Infinite-dimensional Lur'e systems with almost periodic forcing

Max E. Gilmore [*] Chris Guiver[*†] Hartmut Logemann[*]

September 2019

Abstract. We consider forced Lur'e systems in which the linear dynamic component is an infinite-dimensional well-posed system. Numerous physically motivated delay- and partial-differential equations are known to belong to this class of infinite-dimensional systems. We present refinements of recent incremental input-to-state stability results [14] and use them to derive convergence results for trajectories generated by Stepanov almost periodic inputs. In particular, we show that the incremental stability conditions guarantee that for every Stepanov almost periodic input there exists a unique pair of state and output signals which are almost periodic and Stepanov almost periodic, respectively. The almost periods of the state and output signals are shown to be closely related to the almost periods of the input, and a natural module containment result is established. All state and output signals generated by the same Stepanov almost periodic input approach the almost periodic state and the Stepanov almost periodic output in a suitable sense, respectively, as time goes to infinity. The sufficient conditions guaranteeing incremental input-to-state stability and the existence of almost periodic state and Stepanov almost periodic output signals are reminiscent of the conditions featuring in well-known absolute stability criteria such as the complex Aizerman conjecture and the circle criterion.

Keywords. Absolute stability, almost periodic functions, circle criterion, incremental input-to-state stability, infinite-dimensional systems, Lur'e systems, small gain.

MSC(2010). 34C27, 93C10, 93C20, 93C23, 93C25, 93C35, 93D05, 93D09, 93D10, 93D25.

1 Introduction

The analysis of solutions of differential equations the right-hand side of which exhibit almost periodic time dependence has a long history and the relevant literature is vast, see, for example, [1, 8, 9, 12]. Typical questions arising in this context are: does there exist a unique almost periodic solution, and if so, are all solutions asymptotically almost periodic with long term behaviour (in forward time) asymptotically identical to that of the unique almost periodic solution? Whilst the current paper continues this tradition, we use input-to-state stability ideas from control theory which, to the best of our knowledge, have not been employed in this context before.

More specifically, we analyze the asymptotic behaviour of a large class of infinite-dimensional Lur'e systems with Stepanov almost periodic inputs. We remark that the concept of almost periodicity in the sense of Stepanov generalizes that of Bohr, which, in the following, will be simply referred to as almost periodicity. Adopting the set-up considered in [14], we study the forced Lur'e system shown in Figure 1.1, where Σ is a well-posed[‡] linear infinite-dimensional system and f is a static nonlinearity.

[*]Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, UK, email: m.e.gilmore@bath.ac.uk, c.guiver@bath.ac.uk and h.logemann@bath.ac.uk

[†]Corresponding author

[‡] Throughout the paper, "well-posedness" refers to well-posedness in the L^2 sense, which is the natural setting, as frequency-domain methods, familiar from classical absolute stability theory, generalize nicely in this infinite-dimensional framework.

Note that, in Figure 1.1, the signals v, y and u are given by

$$v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \quad y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}, \quad u = \begin{pmatrix} v^1 \\ f(y^2) + v^2 \end{pmatrix}. \tag{1.1}$$

We note that well-posed linear systems allow for considerable unboundedness of the control and

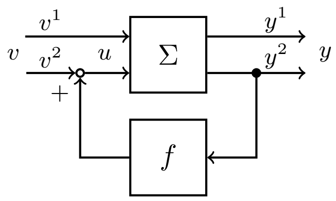

Figure 1.1: Block diagram of forced Lur'e system: the feedback interconnection of the well-posed linear system Σ and the static nonlinearity f.

observation operators and they encompass many of the most commonly studied partial differential equations with boundary control and observation, and a large class of functional differential equations of retarded and neutral type with delays in the inputs and outputs. There exists a highly developed state-space and frequency-domain theory for well-posed infinite-dimensional systems; see, for example, [22, 23, 31, 32, 33, 34, 36, 37, 38].

Lur'e systems are a common and important class of nonlinear control systems, and the study of their stability properties is known as *absolute stability theory* (see, for example, [15, 16, 17, 35, 40]). Classical absolute stability theory comes in two flavours: in a state-space setting, unforced ($v = 0$) finite-dimensional systems are considered and the emphasis is on global asymptotic stability, whilst the input-output approach (initiated by Sandberg and Zames in the 1960s) focusses on L^2-stability and, to a lesser extent, on L^∞-stability, see [11, 35]. A more recent development is the analysis of state-space systems of Lur'e format in an *input-to-state stability* (ISS) context, thereby, in a sense, merging the two strands of the earlier theory [3, 14, 18, 19, 28]. The ISS concept was introduced (for general nonlinear control systems) in [29] and further developed across a huge range of papers, see, for example, the survey articles [10, 30].

So far, the ISS approach to Lur'e systems is very much restricted to finite-dimensional systems with [14] being one of the very few exceptions.† In fact, in [14] a number of incremental ISS results are derived (the underlying concept inspired by that introduced in [2]) and are then applied to obtain convergence properties including the converging-input converging output property and the asymptotic periodicity of the state and output trajectories under periodic forcing. In this paper, we provide a refinement of the incremental ISS results in [14] and use them to analyze the asymptotic behaviour of the Lur'e system shown in Figure 1.1 in response to Stepanov almost periodic inputs.

With regards to stability properties, our main result is Theorem 3.4, which is reminiscent of the complex Aizerman conjecture [16, 17] (familiar from finite-dimensional control theory) and constitutes a refinement of [14, Theorem 4.1]. The main novelty here is that we obtain an incremental ISS estimate which is in terms of the Stepanov norm

$$\|\Delta v\|_{S^q} := \sup_{a \geq 0} \left(\int_a^{a+1} \|(\Delta v)(t)\|^q dt \right)^{1/q}, \quad 2 \leq q < \infty,$$

where Δv denotes the difference of two inputs. Our main concern in this paper is to analyse the behaviour of Lur'e systems subject to Stepanov almost periodic forcing. Based on the incremental ISS result Theorem 3.4, we show that incremental versions of certain classical sufficient conditions for absolute stability such as the complex Aizerman conjecture [16, 17], the small-gain theorem [11, 35]

† See the introduction of [14] for some commentary on the literature on ISS theory for infinite-dimensional systems (not necessarily of Lur'e form).

and the circle criterion [19, 35] (or variations thereof) guarantee that, for a given Stepanov almost periodic input v^*, there exists a corresponding unique state/output trajectory (x^*, y^*) with x^* almost periodic and y^* Stepanov almost periodic, and, furthermore, for any input/state/output trajectory (v, x, y) such that $v(t)$ approaches $v^*(t)$ as $t \to \infty$ in a natural sense, the behaviour of (x, y) is asymptotically identical to that of (x^*, y^*). The almost periods of x^* and y^* are shown to be closely related to the almost periods of v^* in the sense that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that every δ-almost period of v^* is an ε-almost period of x^* and y^*. Furthermore, it is established that the modules generated by the frequency spectra of x^* and y^* are contained in that generated by the frequency spectrum of v^*. Our main results, Theorems 4.5 and 4.6, provide far-reaching generalizations of earlier contributions in [6, 14, 24, 25, 26, 27, 39], see the commentary at the end of Section 4 for more details.

The paper is organized as follows. Section 2 gathers notation and required material from the theory of well-posed linear systems. In Section 3, we introduce the Lur'e system shown in Figure 1.1 in a formal way and then develop the key tool for our analysis of almost periodically forced Lur'e systems, namely a suitably refined version of the incremental ISS result [14, Theorem 4.1]. The main topic of the paper is addressed in Sections 4: after a discussion of relevant background material from the theory of almost periodic functions (in the sense of Bohr and its generalization by Stepanov), we state and prove Theorems 4.5 and 4.6, the main results of this work.

2 Preliminaries

Let $\mathbb{Z}$ be the set of integers and set

$$\mathbb{Z}_+ := \{n \in \mathbb{Z} : n \geq 0\} \quad \text{and} \quad \mathbb{N} := \{n \in \mathbb{Z} : n \geq 1\}.$$

For real or complex Hilbert spaces U and Y, let $\mathcal{L}(U, Y)$ denote the space of all linear bounded operators mapping U to Y. As usual, we set $\mathcal{L}(U) := \mathcal{L}(U, U)$. For $Z \in \mathcal{L}(U, Y)$ and $r > 0$, define

$$\mathbb{B}(Z, r) := \{T \in \mathcal{L}(U, Y) : \|T - Z\| < r\},$$

the open ball in $\mathcal{L}(U, Y)$, with centre Z and radius r.

For $\alpha \in \mathbb{R}$, set $\mathbb{C}_\alpha := \{s \in \mathbb{C} : \text{Re } s > \alpha\}$. The space of all holomorphic and bounded functions $\mathbb{C}_\alpha \to \mathcal{L}(U, Y)$ is denoted by $H^\infty_\alpha(\mathcal{L}(U, Y))$. Endowed with the norm

$$\|\mathbf{H}\|_{H^\infty_\alpha} := \sup_{s \in \mathbb{C}_\alpha} \|\mathbf{H}(s)\|,$$

$H^\infty_\alpha(\mathcal{L}(U, Y))$ is a Banach space. We write $H^\infty(\mathcal{L}(U, Y))$ for $H^\infty_0(\mathcal{L}(U, Y))$.

For an arbitrary Banach space W and $t \geq 0$, define the projection operator $\mathbf{P}_t : L^2_{\text{loc}}(\mathbb{R}_+, W) \to L^2(\mathbb{R}_+, W)$ by

$$(\mathbf{P}_t w)(\tau) = \begin{cases} w(\tau), & \forall \tau \in [0, t] \\ 0, & \forall \tau > t. \end{cases}$$

For $\alpha \in \mathbb{R}$ and $1 \leq q \leq \infty$, we define the weighted L^q-space

$$L^q_\alpha(\mathbb{R}_+, W) := \{w \in L^q_{\text{loc}}(\mathbb{R}_+, W) : \exp_\alpha w \in L^q(\mathbb{R}_+, W)\},$$

where $\exp_\alpha : \mathbb{R} \to \mathbb{R}$ is given by $\exp_\alpha(t) := e^{\alpha t}$. Endowed with the norm

$$\|w\|_{L^q_\alpha} = \|\exp_\alpha w\|_{L^q},$$

$L^q_\alpha(\mathbb{R}_+, W)$ is a Banach space.

In the following, let $R = \mathbb{R}_+$ or $\mathbb{R}$. For $\tau \in R$, the shift operator $\mathbf{S}_\tau : L^2_{\mathrm{loc}}(R, W) \to L^2_{\mathrm{loc}}(R, W)$ is given by $(\mathbf{S}_\tau w)(t) = w(t+\tau)$ for all $t \in R$. For later purposes, we define $BC(R, W)$ and $BUC(R, W)$ as the spaces of all, respectively, bounded continuous and bounded uniformly continuous functions. Endowed with the supremum norm, $BC(R, W)$ and $BUC(R, W)$ are Banach spaces. Moreover, we define the space of uniformly locally q-integrable functions $UL^q_{\mathrm{loc}}(R, W)$ by

$$UL^q_{\mathrm{loc}}(R, W) := \left\{ w \in L^q_{\mathrm{loc}}(R, W) : \sup_{a \in R} \int_a^{a+1} \|w(t)\|^q dt < \infty \right\},$$

where $1 \le q < \infty$. It is straightforward to show that, with the Stepanov norm

$$\|w\|_{S^q} := \sup_{a \in R} \left(\int_a^{a+1} \|w(t)\|^q dt \right)^{1/q},$$

$UL^q_{\mathrm{loc}}(R, W)$ is a Banach space. Furthermore, for every $b > 0$, the functional

$$w \mapsto \sup_{a \in R} \left(\int_a^{a+b} \|w(t)\|^q dt \right)^{1/q}$$

is a norm on $UL^q_{\mathrm{loc}}(R, W)$ and this norm is equivalent to $\| \cdot \|_{S^q}$.

Below we will provide a brief review of some material from the theory of well-posed systems, for more details we refer the reader to [31, 33, 34, 36, 37, 38]. Throughout, we shall be considering a well-posed linear system $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{G})$ with state space X, input space U and output space Y. Here X, U and Y are separable complex Hilbert spaces, $\mathbb{T} = (\mathbb{T}_t)_{t \ge 0}$ is a strongly continuous semigroup on X, $\Phi = (\Phi_t)_{t \ge 0}$ is a family of bounded linear operators from $L^2(\mathbb{R}_+, U)$ to X (input-to-state maps), $\Psi = (\Psi_t)_{t \ge 0}$ is a family of bounded linear operators from X to $L^2(\mathbb{R}_+, Y)$ (state-to-output maps) and $\mathbb{G} = (\mathbb{G}_t)_{t \ge 0}$ is a family of bounded linear operators from $L^2(\mathbb{R}_+, U)$ to $L^2(\mathbb{R}_+, Y)$ (input-to-output maps). In order for Σ to qualify as a well-posed system, these families of operators need to satisfy certain natural conditions, see [31, 34, 36, 37]. Particular consequences of these conditions are the following properties:

$$\Phi_t \mathbf{P}_t = \Phi_t, \quad \mathbf{P}_t \Psi_{t+\tau} = \Psi_t, \quad \mathbf{P}_t \mathbb{G}_{t+\tau} \mathbf{P}_t = \mathbf{P}_t \mathbb{G}_{t+\tau} = \mathbb{G}_t \quad \forall\, t, \tau \ge 0\,.$$

It follows that Φ_t extends in a natural way to $L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ and there exist operators $\Psi_\infty : X \to L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ and $\mathbb{G}_\infty : L^2_{\mathrm{loc}}(\mathbb{R}_+, U) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ such that

$$\mathbf{P}_t \Psi_\infty = \Psi_t, \quad \mathbf{P}_t \mathbb{G}_\infty = \mathbb{G}_t \quad \forall\, t \ge 0\,.$$

The operator $\mathbb{G}_\infty$ is right-shift invariant (and hence causal) and is called the input-output operator of Σ. Given an initial state x^0 and an input $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, the corresponding state and output trajectories x and y of Σ are defined by

$$\left. \begin{aligned} x(t) &= \mathbb{T}_t x^0 + \Phi_t \mathbf{P}_t u \\ \mathbf{P}_t y &= \Psi_t x^0 + \mathbb{G}_t \mathbf{P}_t u \end{aligned} \right\} \quad \forall\, t \ge 0\,, \tag{2.1}$$

respectively.

