for $\lambda\in\mathbb{C}$,

$$\lambda I-(I-g\Gamma)=(\lambda-1)I+g\Gamma=g\left(\left(\frac{\lambda-1}{g}+1\right)I-(I-\Gamma)\right),$$

which in turn yields that $\lambda\in\sigma(I-g\Gamma)$ if, and only if, $((\lambda-1)/g+1)\in\sigma(I-\Gamma)$. By combining this with the fact that $I-\Gamma$ is exponentially stable, we see that if $\lambda\in\sigma(I-g\Gamma)$, then $(\lambda-1)/g+1\in\mathbb{D}$, which in turn implies that $(\lambda-1)/g\in\mathbb{D}(-1,1)$. For such a λ, it is easily checked that since $g\le 1$, $\lambda-1\in\mathbb{D}(-1,1)$ and hence $\lambda\in\mathbb{D}$. We have thus shown that $\sigma(I-g\Gamma)\subseteq\mathbb{D}$, hence yielding that $\widetilde{A}$ is exponentially stable for each $g\in(0,1]$.

For statement (2), fix $\varepsilon>0$ and note that, for all $\xi\in\mathbb{E}$ and $g\in(0,1]$, a routine calculation gives

$$\begin{aligned}\widetilde{\mathbf{G}}(\xi)&=\begin{pmatrix}0 & I\end{pmatrix}\begin{pmatrix}\xi I-A & 0\\ gKC & \xi I-(I-g\Gamma)\end{pmatrix}^{-1}\begin{pmatrix}B\\ g(\Gamma-KD)\end{pmatrix}\\ &=g(\xi I-(I-g\Gamma))^{-1}(\Gamma-K\mathbf{G}(\xi)).\end{aligned}$$

We write $\widetilde{\mathbf{G}}=\mathbf{H}_1+\mathbf{H}_2$, where, for all $\xi\in\mathbb{E}$ and $g\in(0,1]$,

$$\begin{aligned}\mathbf{H}_1(\xi)&:=g(\xi I-(I-g\Gamma))^{-1}K(\mathbf{G}(1)-\mathbf{G}(\xi)),\\ \mathbf{H}_2(\xi)&:=-g(\xi I-(I-g\Gamma))^{-1}(K\mathbf{G}(1)-\Gamma).\end{aligned}$$

We highlight that $\mathbf{H}_1,\mathbf{H}_2\in H^\infty(\mathcal{L}(U))$ for all $g\in(0,1]$. We claim that there exists $g^*\in(0,1]$ such that

$$\|\mathbf{H}_1\|_{H^\infty}\le\varepsilon\quad\forall\, g\in(0,g^*),\tag{2.11}$$

and that

$$\|\mathbf{H}_2\|_{H^\infty}\le\sup_{\xi\in\mathbb{E}}\|(\xi I-(I-\Gamma))^{-1}\|\cdot\|\Gamma-K\mathbf{G}(1)\|\quad\forall\, g\in(0,1].\tag{2.12}$$

The desired estimate (2.10) follows from the conjunction of (2.11) and (2.12). We record that

$$(\xi-1)/g+1\in\mathbb{E}\quad\forall\,\xi\in\mathbb{E},\ \forall\, g\in(0,1],\tag{2.13}$$

which follows from the estimates

$$\begin{aligned}\left|\frac{\xi-1}{g}+1\right|&\ge\left|\frac{\xi-1}{g}+\frac{1}{g}\right|-\left|\frac{1}{g}-1\right|>\frac{1}{g}-\frac{1}{g}+1\quad\text{as }\xi/g\in\mathbb{E}_{1/g}\\ &=1\quad\forall\,\xi\in\mathbb{E},\ \forall\, g\in(0,1].\end{aligned}$$

Consequently, from the exponential stability of $I-\Gamma$

$$\sup_{\xi\in\mathbb{E}}\left\|\left(\left(\frac{\xi-1}{g}+1\right)I-(I-\Gamma)\right)^{-1}\right\|=M_1<\infty\quad\forall\, g\in(0,1].\tag{2.14}$$

To establish (2.11) we express $\mathbf{H}_1$ as

$$\mathbf{H}_1(\xi)=g\left(I+\frac{g}{\xi-1}\Gamma\right)^{-1}K\frac{\mathbf{G}(1)-\mathbf{G}(\xi)}{\xi-1}\quad\forall\,\xi\in\mathbb{E},\ \forall\, g\in(0,1].\tag{2.15}$$

We claim that there exists $M>0$ such that

$$\left\|\left(I+\frac{g}{\xi-1}\Gamma\right)^{-1}\right\|\le M\quad\forall\,\xi\in\mathbb{E},\forall\, g\in(0,1].\tag{2.16}$$

To show this, fix $\rho > 1$ and let $g \in (0,1]$ and $\xi \in \mathbb{E}$. If $|\xi - 1| \le \rho\|\Gamma\|g$, then, from (2.14),

$$\Big\|\Big(I + \frac{g}{\xi-1}\Gamma\Big)^{-1}\Big\| = \frac{|\xi-1|}{g}\Big\|\Big(\Big(\frac{\xi-1}{g}+1\Big)I - (I-\Gamma)\Big)^{-1}\Big\| \\ \le \rho\|\Gamma\|M_1\,.$$

If instead $|\xi - 1| \ge \rho\|\Gamma\|g$, then

$$\Big\|\frac{g}{\xi-1}\Gamma\Big\| = g\frac{\|\Gamma\|}{|\xi-1|} \le \frac{1}{\rho} < 1\,,$$

whence, estimating the convergent Neumann series gives

$$\Big\|\Big(I + \frac{g}{\xi-1}\Gamma\Big)^{-1}\Big\| = \Big\|\Big(I - \Big(-\frac{g}{\xi-1}\Gamma\Big)\Big)^{-1}\Big\| = \Big\|\sum_{k\in\mathbb{Z}_+}\Big(-\frac{g}{\xi-1}\Gamma\Big)^k\Big\| \\ \le \sum_{k\in\mathbb{Z}_+}\Big\|\frac{g}{\xi-1}\Gamma\Big\|^k \le \frac{\rho}{\rho-1} =: M_2\,.$$

Taking $M := \max\{\rho\|\Gamma\|M_1, M_2\}$ gives (2.16). Combining (2.15) and (2.16) yields that

$$\|\mathbf{H}_1(\xi)\| \le gM\|\mathbf{J}\|_{H^\infty} \quad \forall\,\xi\in\mathbb{E},\,\forall\,g\in(0,1], \tag{2.17}$$

where

$$\mathbf{J}(\xi) := \begin{cases} K\frac{\mathbf{G}(\xi)-\mathbf{G}(1)}{\xi-1}, & \text{if } \xi \ne 1 \\ K\mathbf{G}'(1), & \text{if } \xi = 1. \end{cases}$$

The bound (2.11) now follows from (2.17) by taking $g^* := \min\{1, \varepsilon/(M\|\mathbf{J}\|_{H^\infty})\}$.

To establish (2.12), we note that

$$\|\mathbf{H}_2(\xi)\| \le \Big\|\Big(\frac{\xi-1}{g}I + \Gamma\Big)^{-1}\Big\|\cdot\|\Gamma - K\mathbf{G}(1)\| \\ = \Big\|\Big(\Big(\frac{\xi-1}{g}+1\Big)I - (I-\Gamma)\Big)^{-1}\Big\|\cdot\|\Gamma - K\mathbf{G}(1)\| \\ \le \sup_{\lambda\in\mathbb{E}}\|(\lambda I - (I-\Gamma))^{-1}\|\cdot\|\Gamma - K\mathbf{G}(1)\| \quad \forall\,\xi\in\mathbb{E},\,\forall\,g\in(0,1]\,, \tag{2.18}$$

where we have used (2.13) to obtain the final inequality. The estimate (2.12) follows from (2.18). □

We are now in position to prove Theorem 2.1.

Proof of Theorem 2.1. Let $r \in R$, $u^r \in U^r$, $x^0 \in X$, $u^0 \in U$, $g > 0$ and $(v,w) \in V^{\mathbb{Z}_+} \times U^{\mathbb{Z}_+}$ be given. Let $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ be given by (2.9), define

$$\phi_{u^r}(\zeta) := \phi(\zeta + u^r) - \phi(u^r) \quad \forall\,\zeta\in U\,,$$

and consider the Lur'e system of difference equations

$$\left.\begin{aligned} z^+ &= \widetilde{A}z + \widetilde{B}\xi + \begin{pmatrix} I & 0 & 0 \\ 0 & -gK & I \end{pmatrix}\begin{pmatrix} B_{\mathrm{e}}v \\ D_{\mathrm{e}}v \\ w \end{pmatrix}, \quad z(0) = z^0\,, \\ \xi &= \phi_{u^r}(\widetilde{C}z)\,. \end{aligned}\right\} \tag{2.19}$$

It is clear that a unique solution to (2.19) exists. Furthermore, a routine calculation shows that z is a solution to (2.19) if, and only if,

$$z = \begin{pmatrix} \widetilde{x} \\ \widetilde{u} \end{pmatrix} := \begin{pmatrix} x - (I-A)^{-1}B\phi(u^r) \\ u - u^r \end{pmatrix}, \tag{2.20}$$

where (u, x) is a solution of (2.3). Hence unique solutions of (2.3) exist, proving statement (1).

To prove statement (2), we realize that the estimate (2.6) follows from an application of [13, Statement (ii) of Theorem 3.2] to (2.19) and (2.20). We proceed to verify that the hypotheses of [13, Theorem 3.2] hold. To this end, we note that inequality (2.5) implies the existence of an $\varepsilon > 0$ such that

$$\varepsilon + \sup_{\xi \in \mathbb{E}} \|(\xi I - (I - \Gamma))^{-1}\| \cdot \|\Gamma - K\mathbf{G}(1)\| < 1/L. \tag{2.21}$$

Moreover, by invoking Lemma 2.3, we obtain the existence of $g^* \in (0, 1]$ (independent of u, w, v, x, y, r and u^r) such that, for all $g \in (0, g^*)$, the transfer function $\widetilde{\mathbf{G}}$ of $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ satisfies (2.10). This, along with (2.21), implies that

$$\|\widetilde{\mathbf{G}}\|_{H^\infty} < 1/L \quad \forall\, g \in (0, g^*).$$

Finally, since ϕ_{u^r} satisfies

$$\|\phi_{u^r}(\xi)\|_U \leq L\|\xi\|_U \quad \forall\, \xi \in U\,,$$

by (2.4), we see that the hypotheses of [13, Theorem 3.2] are satisfied.

The estimate (2.7) follows from (2.6) and the bounds

$$\begin{aligned} \|y(t) - r\| &\leq \|C\|\|x(t) - (I-A)^{-1}B\phi(u^r)\| + \|D\|\|\phi(u(t)) - \phi(u^r)\| + \|D_{\mathrm{e}}v(t)\| \\ &\leq \|C\|\|x(t) - (I-A)^{-1}B\phi(u^r)\| + L\|D\|\|u(t) - u^r\| \\ &\qquad + \|D_{\mathrm{e}}v(t)\| \quad \forall\, t \in \mathbb{Z}_+\,, \end{aligned}$$

completing the proof. □

3. **Sampled-data integral control.** In this section, we apply Theorem 2.1 in the context of sampled-data low-gain integral control of well-posed linear systems. We will be brief in our setup, since there is much literature concerning these systems. Indeed, we refer the reader to the references [26, 27, 29, 30] for more details. Throughout, we consider an L^2-well-posed system with state space X, input space $U \times U_{\mathrm{e}}$, output space Y (all Hilbert spaces), generating operators $(A, (B, B_{\mathrm{e}}), C)$, and transfer function $(\mathbf{H}, \mathbf{H}_{\mathrm{e}})$. Control inputs will act through B, whilst external disturbances will act through B_{e}.

By definition, A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X, $(B, B_{\mathrm{e}}) \in \mathcal{L}(U \times U_{\mathrm{e}}, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$, where X_1 is the domain of A endowed with the graph norm $\|x\|_1 := \|x\| + \|Ax\|$, and X_{-1} is the completion of X with respect to the norm $\|x\|_{-1} := \|(\beta I - A)^{-1}x\|$, where β is in the resolvent set of A. We recall here that the choice of β is unimportant, since a different choice leads to equivalent norms. It is clear that $X_1 \subset X \subset X_{-1}$ and that the canonical injections are dense. It is well-known that the semigroup $\mathbb{T}$ restricts to a strongly continuous semigroup on X_1 and extends to a strongly continuous semigroup on X_{-1}, with the exponential growth constants being the same on all three spaces X_1, X and X_{-1}. It is also true that the generator of the restricted semigroup is a restriction of A, and the generator of the extended semigroup is an extension of A (to X). We shall use the same symbols, $\mathbb{T}$ and A, for the restriction and extension of the semigroups and their generators, respectively.

The operators $(B, B_{\rm e})$ and C are admissible control and observations operator, respectively, that is, for every $t \geq 0$, there exist $b_t \geq 0$ and $c_t \geq$ such that

$$\left\| \int_0^t \mathbb{T}(t-s)(B, B_{\rm e})v(s)\, ds \right\| \leq b_t \|v\|_{L^2} \quad \forall\, v \in L^2([0,t], U \times U_{\rm e}),$$

and

$$\int_0^t \|C\mathbb{T}(t)\xi\|^2 dt \leq c_t \|\xi\|^2 \quad \forall\, \xi \in X_1.$$

We define the Λ-extension of C as

$$C_\Lambda \xi := \lim_{\lambda \to \infty} C\lambda(\lambda I - A)^{-1}\xi \quad \forall\, \xi \in \mathrm{dom}(C_\Lambda),$$

where $\mathrm{dom}(C_\Lambda)$ is the set of all $\xi \in X$ such that the above limit exists. We note that $X_1 \subseteq \mathrm{dom}(C_\Lambda)$. Moreover, for all $\xi \in X$, it follows that $\mathbb{T}(t)\xi \in \mathrm{dom}(C_\Lambda)$ for almost all $t \geq 0$, and $C_\Lambda \mathbb{T}\xi \in L^2_\alpha(\mathbb{R}_+, Y)$ for all $\alpha > \omega(\mathbb{T})$, where $\omega(\mathbb{T})$ denotes the exponential growth constant of $\mathbb{T}$, viz.

$$\omega(\mathbb{T}) := \lim_{t \to \infty} \frac{1}{t} \ln \|\mathbb{T}(t)\|\,.$$

The transfer function $(\mathbf{H}, \mathbf{H}_{\rm e})$ is a bounded holomorphic function $\mathbb{C}_\alpha \to \mathcal{L}(U \times U_{\rm e}, Y)$ for every $\alpha > \omega(\mathbb{T})$. Finally, throughout we assume that $\omega(\mathbb{T}) < 0$, that is to say that $\mathbb{T}$ is exponentially stable.

For given $\phi : U \to U$ globally Lipschitz, we shall consider the continuous-time system of the form

$$\left.\begin{aligned} &\dot{x} = Ax + B\phi(w) + B_{\rm e}v_1, \quad x(0) = x^0 \in X, \\ &y = C_\Lambda\left(x + A^{-1}(B\phi(w) + B_{\rm e}v_1)\right) + \mathbf{H}(0)\phi(w) + \mathbf{H}_{\rm e}(0)v_1, \end{aligned}\right\} \tag{3.1}$$

where $w \in L^2_{\rm loc}(\mathbb{R}_+, U)$ is the control input and $v_1 \in L^2_{\rm loc}(\mathbb{R}_+, U_{\rm e})$ denotes a disturbance. We interpret the differential equation in (3.1) in the larger space X_{-1}. If the well-posed system is regular, then the output equation (3.1) reduces to the more familar form $y = C_\Lambda x + D\phi(w) + D_{\rm e}v_1$, where $(D, D_{\rm e}) \in \mathcal{L}(U \times U_{\rm e}, Y)$ is the feedthrough operator.

Let $\tau > 0$ denote the sampling period. The (zero-order) hold operator $\mathcal{H}$ is defined as

$$(\mathcal{H}u)(t) := u(k) \quad \forall\, t \in [k\tau, (k+1)\tau), \quad \forall\, u \in U^{\mathbb{Z}_+}\,,$$

which maps $U^{\mathbb{Z}_+}$ into the set of U-valued step-functions (of step length τ) defined on $[0, \infty)$. Furthermore, we let $a \in L^2([0,\tau], \mathbb{R})$ be such that

$$\text{(i)} \int_0^\tau a(t)\, dt = 1 \quad \text{and} \quad \text{(ii)} \int_0^\tau a(t)\mathbb{T}(t)x dt \in X_1 \quad \forall\, x \in X. \tag{3.2}$$

We comment that (ii) holds if a is piecewise absolutely continuous, which follows from integration by parts and the fact that $\int_0^s \mathbb{T}(t)\, x dt = A^{-1}(\mathbb{T}(s) - I)x$ is a continuous X_1-valued function of s for every $x \in X$. A trivial example of a function a satisfying (3.2) is the constant function $a(t) = 1/\tau$ for all $t \in [0, \tau]$.

For $a \in L^2([0,\tau], \mathbb{R})$ satisfying (3.2), the (generalised) sampling operator $\mathcal{S} : L^2_{\rm loc}(\mathbb{R}_+, Y) \to Y^{\mathbb{Z}_+}$ is defined by

$$(\mathcal{S}y)(k) := \int_0^\tau a(t)y(k\tau + t)\, dt \quad \forall\, y \in L^2_{\rm loc}(\mathbb{R}_+, Y), \quad \forall\, k \in \mathbb{Z}_+\,.$$

We say that $r \in Y$ is feasible (with respect to (3.1)), if the set

$$U^r := \left\{ w \in U : \mathbf{H}(0)\phi(w) = r \right\},$$

is non-empty. A subset $R \subseteq Y$ is said to be feasible (with respect to (3.1)) if every $r \in R$ is feasible.

Given feasible $R \subseteq Y$, $r \in R$ and $u^r \in U^r$, consider the discrete-time integral control law

$$u^+ = u + gK(r - \mathcal{S}y) - g\Gamma(u - \phi(u) - u^r + \phi(u^r)) + v_2, \quad u(0) = u^0 \in U\,, \tag{3.3}$$

where y is the output of (3.1), and $g > 0$, $K \in \mathcal{L}(Y, U)$ and $\Gamma \in \mathcal{L}(U)$ are design parameters. The signal $v_2 \in U^{\mathbb{Z}_+}$ is a disturbance.

The feedback interconnection of (3.1) and the control law (3.3) via $w = \mathcal{H}u$, yields the closed-loop system

$$\dot{x} = Ax + B\phi(\mathcal{H}u) + B_{\mathrm{e}}v_1, \quad x(0) = x^0 \in X, \tag{3.4a}$$

$$y = C_\Lambda\left(x + A^{-1}(B\phi(\mathcal{H}u) + B_{\mathrm{e}}v_1)\right) + \mathbf{H}(0)\phi(\mathcal{H}u) + \mathbf{H}_{\mathrm{e}}(0)v_1, \tag{3.4b}$$

$$u^+ = u + gK(r - \mathcal{S}y) - g\Gamma(u - \phi(u) - u^r + \phi(u^r)) + v_2, \quad u(0) = u^0 \in U. \tag{3.4c}$$

For given $v_1 \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U_{\mathrm{e}})$, $v_2 \in U^{\mathbb{Z}_+}$, $x^0 \in X$ and $u^0 \in U$, we say that $(x, y, u) \in C(\mathbb{R}_+, X) \times L^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \times U^{\mathbb{Z}_+}$ is a solution of (3.4) if

$$x(t) = \mathbb{T}(t)x^0 + \int_0^t \mathbb{T}(t-s)\left(B\phi((\mathcal{H}u)(s)) + B_{\mathrm{e}}v_1(s)\right) ds \quad \forall\, t \geq 0,$$

and (3.4b) and (3.4c) are satisfied. It is a routine exercise to show that there exists a unique solution of (3.4)

In the following, we let H and H_{e} denote the input-output operators associated with $\mathbf{H}$ and $\mathbf{H}_{\mathrm{e}}$, respectively. These are causal operators defined on $L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ and $L^2_{\mathrm{loc}}(\mathbb{R}_+, U_{\mathrm{e}})$, respectively, with values in $L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$, and they map $L^2_\alpha(\mathbb{R}_+, U)$ and $L^2_\alpha(\mathbb{R}_+, U_{\mathrm{e}})$, respectively, boundedly into $L^2_\alpha(\mathbb{R}_+, Y)$ for any $\alpha > \omega(\mathbb{T})$. We say that H has a measure impulse response if $U = \mathbb{C}^m$, $Y = \mathbb{C}^p$ and there exists a $\mathbb{C}^{p\times m}$-valued Borel measure on $\mathbb{R}_+$ such that

$$(Hv)(t) = (\mu * v)(t) := \int_0^t \mu(ds)v(t-s) \quad \forall\, t \geq 0, \forall\, v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^m)\,.$$

We now present the main theorem of this section, a stability result for the sampled-data low-gain integral control system (3.4). The strategy for the proof is to extract a discrete-time system from (3.4) which describes the evolution at the sampling points and which allows an application of Theorem 2.1, and then to show that the inter-sampling dynamics are well behaved.

Theorem 3.1. *Let $\omega(\mathbb{T}) < 0$, $K \in \mathcal{L}(Y, U)$, $\Gamma \in \mathcal{L}(U)$ and let $R \subseteq Y$ be feasible. Assume that $I - \Gamma$ is discrete-time exponentially stable, ϕ is globally Lipschitz continuous with Lipschitz constant $L > 0$ and*

$$\sup_{\xi \in \mathbb{E}} \|(\xi I - (I - \Gamma))^{-1}\| \cdot \|\Gamma - K\mathbf{H}(0)\| < 1/L\,. \tag{3.5}$$

Then there exists $g^* \in (0, 1]$ such that, for all $g \in (0, g^*)$, there exist constants $c_1, c_2, c_3, \gamma > 0$, $\alpha < 0$ and $\theta \in (0, 1)$ such that for all $r \in R$, $u^r \in U^r$, all $(x^0, u^0) \in X \times U$ and all $v_1 \in L^\infty(\mathbb{R}_+, U_{\mathrm{e}})$, $v_2 \in \ell^\infty(\mathbb{Z}_+, U)$, the solution (x, y, u) of (3.4) has the following properties.