Let (A, B, C) denote the generating operators of Σ. The operator A is the generator of the strongly continuous semigroup $\mathbb{T} = (\mathbb{T}_t)_{t \ge 0}$ and the operators $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$ are the unique operators satisfying

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\tau} B u(\tau) \mathrm{d}\tau \quad \forall\, u \in L^2(\mathbb{R}_+, U),\ \forall\, t \ge 0\,,$$

and

$$(\Psi_\infty x^0)(t) = C\mathbb{T}_t x^0 \quad \forall\, x^0 \in X_1,\ \forall\, t \geq 0,$$

where the spaces X_1 and X_{-1}, respectively, are the usual interpolation and extrapolation spaces associated with A and X.

The transfer function $\mathbf{G}$ of Σ has the property that $\mathbf{G} \in H^\infty_\alpha(\mathcal{L}(U,Y))$ for every $\alpha > \omega(\mathbb{T})$, where $\omega(\mathbb{T})$ denotes the exponential growth constant of $\mathbb{T}$. The relationship between $\mathbf{G}$ and the operators (A,B,C) is expressed by the formula

$$\frac{1}{s-z}\big(\mathbf{G}(s) - \mathbf{G}(z)\big) = -C(sI-A)^{-1}(zI-A_{-1})^{-1}B \quad \forall\, s,z \in \mathbb{C}_{\omega(\mathbb{T})},\ s \neq z,$$

see [31, equation (4.6.9)], where $A_{-1} \in \mathcal{L}(X, X_{-1})$ extends A to X and, considered as an unbounded operator on X_{-1}, generates a semigroup on X_{-1} which extends $\mathbb{T}$ to X_{-1}. Furthermore, for $\beta \in \mathbb{R}$, the operator $\mathbb{G}_\infty$ is in $\mathcal{L}(L^2_\beta(\mathbb{R}_+,U), L^2_\beta(\mathbb{R}_+,Y))$ if, and only if, $\mathbf{G} \in H^\infty_{-\beta}(\mathcal{L}(U,Y))$, in which case

$$\|\mathbb{G}_\infty\|_\beta = \|\mathbf{G}\|_{H^\infty_{-\beta}},$$

where $\|\cdot\|_\beta$ denotes the L^2_β-induced operator norm. We remark that $\beta < -\omega(\mathbb{T})$ is sufficient for $\mathbb{G}_\infty$ to be in $\mathcal{L}(L^2_\beta(\mathbb{R}_+,U), L^2_\beta(\mathbb{R}_+,Y))$. We also record that, for every $\beta < -\omega(\mathbb{T})$, there exist positive constants φ and ψ such that

$$\|e^{\beta t}\Phi_t u\| \leq \varphi \|\mathbf{P}_t u\|_{L^2_\beta} \ \ \forall\, u \in L^2_{\text{loc}}(\mathbb{R}_+,U),\ \forall\, t \geq 0\,,$$

and

$$\|\Psi_\infty x^0\|_{L^2_\beta} \leq \psi \|x^0\| \quad \forall\, x^0 \in X\,.$$

The system (2.1) is said to be *optimizable* if, for every $x^0 \in X$, there exists $u \in L^2(\mathbb{R}_+,U)$, such that $x \in L^2(\mathbb{R}_+,X)$. Furthermore, we say that (2.1) is *estimatable* if, the "dual" system is optimizable, that is, for every $z^0 \in X$, there exists $v \in L^2(\mathbb{R}_+,Y)$ such that the function $t \mapsto \mathbb{T}^*_t z^0 + \Psi^*_t v$ is in $L^2(\mathbb{R}_+,X)$. We note that, by [20], optimizability is equivalent to exponential stabilizability and estimatability is equivalent to exponential detectability (where exponential stabilizability and detectability are understood in the sense of [31]).

An operator $K \in \mathcal{L}(Y,U)$ is said to be an *admissible feedback operator* for Σ (or for $\mathbf{G}$) if there exists $\alpha \in \mathbb{R}$ such that $I - \mathbf{G}K$ is invertible in $H^\infty_\alpha(\mathcal{L}(Y))$. If $K \in \mathcal{L}(Y,U)$ is an admissible feedback operator, then, for every $t \geq 0$, the operator $I - \mathbb{G}_t K$ is invertible in $\mathcal{L}(L^2(\mathbb{R}_+,Y))$, and, $I - \mathbb{G}_\infty K$ has a causal inverse $(I-\mathbb{G}_\infty K)^{-1}$ (mapping $L^2_{\text{loc}}(\mathbb{R}_+,Y)$ into itself). Furthermore, if $K \in \mathcal{L}(Y,U)$ is an admissible feedback operator for Σ, then there exists a unique well-posed system $\Sigma^K = (\mathbb{T}^K, \Phi^K, \Psi^K, \mathbb{G}^K)$ such that

$$\Sigma^K_t = \Sigma_t + \Sigma_t \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} \Sigma^K_t \qquad \forall\, t \geq 0, \tag{2.2}$$

where

$$\Sigma_t := \begin{pmatrix} \mathbb{T}_t & \Phi_t \\ \Psi_t & \mathbb{G}_t \end{pmatrix}, \quad \Sigma^K_t := \begin{pmatrix} \mathbb{T}^K_t & \Phi^K_t \\ \Psi^K_t & \mathbb{G}^K_t \end{pmatrix}.$$

The interpretation of (2.2) is that Σ^K is the closed-loop system shown in Figure 2.1.

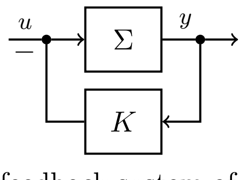

Figure 2.1: Block diagram of closed-loop feedback system of Σ in connection with output feedback K.

We say that an operator $K \in \mathcal{L}(Y,U)$ *stabilizes* $\mathbf{G}$ (or stabilizes Σ in the input-output sense) if $(I-\mathbf{G}K)^{-1}\mathbf{G} \in H^\infty(\mathcal{L}(U,Y))$. The set of all operators stabilizing $\mathbf{G}$ is denoted by $\mathbb{S}(\mathbf{G})$. Trivially, every element in $\mathbb{S}(\mathbf{G})$ is an admissible feedback operator for $\mathbf{G}$.

The following lemma is a special case of [13, Proposition 5.6].

Lemma 2.1. *For $K \in \mathcal{L}(Y,U)$ and $r>0$, $\mathbb{B}(K,r) \subset \mathbb{S}(\mathbf{G})$ if, and only if, $\|(I-\mathbf{G}K)^{-1}\mathbf{G}\|_{H^\infty} \leq 1/r$.*

In particular, if $K \in \mathbb{S}(\mathbf{G})$ and $\|(I-\mathbf{G}K)^{-1}\mathbf{G}\|_{H^\infty} > 0$, then $\rho := 1/\|(I-\mathbf{G}K)^{-1}\mathbf{G}\|_{H^\infty}$ is the largest number such that $\mathbb{B}(K,\rho) \subset \mathbb{S}(\mathbf{G})$.

An immediate consequence of the sufficiency part of Lemma 2.1 is that $\mathbb{S}(\mathbf{G})$ is an open subset of $\mathcal{L}(Y,U)$. Note that the sufficiency part is simply a version of the small-gain theorem. The assumption that the Hilbert spaces U and Y are complex plays an important role in the necessity part which in general does not hold for real Hilbert spaces.

In the following, we shall adopt the four-block setting for Lur'e systems considered in [14], see Figure 1.1. In particular, we assume that the input and output spaces U and Y are of the form $U = U^1 \times U^2$ and $Y = Y^1 \times Y^2$, where U^i and Y^i are complex Hilbert spaces, $i = 1,2$. It is convenient to introduce the following maps

$$P^i : Y \to Y^i, \ \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \mapsto y^i, \quad i = 1,2\,,$$

and

$$E^1 : U^1 \to U, \ u \mapsto \begin{pmatrix} u \\ 0 \end{pmatrix}, \quad E^2 : U^2 \to U, \ u \mapsto \begin{pmatrix} 0 \\ u \end{pmatrix}.$$

If $y \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$, then $P^i y$ is the function in $L^2_{\mathrm{loc}}(\mathbb{R}_+, Y^i)$ given by $(P^i y)(t) = P^i y(t)$. Similarly, for $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U^i)$, the symbol $E^i u$ denotes the function in $L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ given by $(E^i u)(t) = E^i u(t)$. The decompositions of the input and output spaces, $U = U^1 \times U^2$ and $Y = Y^1 \times Y^2$, respectively, induces four well-posed systems, namely,

$$\Sigma^{ij} := (\mathbb{T}, \Phi E^j, P^i \Psi, P^i \mathbb{G} E^j), \quad i,j = 1,2\,.$$

Obviously, the state, input and output spaces of Σ^{ij} are given by X, U^j and Y^i, respectively. For $K^{ij} \in \mathcal{L}(Y^j, U^i)$, let $K \in \mathcal{L}(Y,U)$ be defined by

$$Ky = E^i K^{ij} P^j y \quad \forall\, y \in Y\,.$$

For example, if $i = 1$ and $j = 2$, then

$$K = \begin{pmatrix} 0 & K^{12} \\ 0 & 0 \end{pmatrix}.$$

3 Incremental stability of infinite-dimensional Lur'e systems

We start this section by defining the class of Lur'e systems which we will be considering, thereby formalizing the arrangement depicted in Figure 1.1. Given an initial state x^0 and an input $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, the corresponding state and output trajectories of Σ are given by (2.1). Let $i,j \in \{1,2\}$ and let $f : Y^j \to U^i$ be a nonlinearity. The closed-loop system obtained by applying the feedback

$$u = E^i(f \circ P^j y) + v, \quad \text{where } v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)\,,$$

is then given by

$$\left.\begin{aligned} x(t) &= \mathbb{T}_t x^0 + \Phi_t \mathbf{P}_t\big(E^i(f \circ P^j y) + v\big)\,, \\ \mathbf{P}_t y &= \Psi_t x^0 + \mathbb{G}_t \mathbf{P}_t\big(E^i(f \circ P^j y) + v\big)\,. \end{aligned}\right\} \tag{3.1}$$

As an illustration, Figure 1.1 corresponds to the case $i = j = 2$. Given $x^0 \in X$ and $v \in L^2_{\rm loc}(\mathbb{R}_+, U)$, a *solution* of the Lur'e system (3.1) on $[0, \sigma)$, where $0 < \sigma \le \infty$, is a pair $(x, y) \in C([0, \sigma), X) \times L^2_{\rm loc}([0, \sigma), Y)$ such that $f \circ P^j y \in L^2_{\rm loc}([0, \sigma), U^i)$ and (3.1) holds for all $t \in [0, \sigma)$. Obviously, if (x, y) is a solution of (3.1), then $x(0) = x^0$.

It can be shown (by invoking Zorn's lemma) that, for every solution of (3.1) on $[0, \sigma)$, there exists a *maximally defined* solution (3.1) defined on $[0, \tau)$ with $\sigma \le \tau \le \infty$ which cannot be extended any further (that is, τ is maximal).

The set of all triples (v, x, y) in $L^2_{\rm loc}(\mathbb{R}_+, U) \times C(\mathbb{R}_+, X) \times L^2_{\rm loc}(\mathbb{R}_+, Y)$ such that (3.1) holds with $x^0 = x(0)$ is said to be the *behaviour* of (3.1) and is denoted by $\mathcal{B}$. Elements of $\mathcal{B}$ will sometimes be referred to as *trajectories* of (3.1). In particular, if $(v, x, y) \in \mathcal{B}$, then (x, y) is a solution of (3.1) which is defined on $\mathbb{R}_+$ and with $x^0 = x(0)$. In an ISS context, we consider external inputs v which belong to $L^\infty_{\rm loc}(\mathbb{R}_+, U) \subset L^2_{\rm loc}(\mathbb{R}_+, U)$. More generally, for $2 \le q \le \infty$, we may wish to consider inputs v in $L^q_{\rm loc}(\mathbb{R}_+, U) \subset L^2_{\rm loc}(\mathbb{R}_+, U)$. It is therefore convenient to define the following "sub-behaviour" of $\mathcal{B}$:

$$\mathcal{B}^q := \{(v, x, y) \in \mathcal{B} : v \in L^q_{\rm loc}(\mathbb{R}_+, U)\} .$$

Obviously, we have $\mathcal{B}^2 = \mathcal{B}$. A key property of the behaviour $\mathcal{B}^q$ is its invariance with respect to left translations, that is,

$$(v, x, y) \in \mathcal{B}^q \quad \Longrightarrow \quad (\mathbf{S}_\tau v, \mathbf{S}_\tau x, \mathbf{S}_\tau y) \in \mathcal{B}^q \quad \forall\, \tau \ge 0.$$

In this paper, we are mainly concerned with stability and convergence properties of (3.1) and not with existence and uniqueness questions. However, we state a simple, but important, existence and uniqueness result from [34].