(1) *The output and state variables, y, u and x, respectively satisfy*

$$\big\|u(k)-u^r\big\|+\big\|(\mathcal{S}y)(k)-r\big\|\le c_1\Big(\theta^k\Big\|\begin{pmatrix}x^0+A^{-1}B\phi(u^r)\\ u^0-u^r\end{pmatrix}\Big\|+\big\|v_1\big\|_{L^\infty(0,k\tau)}+\big\|\pi_{k-1}v_2\big\|_{\ell^\infty}\Big),\quad\forall\,k\in\mathbb{N}\,,\tag{3.6}$$

and

$$\|x(k\tau+t)+A^{-1}B\phi(u^r)\|\le c_2\Big(e^{-\gamma(k\tau+t)}\Big\|\begin{pmatrix}x^0+A^{-1}B\phi(u^r)\\ u^0-u^r\end{pmatrix}\Big\|+\big\|\pi_k v_2\big\|_{\ell^\infty}+\big\|v_1\big\|_{L^\infty(0,k\tau+t)}\Big)\quad\forall\,k\in\mathbb{Z}_+,\ \forall\,t\in[0,\tau)\,.\tag{3.7}$$

(2) *If H has measure impulse response and*

$$H_{\rm e}\in\mathcal{L}(L^\infty(\mathbb{R}_+,U_{\rm e}),L^\infty(\mathbb{R}_+,Y)),\tag{3.8}$$

then there exists $q\in L^2_\alpha(\mathbb{R}_+,\mathbb{R}_+)$ such that

$$\|y(k\tau+t)-r\|\le q(t)+c_3\big(\|v_1\|_{L^\infty(0,k\tau+t)}+\|\pi_k v_2\|_{\ell^\infty}\big)\quad\forall\,k\in\mathbb{Z}_+,\ \text{for a.e. } t\in[0,\tau)\,.\tag{3.9}$$

Moreover, under the additional assumption that there exists $t_0\ge 0$ such that

$$\mathbb{T}(t_0)(Ax^0+B\phi(u^r))\in X\,,\tag{3.10}$$

the output y satisfies

$$\limsup_{t\to\infty}\|y(t)-r\|\le c_3\big(\|v_1\|_{L^\infty(0,\infty)}+\|v_2\|_{\ell^\infty}\big)\,.\tag{3.11}$$

(3) *If $v_1=0$ and $v_2=0$, then $u(k)\to u^r$ as $k\to\infty$ and $x(t)\to -A^{-1}B\phi(u^r)$ as $t\to\infty$ exponentially and $r-y\in L^2_\alpha(\mathbb{R}_+,Y)$; furthermore, if additionally (3.10) and*

$$\lim_{t\to\infty}(Hf)(t)=0\quad\forall\,f\in L^2(\mathbb{R}_+,U)\ \text{with}\ \lim_{t\to\infty}f(t)=0\,,\tag{3.12}$$

hold, then $y(t)\to r$ as $t\to\infty$.

Remark 3.2. (i) We note that the conclusions of Theorem 3.1 hold for any sampling period $\tau>0$. Although not performed exhaustively, our numerical simulations suggest that performance (for instance speed of convergence) is slower when τ is larger, which seems reasonable intuitively.

(ii) If $H_{\rm e}$ has measure impulse response, then (3.8) is satisfied. Condition (3.12) holds if H has measure impulse response (see [17, Lemma 6.2.4]).

(iii) Note that if $r-y\in L^2_\alpha(\mathbb{R}_+,Y)$ for some $\alpha<0$, then, for any $\delta>0$,

$$\lim_{\theta\to\infty}\ \text{meas}\,\{t\ge\theta: e^{-\alpha t}\|y(t)-r\|\ge\delta\}=0,\quad\text{where meas}=\text{Lebesgue measure},$$

that is, y tracks r exponentially fast in measure.

(iv) The smoothness condition $x^0+A^{-1}B\phi(u^r)\in X_1$ is sufficient for (3.10) to hold. Furthermore, analyticity of the semigroup $\mathbb{T}(t)$ guarantees that (3.10) is satisfied for every $t_0>0$, $x^0\in X$, $r\in R$ and $u^r\in U^r$. ◊

The remainder of this section is dedicated to proving Theorem 3.1.

Proof of Theorem 3.1. (1) The first aim is to extract a discrete-time system from (3.4) to which Theorem 2.1 may be applied. For which purpose, define

$$Mx:=\int_0^\tau a(t)\mathbb{T}(t)x\,dt\quad\forall\,x\in X\,.$$

Then, $Mx \in X_1$ for all $x \in X$ by property (ii) in (3.2). It is straightforward to show that M is closed, and hence $M \in \mathcal{L}(X, X_1)$ by the closed-graph theorem. Furthermore, set

$$A_\tau := \mathbb{T}(\tau),\ B_\tau := \int_0^\tau \mathbb{T}(s)Bds,\ C_\tau := CM,\ D_\tau := CMA^{-1}B + \mathbf{H}(0)\,. \quad (3.13)$$

Clearly $A_\tau \in \mathcal{L}(X)$, and it easy to show that A_τ is (discrete-time) exponentially stable, since $\mathbb{T}$ is an exponentially stable semigroup. Further, B_τ satisfies

$$B_\tau = (\mathbb{T}(\tau) - I)A^{-1}B \in \mathcal{L}(U, X)\,,$$

and rearranging the above gives

$$(I - A_\tau)^{-1}B_\tau = -A^{-1}B\,. \quad (3.14)$$

By the boundedness of M, it follows that $C_\tau \in \mathcal{L}(X, Y)$ and $D_\tau \in \mathcal{L}(U, Y)$. We denote the transfer function of the discrete-time system given by $(A_\tau, B_\tau, C_\tau,\ D_\tau)$ by $\mathbf{G}_\tau$.

Let $r \in R$, $u^r \in U^r$, $(x^0, u^0) \in X \times U$ and $v_1 \in L^\infty(\mathbb{R}_+, U_\mathrm{e})$, $v_2 \in \ell^\infty(\mathbb{Z}_+, U)$ be given, and let (x, y, u) denote the solution of (3.4). We claim that ξ and ζ given by

$$\xi(k) := x(k\tau) \quad \text{and} \quad \zeta(k) := (\mathcal{S}y)(k) \quad \forall\, k \in \mathbb{Z}_+\,, \quad (3.15a)$$

satisfy the discrete-time system

$$\left.\begin{aligned} \xi^+ &= A_\tau\xi + B_\tau\phi(u) + \eta_1, \\ \zeta &= C_\tau\xi + D_\tau\phi(u) + \eta_2, \end{aligned}\right\} \quad (3.15b)$$

with

$$\eta_1(k) := \int_0^\tau \mathbb{T}(\tau - s)B_\mathrm{e}v_1(s + \tau k)\, ds \quad \forall\, k \in \mathbb{Z}_+\,, \quad (3.15c)$$

and

$$\begin{aligned} \eta_2(k) := &\int_0^\tau a(t)C_\Lambda \left(\int_0^t \mathbb{T}(t - s)B_\mathrm{e}v_1(s + k\tau))ds + A^{-1}B_\mathrm{e}v_1(k\tau + t) \right) dt \\ &+ \int_0^\tau a(t)\mathbf{H}_\mathrm{e}(0)v_1(k\tau + t)dt \quad \forall\, k \in \mathbb{Z}_+. \end{aligned} \quad (3.15d)$$

Further, recall that u in (3.15b) is given by

$$u^+ = u + gK(r - \zeta) - g\Gamma(u - \phi(u) - u^r + \phi(u^r)) + v_2\,. \quad (3.16)$$

We seek to apply Theorem 2.1 to the feedback interconnection of (3.15) and (3.16). We proceed in steps.

Step 1. Extracting a discrete-time system. We show that ξ and ζ in (3.15a) do indeed satisfy (3.15b) with η_1 and η_2 as in (3.15c) and (3.15d), respectively. Straightforward modifications of [18, Propositon 3.1] to incorporate v_1 give the difference equation for ξ. As for the output equation of (3.15b), let us first note that, for all $k \in \mathbb{Z}_+$ and $t \in [0, \tau)$,

$$
\begin{aligned}
y(k\tau + t) &= C_\Lambda \big(x(k\tau + t) + A^{-1}B\phi(u(k)) + A^{-1}B_{\rm e}v_1(k\tau + t)\big) + \mathbf{H}(0)\phi(u(k)) \\
&\quad + \mathbf{H}_{\rm e}(0)v_1(k\tau + t) \\
&= C_\Lambda \Bigg(\mathbb{T}(t)x(k\tau) + \int_{k\tau}^{k\tau+t} \mathbb{T}(k\tau + t - s)(B\phi(u(k)) + B_{\rm e}v_1(s))ds \\
&\qquad + A^{-1}B\phi(u(k)) + A^{-1}B_{\rm e}v_1(k\tau + t)\Bigg) + \mathbf{H}(0)\phi(u(k)) \\
&\qquad + \mathbf{H}_{\rm e}(0)v_1(k\tau + t)\,.
\end{aligned}
$$

By using a change of variables, this becomes, for all $k \in \mathbb{Z}_+$ and $t \in [0, \tau)$,

$$
\begin{aligned}
y(k\tau + t) = C_\Lambda \Bigg(&\mathbb{T}(t)x(k\tau) + \int_0^t \mathbb{T}(t - s)\big(B\phi(u(k)) + B_{\rm e}v_1(s + k\tau)\big)ds \\
&+ A^{-1}B\phi(u(k)) + A^{-1}B_{\rm e}v_1(k\tau + t)\Bigg) + \mathbf{H}(0)\phi(u(k)) \\
&+ \mathbf{H}_{\rm e}(0)v_1(k\tau + t)\,.
\end{aligned}
$$

Since

$$
\int_0^t \mathbb{T}(t - s)B\phi(u(k))ds = A^{-1}(\mathbb{T}(t) - I)B\phi(u(k)) \quad \forall\, k \in \mathbb{Z}_+, \forall\, t \in [0, \tau)\,,
$$

we see that, for all $k \in \mathbb{Z}_+$ and $t \in [0, \tau)$,

$$
\begin{aligned}
y(k\tau + t) = C_\Lambda \Bigg(&\mathbb{T}(t)x(k\tau) + \mathbb{T}(t)A^{-1}B\phi(u(k)) + \int_0^t \mathbb{T}(t - s)B_{\rm e}v_1(s + k\tau)ds \\
&+ A^{-1}B_{\rm e}v_1(k\tau + t)\Bigg) + \mathbf{H}(0)\phi(u(k)) + \mathbf{H}_{\rm e}(0)v_1(k\tau + t)\,.
\end{aligned}
$$

Consequently, for all $k \in \mathbb{Z}_+$,

$$
\begin{aligned}
\zeta(k) = \int_0^\tau a(t)C_\Lambda \Bigg(&\mathbb{T}(t)x(k\tau) + \mathbb{T}(t)A^{-1}B\phi(u(k)) + \int_0^t \mathbb{T}(t - s)B_{\rm e}v_1(s + k\tau)ds \\
&+ A^{-1}B_{\rm e}v_1(k\tau + t)\Bigg)\, dt + \int_0^\tau a(t)\big(\mathbf{H}(0)\phi(u(k)) + \mathbf{H}_{\rm e}(0)v_1(k\tau + t)\big)dt\,.
\end{aligned}
$$

Recalling that, for all $\xi \in X$, $\mathbb{T}(t)\xi \in \operatorname{dom}(C_\Lambda)$ for almost all $t \geq 0$, we see that, for all $k \in \mathbb{Z}_+$,

$$
\begin{aligned}
\zeta(k) &= \int_0^\tau a(t)C_\Lambda\mathbb{T}(t)(x(k\tau) + A^{-1}B\phi(u(k)))dt \\
&\quad + \int_0^\tau a(t)C_\Lambda \left(\int_0^t \mathbb{T}(t - s)B_{\rm e}v_1(s + k\tau))ds + A^{-1}B_{\rm e}v_1(k\tau + t)\right) dt \\
&\quad + \int_0^\tau a(t)\big(\mathbf{H}(0)\phi(u(k)) + \mathbf{H}_{\rm e}(0)v_1(k\tau + t)\big)dt\,.
\end{aligned}
$$

Invoking (3.2), (3.13) and (3.15d), we obtain that, for all $k \in \mathbb{Z}_+$,

$$
\begin{aligned}
\zeta(k) &= CMx(k\tau) + CMA^{-1}B\phi(u(k)) + \mathbf{H}(0)\phi(u(k)) + \eta_2(k) \\
&= C_\tau\xi(k) + D_\tau\phi(u(k)) + \eta_2(k)\,.
\end{aligned}
$$

We have now shown that (3.15b) holds.

STEP 2. ESTIMATING THE DISCRETE-TIME FORCING TERMS η_1 AND η_2. By the admissibility of $B_{\rm e}$, there exists $\kappa > 0$ such that

$$\Big\| \int_0^t \mathbb{T}(t-s)B_{\rm e}v_1(k\tau+s)\,ds\Big\| \le \kappa\|v_1\|_{L^2([k\tau,k\tau+t])} \le \kappa\sqrt{\tau}\|v_1\|_{L^\infty([k\tau,k\tau+t])} \quad \forall\, k \in \mathbb{Z}_+,\ \forall\, t \in [0,\tau)\,. \tag{3.17}$$

Whence,

$$\|\pi_{k-1}\eta_1\|_{\ell^\infty} \le \kappa\sqrt{\tau}\|v_1\|_{L^\infty([0,k\tau])} \quad \forall\, k \in \mathbb{N}\,. \tag{3.18}$$

We now seek to prove the existence of $d > 0$ such that

$$\|\pi_{k-1}\eta_2\|_{\ell^\infty} \le d\|v_1\|_{L^\infty([0,k\tau])} \quad \forall\, k \in \mathbb{N}\,. \tag{3.19}$$

To this end, by writing

$$w_{\rm e}(t) := C_\Lambda\left(\int_0^t \mathbb{T}(t-s)B_{\rm e}v_1(s+k\tau))ds + A^{-1}B_{\rm e}v_1(k\tau+t)\right) + \mathbf{H}_{\rm e}(0)v_1(k\tau+t),$$

we see that $w_{\rm e}$ is the output of an exponentially stable well-posed system with zero initial condition and input $v_1(k\tau+\cdot)$. Hence, there exists $d_1 > 0$ such that

$$\|w_{\rm e}\|_{L^2([0,\tau])} \le d_1\|v_1\|_{L^2([k\tau,(k+1)\tau])} \le d_1\sqrt{\tau}\|v_1\|_{L^\infty([k\tau,(k+1)\tau])}.$$

By use of Hölder's inequality, we then obtain from (3.15d) that, for all $k \in \mathbb{Z}_+$,

$$\|\eta_2(k)\| \le \|a\|_{L^2([0,\tau])}\|w_{\rm e}\|_{L^2([0,\tau])} \le d\|v_1\|_{L^\infty([k\tau,(k+1)\tau])},$$

where $d := d_1\sqrt{\tau}\|a\|_{L^2([0,\tau])} > 0$. It is thus evident that (3.19) holds.

STEP 3. INVOKING THEOREM 2.1. An application of [18, Propositon 3.1] yields that $\mathbf{G}_\tau(1) = \mathbf{H}(0)$. Thus, $r \in Y$ is feasible with respect to (3.1) if, and only if, it is feasible with respect to (3.15b). Moreover, from (3.5), (2.5) is satisfied in the context of (3.15). Therefore, in light of steps 1 and 2 above, an application of statement (2) of Theorem 2.1 to (3.15) and (3.16) yields the existence of $g^* \in (0,1]$ such that, for all $g \in (0,g^*)$, there exist $c_1 > 0$ and $\theta \in (0,1)$ such that

$$\left\|\begin{pmatrix}\xi(k)-(I-A_\tau)^{-1}B_\tau\phi(u^r)\\ u(k)-u^r\\ (\mathcal{S}y)(k)-r\end{pmatrix}\right\| \le c_1\left(\theta^k\left\|\begin{pmatrix}x^0-(I-A_\tau)^{-1}B_\tau\phi(u^r)\\ u^0-u^r\end{pmatrix}\right\| + \left\|\pi_{k-1}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\right\|_{\ell^\infty}\right) \quad \forall\, k \in \mathbb{N}\,. \tag{3.20}$$

In light of (3.14), (3.18) and (3.19) (and the definition of the norm on product spaces), the estimate (3.6) follows.

STEP 4. DERIVING (3.7). We note that, for each $k \in \mathbb{N}$ and $t \in [0,\tau)$,

$$\begin{aligned}x(k\tau+t) &= \mathbb{T}(t)x(k\tau) + \int_{k\tau}^{t+k\tau}\mathbb{T}(t+k\tau-s)(B\phi(u(k))+B_{\rm e}v_1(s))\,ds\\ &= \mathbb{T}(t)x(k\tau) + \int_0^t \mathbb{T}(s)B\phi(u(k))\,ds + \int_0^t \mathbb{T}(t-s)B_{\rm e}v_1(k\tau+s)\,ds\\ &= \mathbb{T}(t)x(k\tau) + (\mathbb{T}(t)-I)A^{-1}B\phi(u(k))\\ &\quad + \int_0^t \mathbb{T}(t-s)B_{\rm e}v_1(k\tau+s)\,ds\,.\end{aligned} \tag{3.21}$$

Therefore, taking norms in (3.21) and invoking (3.17), we see that

$$\begin{aligned}\|x(k\tau+t)+A^{-1}B\phi(u^r)\| \leq \|\mathbb{T}(t)\|\|x(k\tau)+A^{-1}B\phi(u(k))\|\\ +\|A^{-1}B(\phi(u^r)-\phi(u(k)))\|+\kappa\sqrt{\tau}\|v_1\|_{L^\infty([k\tau,k\tau+t])}\\ \forall\, k\in\mathbb{Z}_+,\ \forall\, t\in[0,\tau)\,.\end{aligned} \tag{3.22}$$

Define the positive constants

$$\nu_1 := \sup_{t\in[0,\tau]}\|\mathbb{T}(t)\|<\infty \quad\text{and}\quad \nu_2 := \|A^{-1}B\|<\infty\,. \tag{3.23}$$

The Lipschitz property of ϕ together with (3.22) and (3.23) leads to

$$\begin{aligned}\|x(k\tau+t)+A^{-1}B\phi(u^r)\| \leq \nu_1\|x(k\tau)+A^{-1}B\phi(u^r)\|+\nu_2 L(1+\nu_1)\|u(k)-u^r\|\\ +\kappa\sqrt{\tau}\|v_1\|_{L^\infty([k\tau,k\tau+t])}\\ \forall\, k\in\mathbb{Z}_+,\ \forall\, t\in[0,\tau)\,.\end{aligned} \tag{3.24}$$

Setting $\gamma := -(\ln\theta)/\tau>0$ and $\nu_3 := e^{\gamma\tau}$, we have that

$$\theta^k = \nu_3 e^{-\gamma(k+1)\tau} \leq \nu_3 e^{-\gamma(k\tau+t)} \quad \forall\, k\in\mathbb{Z}_+,\ \forall\, t\in[0,\tau)\,. \tag{3.25}$$

Combining (3.20) with (3.14), (3.24) and (3.25) gives (3.7), as required.

(2) Set

$$f(t) := C_\Lambda\mathbb{T}(t)x^0 + H(\phi(u^r)) - \mathbf{H}(0)\phi(u^r),$$

where we view $\phi(u^r)$ as the constant function with value $\phi(u^r)$. The Laplace transform of $\mathcal{H}(\phi(u^r))-\mathbf{H}(0)\phi(u^r)$ is given by $(\mathbf{H}(s)-\mathbf{H}(0))\phi(u^r)/s$. As this function is in $H^2(\mathbb{C}_\beta,Y)$ for every $\beta>\omega(\mathbb{T})$, the Payley-Wiener theorem [27, Theorem 10.3.4] then guarantees that

$$H(\phi(u^r))-\mathbf{H}(0)\phi(u^r)\in L^2_\beta(\mathbb{R}_+,Y) \quad \forall\,\beta>\omega(\mathbb{T}),$$

and so $f\in L^2_\beta(\mathbb{R}_+,Y)$ for all $\beta>\omega(\mathbb{T})$. As $r=\mathbf{H}(0)\phi(u^r)$, the error $y-r$ can be written in the form

$$y-r = f + H(\mathcal{H}(\phi(u))-\phi(u^r)) + H_{\rm e}v_1. \tag{3.26}$$

Assume now that H has a measure impulse response and that (3.8) holds. Denoting the measure impulse response of H by μ and the total variation of μ by $|\mu|$, we have that

$$\big\|\big(H(\mathcal{H}(\phi(u))-\phi(u^r))\big)(t)\big\| \leq L(|\mu|*\|\mathcal{H}(u)-u^r\|)(t) \quad \forall\, t\geq 0,$$

where we recall that L is the Lipschitz constant of ϕ. Consequently, it follows from statement (1) that

$$\begin{aligned}\big\|\big(H(\mathcal{H}(\phi(u))-\phi(u^r))\big)(k\tau+t)\big\| \leq c_1 L(|\mu|*q_0)(k\tau+t)\\ +|\mu|(\mathbb{R}_+)(\|v_1\|_{L^\infty}+\|v_2\|_{\ell^\infty})\\ \forall\, k\in\mathbb{Z}_+,\ \forall\, t\in[0,\tau),\end{aligned} \tag{3.27}$$

where

$$q_0(k\tau+t) := \theta^k\left\|\begin{pmatrix} x^0+A^{-1}B\phi(u^r)\\ u^0-u^r\end{pmatrix}\right\| \quad \forall\, k\in\mathbb{Z}_+,\ \forall\, t\in[0,\tau).$$

Noting that $q_0\in L^2_{\alpha_0}(\mathbb{R}_+,\mathbb{R})$, where $\alpha_0 := (\ln\theta)/\tau<0$, and choosing $0>\alpha>\max(\omega(\mathbb{T}),\alpha_0)$, we conclude that $|\mu|*q_0\in L^2_\alpha(\mathbb{R}_+,\mathbb{R})$, and so, the function q defined by

$$q(t) := \|f(t)\| + c_1 L(|\mu|*q_0)(t) \ \ \forall\, t\geq 0\,,$$

is in $L^2_\alpha(\mathbb{R}_+,\mathbb{R})$. It follows from (3.26) and (3.27) that there exists $c_3 > 0$ such that (3.9) holds.

Let us now assume that (3.10) is satisfied. It is clear from (3.9) that, to establish (3.11), it is sufficient to show that $q(t) \to 0$ as $t \to \infty$. Since

$$\lim_{t\to\infty} (|\mu| * q_0)(t) = 0,$$

we need to show that $f(t) \to 0$ as $t \to \infty$. The Laplace transform $\hat{f}$ of f is given by

$$\hat{f}(s) = C(sI - A)^{-1}x^0 + \frac{1}{s}(\mathbf{H}(s) - \mathbf{H}(0))\phi(u^r) \quad \forall\, s \in \mathbb{C}_{\omega(\mathbb{T})}\,. \tag{3.28}$$

By [27, Theorem 4.6.7]

$$\frac{1}{s}(\mathbf{H}(s) - \mathbf{H}(0))\phi(u^r) = C(sI - A)^{-1}A^{-1}B\phi(u^r) \quad \forall\, s \in \mathbb{C}_{\omega(\mathbb{T})}\,,$$

and thus, it follows from (3.28) that

$$\hat{f}(s) = C(sI - A)^{-1}A^{-1}(Ax_0 + B\phi(u^r)) \quad \forall s \in \mathbb{C}_{\omega(\mathbb{T})}\,.$$

Consequently,

$$\begin{aligned} f(t) &= C_\Lambda \mathbb{T}(t) A^{-1}(Ax_0 + B\phi(u^r)) \\ &= CA^{-1}\mathbb{T}(t - t_0)\mathbb{T}(t_0)(Ax_0 + B\phi(u^r)) \quad \forall\, t \geq t_0\,, \end{aligned}$$

and the exponential stability of $\mathbb{T}$ implies that $\lim_{t\to\infty} f(t) = 0$.