Proposition 3.1. *If $f : Y^j \to U^i$ is globally Lipschitz with Lipschitz constant $\lambda \ge 0$ and*

$$\lambda \liminf_{\alpha \to \infty} \|P^j \mathbf{G} E^i\|_{H^\infty_\alpha} < 1 ,$$

then, for all $x^0 \in X$ and $v \in L^2_{\rm loc}(\mathbb{R}_+, U)$, the Lur'e system (3.1) *has a unique solution on $\mathbb{R}_+$.*

For later purposes, we define the *bi-lateral behaviour* $\mathcal{BB}$ of (3.1) as the set of all triples $(v, x, y) \in L^2_{\rm loc}(\mathbb{R}, U) \times C(\mathbb{R}, X) \times L^2_{\rm loc}(\mathbb{R}, Y)$ such that, for every $t_0 \in \mathbb{R}$,

$$\left.\begin{aligned} x(t) &= \mathbb{T}_{t-t_0} x(t_0) + \Phi_{t-t_0} \mathbf{P}_{t-t_0}\big(E^i(f \circ P^j \mathbf{S}_{t_0} y) + \mathbf{S}_{t_0} v\big) \\ \mathbf{P}_{t-t_0} \mathbf{S}_{t_0} y &= \Psi_{t-t_0} x(t_0) + \mathbb{G}_{t-t_0} \mathbf{P}_{t-t_0}\big(E^i(f \circ P^j \mathbf{S}_{t_0} y) + \mathbf{S}_{t_0} v\big) \end{aligned}\right\} \quad \forall\, t \ge t_0.$$

We refer to the elements of $\mathcal{BB}$ as *bi-trajectories* of (3.1). Obviously, a bi-trajectory restricted to $\mathbb{R}_+$ is an element in $\mathcal{B}$. Furthermore, the bi-lateral behaviour $\mathcal{BB}$ is invariant with respect to all translations, that is,

$$(v, x, y) \in \mathcal{BB} \quad \Longrightarrow \quad (\mathbf{S}_\tau v, \mathbf{S}_\tau x, \mathbf{S}_\tau y) \in \mathcal{BB} \quad \forall\, \tau \in \mathbb{R}.$$

The next lemma (which can be found in [14]) shows that the behaviour $\mathcal{B}$ of (3.1) is identical to the behaviour of the feedback interconnection obtained when the linear system Σ^K is subjected to the feedback law $u = E^i f(P^j y) - Ky + v$, where $K \in \mathcal{L}(Y, U)$ is an admissible feedback operator for Σ.

Lemma 3.2. *Let $K \in \mathcal{L}(Y, U)$ be an admissible feedback operator for Σ and let $(v, x, y) \in L^2_{\rm loc}(\mathbb{R}_+, U) \times C(\mathbb{R}_+, X) \times L^2_{\rm loc}(\mathbb{R}_+, Y)$. The triple (v, x, y) is in $\mathcal{B}$ if, and only if,*

$$\left.\begin{aligned} x(t) &= \mathbb{T}^K_t x(0) + \Phi^K_t \mathbf{P}_t\big(E^i(f \circ P^j y) + v - Ky\big) \\ \mathbf{P}_t y &= \Psi^K_t x(0) + \mathbb{G}^K_t \mathbf{P}_t\big(E^i(f \circ P^j y) + v - Ky\big) \end{aligned}\right\} \quad \forall\, t \ge 0 .$$

A triple $(v^{\rm eq}, x^{\rm eq}, y^{\rm eq}) \in U \times X \times Y$ is said to be an *equilibrium* or *equilibrium triple* of the Lur'e system (3.1) if the constant trajectory $t \mapsto (v^{\rm eq}, x^{\rm eq}, y^{\rm eq})$ belongs to $\mathcal{B}$ (in which case it is also a bi-trajectory). The next result provides formulas relating the components of an equilibrium triple $(v^{\rm eq}, x^{\rm eq}, y^{\rm eq})$.

Proposition 3.3. *Let $(v^{\rm eq}, x^{\rm eq}, y^{\rm eq}) \in U \times X \times Y$, let $\eta \in \mathbb{C}$ such that $\operatorname{Re}\eta > \omega(\mathbb{T})$ and set $u^{\rm eq} := E^i f(P^j y^{\rm eq}) + v^{\rm eq}$. The triple $(v^{\rm eq}, x^{\rm eq}, y^{\rm eq})$ is an equilibrium of (3.1) if, and only if,*

$$Ax^{\rm eq} + Bu^{\rm eq} = 0 \quad \text{and} \quad y^{\rm eq} = C\big(x^{\rm eq} - (\eta I - A)^{-1}Bu^{\rm eq}\big) + \mathbf{G}(\eta)u^{\rm eq}\,.$$

We refer to [14] for a proof of Proposition 3.3.

Note that the identity $Ax^{\rm eq} + Bu^{\rm eq} = 0$ implies that $x^{\rm eq} - (\eta I - A)^{-1}Bu^{\rm eq} \in X_1$ and thus, the expression $C\big(x^{\rm eq} - (\eta I - A)^{-1}Bu^{\rm eq}\big)$ is well defined. Furthermore, Proposition 3.3 shows that that the triple $(-E^i f(0), 0, 0)$ is always an equilibrium of (3.1).

Let $2 \le q \le \infty$. An equilibrium triple $(v^{\rm eq}, x^{\rm eq}, y^{\rm eq})$ of (3.1) is said to be *exponentially L^q-input-to-state stable* (*exponentially L^q-ISS*) if there exist positive constants Γ and γ such that

$$\|x(t) - x^{\rm eq}\| \le \Gamma\big(e^{-\gamma t}\|x(0) - x^{\rm eq}\| + \|\mathbf{P}_t(v - v^{\rm eq}\vartheta)\|_{L^q}\big) \quad \forall\, t \ge 0,\ \forall\,(v, x, y) \in \mathcal{B}^q\,,$$

where $\vartheta(t) = 1$ for all $t \ge 0$. Furthermore, (3.1) is said to be *exponentially incrementally L^q-input-to-state stable* (*exponentially L^q-δISS*) if there exist positive constants Γ and $\gamma > 0$ such that, for all $(v_1, x_1, y_1), (v_2, x_2, y_2) \in \mathcal{B}^q$,

$$\|x_1(t) - x_2(t)\| \le \Gamma\big(e^{-\gamma t}\|x_1(0) - x_2(0)\| + \|\mathbf{P}_t(v_1 - v_2)\|_{L^\infty}\big) \quad \forall\, t \ge 0.$$

Here v_i and y_i should not be confused with v^i and y^i, $i = 1, 2$, which appear in (1.1) and Figure 1.1.

We introduce a further type of "sub-behaviour" which shall be useful in formulating our stability results. For a non-empty subset $Z \subset Y^j$ and $2 \le q \le \infty$, we set

$$\mathcal{B}^q_Z := \{(v, x, y) \in \mathcal{B}^q : P^j y(t) \in Z \ \text{ for a.e } t \ge 0\}\,.$$

Furthermore, $\mathcal{B}_Z := \mathcal{B}^2_Z$.

The following theorem, a refinement of [14, Theorem 4.1], is reminiscent of the complex Aizerman conjecture in finite dimensions (which is known to be true, see [16, 17, 19]): incremental stability properties of the nonlinear system (3.1) are guaranteed by the assumption that a corresponding linear feedback system is stable for all linear complex feedback operators belonging to a certain ball, provided the nonlinearity satisfies, in a suitable and natural sense, an incremental version of the same boundedness condition. Before we state the results, it is convenient to define

$$\Delta := \{(t, s) \in \mathbb{R}^2 : t \ge s \ge 0\} \subset \mathbb{R}^2_+.$$

Theorem 3.4. *Let $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{G})$ be a well-posed linear system, let $i, j \in \{1, 2\}$, $K^{ij} \in \mathcal{L}(Y^j, U^i)$, $r > 0$, $2 \le q \le \infty$, and let $Z_1, Z_2 \subset Y^j$ be non-empty subsets. Assume that $\Sigma^{ji} = (\mathbb{T}, \Phi E^i, P^j\Psi, P^j\mathbb{G}E^i)$ is optimizable and estimatable and $\mathbb{B}(K^{ij}, r) \subset \mathbb{S}(P^j\mathbf{G}E^i)$. If $f : Y^j \to U^i$ satisfies*

$$\sup_{(z_1,z_2)\in Z_1\times Z_2,\, z_1\neq z_2} \frac{\|f(z_1) - f(z_2) - K^{ij}(z_1 - z_2)\|}{\|z_1 - z_2\|} < r, \tag{3.2}$$

then the following statements hold.

(1) *There exist constants $\Gamma_q > 0$ and $\gamma > 0$ such that, for all $(v_1, x_1, y_1) \in \mathcal{B}^q_{Z_1}$ and all $(v_2, x_2, y_2) \in \mathcal{B}^q_{Z_2}$,*

$$\|x_1(t) - x_2(t)\| \le \Gamma_q\big(e^{-\gamma(t-t_0)}\|x_1(t_0) - x_2(t_0)\| + \|v_1 - v_2\|_{L^q(t_0,t)}\big) \quad \forall\,(t, t_0) \in \Delta.$$

Here Γ_q depends on q, but γ does not.

(2) *There exist constants* $\Gamma > 0$ *and* $\varepsilon > 0$ *such that, for all* $(v_1, x_1, y_1) \in \mathcal{B}_{Z_1}$*, all* $(v_2, x_2, y_2) \in \mathcal{B}_{Z_2}$ *and all* $\alpha \in [0, \varepsilon]$*,*

$$\|x_1 - x_2\|_{L^2_\alpha(t_0,t)} + \|y_1 - y_2\|_{L^2_\alpha(t_0,t)} \leq \Gamma\big(\|x_1(t_0) - x_2(t_0)\| + \|v_1 - v_2\|_{L^2_\alpha(t_0,t)}\big) \quad \forall\, (t, t_0) \in \Delta.$$

(3) *There exist constants* $\tilde{\Gamma}_q > 0$ *and* $\tilde{\gamma} > 0$ *such that, for all* $(v_1, x_1, y_1) \in \mathcal{B}^q_{Z_1}$ *and all* $(v_2, x_2, y_2) \in \mathcal{B}^q_{Z_2}$ *with* $v_1, v_2 \in UL^q_{\mathrm{loc}}(\mathbb{R}_+, U)$

$$\|x_1(t) - x_2(t)\| \leq \tilde{\Gamma}_q\big(e^{-\tilde{\gamma}(t-t_0)}\|x_1(t_0) - x_2(t_0)\| + \|\mathbf{S}_{t_0}(v_1 - v_2)\|_{S^q}\big) \quad \forall\, (t, t_0) \in \Delta.$$

(4) *There exists a constant* $\tilde{\Gamma} > 0$ *such that, for all* $(v_1, x_1, y_1) \in \mathcal{B}_{Z_1}$ *and all* $(v_2, x_2, y_2) \in \mathcal{B}_{Z_2}$ *with* $v_1, v_2 \in UL^2_{\mathrm{loc}}(\mathbb{R}_+, U)$*,*

$$\|\mathbf{S}_{t_0}(x_1 - x_2)\|_{S^2} + \|\mathbf{S}_{t_0}(y_1 - y_2)\|_{S^2} \leq \tilde{\Gamma}\big(\|x_1(t_0) - x_2(t_0)\| + \|\mathbf{S}_{t_0}(v_1 - v_2)\|_{S^2}\big) \quad \forall\, t_0 \geq 0.$$

We highlight two important special cases.

Special case 1. Assume that $(v^{\mathrm{eq}}, x^{\mathrm{eq}}, y^{\mathrm{eq}}) \in U \times X \times Y$ is an equilibrium triple of the Lur'e system (3.1) and the assumptions of Theorem 3.4 hold with $Z_1 = Y^j$ and $Z_2 = \{P^j y^{\mathrm{eq}}\}$. Then the constant trajectory $(v^{\mathrm{eq}}, x^{\mathrm{eq}}, y^{\mathrm{eq}})$ is in $\mathcal{B}^\infty_{Z_2}$ and statement (1) implies that, for every $2 \leq q \leq \infty$, the equilibrium $(v^{\mathrm{eq}}, x^{\mathrm{eq}}, y^{\mathrm{eq}})$ is exponentially L^q-ISS. Furthermore, statement (3) guarantees that, for any $(v, x, y) \in \mathcal{B}^q$ with $v \in UL^q_{\mathrm{loc}}(\mathbb{R}_+, U)$ and $2 \leq q < \infty$, the state x is bounded.

Special case 2. Assume that the hypotheses of Theorem 3.4 hold with $Z_1 = Z_2 = Y^j$ (and so (3.2) is equivalent to $z \mapsto f(z) - K^{ij}z$ being globally Lipschitz with Lipschitz constant smaller than r). In this case, statement (1) of Theorem 3.4 implies that the Lur'e system (3.1) is exponentially L^q-δISS for every q such that $2 \leq q \leq \infty$. Furthermore, as a consequence of Proposition 3.1 and Lemma 3.2, for every pair $(x^0, v) \in X \times L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, there exists a unique triple $(v, x, y) \in \mathcal{B}$ such that $x(0) = x^0$.

As compared to [14, Theorem 4.1], the new contribution of Theorem 3.4 are statements (3) and (4) which provide bounds in terms of the Stepanov norm of $\mathbf{S}_{t_0}(v_1 - v_2)$.

Proof of Theorem 3.4. To prove statement (1), let $(v_1, x_1, y_1) \in \mathcal{B}^q_{Z_1}$ and $(v_2, x_2, y_2) \in \mathcal{B}^q_{Z_2}$ and note that, for any $t_0 \geq 0$, $(\mathbf{S}_{t_0}v_1, \mathbf{S}_{t_0}x_1, \mathbf{S}_{t_0}y_1) \in \mathcal{B}^q_{Z_1}$ and $(\mathbf{S}_{t_0}v_2, \mathbf{S}_{t_0}x_2, \mathbf{S}_{t_0}y_2) \in \mathcal{B}^q_{Z_2}$, and thus, by [14, Theorem 4.1], there exist constants $\Gamma_q > 0$ and $\gamma > 0$, such that

$$\|(\mathbf{S}_{t_0}x_1)(s) - (\mathbf{S}_{t_0}x_2)(s)\| \leq \Gamma_q\big(e^{-\gamma s}\|(\mathbf{S}_{t_0}x_1)(0) - (\mathbf{S}_{t_0}x_2)(0)\| + \|\mathbf{P}_s(\mathbf{S}_{t_0}v_1 - \mathbf{S}_{t_0}v_2)\|_{L^q}\big) \quad \forall\, s \geq 0.$$

Setting $t := s + t_0 \geq 0$, it follows that

$$\|x_1(t) - x_2(t)\| \leq \Gamma_q\big(e^{-\gamma(t-t_0)}\|x_1(t_0) - x_2(t_0)\| + \|v_1 - v_2\|_{L^q(t_0,t)}\big) \quad \forall\, t \geq t_0,$$

establishing statement (1).

Statement (2) can be derived from [14, Theorem 4.1] in a similar manner.

We proceed to prove statement (3). Let $2 \leq q < \infty$, $t_0 \geq 0$, $(v_1, x_1, y_1) \in \mathcal{B}^q_{Z_1}$ and $(v_2, x_2, y_2) \in \mathcal{B}^q_{Z_2}$ with $v_1, v_2 \in UL^q_{\mathrm{loc}}(\mathbb{R}_+, U)$. Setting $x := x_1 - x_2$ and $v := v_1 - v_2$, we obtain from statement (1) that

$$\|x(t)\| \leq \Gamma_q\big(e^{-\gamma(t-s)}\|x(s)\| + \|v\|_{L^q(s,t)}\big) \quad \forall\, (t, s) \in \Delta. \tag{3.3}$$

Choose $\tau > 0$ such that $\Gamma_q e^{-\gamma\tau} < 1$ and let m be the smallest integer such that $m \geq \tau$. A straightforward argument shows that

$$\|v\|_{L^q(a,a+\tau)} \leq m^{1/q}\|\mathbf{S}_{t_0}v\|_{S^q} \quad \forall\, (a, t_0) \in \Delta.$$

In particular,

$$b := \sup_{k\in\mathbb{Z}_+} \|v\|_{L^q(t_0+k\tau, t_0+(k+1)\tau)} \le m^{1/q}\|\mathbf{S}_{t_0}v\|_{S^q},$$

and so, by (3.3) with $t = t_0 + (k+1)\tau$ and $s = t_0 + k\tau$,

$$\|x(t_0 + (k+1)\tau)\| \le \Gamma_q\big(e^{-\gamma\tau}\|x(t_0 + k\tau)\| + b\big) \le \theta\|x(t_0 + k\tau)\| + \Gamma_q b \quad \forall\, k \in \mathbb{Z}_+,$$

where $\theta := \Gamma_q e^{-\gamma\tau} < 1$. Consequently,

$$\|x(t_0 + k\tau)\| \le \theta^k\|x(t_0)\| + \Gamma_q b\sum_{j=0}^{k-1}\theta^j \le \theta^k\|x(t_0)\| + \frac{\Gamma_q b}{1-\theta} \quad \forall\, k \in \mathbb{N}. \tag{3.4}$$

Appealing to (3.3) with $s = t_0 + k\tau$ we obtain

$$\|x(t)\| \le \Gamma_q\big(\|x(t_0 + k\tau)\| + \|v\|_{L^2(t_0+k\tau,t)}\big) \quad \forall\, t \in [t_0 + k\tau, t_0 + (k+1)\tau],\ \forall\, k \in \mathbb{Z}_+.$$

Now $\|v\|_{L^2(t_0+k\tau,t)} \le b$ for all $t \in [t_0 + k\tau, t_0 + (k+1)\tau]$ and all $k \in \mathbb{Z}_+$, and so, invoking (3.4),

$$\|x(t)\| \le \Gamma_q\big(\theta^k\|x(t_0)\| + (\Gamma_q + 1 - \theta)b/(1-\theta)\big) \quad \forall\, t \in [t_0 + k\tau, t_0 + (k+1)\tau],\ \forall\, k \in \mathbb{Z}_+.$$

Consequently, setting $\tilde\gamma := -(\ln\theta)/\tau > 0$, we have that

$$\|x(t)\| \le \tilde\Gamma_q\big(e^{-\tilde\gamma(t-t_0)}\|x(t_0)\| + \|\mathbf{S}_{t_0}v\|_{S^q}\big) \quad \forall\,(t,t_0) \in \Delta,$$

where $\tilde\Gamma_q := \Gamma_q \max(e^{\tilde\gamma\tau}, m^{1/q}(\Gamma_q + 1 - \theta)/(1-\theta))$, completing the proof of statement (3).

To prove statement (4), let $(v_1, x_1, y_1) \in \mathcal{B}_{Z_1}$ and $(v_2, x_2, y_2) \in \mathcal{B}_{Z_2}$ with $v_1, v_2 \in UL^2_{\mathrm{loc}}(\mathbb{R}_+, U)$. Then, for every $t_0 \ge 0$,

$$(\mathbf{S}_{t_0}v_j, \mathbf{S}_{t_0}x_j, \mathbf{S}_{t_0}y_j) \in \mathcal{B}_{Z_j}, \quad j = 1, 2.$$

Therefore, setting $v := v_1 - v_2$, $x := x_1 - x_2$ and $y := y_1 - y_2$, we obtain from statement (2) that

$$\|\mathbf{S}_{t_0}x\|_{L^2(\tau,\tau+1)} + \|\mathbf{S}_{t_0}y\|_{L^2(\tau,\tau+1)} \le \Gamma\big(\|(\mathbf{S}_{t_0}x)(\tau)\| + \|\mathbf{S}_{t_0}v\|_{L^2(\tau,\tau+1)}\big) \quad \forall\,\tau, t_0 \ge 0.$$

The above inequality implies that, for every $t_0 \ge 0$,

$$\|\mathbf{S}_{t_0}x\|_{S^2} + \|\mathbf{S}_{t_0}y\|_{S^2} \le 2\Gamma\big(\sup_{\tau\ge 0}\|(\mathbf{S}_{t_0}x)(\tau)\| + \|\mathbf{S}_{t_0}v\|_{S^2}\big).$$

Now, as $(\mathbf{S}_{t_0}v_j, \mathbf{S}_{t_0}x_j, \mathbf{S}_{t_0}y_j) \in \mathcal{B}_{Z_j}$, $j = 1, 2$, an application of statement (3) yields,

$$\|(\mathbf{S}_{t_0}x)(s)\| \le \tilde\Gamma_2\big(e^{-\tilde\gamma s}\|(\mathbf{S}_{t_0}x)(0)\| + \|\mathbf{S}_{t_0}v\|_{S^2}\big) \quad \forall\, s, t_0 \ge 0,$$

and thus, for every $t_0 \ge 0$,

$$\|\mathbf{S}_{t_0}x\|_{S^2} + \|\mathbf{S}_{t_0}y\|_{S^2} \le \tilde\Gamma(\|x(t_0)\| + \|\mathbf{S}_{t_0}v\|_{S^2}),$$

where $\tilde\Gamma := 2\Gamma(\tilde\Gamma_2 + 1)$, completing the proof. □

Assuming that $K^{ij} \in \mathbb{S}(P^j\mathbf{G}E^i)$ and setting $r := 1/\|(P^j\mathbf{G}E^i)^{K^{ij}}\|_{H^\infty}$, it follows from Lemma 2.1 that $\mathbb{B}(K^{ij}, r) \subset \mathbb{S}(P^j\mathbf{G}E^i)$, and hence, the following small-gain result is an immediate consequence of Theorem 3.4.

Corollary 3.5. *Let Σ, f, Z_1 and Z_2 be as in* Theorem 3.4*, let $i, j \in \{1, 2\}$ and let $K^{ij} \in \mathbb{S}(P^j\mathbf{G}E^i)$. Assume that $\Sigma^{ji} = (\mathbb{T}, \Phi E^i, P^j\Psi, P^j\mathbb{G}E^i)$ is optimizable and estimatable. If*

$$\sup_{(z_1,z_2)\in Z_1\times Z_2,\, z_1\ne z_2} \frac{\|f(z_1) - f(z_2) - K^{ij}(z_1 - z_2)\|}{\|z_1 - z_2\|} \cdot \|(P^j\mathbf{G}E^i)^{K^{ij}}\|_{H^\infty} < 1, \tag{3.5}$$

then statements (1)–(4) *of* Theorem 3.4 *hold.*

Let H be a complex Hilbert space. We say that $\mathbf{H} : \mathbb{C}_0 \to \mathcal{L}(H)$ is *positive real* if $\mathbf{H}$ is holomorphic with the exception of isolated singularities and $\mathbf{H}(s) + \mathbf{H}^*(s)$ is positive semi-definite for all $s \in \mathbb{C}_0$ which are not singularities of $\mathbf{H}$. In fact, if $\mathbf{H}$ is positive real, then $\mathbf{H}$ is holomorphic on $\mathbb{C}_0$ [13, Proposition 3.3].

The following result can be considered as an incremental version of the circle criterion.

Corollary 3.6. *Let Σ, f, Z_1 and Z_2 be as in* Theorem 3.4*, let $i, j \in \{1, 2\}$ and let $K_1, K_2 \in \mathcal{L}(Y^j, U^i)$. Assume that $\Sigma^{ji} = (\mathbb{T}, \Phi E^i, P^j\Psi, P^j\mathbb{G}E^i)$ is optimizable and estimatable, K_1 is admissible feedback operator for Σ^{ji} and $Z_2 = Y^j$. If $(I - K_2P^j\mathbf{G}E^i)(I - K_1P^j\mathbf{G}E^i)^{-1}$ is positive real and there exists $\varepsilon > 0$ such that*

$$\mathrm{Re}\,\langle f(z_1) - f(z_2) - K_1(z_1 - z_2), f(z_1) - f(z_2) - K_2(z_1 - z_2)\rangle \le -\varepsilon\|z_1 - z_2\|^2 \quad \forall\,(z_1, z_2) \in Z_1 \times Y^j,$$

then statements (1)–(4) *of* Theorem 3.4 *hold (with $Z_2 = Y^j$).*

The above corollary can be derived from Theorem 3.4 in the same way as [14, Corollary 4.5] is obtained from [14, Theorem 4.1] and we do not repeat the details here.

4 Lur'e systems with almost periodic inputs

Before we come to the main result of this paper, we provide some relevant background on almost periodic functions (in the sense of Bohr and its generalization by Stepanov).

Let $R = \mathbb{R}$ or $\mathbb{R}_+$ and let W be a Banach space. A set $S \subseteq R$ is said to be *relatively dense* (in R) if there exists $l > 0$ such that

$$[a, a + l] \cap S \neq \emptyset \quad \forall\, a \in R.$$

For $\varepsilon > 0$, we say that $\tau \in R$ is an *ε-period* of $v \in C(R, W)$ if

$$\|v(t) - v(t + \tau)\| \le \varepsilon \quad \forall\, t \in R.$$

We denote by $P(v, \varepsilon) \subseteq R$ the set of ε-periods of v and we say that $v \in C(R, W)$ is *almost periodic* (in the sense of Bohr) if $P(v, \varepsilon)$ is relatively dense in R for every $\varepsilon > 0$. We denote the set of almost periodic functions $v \in C(R, W)$ by $AP(R, W)$ and mention that $AP(R, W)$ is a closed subspace of $BUC(R, W)$. Obviously, any periodic continuous function is almost periodic.

The straightforward proof of the following lemma is left to the reader.

Lemma 4.1. *If $v \in AP(R, W)$, then, for every $\tau \in R$,* $\sup_{t \ge \tau} \|v(t)\| = \|v\|_\infty$.

The above lemma shows that functions in $AP(R, W)$ are completely determined by their "infinite right tails": if $v, w \in AP(R, W)$ and there exists $\tau \in R$ such that $v(t) = w(t)$ for all $t \ge \tau$, then $v = w$. A similar result holds in the context of "infinite left tails" of almost periodic functions defined on $\mathbb{R}$, but since it is not needed in what follows, we omit the details.

We say that a function $v \in C(\mathbb{R}_+, W)$ is *asymptotically almost periodic* if it is of the form $v = v^{\mathrm{ap}} + w$ with $v^{\mathrm{ap}} \in AP(\mathbb{R}_+, W)$ and $w \in C_0(\mathbb{R}_+, W)$, where $C_0(\mathbb{R}_+, W)$ is the space of functions $u \in C(\mathbb{R}_+, W)$ such that $\lim_{t\to\infty} u(t) = 0$. The space of all asymptotically almost periodic functions is denoted by $AAP(\mathbb{R}_+, W)$, that is,

$$AAP(\mathbb{R}_+, W) = AP(\mathbb{R}_+, W) + C_0(\mathbb{R}_+, W).$$

Noting that, by Lemma 4.1,

$$\|v + w\|_\infty \ge \|v\|_\infty \quad \forall\, v \in AP(\mathbb{R}_+, W), \forall\, w \in C_0(\mathbb{R}_+, W), \tag{4.1}$$

it is easy to see that $AAP(\mathbb{R}_+, W)$ is a closed subspace of $BUC(\mathbb{R}_+, W)$.

As an immediate consequence of Lemma 4.1, we obtain the following result.

Lemma 4.2. *If $v \in AAP(\mathbb{R}_+, W)$, then the decomposition $v = v^{\mathrm{ap}} + w$, where $v^{\mathrm{ap}} \in AP(\mathbb{R}_+, W)$ and $w \in C_0(\mathbb{R}_+, W)$, is unique.*

In the following, for $v \in AAP(\mathbb{R}_+, W)$, we let v^{ap} denote the unique function in $AP(\mathbb{R}_+, W)$ such that $v - v^{\mathrm{ap}} \in C_0(\mathbb{R}_+, W)$.

It is well-known that $v \in C(\mathbb{R}, W)$ is almost periodic if, and only if, the set of translates $\{\mathbf{S}_\tau v : \tau \in \mathbb{R}\}$ is relatively compact in $BC(\mathbb{R}, W)$. Since, for any $v \in C_0(\mathbb{R}_+, W)$, the set of left-translates $\{\mathbf{S}_\tau v : \tau \in \mathbb{R}_+\}$ is relatively compact in $BC(\mathbb{R}_+, W)$, it is clear that the above characterisation of almost periodicity on $\mathbb{R}$ is not valid for functions in $C(\mathbb{R}_+, W)$. Interestingly, the elements of $AAP(\mathbb{R}_+, W)$ are precisely the functions for which the set $\{\mathbf{S}_\tau v : \tau \in \mathbb{R}_+\}$ is relatively compact in $BUC(\mathbb{R}_+, W)$, see [21]). For more information on and further characterisations of almost periodicity, we refer the reader to the literature, see, for example, [1, 7, 8].