(3) Assume that $v_1 = 0$ and $v_2 = 0$. It is clear from statement (1) that $u(k) \to u^r$ as $k \to \infty$ and $x(t) \to -A^{-1}B\phi(u^r)$ as $t \to \infty$ and that the convergences are exponentially fast. To show that $y - r \in L^2_\alpha(\mathbb{R}_+, Y)$, we note that, by (3.26),

$$y - r = f + H(\mathcal{H}(\phi(u)) - \phi(u^r)). \tag{3.29}$$

Set $\rho := e^{\alpha\tau}$ and note that

$$\begin{aligned} \int_0^\infty \|e^{-\alpha t}(\mathcal{H}(\phi(u))(t) - \phi(u^r))\|^2\, dt &= \int_0^\tau e^{-2\alpha t}\, dt \sum_{k=0}^\infty e^{-2\alpha k\tau}\|\phi(u(k)) - \phi(u^r)\|^2 \\ &= \frac{1}{2|\alpha|}(e^{-2\alpha\tau} - 1)\sum_{k=0}^\infty \rho^{-2k}\|\phi(u(k)) - \phi(u^r)\|^2 \\ &< \infty\,, \end{aligned} \tag{3.30}$$

where the convergence of the infinite series follows from (3.6) and the fact that $\rho > \theta$. Consequently, $\mathcal{H}(\phi(u)) - \phi(u^r)) \in L^2_\alpha(\mathbb{R}_+, Y)$. We know that $f \in L^2_\alpha(\mathbb{R}_+, Y)$ and therefore, invoking (3.29), we conclude that $y - r \in L^2_\alpha(\mathbb{R}_+, Y)$.

Finally assume that (3.10) and (3.12) hold. We know from the proof of statement (2) that $f(t) \to 0$ as $t \to \infty$. By (3.30), $\mathcal{H}(\phi(u)) - \phi(u^r)$ is in $L^2(\mathbb{R}_+, U)$ and, by statement (1), $\lim_{t\to\infty}(\mathcal{H}(\phi(u))(t) = \phi(u^r)$, and thus, invoking (3.12), $H(\mathcal{H}(\phi(u)) - \phi(u^r))(t) \to 0$ as $t \to \infty$. It folows now from that $y(t) \to r$ as $t \to \infty$. □

4. **Examples.** We conclude with two examples. The first considers an application of Theorem 3.1 to a system with output quantization, and in the second example we apply Theorem 3.1 to to a heat equation on a square domain.

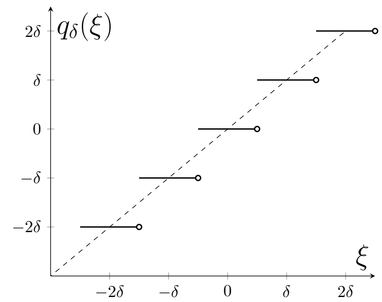

FIGURE 4.1. Quantization function q_δ.

Example 4.1. For $\delta > 0$, the quantization function $q_\delta : \mathbb{R} \to \mathbb{R}$ is defined by

$$q_\delta\left(\delta k + \xi\right) := \delta k \quad \forall\, k \in \mathbb{Z},\ \forall\, \xi \in [-\delta/2, \delta/2)\,.$$

The graph of q_δ is plotted in Figure 4.1.

We consider the sampled-data system (3.4) with $K \in \mathcal{L}(Y,U)$, $\Gamma \in \mathcal{L}(U)$, $g > 0$ and $r \in R \subseteq Y$, where R is feasible. Let $E \subset Y$ be an orthonormal basis of Y and let $\eta = (\eta_e)_{e\in E} \in \ell^2(E)$ be positive, that is, $\eta_e > 0$ for every $e \in E$. If Y is not separable, then E is uncountable, in which case $\ell^2(E)$ is defined to be the space of all functions $\xi : E \to \mathbb{R}$ such that $\xi(e) \neq 0$ for at most countably many $e \in E$ and $\sum_{e\in E} |\xi(e)|^2 < \infty$. Obviously, if Y is of finite dimension p, then $\ell^2(E)$ can identified with $\mathbb{R}^p$. Given $\zeta \in Y$, we have that

$$|q_{\eta_e}(\langle \zeta, e\rangle)| \leq |\langle \zeta, e\rangle| + \eta_e/2 \quad \forall\, e \in E,$$

and so the function $e \mapsto q_{\eta_e}(\langle \zeta, e\rangle)$ is in $\ell^2(E)$. Consequently, the quantization operator $Q_\eta : Y \to Y$ given by

$$Q_\eta(\zeta) := \sum_{e\in E} q_{\eta_e}(\langle \zeta, e\rangle)e \quad \forall\, \zeta \in Y\,,$$

is well defined. We will study the situation in which only the quantized version $Q_\eta(y)$ is available for feedback, but not y itself. That is, the feedback system under consideration is given by

$$\dot{x} = Ax + B\phi(\mathcal{H}u) + B_{\mathrm{e}}v_1, \quad x(0) = x^0 \in X, \tag{4.1a}$$
$$y = C_\Lambda\left(x + A^{-1}(B\phi(\mathcal{H}u) + B_{\mathrm{e}}v_1)\right) + \mathbf{H}(0)\phi(\mathcal{H}u) + \mathbf{H}_{\mathrm{e}}(0)v_1, \tag{4.1b}$$
$$u^+ = u + gK(r - \mathcal{S}(Q_\eta \circ y)) - g\Gamma(u - \phi(u) - u^r + \phi(u^r)) + v_2,$$
$$u(0) = u^0 \in U. \tag{4.1c}$$

As

$$\|(\mathcal{S}y)(k) - (\mathcal{S}(Q_\eta \circ y))(k)\| \leq \|a\|_{L^2}\sqrt{\tau}\|\eta\|_{\ell^2}/2 \quad \forall\, k \in \mathbb{Z}_+,$$

where $\|\eta\|_{\ell^2} := \left(\sum_{e\in E} |\xi(e)|^2\right)^{1/2}$, it is clear that we can write (4.1c) in the form

$$u^+ = u + gK(r - \mathcal{S}y) - g\Gamma(u - \phi(u) - u^r + \phi(u^r)) + v_2 + w, \quad u(0) = u^0 \in U, \tag{4.2}$$

where $w \in U^{\mathbb{Z}_+}$ satisfies $\|w\|_{\ell^\infty} \leq g\|K\|\|a\|_{L^2}\sqrt{\tau}\|\eta\|_{\ell^2}/2$.

Assuming that (3.5) holds, then the conclusions of statement (1) of Theorem 3.1 apply to the sampled-data system given by (4.1a), (4.1b) and (4.2). If additionally, H has measure impulse response, $v_1 = 0$, $v_2 = 0$ and (3.10) holds, then, by statement (2) of Theorem 3.1[2], there exists $c > 0$ such that

$$\limsup_{t\to\infty} \|y(t) - r\| \leq c\|\eta\|_{\ell^2} .$$

Hence, the smaller $\|\eta\|_{\ell^2}$ (the size of the quantization parameter), the more accurate the tracking behaviour of the sampled-date system. $\Diamond$

Example 4.2. We consider sampled-data low-gain integral control in the presence of input saturation for a controlled and observed heat equation on a square domain with disturbances. Let $\Omega := (0,1)\times(0,1) \subseteq \mathbb{R}^2$ denote the unit square. We define the sections of the boundary

$$\begin{aligned}
\partial\Omega_1 &:= \left\{(\xi_1,\xi_2) \in \mathbb{R}^2 \;:\; \; 0 < \xi_1 < 1,\; \xi_2 = 0\right\},\\
\partial\Omega_2 &:= \left\{(\xi_1,\xi_2) \in \mathbb{R}^2 \;:\; \; 0 < \xi_1 < 1,\; \xi_2 = 1\right\},\\
\partial\Omega_3 &:= \left\{(\xi_1,\xi_2) \in \mathbb{R}^2 \;:\; \xi_1 = 0,\; 0 < \xi_2 < 1\right\},\\
\partial\Omega_4 &:= \left\{(\xi_1,\xi_2) \in \mathbb{R}^2 \;:\; \xi_1 = 1,\; 0 < \xi_2 < 1\right\},
\end{aligned}$$

the bottom, top, left and right sides, respectively, see Figure 4.2.

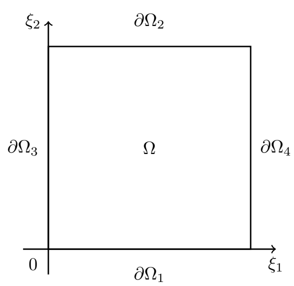

FIGURE 4.2. Square domain $\Omega \subseteq \mathbb{R}^2$.

The plant inputs u_1 and u_2 are boundary controls applied on $\partial\Omega_1$ and $\partial\Omega_2$, respectively, and the boundaries $\partial\Omega_3$ and $\partial\Omega_4$ are maintained at a constant temperature, leading to

$$\frac{\partial z}{\partial t} = \Delta z \quad \text{in } (0,\infty)\times\Omega, \tag{4.3a}$$

$$z = z^0 \quad \text{on } \{0\}\times\Omega, \tag{4.3b}$$

$$z = 0 \quad \text{on } (0,\infty)\times\partial\Omega_i,\ \text{ for } i \in \{3,4\}, \tag{4.3c}$$

$$\frac{\partial z}{\partial n} = q_i \quad \text{on } (0,\infty)\times\partial\Omega_i,\ \text{ for } i \in \{1,2\}, \tag{4.3d}$$

where $z = z(t,\xi)$ and

- z^0 is the initial condition;
- Δ denotes the Laplacian;
- $\frac{\partial z}{\partial n} = \nabla z \cdot n$ is the outward normal derivative of z;
- $q_i = u_i + v_i$ is the sum of the plant input u_i and disturbance v_i.

[2] Since we are assuming that $v_1 = 0$, the condition (3.8) is irrelevant and does not need to be imposed.

The plant outputs y_1 and y_2 are the averaged temperature over the two boundary sections $\partial\Omega_1$ and $\partial\Omega_2$, respectively, that is,

$$y_i(t) = \int_{\partial\Omega_i} z(t,\xi)\, d\xi \quad \text{for } t > 0 \text{ and } i \in \{1,2\}\,. \tag{4.3e}$$

We claim that (4.3) gives rise to a L^2 well-posed linear system on the state-space $X = L^2(\Omega)$ with input and disturbance space $U = U_\mathrm{e} = \mathbb{R}^2$, $B = B_\mathrm{e}$, and output space $Y = \mathbb{R}^2$. The details are somewhat technical, require some material from the theory of PDEs, and are not required to illustrate our results, which is the purpose of the present example. Therefore, we have chosen to relegate the details to the Appendix.

The semigroup associated with (4.3) is exponentially stable and analytic, so that (3.10) holds for every $t_0 > 0$, $x^0 \in X$ and every feasible r. Furthermore, the resulting well-posed linear system has an L^1 impulse response, and so properties (3.8) and (3.12) hold as well.

Since $B = B_\mathrm{e}$, the transfer functions from input to output $\mathbf{H}$ in (4.3), and from disturbance to output $\mathbf{H}_\mathrm{e}$ in (4.3) are equal. Thus, the transfer function of (4.3) can be identified with $\mathbf{H}$. In the Appendix we show that

$$\mathbf{H}(0) = \begin{pmatrix} c & d \\ d & c \end{pmatrix}\,, \tag{4.4}$$

for certain constants $c > d > 0$. In particular, $\mathbf{H}(0)$ is invertible.

For the purpose of output regulation by sampled-data low-gain integral control in the presence of input saturation, we fix

$$a(t) = \frac{1}{\tau} \quad \forall\, t \in [0,\tau], \quad K := \mathbf{H}(0)^{-1} \quad \text{and} \quad \Gamma := I\,. \tag{4.5}$$

The choice of a in (4.5) leads to the sampled output $\mathcal{S}y$ given by

$$(\mathcal{S}y)(k) = \frac{1}{\tau}\int_0^\tau y(t+k\tau)\, dt, \quad \forall\, k \in \mathbb{N}, \quad \text{where } y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The choice of K and Γ in (4.5) ensures that the condition (3.5) holds for any $L > 0$, and hence for any globally Lipschitz saturation term ϕ. Therefore, the conclusions of Theorem 3.1 apply to (3.4) with plant given by (4.3) and controller data as specified in (4.5).