There is a close relationship between the spaces $AP(\mathbb{R}_+, W)$ and $AP(\mathbb{R}, W)$ which we now briefly explain. Following an idea in [5, Remark on p. 318], for every $v \in AP(\mathbb{R}_+, W)$, we define a function $v_{\mathrm{e}} : \mathbb{R} \to W$ by

$$v_{\mathrm{e}}(t) := \lim_{k \to \infty} v(t + \tau_k) \quad \forall\, t \in \mathbb{R},$$

where $\tau_k \in P(v, 1/k)$ for each $k \in \mathbb{N}$ and $\tau_k \to \infty$ as $k \to \infty$. For given $t \in \mathbb{R}$, we have

$$\|v(t + \tau_k) - v(t + \tau_l)\| \le \|v(t + \tau_k) - v(t + \tau_k + \tau_l)\| + \|v(t + \tau_k + \tau_l) - v(t + \tau_l)\| \le \frac{1}{l} + \frac{1}{k},$$

for all $k, l \in \mathbb{N}$ sufficiently large, and so $(v(t + \tau_k))_k$ is a Cauchy sequence. Hence $v_{\mathrm{e}}(t)$ is well-defined for each $t \in \mathbb{R}$. It is clear that $v_{\mathrm{e}}(t) = v(t)$ for all $t \ge 0$, that is, v_{e} extends v to $\mathbb{R}$. Furthermore, it is not difficult to show that v_{e} is continuous and $P(v_{\mathrm{e}}, \varepsilon) = \{\pm\tau : \tau \in P(v, \varepsilon)\}$. In particular, $v_{\mathrm{e}} \in AP(\mathbb{R}, W)$. Moreover, there is no other function in $AP(\mathbb{R}, W)$ which extends v to $\mathbb{R}$, and Lemma 4.1 guarantees that

$$\sup_{t \in \mathbb{R}} \|v_{\mathrm{e}}(t)\| = \sup_{t \in \mathbb{R}_+} \|v(t)\|.$$

It is now clear that the map $AP(\mathbb{R}_+, W) \to AP(\mathbb{R}, W)$, $v \mapsto v_{\mathrm{e}}$ is an isometric isomorphism. We remark that, by invoking the translation semigroup acting on $AP(\mathbb{R}_+, W)$, [4] provides an alternative approach to establishing that every element in $AP(\mathbb{R}_+, W)$ has an almost periodic extension to $\mathbb{R}$.

For a function $v \in AP(\mathbb{R}, W)$, the generalized Fourier coefficients of v are defined by

$$\hat{v}(\lambda) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} v(t)\, dt \quad \forall\, \lambda \in \mathbb{R}.$$

It is well-known that the above limit exists for all $\lambda \in \mathbb{R}$ and the frequency spectrum

$$\sigma_{\mathrm{f}}(v) := \{\lambda \in \mathbb{R} : \hat{v}(\lambda) \ne 0\}$$

of v is countable, see, for example, [1, 8]. The module $\mathrm{mod}(v)$ of $v \in AP(\mathbb{R}, W)$ is the set of all numbers of the form $\sum_{\lambda \in \sigma_{\mathrm{f}}(v)} m(\lambda)\lambda$, where $m : \sigma_{\mathrm{f}}(v) \to \mathbb{Z}$ has finite support, that is, $m(\lambda) \ne 0$ for at most finitely many $\lambda \in \sigma_{\mathrm{f}}(v)$. It is clear that $\mathrm{mod}(v)$ carries the structure of a $\mathbb{Z}$-module and is the smallest additive subgroup of $\mathbb{R}$ containing $\sigma_{\mathrm{f}}(v)$.

We recall another concept of almost periodicity which is weaker than that of Bohr. To this end, let $v \in L^q_{\mathrm{loc}}(R, W)$, where $1 \le q < \infty$, and $\varepsilon > 0$. We say that $\tau \in R$ is an *ε-period of v* (in the sense of Stepanov) if

$$\sup_{a \in R} \left(\int_a^{a+1} \|v(s + \tau) - v(s)\|^q ds \right)^{1/q} \le \varepsilon.$$

The set of ε-periods of v (in the sense of Stepanov) is denoted by $P_q(v,\varepsilon)$. We say that v is *almost periodic* in the sense of Stepanov if, for every $\varepsilon > 0$, the set $P_q(v,\varepsilon)$ is relatively dense in R. The set of all functions in $L^q_{\rm loc}(R,W)$ which are almost periodic in the sense of Stepanov is denoted by $S^q(R,W)$. It is clear that $AP(R,W) \subset S^q(R,W)$ (where the inclusion is strict), and, for every $v \in AP(R,W)$ and every $\varepsilon > 0$, $P(v,\varepsilon) \subset P_q(v,\varepsilon)$. It is a routine exercise to prove that $S^q(R,W)$ is a closed subspace of $UL^1_{\rm loc}(R,W)$ with respect to the Stepanov norm $\|\cdot\|_{S^q}$. Sometimes it will be convenient to associate with a function $v \in L^q_{\rm loc}(R,W)$ another function $\tilde v : R \to L^q([0,1],W)$ defined by

$$\big(\tilde v(t)\big)(s) := v(t+s) \quad \forall\, t \in R,\ \forall\, s \in [0,1],$$

the so-called *Bochner transform* of v. Then $\tilde v \in C(R, L^q([0,1],W))$, and,

$$\|v\|_{S^q} = \|\tilde v\|_\infty \quad \forall\, v \in UL^q_{\rm loc}(R,W), \tag{4.2}$$

that is, the Bochner transform restricted to $UL^q_{\rm loc}(R,W)$ is an isometry. Furthermore, a function $v \in UL^q_{\rm loc}(R,W)$ is in $S^q(R,W)$ if, and only if, $\tilde v \in AP(R, L^q([0,1],W))$. We remark that the Bochner transform is not surjective.†

The following simple lemma is a consequence of Lemma 4.1 and (4.2).

Lemma 4.3. *If $v \in S^q(\mathbb{R}_+,W)$, then, for every $\tau \in \mathbb{R}_+$, $\|\mathbf{S}_\tau v\|_{S^q} = \|v\|_{S^q}$.*

The space $AS^q(\mathbb{R}_+,W)$ of asymptotically almost periodic functions in the sense of Stepanov is defined as follows

$$AS^q(\mathbb{R}_+,W) := S^q(\mathbb{R}_+,W) + U_0L^q_{\rm loc}(\mathbb{R}_+,W),$$

where $U_0L^q_{\rm loc}(\mathbb{R}_+,W) := \{v \in UL^q_{\rm loc}(\mathbb{R}_+,W) : \|\mathbf{S}_t v\|_{S^q} \to 0 \text{ as } t \to \infty\}$. Obviously, $AAP(\mathbb{R}_+,W) \subset AS^q(\mathbb{R}_+,W)$. Noting that

$$v \in U_0L^q_{\rm loc}(\mathbb{R}_+,W) \Leftrightarrow \tilde v \in C_0(\mathbb{R}_+, L^q([0,1],W)) \quad \forall\, v \in UL^q_{\rm loc}(\mathbb{R}_+,W) \tag{4.3}$$

and

$$v \in AS^q(\mathbb{R}_+,W) \Leftrightarrow \tilde v \in AAP(\mathbb{R}_+, L^q([0,1],W)) \quad \forall\, v \in UL^q_{\rm loc}(\mathbb{R}_+,W), \tag{4.4}$$

it follows from (4.1) and (4.2) that

$$\|v+w\|_{S^q} \ge \|v\|_{S^q} \quad \forall\, v \in S^q(\mathbb{R}_+,W),\ \forall\, w \in U_0L^q_{\rm loc}(\mathbb{R}_+,W).$$

It is an easy consequence of this inequality that $AS^q(\mathbb{R}_+,W)$ is a closed subspace of $UL^q_{\rm loc}(\mathbb{R}_+,W)$. Furthermore, (4.3) and (4.4) together with Lemma 4.2 and (4.2) yield the following result.

Lemma 4.4. *If $v \in AS^q(\mathbb{R}_+,W)$, then the decomposition $v = v^{\rm s} + w$, where $v^{\rm s} \in S^q(\mathbb{R}_+,W)$ and $w \in U_0L^q_{\rm loc}(\mathbb{R}_+,W)$, is unique.*

In the following, for $v \in AS^q(\mathbb{R}_+,W)$, we let $v^{\rm s}$ denote the unique function in $S^q(\mathbb{R}_+,W)$ such that $v - v^{\rm s} \in U_0L^q_{\rm loc}(\mathbb{R}_+,W)$.

Let $v \in S^q(\mathbb{R}_+,W)$ and let $\tau_k \in P_q(v,1/k)$ for all $k \in \mathbb{N}$ and $\tau_k \to \infty$ as $k \to \infty$. Then it can be shown that, for each $\tau > 0$, $\big(v(\cdot + \tau_k)\big)_k$ is a Cauchy sequence in $L^q([-\tau,\tau],W)$ and hence defines a function $v_{\rm e} \in L^q_{\rm loc}(\mathbb{R},W)$. A straightforward argument shows that $v_{\rm e}|_{\mathbb{R}_+} = v$ (i.e., $v_{\rm e}$ extends v to $\mathbb{R}$), $v_{\rm e} \in S^q(\mathbb{R},W)$, $P_q(v_{\rm e},\varepsilon) = \{\pm\tau : \tau \in P_q(v,\varepsilon)\}$ for every $\varepsilon > 0$, and the map $S^q(\mathbb{R}_+,W) \mapsto S^q(\mathbb{R},W)$, $v \mapsto v_{\rm e}$ is an isometric isomorphism.

We are now in the position to state and prove the main result of this paper.

† Consider the constant function $F \in AP(R, L^q([0,1],W))$ given by $F(t) = \lambda$, where $\lambda \in L^q([0,1],W)$ is such that $\lambda|_{[0,1/2]} = 0$ and $\lambda|_{[1/2,1]} \neq 0$. Seeking a contradiction, suppose that there exists $f \in L^q_{\rm loc}(R,W)$ with $\tilde f = F$. Then $f(t+s) = 0$ for every $t \in R$ and almost every $s \in [0,1/2]$, implying that $f = 0$, and thus, $F = \tilde f = 0$ which is absurd.

Theorem 4.5. *Let $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{G})$ be a well-posed linear system, let $i, j \in \{1, 2\}$, $K^{ij} \in \mathcal{L}(Y^j, U^i)$ and let $v^* \in S^2(\mathbb{R}_+, U)$. Assume that $\Sigma^{ji} = (\mathbb{T}, \Phi E^i, P^j\Psi, P^j\mathbb{G}E^i)$ is optimizable and estimatable and $K^{ij} \in \mathbb{S}(P^j\mathbf{G}E^i)$. If $f : Y^j \to U^i$ satisfies (3.5) with $Z_1 = Z_2 = Y^j$, then there exists a unique pair $(x^*, y^*) \in AP(\mathbb{R}_+, X) \times S^2(\mathbb{R}_+, Y)$ such that $(v^*, x^*, y^*) \in \mathcal{B}$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $P_2(v^*, \delta) \subset P(x^*, \varepsilon) \cap P_2(y^*, \varepsilon)$ and the following statements hold.*

(1) *If $(v, x, y) \in \mathcal{B}$ is such that $v \in AS^2(\mathbb{R}_+, U)$ with $v^{\mathrm{s}} = v^*$, then*

$$\lim_{t\to\infty} \big(x(t) - x^*(t)\big) = 0, \;\; y \in UL^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \quad \textit{and} \quad \|\mathbf{S}_t(y - y^*)\|_{S^2} \to 0 \;\; \textit{as } t \to \infty,$$

that is, $x \in AAP(\mathbb{R}_+, X)$ with $x^{\mathrm{ap}} = x^*$ and $y \in AS^2(\mathbb{R}_+, Y)$ with $y^{\mathrm{s}} = y^*$.

(2) *If v^* is periodic with period τ, then (x^*, y^*) is τ-periodic.*

(3) *$(v^*_{\mathrm{e}}, x^*_{\mathrm{e}}, y^*_{\mathrm{e}}) \in \mathcal{BB}$ and there is no other pair $(\hat{x}, \hat{y}) \in BC(\mathbb{R}, X) \times UL^2_{\mathrm{loc}}(\mathbb{R}, Y)$ such that the triple $(v^*_{\mathrm{e}}, \hat{x}, \hat{y})$ is in $\mathcal{BB}$.*

(4) *$\mathrm{mod}(\widetilde{v^*_{\mathrm{e}}}) \supset \mathrm{mod}(x^*_{\mathrm{e}}) \cup \mathrm{mod}(\widetilde{y^*_{\mathrm{e}}})$.*

Note that if $v^* \in AP(\mathbb{R}_+, U)$ and $v \in AAP(\mathbb{R}_+, U)$ with $v^{\mathrm{ap}} = v^*$, then $v^* \in S^2(\mathbb{R}_+, U)$ and $v \in AS^2(\mathbb{R}_+, U)$ with $v^{\mathrm{s}} = v^{\mathrm{ap}} = v^*$, and Theorem 4.5 applies. Furthermore, in this case, it can be shown that $\sigma_{\mathrm{f}}(v^*_{\mathrm{e}}) = \sigma_{\mathrm{f}}(\widetilde{v^*_{\mathrm{e}}})$, and so, statement (4) can be written in the form $\mathrm{mod}(v^*_{\mathrm{e}}) \supset \mathrm{mod}(x^*_{\mathrm{e}}) \cup \mathrm{mod}(\widetilde{y^*_{\mathrm{e}}})$. Of course, the extra regularity in the forcing provided by assuming that $v^* \in AP(\mathbb{R}_+, U)$ and $v \in AAP(\mathbb{R}_+, U)$ is not sufficient to guarantee that $y^* \in AP(\mathbb{R}_+, Y)$ and/or $y \in AAP(\mathbb{R}_+, Y)$.

As regards to statement (1), note that if $v - v^* \in L^2(\mathbb{R}_+, U)$ or if $\|v - v^*\|_{L^\infty(t,\infty)} \to 0$ as $t \to \infty$, then $v \in AS^2(\mathbb{R}_+, U)$ with $v^{\mathrm{s}} = v^*$. In statement (2), τ-periodicity of the L^2_{loc}-function y^* means that $y^*(t + \tau) = y^*(t)$ for almost every $t \geq 0$. We remark that statement (2) is not new: it was first proved in [14, Theorem 5.4]. Here we recover it as a special case of the results on almost periodic forcing, see the proof of Theorem 4.5 given below.

Proof of Theorem 4.5. Let $v^* \in S^2(\mathbb{R}_+, U)$. It follows from (3.5), Proposition 3.1 and Lemma 3.2 (with $K = E^iK^{ij}P^j$) that there exists a pair $(x, y) \in C(\mathbb{R}_+, X) \times L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ such that $(v^*, x, y) \in \mathcal{B}$. Setting $r := 1/\|(P^j\mathbf{G}E^i)^{K^ij}\|_{H^\infty}$, it follows from Lemma 2.1 that $\mathbb{B}(K^{ij}, r) \subset \mathbb{S}(P^j\mathbf{G}E^i)$, and hence, the hypotheses of Theorem 3.4 are satisfied. Therefore, applying statements (3) and (4) of Theorem 3.4 with $(v_1, x_1, y_1) = (v^*, x, y)$ and $(v_2, x_2, y_2) = (v^{\mathrm{eq}}\vartheta, x^{\mathrm{eq}}\vartheta, y^{\mathrm{eq}}\vartheta)$, where $(v^{\mathrm{eq}}, x^{\mathrm{eq}}, y^{\mathrm{eq}})$ is an arbitrary equilibrium triple of (3.1) and ϑ is the constant function $\vartheta(t) \equiv 1$, it follows that x is bounded, and so $x \in BC(\mathbb{R}_+, X)$, and, furthermore, $y \in UL^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$. We set

$$\rho := 2\|x\|_\infty,$$

and choose a non-decreasing sequence $(\tau_k)_{k\in\mathbb{N}}$ such that

$$\tau_k \in P_2(v^*, 1/k^2) \quad \text{and} \quad \tau_k > k \qquad \forall\, k \in \mathbb{N}.$$

We proceed in several steps.

Step 1: Construction of x^*. We are going to show that $(\mathbf{S}_{\tau_k}x)_k$ is a Cauchy sequence in $BC(\mathbb{R}_+, X)$.[†] To this end, we note that

$$\begin{aligned}\left(\int_a^{a+k} \|v^*(t + \tau_k) - v^*(t)\|^2 dt\right)^{1/2} &\leq \sum_{j=1}^{k} \left(\int_{a+j-1}^{a+j} \|v^*(t + \tau_k) - v^*(t)\|^2 dt\right)^{1/2} \\ &\leq \frac{1}{k} \quad \forall\, a \geq 0, \;\forall\, k \in \mathbb{N}.\end{aligned}$$

[†] Thereby extending an idea from [2] where a similar argument is used to establish the existence of a periodic solution of periodically forced finite-dimensional systems.

Consequently,

$$\sup_{a\geq 0}\left(\int_a^{a+k}\|v^*(t+\tau_k)-v^*(t)\|^2\right)^{1/2}\leq\frac{1}{k}. \tag{4.5}$$

Since $(\mathbf{S}_\tau v^*, \mathbf{S}_\tau x, \mathbf{S}_\tau y)\in\mathcal{B}$ for all $\tau\geq 0$, it follows from statement (1) of Theorem 3.4 that there exist constants $\Gamma_2, \gamma>0$ such that

$$\|(\mathbf{S}_\sigma x)(s)-(\mathbf{S}_{\sigma+\tau}x)(s)\|\leq\Gamma_2\big(\rho e^{-\gamma(s-s_0)}+\|\mathbf{S}_\sigma v^*-\mathbf{S}_{\sigma+\tau}v^*\|_{L^2(s_0,s)}\big)\quad\forall\,(s,s_0)\in\Delta,\,\forall\,\sigma,\tau\geq 0. \tag{4.6}$$

Trivially, for $k,\ell\in\mathbb{N}$ with $k\geq\ell$,

$$(\mathbf{S}_{\tau_\ell}x)(t)-(\mathbf{S}_{\tau_k}x)(t)=(\mathbf{S}_t x)(\tau_\ell)-(\mathbf{S}_{t+\tau_k-\tau_\ell}x)(\tau_\ell),\quad\forall\,t\geq 0,$$

and so, setting

$$I(t;k,\ell):=\|\mathbf{S}_t v^*-\mathbf{S}_{t+\tau_k-\tau_\ell}v^*\|_{L^2(\tau_\ell-\ell,\tau_\ell)}\quad\forall\,t\geq 0,$$

and invoking (4.6) with $s=\tau_\ell$, $s_0=\tau_\ell-\ell$, $\sigma=t$ and $\tau=\tau_k-\tau_\ell$, we arrive at

$$\|(\mathbf{S}_{\tau_\ell}x)(t)-(\mathbf{S}_{\tau_k}x)(t)\|\leq\Gamma_2\big(\rho e^{-\gamma\ell}+I(t;k,\ell)\big)\quad\forall\,t\geq 0,\ \forall\,k,\ell\in\mathbb{N}\text{ s.t. }k\geq\ell. \tag{4.7}$$

Now

$$I(t;k,\ell)\leq\|\mathbf{S}_t v^*-\mathbf{S}_{t+\tau_k}v^*\|_{L^2(\tau_\ell-\ell,\tau_\ell)}+\|\mathbf{S}_{t+\tau_k}v^*-\mathbf{S}_{t+\tau_k-\tau_\ell}v^*\|_{L^2(\tau_\ell-\ell,\tau_\ell)},$$

and so, changing variables in the two terms on the right-hand side, we obtain that, for all $t\geq 0$ and all $k,\ell\in\mathbb{N}$ such that $k\geq\ell$,

$$\begin{aligned}I(t;k,\ell)&\leq\|v^*-\mathbf{S}_{\tau_k}v^*\|_{L^2(t+\tau_\ell-\ell,t+\tau_\ell)}+\|\mathbf{S}_{\tau_\ell}v^*-v^*\|_{L^2(t+\tau_k-\ell,t+\tau_k)}\\&\leq\|v^*-\mathbf{S}_{\tau_k}v^*\|_{L^2(t+\tau_\ell-\ell,t+\tau_\ell-\ell+k)}+\|\mathbf{S}_{\tau_\ell}v^*-v^*\|_{L^2(t+\tau_k-\ell,t+\tau_k)}.\end{aligned}$$

Consequently, by (4.5),

$$I(t;k,\ell)\leq\frac{1}{k}+\frac{1}{\ell}\quad\forall\,t\geq 0,\ \forall\,k,\ell\in\mathbb{N}\text{ s.t. }k\geq\ell,$$

and it follows from (4.7) that

$$\|(\mathbf{S}_{\tau_\ell}x)(t)-(\mathbf{S}_{\tau_k}x)(t)\|\leq\Gamma_2\big(\rho e^{-\gamma\ell}+1/k+1/\ell\big)\quad\forall\,t\geq 0,\ \forall\,k,\ell\in\mathbb{N}\text{ s.t. }k\geq\ell.$$

This shows that $(\mathbf{S}_{\tau_k}x)_k$ is a Cauchy sequence in $BC(\mathbb{R}_+,X)$, the limit of which we denote by x^*. To show that $x^*\in AP(\mathbb{R}_+,X)$, let $\varepsilon>0$, choose $k_\varepsilon\in\mathbb{N}$ such that $\rho e^{-\gamma k_\varepsilon}\leq\varepsilon/(2\Gamma_2)$ and set

$$\eta_\varepsilon:=\frac{\varepsilon}{2k_\varepsilon\Gamma_2}.$$

Let $\tau\in P_2(v^*,\eta_\varepsilon)$. We will show that $P_2(v^*,\eta_\varepsilon)\subset P(x^*,\varepsilon)$. Obviously,

$$(\mathbf{S}_{\tau_k}x)(t+\tau)-(\mathbf{S}_{\tau_k}x)(t)=(\mathbf{S}_{t+\tau}x)(\tau_k)-(\mathbf{S}_t x)(\tau_k)\quad\forall t\geq 0,\ \forall\,k\in\mathbb{N},$$

and so, by (4.6) with $s=\tau_k$, $\sigma=t$ and $s_0=\tau_k-k_\varepsilon$,

$$\|(\mathbf{S}_{\tau_k}x)(t+\tau)-(\mathbf{S}_{\tau_k}x)(t)\|\leq\Gamma_2\big(\rho e^{-\gamma k_\varepsilon}+\|\mathbf{S}_t v^*-\mathbf{S}_{t+\tau}v^*\|_{L^2(\tau_k-k_\varepsilon,\tau_k)}\big)\quad\forall\,t\geq 0,\ \forall\,k\geq k_\varepsilon.$$

Now

$$\|\mathbf{S}_t v^*-\mathbf{S}_{t+\tau}v^*\|_{L^2(\tau_k-k_\varepsilon,\tau_k)}=\|v^*-\mathbf{S}_\tau v^*\|_{L^2(t+\tau_k-k_\varepsilon,t+\tau_k)}\leq k_\varepsilon\eta_\varepsilon=\frac{\varepsilon}{2\Gamma_2}\quad\forall\,t\geq 0,\ \forall\,k\geq k_\varepsilon,$$

and thus,

$$\|(\mathbf{S}_{\tau_k}x)(t+\tau)-(\mathbf{S}_{\tau_k}x)(t)\| \le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \quad \forall\, t\ge 0,\ \forall\, k\ge k_\varepsilon.$$

Letting $k\to\infty$, we obtain

$$\|x^*(t+\tau)-x^*(t)\|\le\varepsilon \quad \forall\, t\ge 0,$$

establishing that $P_2(v^*,\eta_\varepsilon)\subset P(x^*,\varepsilon)$. The set $P_2(v^*,\eta_\varepsilon)$ is relatively dense in $\mathbb{R}_+$, and, *a fortiori*, $P(x^*,\varepsilon)$ is also relatively dense in $\mathbb{R}_+$. Since ε was arbitrary, we conclude that $x^*\in AP(\mathbb{R}_+,X)$.

Step 2: Construction of y^*. By statement (4) of Theorem 3.4 there exists a constant $\tilde\Gamma>0$ such that

$$\|\mathbf{S}_{t+\tau_k}y-\mathbf{S}_{\tau_\ell}y\|_{S^2}\le\tilde\Gamma\big(\|(\mathbf{S}_{t+\tau_k}x)(0)-(\mathbf{S}_{\tau_\ell}x)(0)\|+\|\mathbf{S}_{t+\tau_k}v^*-\mathbf{S}_{\tau_\ell}v^*\|_{S^2}\big) \quad \forall\, t\ge 0,\ \forall\, k,\ell\in\mathbb{N}. \tag{4.8}$$

Obviously, $\mathbf{S}_{\tau_k}v^*\to v^*$ in $S^2(\mathbb{R}_+,U)$ and $(\mathbf{S}_{\tau_k}x)(0)\to x^*(0)$ as $k\to\infty$, and so it follows from (4.8) with $t=0$ that $(\mathbf{S}_{\tau_k}y)_k$ is a Cauchy sequence in $UL^2_{\rm loc}(\mathbb{R}_+,Y)$, the limit of which we denote by y^*. Letting $k\to\infty$ and $\ell\to\infty$ in (4.8) we arrive at

$$\|\mathbf{S}_t y^*-y^*\|_{S^2}\le\tilde\Gamma\big(\|(\mathbf{S}_t x^*)(0)-x^*(0)\|+\|\mathbf{S}_t v^*-v^*\|_{S^2}\big) \quad \forall\, t\ge 0. \tag{4.9}$$

Now let $\varepsilon>0$, set $\tilde\varepsilon:=\varepsilon/(2\tilde\Gamma)$ and

$$\delta_\varepsilon:=\min\{\eta_\varepsilon,\eta_{\tilde\varepsilon},\tilde\varepsilon\},$$

and let $\tau\in P_2(v^*,\delta_\varepsilon)$. Then, $\tau\in P_2(v^*,\eta_{\tilde\varepsilon})$, and consequently, by what we proved in Step 1, $\tau\in P(x^*,\tilde\varepsilon)$. An application of (4.9) with $t=\tau$ yields

$$\|\mathbf{S}_\tau y^*-y^*\|_{S^2}\le\tilde\Gamma\big(\tilde\varepsilon+\delta_\varepsilon\big)\le\varepsilon.$$

Hence, $P_2(v^*,\delta_\varepsilon)\subset P_2(y^*,\varepsilon)$. Therefore, the relative denseness of $P_2(v^*,\delta_\varepsilon)$ implies that of $P_2(y^*,\varepsilon)$, showing that $y^*\in S^2(\mathbb{R}_+,Y)$. By the definition of δ_ε, we have that $P_2(v^*,\delta_\varepsilon)\subset P_2(v^*,\eta_\varepsilon)$, and so, by Step 1, $P_2(v^*,\delta_\varepsilon)\subset P(x^*,\varepsilon)$. Consequently, $P_2(v^*,\delta_\varepsilon)\subset P(x^*,\varepsilon)\cap P_2(y^*,\varepsilon)$.

Step 3: $(v^*,x^*,y^*)\in\mathcal{B}$ *and uniqueness of* (x^*,y^*) *within* $AP(\mathbb{R}_+,X)\times S^2(\mathbb{R}_+,Y)$. Since the triple $(\mathbf{S}_{\tau_k}v^*,\mathbf{S}_{\tau_k}x,\mathbf{S}_{\tau_k}y)$ is in $\mathcal{B}$ for all $k\in\mathbb{N}$, $\|\mathbf{S}_{\tau_k}v^*-v^*\|_{S^2}\to 0$, $\|\mathbf{S}_{\tau_k}y-y^*\|_{S^2}\to 0$ and $\|\mathbf{S}_{\tau_k}x-x^*\|_\infty\to 0$ as $k\to\infty$, it follows from (3.1), the continuity properties of well-posed linear systems and the global Lipschitz property of f that $(v^*,x^*,y^*)\in\mathcal{B}$.