For a numerical simulation, we discretise (4.3) in space with a finite-difference approximation, the details of which are given in the Appendix. Common to all our simulations are the data

$$g = 0.1, \quad z^0(\xi) = \sin(\pi\xi_1) \quad \forall\, \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathrm{cl}\,\Omega\,, \tag{4.6}$$

where $\mathrm{cl}\,\Omega$ denotes the closure of Ω. For fixed $a < b$, define $\psi(\cdot\,; a, b) : \mathbb{R} \to \mathbb{R}$ by $\psi(w\,; a,b) = \max\{a, \min\{w, b\}\}$. We consider two saturation functions $\phi = \phi_i$ for $i = 1,2$ given by

$$\phi_1(w) = \begin{pmatrix} \psi(w_1\,; -1, 4) \\ \psi(w_2\,; -1, 2) \end{pmatrix} \quad \text{and} \quad \phi_2(w) = \begin{pmatrix} \psi(w_1\,; -1, 4) \\ \psi(w_2\,; -2, 3) \end{pmatrix} \quad \forall\, w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{R}^2\,. \tag{4.7}$$

The sets

$$\big\{\mathbf{H}(0)\phi_i(w) \,:\, w \in \mathbb{R}^2\big\} \quad i = 1,2\,,$$

are the largest sets of feasible references for (4.3) and (4.4) with $\phi = \phi_i$, and are plotted in Figure 4.3. The reference

$$r := \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}, \tag{4.8}$$

is clearly feasible in both cases.

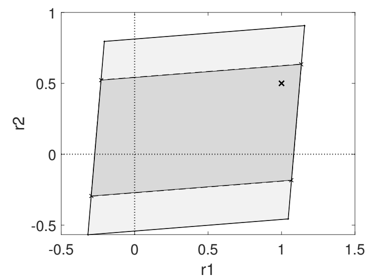

FIGURE 4.3. Feasible sets of reference vectors $r = \begin{pmatrix} r_1 & r_2 \end{pmatrix}^T$ for ϕ_1 (darker grey) and ϕ_2 (lighter gray) regions. The reference r in (4.8) is marked with a cross.

We will use the notation

$$v := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ and } y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

For the simulation plotted in Figure 4.4, we choose

$$u(0) = 0, \quad \tau \in \{0.25, 0.5\} \quad \text{and} \quad \phi = \phi_1\,. \tag{4.9}$$

The simulations shown in Figure 4.4 illustrate the convergence of the sampled-data system when no disturbance is present, that is, when $v = 0$. Moreover, in this example, we see that the saturation bounds are not reached (meaning $\phi_1(u(t)) = u(t)$ for all $t \in \mathbb{Z}_+$). As expected, the convergence appears faster when the sampling-period τ is smaller.

For the simulation presented in Figure 4.5, the data in (4.9) are replaced by

$$u(0) = \begin{pmatrix} 6 \\ -4 \end{pmatrix}, \quad \tau = 0.4 \quad \text{and} \quad \phi = \phi_2\,. \tag{4.10}$$

Additionally, as nonzero forcing we choose

$$v = 0.1 \begin{pmatrix} \sin(t) \\ \sin(3t/4) + \sin(\sqrt{2}t) \end{pmatrix} \quad \forall\, t \geq 0\,. \tag{4.11}$$

The choice of initial integrator state $u(0)$ in the saturation region of ϕ is deliberate and is to illustrate the saturation effects. Two simulations are shown with forcing terms v and $3v$. Not surprisingly, the tracking error seems to increase as $\|v\|_{L^\infty}$ increases. $\Diamond$

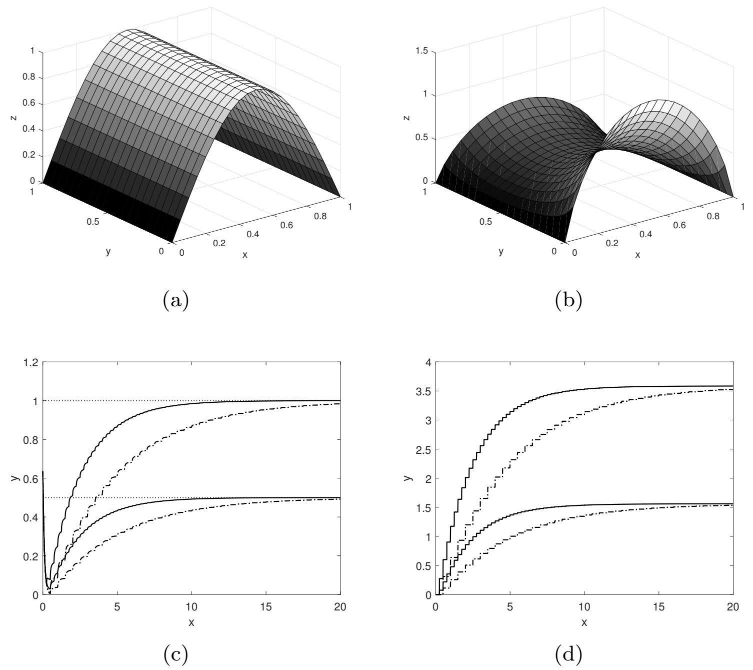

FIGURE 4.4. Model data as in (4.5), (4.6), (4.8) and (4.9). (a) Initial temperature profile z^0. (b) Temperature profile of solution z at time $t = 20$. (c) Outputs. (d) Held inputs. In panels (c) and (d), the solid and dashed lines correspond to $\tau = 0.25$ and $\tau = 0.5$, respectively. The dotted lines in panel (c) are the components of the reference.

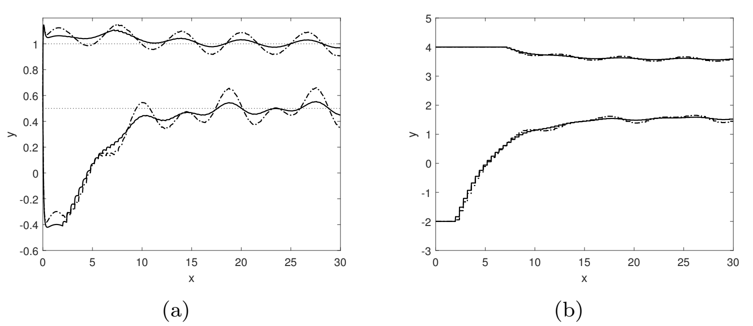

FIGURE 4.5. Model data as in (4.5), (4.6), (4.8), (4.10) and (4.11). (a) Outputs. (b) Held inputs. In panels (a) and (b), the solid and dashed lines correspond to the external forcing v and $3v$, respectively. The dotted lines in panel (a) are the components of the reference.

Appendix

Further details for example 4.2. We provide further details for Example 4.2 not given in the main text. We first describe the theoretical setting, and then our numerical approximation.

Theoretical setting

The following arguments are somewhat similar to those in [3]. There the authors assume that the boundary $\partial\Omega$ has a smoothness called piecewise C^2, which (despite the name) is not the case here. To describe the model requires some standard material from the theory of PDEs. We make use of notation and results from [14], and from the theory of well-posed linear systems [27]. First note that $\Omega := (0,1) \times (0,1) \subseteq \mathbb{R}^2$ clearly has a so-called curvilinear polygonal boundary [14, Definition 1.4.5.1]. We let $H^s(\Omega)$ for $s \in \mathbb{R}$ denote the usual (fractional) Sobolev space. For open $\Gamma \subseteq \mathbb{R}$ and a given function $z : \Gamma \to \mathbb{R}$, we let $\widetilde{z}$ denote the extension of z to all $\mathbb{R}$ by zero. Thus for $s > 0$, we let

$$\widetilde{H}^s(\Gamma) := \left\{ z \in H^s(\Gamma) \,:\, \widetilde{z} \in H^s(\mathbb{R}) \right\}.$$

Further, $C_c^\infty(\operatorname{cl}\Omega)$ (denoted by $\mathcal{D}(\overline{\Omega})$ in [14]) denotes the set of infinitely-differentiable functions $\mathbb{R}^2 \to \mathbb{R}$ with compact support, restricted to $\operatorname{cl}\Omega$. Next, let $\gamma_i : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega_i)$ denote the usual trace operators, which are continuous extensions of γ_i defined on $C_c^\infty(\operatorname{cl}\Omega)$ given by

$$\gamma_i z = z|_{\partial\Omega_i} \quad \forall\, i \in \{1,2,3,4\},$$

see [14, Theorem 1.5.2.1]. Let

$$E := \{ z \in H^1(\Omega) \,:\, \Delta z \in L^2(\Omega) \}, \tag{A.1}$$

which is a Banach space with the norm

$$\|z\|_E := \|z\|_{H^1(\Omega)} + \|\Delta z\|_{L^2(\Omega)} \quad \forall\, z \in E.$$

Furthermore, by [14, Theorem 1.5.3.10], the Neumann trace $\gamma_j \frac{\partial}{\partial n}$ defined on C_c^∞ $(\operatorname{cl}\Omega)$ has a continuous extension $E \to (\widetilde{H}^{\frac{1}{2}}(\partial\Omega_j))^*$. The following Green's identity for Ω holds

$$\int_\Omega (\Delta u) v \, d\xi = -\int_\Omega \langle \nabla u, \nabla v \rangle \, d\xi + \sum_{j=1}^4 \int_{\partial\Omega} \Big(\gamma_j \frac{\partial u}{\partial n} \Big) \gamma_j v d\sigma$$
$$\forall\, u \in H^2(\Omega),\ \forall\, v \in H^1(\Omega), \tag{A.2}$$

and also for all $u \in E$, and all $v \in H^1(\Omega)$ such that $\gamma_i v \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_i)$ for all $i = 1,2,3,4$. These claims are [14, Lemma 1.5.3.8] and [14, Theorem 1.5.3.11], respectively. Note that $\widetilde{H}^{\frac{1}{2}}(\partial\Omega_i)$ contains the image under γ_i of $\{v \in H^1(\Omega) : \gamma_k v = 0,\ k \neq i\}$, see [14, p. 61].

Noting that the constant function with value one, denoted by 1, satisfies $1 \in L^2(\partial\Omega_i) \hookrightarrow H^{-\frac{1}{2}}(\partial\Omega_i)$, and that the $L^2(\partial\Omega_i)$ inner product is the duality product between $H^{-\frac{1}{2}}(\partial\Omega_i)$ and $H^{\frac{1}{2}}(\partial\Omega_i)$, we obtain the estimates

$$\left| \int_{\partial\Omega_i} \gamma_i(w) \right| = \left| \langle 1, \gamma_i(w) \rangle_{L^2(\partial\Omega_i)} \right| \leq \|1\|_{H^{-\frac{1}{2}}(\partial\Omega_i)} \|\gamma_i(w)\|_{H^{\frac{1}{2}}(\partial\Omega_i)}$$
$$\leq m_0 \|w\|_{H^1(\Omega)} \quad \forall\, w \in H^1(\Omega),\ \forall\, i \in \{1,2,3,4\}, \tag{A.3}$$

for some $m_0 > 0$.