To prove uniqueness of (x^*,y^*), assume that $(x^\sharp,y^\sharp)\in AP(\mathbb{R}_+,X)\times S^2(\mathbb{R}_+,Y)$ is such that $(v^*,x^\sharp,y^\sharp)\in\mathcal{B}$. Then, appealing to statement (1) of Theorem 3.4, we see that $x^*(t)-x^\sharp(t)\to 0$ as $t\to\infty$. But the function $x^*-x^\sharp$ is in $AP(\mathbb{R}_+,X)$, and so, invoking Lemma 4.1, we conclude that $x^*=x^\sharp$. Statement (2) of Theorem 3.4 now implies that $y^*=y^\sharp$.

Step 4: Proof of statements (1) *and* (2). Let $(v,x,y)\in\mathcal{B}$ be such that $v\in AS^2(\mathbb{R}_+,U)$ with $v^{\rm s}=v^*$. An application of statement (3) of Theorem 3.4 shows that there exists $\tilde\Gamma_2>0$ and $\tilde\gamma>0$ such that

$$\|x(t)-x^*(t)\|\le\tilde\Gamma_2\big(e^{-\tilde\gamma(t-t_0)}\|x(t_0)-x^*(t_0)\|+\|\mathbf{S}_{t_0}(v-v^*)\|_{S^2}\big) \quad \forall\,(t,t_0)\in\Delta.$$

In particular, $x-x^*$ is bounded, and so $\mu:=\sup_{t\ge0}\|x(t)-x^*(t)\|<\infty$. Let $\varepsilon>0$. Since $\|\mathbf{S}_t(v-v^*)\|_{S^2}\to 0$ as $t\to\infty$, there exists $\sigma\ge 0$ such that $\|\mathbf{S}_\sigma(v-v^*)\|_{S^2}\le\varepsilon/(2\tilde\Gamma_2)$. Choosing $\tau\ge 0$ such that $e^{-\tilde\gamma\tau}\le\varepsilon/(2\mu\tilde\Gamma_2)$, it follows from the above inequality with $t_0=\sigma$ that $\|x(t)-x^*(t)\|\le\varepsilon$ for all $t\ge\sigma+\tau$, showing that $(x(t)-x^*(t))\to 0$ as $t\to\infty$. Furthermore, statement (4) of Theorem 3.4 shows that $y\in UL^2_{\rm loc}(\mathbb{R}_+,Y)$ and

$$\|\mathbf{S}_t(y-y^*)\|_{S^2}\le\tilde\Gamma\big(\|x(t)-x^*(t)\|+\|\mathbf{S}_t(v-v^*)\|_{S^2}\big) \quad \forall\, t\ge 0.$$

Letting $t\to\infty$, we see that $\|\mathbf{S}_t(y-y^*)\|_{S^2}\to 0$, completing the proof of statement (1).

To prove statement (2), assume that v^* is τ-periodic for some $\tau>0$. Then $\tau\in P_2(v^*,\delta)$ for every $\delta>0$ and so, $\tau\in P(x^*,\varepsilon)\cap P_2(y^*,\varepsilon)$ for every $\varepsilon>0$, implying that x^* and y^* are τ-periodic.

Step 5: Proof of statement (3). To show that $(v_{\rm e}^*, x_{\rm e}^*, y_{\rm e}^*) \in \mathcal{BB}$, we choose $\delta_k > 0$ such that

$$P_2(v^*, \delta_k) \subset P(x^*, 1/k) \cap P_2(y^*, 1/k) \quad \forall\, k \in \mathbb{N}.$$

The existence of such numbers δ_k is guaranteed by Steps 1 and 2. Setting $\eta_k := \min(\delta_k, 1/k)$, we have that $\eta_k \to 0$ as $k \to \infty$ and

$$P_2(v^*, \eta_k) \subset P(x^*, 1/k) \cap P_2(y^*, 1/k) \quad \forall\, k \in \mathbb{N}.$$

Let $t_0 \in \mathbb{R}$ and $\tau_k \in P_2(v^*, \eta_k)$ such that $\tau_k \geq \max(0, -t_0)$ for all $k \in \mathbb{N}$. The latter ensures that $t_0 + \tau_k \geq 0$ for all $k \in \mathbb{N}$. Noting that

$$x_{\rm e}^*(t + \tau_k) = x_{\rm e}^*(t - t_0 + \tau_k + t_0) = x^*(t - t_0 + \tau_k + t_0) = (\mathbf{S}_{t_0+\tau_k} x^*)(t - t_0) \quad \forall\, t \geq t_0,\ \forall\, k \in \mathbb{N},$$

we conclude that

$$x_{\rm e}^*(t + \tau_k) = (\mathbf{S}_{t_0+\tau_k} x^*)(t - t_0) = \mathbb{T}_{t-t_0}(\mathbf{S}_{t_0+\tau_k} x^*)(0) + \Phi_{t-t_0}\mathbf{P}_{t-t_0}\mathbf{S}_{t_0+\tau_k} u^* \quad \forall\, t \geq t_0,\ \forall\, k \in \mathbb{N}, \tag{4.10}$$

where $u^* := E^i(f \circ P^j y^*) + v^*$. Since $v^* \in S^2(\mathbb{R}_+, U)$, $y^* \in S^2(\mathbb{R}_+, Y)$ and f is globally Lipschitz, it follows that $u^* \in S^2(\mathbb{R}_+, U)$. Trivially, by (4.10),

$$x_{\rm e}^*(t + \tau_k) = \mathbb{T}_{t-t_0} x_{\rm e}^*(t_0 + \tau_k) + \Phi_{t-t_0}\mathbf{P}_{t-t_0}\mathbf{S}_{t_0+\tau_k} u_{\rm e}^* \quad \forall\, t \geq t_0,\ \forall\, k \in \mathbb{N}. \tag{4.11}$$

Obviously, $u_{\rm e}^* = E^i(f \circ P^j y_{\rm e}^*) + v_{\rm e}^*$. As $\tau_k \in P_2(v_{\rm e}^*, \eta_k) \subset P(x_{\rm e}^*, 1/k) \cap P_2(y_{\rm e}^*, 1/k)$, we have

$$\|\mathbf{S}_{\tau_k} v_{\rm e}^* - v_{\rm e}^*\|_{S^2} \to 0,\ \ \|\mathbf{S}_{\tau_k} x_{\rm e}^* - x_{\rm e}^*\|_\infty \to 0 \quad \text{and} \quad \|\mathbf{S}_{\tau_k} y_{\rm e}^* - y_{\rm e}^*\|_{S^2} \to 0 \quad \text{as } k \to \infty, \tag{4.12}$$

which in turn implies that

$$\|\mathbf{S}_{\tau_k} u_{\rm e}^* - u_{\rm e}^*\|_{S^2} \to 0 \quad \text{as } k \to \infty. \tag{4.13}$$

Therefore, letting $k \to \infty$ in (4.11), we arrive at

$$x_{\rm e}^*(t) = \mathbb{T}_{t-t_0} x_{\rm e}^*(t_0) + \Phi_{t-t_0}\mathbf{P}_{t-t_0}\mathbf{S}_{t_0} u_{\rm e}^* \quad \forall\, t \geq t_0. \tag{4.14}$$

Furthermore, on $\mathbb{R}_+$, $\mathbf{S}_{t_0+\tau_k} u_{\rm e}^* = \mathbf{S}_{t_0+\tau_k} u^*$, $\mathbf{S}_{t_0+\tau_k} x_{\rm e}^* = \mathbf{S}_{t_0+\tau_k} x^*$ and $\mathbf{S}_{t_0+\tau_k} y_{\rm e}^* = \mathbf{S}_{t_0+\tau_k} y^*$, and thus, as

$$\mathbf{P}_{t-t_0}\mathbf{S}_{t_0+\tau_k} y^* = \Psi_{t-t_0}(\mathbf{S}_{t_0+\tau_k} x^*)(0) + \mathbb{G}_{t-t_0}\mathbf{P}_{t-t_0}\mathbf{S}_{t_0+\tau_k} u^* \quad \forall\, t \geq t_0,$$

we obtain

$$\mathbf{P}_{t-t_0}\mathbf{S}_{\tau_k}\mathbf{S}_{t_0} y_{\rm e}^* = \Psi_{t-t_0}(\mathbf{S}_{\tau_k}\mathbf{S}_{t_0} x_{\rm e}^*)(0) + \mathbb{G}_{t-t_0}\mathbf{P}_{t-t_0}\mathbf{S}_{\tau_k}\mathbf{S}_{t_0} u_{\rm e}^* \quad \forall\, t \geq t_0. \tag{4.15}$$

By (4.12) and (4.13),

$$\|\mathbf{S}_{\tau_k}\mathbf{S}_{t_0} u_{\rm e}^* - \mathbf{S}_{t_0} u_{\rm e}^*\|_{S^2} \to 0,\ \ \|\mathbf{S}_{\tau_k}\mathbf{S}_{t_0} x_{\rm e}^* - \mathbf{S}_{t_0} x_{\rm e}^*\|_\infty \to 0 \quad \text{and} \quad \|\mathbf{S}_{\tau_k}\mathbf{S}_{t_0} y_{\rm e}^* - \mathbf{S}_{t_0} y_{\rm e}^*\|_{S^2} \to 0 \quad \text{as } k \to \infty,$$

and thus, letting $k \to \infty$ in (4.15) leads to

$$\mathbf{P}_{t-t_0}\mathbf{S}_{t_0} y_{\rm e}^* = \Psi_{t-t_0} x_{\rm e}^*(t_0) + \mathbb{G}_{t-t_0}\mathbf{P}_{t-t_0}\mathbf{S}_{t_0} u_{\rm e}^* \quad \forall\, t \geq t_0. \tag{4.16}$$

Since $t_0 \in \mathbb{R}$ was arbitrary and $\mathbf{S}_{t_0} u_{\rm e}^* = E^i(f \circ P^j \mathbf{S}_{t_0} y_{\rm e}^*) + \mathbf{S}_{t_0} v_{\rm e}^*$, it follows from (4.14) and (4.16) that $(v_{\rm e}^*, x_{\rm e}^*, y_{\rm e}^*) \in \mathcal{BB}$.

To show that $(x_{\rm e}^*, y_{\rm e}^*)$ is the unique pair in $BC(\mathbb{R}, X) \times UL^2_{\rm loc}(\mathbb{R}, Y)$ satisfying $(v_{\rm e}^*, x_{\rm e}^*, y_{\rm e}^*) \in \mathcal{BB}$, let $(\hat{x}, \hat{y}) \in BC(\mathbb{R}, X) \times UL^2_{\rm loc}(\mathbb{R}, Y)$ be such that $(v_{\rm e}^*, \hat{x}, \hat{y}) \in \mathcal{BB}$. We have to show that $(\hat{x}, \hat{y}) = (x_{\rm e}^*, y_{\rm e}^*)$. To this end, note that, for any $\sigma \in \mathbb{R}$, the restrictions of $(\mathbf{S}_\sigma v_{\rm e}^*, \mathbf{S}_\sigma x_{\rm e}^*, \mathbf{S}_\sigma y_{\rm e}^*)$ and $(\mathbf{S}_\sigma v_{\rm e}^*, \mathbf{S}_\sigma \hat{x}, \mathbf{S}_\sigma \hat{y})$ to $\mathbb{R}_+$ are in $\mathcal{B}$. Hence, by statement (1) of Theorem 3.4, there exist $\Gamma_2 > 0$ and $\gamma > 0$ such that

$$\|(\mathbf{S}_\sigma x_{\rm e}^*)(s) - (\mathbf{S}_\sigma \hat{x})(s)\| \leq \Gamma_2 e^{-\gamma s}\|x_{\rm e}^*(\sigma) - \hat{x}(\sigma)\| \quad \forall\, s \geq 0,\ \forall\, \sigma \in \mathbb{R}. \tag{4.17}$$

Let $t \in \mathbb{R}$ and $\varepsilon > 0$. Choose $\sigma \le t$ such that

$$\Gamma_2 e^{-\gamma(t-\sigma)} \|x_{\rm e}^* - \hat{x}\|_\infty \le \varepsilon.$$

An application of (4.17) wit $s = t - \sigma$ yields

$$\|x_{\rm e}^*(t) - \hat{x}(t)\| = \|(\mathbf{S}_\sigma x_{\rm e}^*)(t-\sigma) - (\mathbf{S}_\sigma \hat{x})(t-\sigma)\| \le \Gamma_2 e^{-\gamma(t-\sigma)} \|x_{\rm e}^* - \hat{x}\|_\infty \le \varepsilon.$$

Now $t \in \mathbb{R}$ and $\varepsilon > 0$ were arbitrary, and consequently, $\hat{x} = x_{\rm e}^*$. An application of statement (4) of Theorem 3.4 (with $t_0 = 0$) to the restrictions of $(\mathbf{S}_\sigma v_{\rm e}^*, \mathbf{S}_\sigma x_{\rm e}^*, \mathbf{S}_\sigma y_{\rm e}^*)$ and $(\mathbf{S}_\sigma v_{\rm e}^*, \mathbf{S}_\sigma x_{\rm e}^*, \mathbf{S}_\sigma \hat{y})$ to $\mathbb{R}_+$, where $\sigma \in \mathbb{R}$, shows that $(\mathbf{S}_\sigma \hat{y})(t) = (\mathbf{S}_\sigma y_{\rm e}^*)(t)$ for almost every $t \ge 0$. Therefore, $\hat{y}(t) = y_{\rm e}^*(t)$ for almost every $t \ge \sigma$. Letting $\sigma \to -\infty$ yields that $\hat{y} = y_{\rm e}^*$.