Next, we define

$$H^1_{\Gamma_0}(\Omega) := \{z \in H^1(\Omega) \,:\, \gamma_3 z = 0,\ \gamma_4 z = 0\}.$$

We shall make extensive use of the Poincaré-type estimate [29, Theorem 13.6.9], namely, that there exists $\delta > 0$ such that

$$\delta \|z\|_{L^2(\Omega)} \leq \|\nabla z\|_{L^2(\Omega)} \quad \forall\, z \in H^1_{\Gamma_0}(\Omega). \tag{A.4}$$

Therefore, $H^1_{\Gamma_0}(\Omega)$ is a Hilbert space when equipped with the inner product

$$\langle z_1, z_2\rangle_{H^1_{\Gamma_0}(\Omega)} := \langle \nabla z_1, \nabla z_2\rangle_{L^2(\Omega)} \quad \forall\, z_1, z_2 \in H^1_{\Gamma_0}(\Omega),$$

and $\|\cdot\|_{H^1_{\Gamma_0}(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent norms on $H^1_{\Gamma_0}(\Omega)$. We proceed to describe the generators A, B and C of Σ. Recall that $X = L^2(\Omega)$. Define A by

$$A : X \supseteq D(A) \to X, \quad Az = \Delta z,$$

with domain

$$D(A) := \{z \in H^1_{\Gamma_0}(\Omega) \,:\, \Delta z \in X\} \cap \ker\Big(\gamma_1 \frac{\partial}{\partial n}\Big) \cap \ker\Big(\gamma_2 \frac{\partial}{\partial n}\Big).$$

Note that $\{z \in H^1_{\Gamma_0}(\Omega) \,:\, \Delta z \in X\} \subseteq E$ given by (A.1), and the Neumann trace is continuous on E, so $D(A)$ is well-defined. We claim that A generates an analytic and exponentially stable contraction semigroup. The following is loosely based on [29, Section 3.6] which considers the Dirichlet Laplacian. First note that A is symmetric as, by Green's identity (A.2) (with second set of hypotheses on z_1 and z_2)

$$\langle z_1, Az_2\rangle_X = -\langle \nabla z_1, \nabla z_2\rangle_X = \langle Az_1, z_2\rangle_X \quad \forall\, z_1, z_2 \in D(A). \tag{A.5}$$

Second, A is surjective since, for every $f \in L^2(\Omega)$, there exists a unique solution $z \in D(A)$ to the elliptic problem

$$\left.\begin{aligned} \Delta z &= f \\ \gamma_i z &= 0 \quad i \in \{3,4\}, \\ \gamma_j \frac{\partial z}{\partial n} &= 0 \quad j \in \{1,2\}, \end{aligned}\right\}$$

see [14, Lemma 4.4.3.1]. Consequently, A is self-adjoint by [29, Proposition 3.2.4]. Taking $z = z_1 = z_2$ in (A.5) gives that A is dissipative and so, by the Lumer-Phillips Theorem, A generates a strongly continuous contraction semigroup, denoted $\mathbb{T}$. By [25, Theorem 13.31 (a)], we have that $\sigma(A) \subseteq (-\infty, 0]$ and since X is a Hilbert space and A is self-adjoint, analyticity of $\mathbb{T}$ follows from, for example, [8, Corollary II. 4.7].

For exponential stability, fix $z_0 \in X$. Taking $z_1 = z_2 = \mathbb{T}(t)z_0 \in D(A)$ for $t > 0$ in (A.5) gives

$$\langle \mathbb{T}(t)z_0, A\mathbb{T}(t)z_0\rangle_X = -\|\nabla \mathbb{T}(t)z_0\|_X^2 \leq -\delta^2 \|\mathbb{T}(t)z_0\|_X^2 \quad \forall\, t > 0,$$

by (A.4) again. Therefore, since $z := z_1 = z_2$ is classically differentiable, the above reads

$$\frac{d}{dt}\frac{1}{2}\|z(t)\|_X^2 = -\|\nabla z(t)\|_X^2 \leq -\delta^2 \|z(t)\|_X^2 \quad \forall\, t > 0.$$

Rearranging and integrating gives

$$\int_{\mathbb{R}_+} \|\mathbb{T}(t)z_0\|_X^2 \, dt \leq \frac{1}{2\delta^2}\|z_0\|^2 \quad \forall\, z_0 \in X.$$

Hence, exponential stability of $\mathbb{T}$ follows by Datko's Theorem.

For the sequel we shall require that $-A$ is positive, as

$$\langle z, (-A)z\rangle_X = \|\nabla z\|_X^2 \geq \delta^2 \|z\|_X^2 \quad \forall\, z \in D(A)\,,$$

by (A.4). Consequently, $(-A)$ has a square root, denoted $(-A)^{\frac{1}{2}}$.

We define the control operator $B : U \to (H^1_{\Gamma_0}(\Omega))^*$ by

$$B = \begin{pmatrix} B_1 & B_2 \end{pmatrix} \quad \text{where} \quad \langle B_j u, w\rangle_X = u \int_{\partial\Omega_j} \gamma_j(w) \quad \forall\, w \in H^1_{\Gamma_0}(\Omega), \quad \forall\, u \in \mathbb{C}\,,$$

which is bounded by (A.3), and set $B_{\mathrm{e}} := B$. Similarly, we define the observation operator $C : H^1_{\Gamma_0}(\Omega) \to Y$ by

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \quad \text{where} \quad C_j w := \int_{\partial\Omega_j} \gamma_j(w) \quad \forall\, w \in H^1_{\Gamma_0}(\Omega),\ \forall\, j \in \{1,2\}\,,$$

which again is bounded by (A.3).

Given $X = L^2(\Omega)$, recall the usual interpolation and extrapolation spaces X_1 and X_{-1}, respectively, (see the start of Section 3 or [27, Section 3.6]). For $\lambda \geq 0$, we also require the fractional powers $(\lambda I - A)^\gamma$, where $\gamma \in [-1,1]$, and the associated fractional spaces X_γ for $\gamma \in [-1,1]$, described in [27, Section 3.9]. Since A generates an exponentially stable semigroup, for simplicity we consider $\lambda = 0$.

We claim that, for all $\varepsilon \in (0, 1/4)$,

$$C \in \mathcal{L}(X_{1/4+\varepsilon}, Y) \quad \text{and} \quad B \in \mathcal{L}(U, X_{-(1/4+\varepsilon)})\,. \tag{A.6}$$

Since $B = C^*$, the second claim in (A.6) follows from the first. Assuming (A.6), it follows from [27, Theorem 5.7.3, statement (ii)] (with, using the notation of that result $\gamma = 0$) that (A, B, C) generates a L^2 well-posed linear system on (Y, X, U). That the impulse response belongs to L^1 follows from the same result.

To establish (A.6), fix $\varepsilon \in (0, 1/4)$, $j \in \{1,2\}$, and set $\theta := 1/4 + \varepsilon$. It is sufficient to show that

$$C_j(-A)^{-\theta} : X \to \mathbb{C} \quad \text{is bounded}\,, \tag{A.7}$$

that is

$$|C_j(-A)^{-\theta} z| \lesssim \|z\|_X \quad \forall\, z \in X\,.$$

Here the symbol $\lesssim$ means less than or equal to, up to a multiplicative constant which is independent of the other variables appearing. Its use is intended to clarify the exposition by reducing the number of constants which appear in estimates.

Since $(-A)^\theta$ is an isometry $X_\theta \to X$ (cf. [27, p.149]), we have that $(-A)^\theta w =: z \in X$ for all $w \in X_\theta$. Whence, if (A.7) holds, then

$$|C_j w| \lesssim \|(-A)^\theta w\|_X = \|w\|_{X_\theta} \quad \forall\, w \in X_\theta\,,$$

and so (A.6) holds.

We proceed to establish (A.7). Fix $z \in X = L^2(\Omega)$. Defining $x := (sI - A)^{-1} z \in X_1 = D(A)$ for all $s \geq 0$, we have that $-Ax + sx = z$. Taking the inner product in X of the previous equality with x and using (A.5) gives

$$\|\nabla x\|_X^2 + s\|x\|_X^2 = \langle z, x\rangle_X\,,$$

so that, by the inequality (A.4)

$$\|x\|^2_{H^1(\Omega)} \lesssim \|\nabla x\|_X^2 \leq \|\nabla x\|_X^2 + s\|x\|_X^2 \leq \|z\|_X \cdot \|x\|_X \leq \|z\|_X \cdot \|x\|_{H^1(\Omega)}\,.$$

Therefore, cancelling $\|x\|_{H^1(\Omega)} \neq 0$ from both sides of the inequality above gives

$$\|(sI - A)^{-1} z\|_{H^1(\Omega)} \leq \|z\|_X \quad \forall\, s \in (0,1)\,. \tag{A.8}$$

(Note that (A.8) trivially holds if $\|x\|_{H^1(\Omega)} = 0$.) Combining the continuous embedding $H^1(\Omega) \hookrightarrow H^{\frac{1}{2}}(\Omega)$ with (A.8) yields

$$\|(sI - A)^{-1}z\|_{H^{\frac{1}{2}}(\Omega)} \lesssim \|(sI - A)^{-1}z\|_{H^1(\Omega)} \leq \|z\|_X \quad \forall\, s \in (0,1)\,. \tag{A.9}$$

Since A generates an exponentially stable analytic semigroup, we have the resolvent estimate

$$\|(sI - A)^{-1}z\|_{L^2(\Omega)} = \|(sI - A)^{-1}z\|_X \lesssim \frac{1}{s}\|z\|_X \quad \forall\, s > 0\,, \tag{A.10}$$

from, for example, [27, Theorem 3.10.6]. Again with $x := (sI - A)^{-1}z \in D(A)$ for all $s \geq 0$, it follows from (A.5) and the inequality (A.4) that

$$\|x\|^2_{H^1(\Omega)} \lesssim \|\nabla x\|^2_X = \langle (-A)x, x\rangle_X = \|(-A)^{\frac{1}{2}}x\|^2_{L^2(\Omega)}\,,$$

that is

$$\begin{aligned}\|(sI - A)^{-1}z\|_{H^1(\Omega)} &\lesssim \|(-A)^{\frac{1}{2}}(sI - A)^{-1}z\|_X \\ &\lesssim \frac{1 + (s+\eta)^{\frac{1}{2}}}{s+\eta}\|z\|_X \lesssim \frac{1}{s^{\frac{1}{2}}}\|z\|_X \quad \forall\, s \geq 1\,.\end{aligned} \tag{A.11}$$

The second inequality above is an application of [27, Lemma 3.10.9] (with, using the notation of that result: $\gamma = 0$, $\lambda = s$, $\gamma' = -\eta > 0$, for some small $\eta > 0$, and $\alpha = 1/2$). Here it is crucial that $(-A)^{\frac{1}{2}}$ is both the square root of $-A$, and the fractional operator in the sense of [27, Sections 3.9–3.10].

Since Ω has Lipschitz boundary it follows from, for example [4, Corollary 4.7], that $H^{\frac{1}{2}}(\Omega)$ is the interpolation space between $L^2(\Omega)$ and $H^1(\Omega)$ (up to equivalent norms), and so the interpolation inequality

$$\|x\|_{H^{\frac{1}{2}}(\Omega)} \lesssim \|x\|^{\frac{1}{2}}_{L^2(\Omega)} \cdot \|x\|^{\frac{1}{2}}_{H^1(\Omega)} \quad \forall\, x \in H^1(\Omega)\,,$$

holds. Thus, combining (A.10) and (A.11) gives

$$\begin{aligned}\|(sI - A)^{-1}z\|_{H^{\frac{1}{2}}(\Omega)} &\lesssim \|(sI - A)^{-1}z\|^{\frac{1}{2}}_{L^2(\Omega)} \cdot \|(sI - A)^{-1}z\|^{\frac{1}{2}}_{H^1(\Omega)} \\ &\leq \frac{1}{s^{\frac{3}{4}}}\|z\|_{L^2(\Omega)} \quad \forall\, s \geq 1\,.\end{aligned} \tag{A.12}$$

From the expression for $(-A)^{-\theta}$ from [27, Lemma 3.9.9], namely,

$$(-A)^{-\theta} = \frac{\sin(\theta\pi)}{\pi}\int_0^\infty s^{-\theta}(sI - A)^{-1}\,ds\,,$$

and the bounds (A.9) and (A.12), we estimate that

$$\begin{aligned}\|(-A)^{-\theta}z\|_{H^{\frac{1}{2}}(\Omega)} &\lesssim \int_0^\infty s^{-\theta}\|(sI - A)^{-1}z\|_{H^{\frac{1}{2}}(\Omega)}\,ds \\ &\lesssim \Big(\int_0^1 s^{-\theta}\,ds + \int_1^\infty s^{-(\theta+\frac{3}{4})}\,ds\Big)\|z\|_X \\ &\lesssim \|z\|_X\,,\end{aligned} \tag{A.13}$$

where both integrals are finite since $\theta = 1/4 + \varepsilon$ with $\varepsilon \in (0, 1/4)$.