Step 6: Proof of statement (4). Let $(\sigma_k)_k$ be a sequence in $\mathbb{R}$ such that $(\mathbf{S}_{\sigma_k} \widetilde{v_{\rm e}^*})_k$ converges in $AP(\mathbb{R}, L^1([0,1), U))$. By [1, Statement X on p. 34], it is sufficient to prove that the sequences $(\mathbf{S}_{\sigma_k} x_{\rm e}^*)_k$ and $(\mathbf{S}_{\sigma_k} \widetilde{y_{\rm e}^*})_k$ converge in $AP(\mathbb{R}, X)$ and $AP(\mathbb{R}, L^1([0,1), Y))$, respectively. To this end, let $\varepsilon > 0$ and set $r := 2\|x_{\rm e}^*\|_\infty = 2\|x^*\|_\infty$. Obviously, for each $k \in \mathbb{N}$, the restriction of $(\mathbf{S}_{\sigma_k} v_{\rm e}^*, \mathbf{S}_{\sigma_k} x_{\rm e}^*, \mathbf{S}_{\sigma_k} y_{\rm e}^*)$ to $\mathbb{R}_+$ is in $\mathcal{B}$. Consequently, by statements (3) and (4) of Theorem 3.4,

$$\|(\mathbf{S}_{\sigma_k} x_{\rm e}^*)(t) - (\mathbf{S}_{\sigma_\ell} x_{\rm e}^*)(t)\| \le \tilde{\Gamma}_2 \big(e^{-\tilde{\gamma} t} r + \|\mathbf{S}_{\sigma_k} v_{\rm e}^* - \mathbf{S}_{\sigma_\ell} v_{\rm e}^*\|_{S^2}\big) \quad \forall\, t \ge 0, \tag{4.18}$$

and

$$\|\mathbf{S}_{\sigma_k} y_{\rm e}^* - \mathbf{S}_{\sigma_\ell} y_{\rm e}^*\|_{S^2} \le \tilde{\Gamma} \big(\|(\mathbf{S}_{\sigma_k} x_{\rm e}^*)(0) - (\mathbf{S}_{\sigma_\ell} x_{\rm e}^*)(0)\| + \|\mathbf{S}_{\sigma_k} v_{\rm e}^* - \mathbf{S}_{\sigma_\ell} v_{\rm e}^*\|_{S^2}\big), \tag{4.19}$$

where $\tilde{\Gamma}_2$, $\tilde{\Gamma}$ and $\tilde{\gamma}$ are positive constants. Since $(\mathbf{S}_{\sigma_k} \widetilde{v_{\rm e}^*})_k$ converges in $AP(\mathbb{R}, L^1([0,1), U))$, it is clear that $(\mathbf{S}_{\sigma_k} v_{\rm e}^*)_k$ is a Cauchy sequence in $S^2(\mathbb{R}, U)$. Consequently, there exists $N \in \mathbb{N}$ such that $\tilde{\Gamma}_2 \|\mathbf{S}_{\sigma_k} v_{\rm e}^* - \mathbf{S}_{\sigma_\ell} v_{\rm e}^*\|_{S^2} \le \varepsilon/2$ for all $k, \ell \ge N$. Choosing $\tau \ge 0$ such that $\tilde{\Gamma}_2 r e^{-\tilde{\gamma}\tau} \le \varepsilon/2$, it follows from (4.18) that

$$\|(\mathbf{S}_{\sigma_k} x_{\rm e}^*)(t) - (\mathbf{S}_{\sigma_\ell} x_{\rm e}^*)(t)\| \le \varepsilon \quad \forall\, t \ge \tau,\ \forall k, \ell \ge N.$$

The function $\mathbf{S}_{\sigma_k} x_{\rm e}^* - \mathbf{S}_{\sigma_\ell} x_{\rm e}^*$ is in $AP(\mathbb{R}, X)$, and thus, invoking Lemma 4.1,

$$\|\mathbf{S}_{\sigma_k} x_{\rm e}^* - \mathbf{S}_{\sigma_\ell} x_{\rm e}^*\|_\infty \le \varepsilon \quad \forall k, \ell \ge N.$$

This shows that $(\mathbf{S}_{\sigma_k} x_{\rm e}^*)_k$ is a Cauchy sequence in $AP(\mathbb{R}, X)$ and thus converges in $AP(\mathbb{R}, X)$.

Finally, since $(\mathbf{S}_{\sigma_k} x_{\rm e}^*)_k$ and $(\mathbf{S}_{\sigma_k} v_{\rm e}^*)_k$ are Cauchy sequences in $AP(\mathbb{R}, X)$ and $S^2(\mathbb{R}, U)$, respectively, it follows from (4.19) that $(\mathbf{S}_{\sigma_k} y_{\rm e}^*)_k$ is a Cauchy sequences in $S^2(\mathbb{R}, Y)$, and hence $(\mathbf{S}_{\sigma_k} \widetilde{y_{\rm e}^*})_k$ converges in $AP(\mathbb{R}, L^1([0,1], Y))$, completing the proof. □

We continue by stating a circle-criterion version of Theorem 4.5.

Theorem 4.6. *Let $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{G})$ be a well-posed linear system, let $i, j \in \{1, 2\}$, $K_1, K_2 \in \mathcal{L}(Y^j, U^i)$ and let $v^* \in S^2(\mathbb{R}_+, U)$. Assume that $\Sigma^{ji} = (\mathbb{T}, \Phi E^i, P^j \Psi, P^j \mathbb{G} E^i)$ is optimizable and estimatable and $K_1 \in \mathbb{S}(P^j \mathbf{G} E^i)$ is an admissible feedback operator for Σ^{ji}. If $(I - K_2 P^j \mathbf{G} E^i)(I - K_1 P^j \mathbf{G} E^i)^{-1}$ is positive real and there exists $\varepsilon > 0$ such that $f : Y^j \to U^i$ satisfies*

$$\mathrm{Re}\, \langle f(z_1) - f(z_2) - K_1(z_1 - z_2), f(z_1) - f(z_2) - K_2(z_1 - z_2) \rangle \le -\varepsilon \|z_1 - z_2\|^2 \quad \forall\, (z_1, z_2) \in Y^j \times Y^j,$$

then there exists a unique pair $(x^*, y^*) \in AP(\mathbb{R}_+, X) \times S^2(\mathbb{R}_+, Y)$ such that $(v^*, x^*, y^*) \in \mathcal{B}$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $P_2(v^*, \delta) \subset P(x^*, \varepsilon) \cap P_2(y^*, \varepsilon)$ and statements (1)- (4) *of* Theorem 4.5 *hold.*

Proof. Let $v^* \in S^2(\mathbb{R}_+, U)$. Combining the methods used in the proof of [14, Corollary 4.5] with Lemma 3.2 and Proposition 3.1 shows that there exists a pair $(x, y) \in C(\mathbb{R}_+, X) \times L^2_{\rm loc}(\mathbb{R}_+, Y)$ such that $(v^*, x, y) \in \mathcal{B}$. Invoking Corollary 3.6, it is clear that statements (1)–(4) of Theorem 3.4 hold.

These formed the basis for the proof of Theorem 4.5, and the conclusions of Theorem 4.6 can now be derived by arguments identical to those used in the proof of Theorem 4.5. □

We conclude this section with a brief comparison of Theorems 4.5 and 4.6 to related results in the literature. As for the case of periodic forcing, the most relevant results in this context are [24, Theorem 4] and the first part of [39, Theorem 1], both of which are special cases of [14, Corollary 5.6] (which in turn is essentially identical to statement (2) of Theorem 4.5). Earlier contributions to the analysis of the asymptotic behaviour of Lur'e systems with almost periodic forcing can be found in [6, 26, 27, 39]. The papers [6, 26, 27] adopt an input-output approach, whilst a standard finite-dimensional state space setting is used in [39]. All of these contributions consider input signals which are almost periodic in the sense of Bohr, but do not cover the more general case of Stepanov almost periodic forcing functions. The structure of the feedback systems and the classes of underlying linear systems considered in [6, 26, 27, 39] are considerably less general than those studied in this paper (in particular, [6, 27, 39] are restricted to the single-input single-output case, that is U and Y are one-dimensional and f is a "scalar" nonlinearity). Theorems 4.5 and 4.6 can be considered as far reaching generalizations and refinements of the relevant results in [6, 26, 27, 39].

References

[1] L. Amerio and G. Prouse. *Almost Periodic Functions and Functional Equations*, Springer Verlag, New York, 1971.

[2] D. Angeli. A Lyapunov stability approach to incremental stability properties, *IEEE Trans. Automatic Control*, **37** (2002), 410–421.

[3] M. Arcak and A. Teel. Input-to-state stability for a class of Lurie systems, *Automatica*, **38** (2002), 1945–1949.

[4] H. Bart and S. Goldberg. Characterizations of almost periodic strongly continuous groups and semigroups, *Math. Ann.*, **236** (1978), 105–116.

[5] A. Bellow and V. Losert. The weighted pointwise ergodic theorem and the individual ergodic theorem along subsequences, *Trans. Am. Math. Soc.*, **288** (1985), 307–345.

[6] V.E. Běnes and I.W. Sandberg. On the response of time-variable nonlinear systems to almost periodic signals, *J. Math. Analysis and its Appl.*, **10** (1965), 245–268.

[7] H. Bohr. *Almost Periodic Functions.* Chelsea Publishing Company, New York, 1947.

[8] C. Corduneanu. *Almost Periodic Functions*, 2nd edition, Chelsea Publishing Company, New York, 1989.

[9] C. Corduneanu. *Almost Periodic Oscillations and Waves*, Springer-Verlag, New York, 2009.

[10] S.N. Dashkovskiy, D.V. Efimov and E.D. Sontag. Input-to-state stability and allied system properties, *Autom. Remote Control*, **72** (2011), 1579–1614.

[11] C.A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties*, Academic Press, New York, 1975.

[12] A.M. Fink. *Almost Periodic Differential Equations*, Springer-Verlag, Berlin, 1974.

[13] C. Guiver, H. Logemann and M.R. Opmeer. Transfer functions of infinite-dimensional systems: positive realness and stabilization, *Math. Control Signals Systems*, **29** (2017), https://doi.org/10.1007/s00498-017-0203-z.

[14] C. Guiver, H. Logemann and M.R. Opmeer. Infinite-dimensional Lur'e systems: input-to-state stability and convergence properties, *SIAM J. Control Optim.*, **57** (2019), 334–365.

[15] W.M. Haddad and V. Chellaboina. *Nonlinear Dynamical Systems and Control*, Princeton University Press, Princeton, 2008.

[16] D. Hinrichsen and A.J. Pritchard. Destabilization by output feedback, *Differ. Integral Eqn*, **5** (1995), 357–386.

[17] D. Hinrichsen and A.J. Pritchard. *Mathematical Systems Theory I,* Springer-Verlag, Berlin, 2005.

[18] B. Jayawardhana, H. Logemann and E.P. Ryan. Input-to-state stability of differential inclusions with applications to hysteretic and quantized feedback systems, *SIAM J. Control Optim.*, **48** (2009), 1031–1

[19] B. Jayawardhana, H. Logemann and E.P. Ryan. The circle criterion and input-to-state stability: new perspectives on a classical result, *IEEE Control Systems Mag.*, **31** (2011), 32–67.

[20] K.M. Mikkola. State-feedback stabilization of well-posed linear systems, *Integral Equations and Operator Theory*, **55** (2006), 249–271.

[21] W.M. Ruess and W.H. Summers. Compactness in spaces of vector-valued continuous functions and asymptotic almost periodicity, *Math. Nachr.*, **135** (1988), 7–33.

[22] D. Salamon. Infinite-dimensional linear systems with unbounded control and observation: a functional analytic approach. *Trans. Amer. Math. Soc.*, **300** (1987), 383–431.

[23] D. Salamon. Realization theory in Hilbert space. *Math. Systems Theory*, **21** (1989), 147–164.

[24] I.W. Sandberg. Some results on the theory of physical systems governed by nonlinear functional equations, *Bell System Tech. J.*, **44** (1965), 871–898.

[25] I.W. Sandberg. Criteria for the response of nonlinear systems to be L-asymptotically periodic, *Bell System Tech. J.*, **60** (1981), 2359–2371.

[26] I.W. Sandberg. The circle criterion and almost periodic inputs, *IEEE Trans. Circuits Systems I Fund. Theory Appl.*, **47**, 825–829, 2000.

[27] I. W. Sandberg and G.J.J. van Zyl. The spectral coefficients of the response of nonlinear systems to asymptotically almost periodic inputs, *IEEE Trans. Circuits Systems I Fund. Theory Appl.*, **48**, 170–176, 2001.

[28] E. Sarkans and H. Logemann. Input-to-state stability of Lur'e systems, *Math. Control Signals Systems*, **27** (2015), 439–465.

[29] E.D. Sontag. Smooth stabilization implies coprime factorization, *IEEE Trans. Automatic Control*, **34** (1989), 435–443.

[30] E.D. Sontag. Input to state stability: basic concepts and results, in P. Nistri and G. Stefani (eds.) *Nonlinear and Optimal Control Theory*, pp. 163-220, Springer Verlag, Berlin, 2006.

[31] O.J. Staffans. *Well-Posed Linear Systems*, Cambridge University Press, Cambridge, 2005.

[32] O.J. Staffans and G. Weiss. Transfer functions of regular linear systems. Part II: the systems operator and the Lax-Phillips semigroup, *Trans. American Math. Soc.*, **354**, 3229-3262.

[33] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*, Birkhäuser, Berlin, 2009.

[34] M. Tucsnak and G. Weiss. Well-posed systems — the LTI case and beyond, *Automatica*, **50** (2014), 1757–1779.

[35] M. Vidyasagar. *Nonlinear Systems Analysis*, 2nd edition, Prentice-Hall, Englewood Cliffs, NJ,1993.

[36] G. Weiss. Transfer functions of regular linear systems. Part I: characterizations of regularity *Trans. American Math. Soc.*, **342** (1994), 827–854.

[37] G. Weiss. Regular linear systems with feedback, *Math. Control Signals Systems*, **7** (1994), 23–57.

[38] G. Weiss and R. Rebarber. Optimizability and estimatability for infinite-dimensional linear systems, *SIAM J. Control Optim.*, **39** (2000), 1204–1232.

[39] V.A. Yakubovich. Matrix inequalities method in stability theory for nonlinear control systems: I. Absolute stability of forced vibrations, *Autom. Remote Control*, **7** (1964), 905–917.

[40] V.A. Yakubovich, G.A. Leonov and A.Kh. Gelig. *Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities*, World Scientific, Singapore, 2004.