Finally, recalling that the Dirichlet trace γ_j is continuous $H^{\frac{1}{2}}(\Omega) \to L^2(\partial\Omega_j)$, we invoke (A.13) to estimate that

$$|C_j(-A)^{-\theta}z| = \Big|\int_{\partial\Omega_j}\gamma_j\big((-A)^{-\theta}z\big)\Big| \lesssim \|\gamma_j\big((-A)^{-\theta}z\big)\|_{L^2(\partial\Omega_j)} \lesssim \|(-A)^{-\theta}z\|_{H^{\frac{1}{2}}(\Omega)} \lesssim \|z\|_X\,,$$

as required.

We next establish the expression (4.4) for $\mathbf{H}(0)$, which we do by solving by separation of variables the elliptic problem

$$\left.\begin{aligned} \Delta z &= 0 \quad \text{on } \Omega\,, \\ z &= 0 \quad \text{on } \partial\Omega_i, \text{ for } i = 3,4\,, \\ \frac{\partial z}{\partial n} &= u_i \quad \text{on } \partial\Omega_i, \text{ for } i = 1,2\,, \end{aligned}\right\} \tag{A.14}$$

associated with (4.3) for fixed $u_i \in \mathbb{R}$.

For $i = 1, 2$, we let z_i denote the solution of

$$\Delta z_i = 0 \quad \text{on } \Omega, \quad z_i = 0 \quad \text{on } \partial\Omega_3 \cup \partial\Omega_4, \quad \frac{\partial z_i}{\partial n} = 0 \quad \text{on } \partial\Omega_{3-i}, \quad \frac{\partial z_i}{\partial n} = u_i \quad \text{on } \partial\Omega_i\,,$$

so that by superposition the solution of (A.14) is given by $z_1 + z_2$. For the following calculation, it is useful to note that

$$\int_0^1 \sin(\pi k \xi_1)\, d\xi_1 = \frac{2}{\pi k} \left.\begin{cases} 0 & k \text{ even}, \\ 1 & k \text{ odd}, \end{cases}\right\} = \frac{2}{\pi k} \chi(k) \quad \forall\, k \in \mathbb{N}\,,$$

where $\chi : \mathbb{N} \to \{0, 1\}$ is given by $\chi(m) = 0$ if m is even, and $\chi(m) = 1$ if m is odd. Routine calculations give that

$$z_1(\xi_1, \xi_2) = \sum_{k \in \mathbb{N}} a_k \sin(\pi k \xi_1) \cosh(\pi k(\xi_2 - 1)) \quad \forall\, \xi \in \Omega\,,$$

where a_k are the Fourier coefficients

$$a_k = \frac{-u_1}{\pi k \sinh(-k\pi)} 2 \int_0^1 \sin(\pi k \xi_1)\, d\xi_1 = 4 \frac{u_1}{(\pi k)^2 \sinh(k\pi)} \chi(k) \quad \forall\, k \in \mathbb{N}\,. \tag{A.15}$$

Similarly, routine calculations give that

$$z_2(\xi_1, \xi_2) = \sum_{k \in \mathbb{N}} b_k \sin(\pi k \xi_1) \cosh(\pi k \xi_2) \quad \forall\, \xi \in \Omega\,,$$

where b_k are the Fourier coefficients

$$b_k = \frac{u_2}{\pi k \sinh(k\pi)} 2 \int_0^1 \sin(\pi k \xi_1)\, d\xi_1 = 4 \frac{u_2}{(\pi k)^2 \sinh(k\pi)} \chi(k) \quad \forall\, k \in \mathbb{N}\,. \tag{A.16}$$

Thus

$$z = z_1 + z_2 = \sum_{k \in \mathbb{N}} \sin(\pi k \xi_1) \big(a_k \cosh(\pi k(\xi_2 - 1)) + b_k \cosh(\pi k \xi_2) \big) \quad \forall \xi \in \Omega\,, \tag{A.17}$$

solves (A.14). In light of (A.15), (A.16) and (A.17), the steady-state outputs y_j of (4.3) are given by

$$\begin{aligned} y_1 &= \int_0^1 z(\xi_1, 0)\, d\xi_1 = \sum_{k \in \mathbb{N}} \frac{2}{\pi k} \chi(k) \big(a_k \cosh(-\pi k) + b_k \big) \\ &= \sum_{k \in \mathbb{N}} \frac{8}{\sinh(\pi k)(\pi k)^3} \chi(k) \big(\cosh(\pi k) u_1 + u_2 \big)\,. \end{aligned} \tag{A.18}$$

Similarly,

$$\begin{aligned} y_2 &= \int_0^1 z(\xi_1, 1)\, d\xi_1 = \sum_{k \in \mathbb{N}} \frac{2}{\pi k} \chi(k) \big(a_k + b_k \cosh(\pi k) \big) \\ &= \sum_{k \in \mathbb{N}} \frac{8}{\sinh(\pi k)(\pi k)^3} \chi(k) \big(u_1 + \cosh(\pi k) u_2 \big)\,. \end{aligned} \tag{A.19}$$

Define the series

$$c := \sum_{k\in\mathbb{N}} \frac{8\cosh(\pi k)}{\sinh(\pi k)(\pi k)^3}\chi(k) \quad \text{and} \quad d := \sum_{k\in\mathbb{N}} \frac{8}{\sinh(\pi k)(\pi k)^3}\chi(k)\,,$$

which converge absolutely, and satisfy $c > d > 0$. Since $y = \mathbf{H}(0)u$, an inspection of (A.18) and (A.19) yields (4.4).

Numerical approximation

For notational convenience in this section, we define $\underline{N} := \{1, 2, \dots, N\}$ for $N \in \mathbb{N}$. For the numerical simulations, let $N \in \mathbb{N}$, $N \geq 2$, and define $\zeta_i := (i-1)/N$ for $i \in \underline{N+1}$. We partition $\operatorname{cl}\Omega$ into $(N+1)^2$ equally-spaced points (ζ_i, ζ_j) and set $h := 1/N > 0$. For given $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^2)$ and $z^0 \in L^2(\Omega)$, we approximate the solution z of (4.3) by $T_{ij}(t) = z(t, (\zeta_i, \zeta_j))$ for $t \geq 0$ and $i, j \in \underline{N+1}$. Approximating the Laplacian by centered finite-differences, we obtain the following differential equations for T_{ij}, for $i, j \in \{2, \dots, N\}$,

$$\begin{aligned}
\dot{T}_{ij} &= \frac{(T_{ij-1} - 2T_{ij} + T_{ij+1})}{h^2} + \frac{(T_{i-1j} - 2T_{ij} + T_{i+1j})}{h^2}\,,\\
T_{i1} &= T_{iN} = 0\,,\\
\dot{T}_{1j} &= \frac{(T_{1j-1} - 2T_{1j} + T_{1j+1})}{h^2} + \frac{(2hu_1 - 2T_{1j} + 2T_{2j})}{h^2}\,,\\
\dot{T}_{N+1j} &= \frac{(T_{N+1j-1} - 2T_{N+1j} + T_{N+1j+1})}{h^2} + \frac{(2T_{Nj} - 2T_{N+1j} + 2hu_2)}{h^2}\,,
\end{aligned}$$

with initial condition

$$T_{ij}(0) = z^0(\zeta_i, \zeta_j) \quad \forall\, i, j \in \underline{N+1}\,.$$

Letting the (i,j)-th entry of $T \in \mathbb{R}^{(N+1)\times N-1}$ equal T_{ij+1}, the above differential equations may be written in matrix form

$$\dot{T} = A_L T + T A_R + B_1 u_1 + B_2 u_2\,. \tag{A.20}$$

The matrices $A_L \in \mathbb{R}^{(N+1)\times(N+1)}$, $A_R \in \mathbb{R}^{(N-1)\times(N-1)}$, $B_1, B_2 \in \mathbb{R}^{(N+1)\times N-1}$ in (A.20) are given by

$$(A_L)_{ij} = N^2 \begin{cases} 2 & (i,j) \in \{(1,2), (N+1, N)\} \\ -2 & i = j \\ 1 & i \notin \{1, N+1\},\ j = i \pm 1 \\ 0 & \text{else,} \end{cases}$$

$$(A_R)_{ij} = N^2 \begin{cases} 1 & (i,j) \in \{(2,1), (N-2, N-1)\} \\ -2 & i = j \\ 1 & j \notin \{1, N-1\},\ i = j \pm 1 \\ 0 & \text{else,} \end{cases}$$

and

$$(B_1)_{ij} = \begin{cases} 2N & i = 1, j \in \underline{N-1} \\ 0 & \text{else} \end{cases} \quad \text{and} \quad (B_2)_{ij} = \begin{cases} 2N & i = N+1, j \in \underline{N-1} \\ 0 & \text{else.} \end{cases}$$

The vec operation denotes columnwise stacking of columns, from left to right, of a matrix into a vector. Taking vec of both sides of (A.20), and using standard vec properties, gives

$$\begin{aligned}\mathrm{vec}\left(\dot{T}\right) &= \mathrm{vec}\,(A_L T + T A_R + B_1 u_1 + B_2 u_2) \\ &= \big(I_{N-2} \otimes A_L + A_R \otimes I_N\big)\mathrm{vec}\,(T) + \mathrm{vec}\,(B_1)\,u_1 + \mathrm{vec}\,(B_2)\,u_2 \\ &= A\,\mathrm{vec}\,(T) + Bu\,, \end{aligned} \tag{A.21a}$$

where $\otimes$ denotes the Kronecker product, and

$$A := I_{N-2} \otimes A_L + A_R \otimes I_N, \quad B := \begin{pmatrix}\mathrm{vec}\,(B_1) & \mathrm{vec}\,(B_2)\end{pmatrix}.$$

Further, by the trapezoidal rule,

$$\begin{aligned} y_1(t) &= \int_0^1 z(t,(p,0))\,dp \approx \sum_{j=1}^{N} \frac{z(t,(\zeta_j,0)) + z(t,(\zeta_{j+1},0))}{2N} \\ &= \sum_{j=2}^{N} \frac{1}{N} T_{1j}(t) = C_1 \mathrm{vec}\,(T(t)) =: Y_1(t), \quad \forall\, t \geq 0\,, \end{aligned} \tag{A.21b}$$

where $C_1 \in \mathbb{R}^{N(N-1)}$ is given by

$$(C_1)_i := \begin{cases} \dfrac{1}{N} & i = 1 + k(N+1),\ k \in \{0,1,\ldots,N-2\} \\ 0 & \text{else.} \end{cases}$$

Similarly,

$$y_2(t) = \int_0^1 z(t,(p,1))\,dp \approx \sum_{j=2}^{N} \frac{1}{N} T_{N+1j}(t) = C_2 \mathrm{vec}\,(T(t)) =: Y_2(t), \quad \forall\, t \geq 0\,, \tag{A.21c}$$

where $C_2 \in \mathbb{R}^{N(N-1)}$ is given by

$$(C_2)_i := \begin{cases} \dfrac{1}{N} & i = (k-1)(N+1),\ k \in \{2\ldots,N\} \\ 0 & \text{else.} \end{cases}$$

The controlled and observed linear system (A.21) is a finite-dimensional approximation of the controlled and observed PDE (4.3) used in our numerical simulations.

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