Semi-global incremental input-to-state stability of discrete-time Lur'e systems

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Abstract

We present sufficient conditions for semi-global incremental input-to-state stability of a class of forced discrete-time Lur'e systems. The results derived are reminiscent of well-known absolute stability criteria such as the small gain theorem and the circle criterion. We derive a natural sufficient condition which guarantees that asymptotically (almost) periodic inputs generate asymptotically (almost) periodic state trajectories. As a corollary, we obtain sufficient conditions for the converging-input converging-state property to hold.

Keywords: Absolute stability, almost periodicity, discrete-time systems, incremental stability, Lur'e systems.
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1. Introduction

We consider stability and convergence properties of the feedback interconnection shown in Figure 1.1, which comprises a linear system in the forward path and a static nonlinearity in the feedback path. Such systems are often termed Lur'e systems, and their stability and convergence properties is a well-researched area. The study of the stability of Lur'e systems is called absolute stability theory, which seeks to conclude stability of the feedback system given in Figure 1.1, via the interplay of frequency-domain properties of the linear component and sector properties of the nonlinearity. Lyapunov approaches have been used to deduce global asymptotic stability of unforced Lur'e systems (see, for example, [16, 23, 24]), and input-output methods, pioneered by Sandberg and Zames in the 1960s, have been used to infer L^2 and L^∞ stability (see, for example, [11, 33]). More recently, forced Lur'e systems have been analysed in the context of input-to-state stability (ISS) theory, with attention focussed on the extent to which results from classical absolute stability theory can be generalised to ensure ISS [3, 17, 18, 27, 28, 29]. Originating in the paper [31], ISS is a property of general controlled nonlinear systems and, roughly, ensures a natural boundedness property of the state, in terms of initial conditions and inputs. ISS theory has been, and still is, a very active research area, see, in addition to the previous references, for example [9, 19, 20, 21] and the survey papers [8, 32].

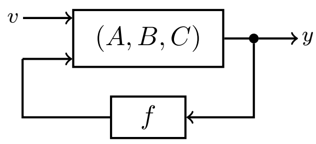

Figure 1.1: Lur'e system with linear part (A, B, C), nonlinearity f, output y and input v

Incremental ISS is concerned with bounding the difference of two state trajectories in terms of the difference of initial conditions and the difference of inputs. For background information regarding incremental stability notions for general nonlinear systems, we refer the reader to [2]. Related ideas, which have been explored in the contexts of contraction methods and convergent systems, can be found in [1, 22, 25] and the references therein. Recently, in [13, 14], sufficient conditions have been

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determined which guarantee that certain infinite-dimensional Lur'e systems are *exponentially incrementally ISS*. Moreover, in [13, 14], exponential incremental ISS is used to infer convergence properties such as the *converging-input converging-state* (CICS) property (see, for example, [5]) and the asymptotic periodicity of the response to asymptotically periodic inputs.

Here we consider discrete-time, finite-dimensional Lur'e systems and the incremental stability notion termed *semi-global incremental ISS*, which is considerably weaker than exponential incremental ISS. Our main result is Theorem 3.2 which provides sufficient conditions, reminiscent of well-known absolute stability criteria, for semi-global incremental ISS. Theorem 3.2 underpins our subsequent investigation of the asymptotic properties of the response of discrete-time Lur'e systems to asymptotically almost periodic inputs. Theorem 4.3 provides sufficient conditions under which, for every almost periodic input v^{ap}, there exists a unique almost periodic state trajectory x^{ap} such that any state trajectory x generated by an input of the form $v^{\mathrm{ap}} + w$ with $\lim_{t\to\infty} w(t) = 0$ satisfies $\lim_{t\to\infty}(x(t) - x^{\mathrm{ap}}(t)) = 0$. In particular, under the assumptions of Theorem 4.3, state trajectories generated by asymptotically almost periodic inputs are asymptotically almost periodic. Sufficient conditions for the CICS property are obtained as a corollary. The relation between the present paper and our earlier work [5, 13, 14] is discussed in Remarks 3.3, 3.4 and 4.7.

The layout of the paper is as follows. In Section 2, we gather some preliminary results and definitions and, in Section 3, these are used to derive our main results. The response to asymptotically almost periodic inputs is addressed in Section 4, and an example is presented in Section 5.

Notation. Throughout, we denote the set of integers by $\mathbb{Z}$, the set of positive integers by $\mathbb{N}$ and we define $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We denote the sets of real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively, and define $\mathbb{E} := \{z \in \mathbb{C} : |z| > 1\}$ and $\mathbb{R}_+ := [0, \infty)$. We fix $n, m, p \in \mathbb{N}$. We let $\mathbb{C}^{m\times p}$ denote the set of complex $m \times p$ matrices, which is a normed space when equipped with the usual induced operator norm

$$\|K\| := \sup_{\|\xi\|=1} \|K\xi\| \quad \forall\, K \in \mathbb{C}^{m\times p}.$$

Here the norms in $\mathbb{C}^m$ and $\mathbb{C}^p$ are the 2-norms. Furthermore, for $r > 0$, we define

$$\mathbb{B}_{\mathbb{C}}(K, r) := \{L \in \mathbb{C}^{m\times p} : \|K - L\| < r\}.$$

A square matrix A is said to be Schur if the eigenvalues of A are contained in the set $\{z \in \mathbb{C} : |z| < 1\}$. The transpose and conjugate transpose of A are denoted by A^T and A^*, respectively.

The Hardy space $H^\infty_{p\times m}$ is the set of all holomorphic functions $\mathbf{H} : \mathbb{E} \to \mathbb{C}^{p\times m}$ with

$$\|\mathbf{H}\|_{H^\infty} := \sup_{z\in\mathbb{E}} \|\mathbf{H}(z)\| < \infty.$$

As usual, for $Z = \mathbb{Z}$ or $\mathbb{Z}_+$, $(\mathbb{R}^n)^Z$ is the vector space of functions $v : Z \to \mathbb{R}^n$, and $\ell^\infty(Z, \mathbb{R}^n)$ is the space of all $v \in (\mathbb{R}^n)^Z$ such that $\|v\|_{\ell^\infty} := \sup_{t\in Z} \|v(t)\| < \infty$. We further define $\mathcal{K}$ as the set of strictly increasing, continuous functions $\mathbb{R}_+ \to \mathbb{R}_+$ that are zero at zero. The subset of $\mathcal{K}$ comprising all unbounded functions is denoted $\mathcal{K}_\infty$. We let $\mathcal{KL}$ stand for the set of functions $\psi : \mathbb{R}_+ \times \mathbb{Z}_+ \to \mathbb{R}_+$ such that for each fixed $t \in \mathbb{Z}_+$, $\psi(\cdot, t) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\psi(s, \cdot)$ is non-increasing and $\lim_{t\to\infty} \psi(s, t) = 0$.

For $t \in \mathbb{R}$, we define $\lfloor t \rfloor$ to be the greatest integer less than or equal to t and $\lceil t \rceil$ to be the smallest integer greater than or equal to t. Moreover, we define $\underline{t} := \{0, 1, \ldots, t\}$ and $\overline{t} := \{t, t+1, \ldots\}$. Finally, for $\tau \in \mathbb{Z}_+$, the left-shift operator $\Lambda_\tau : (\mathbb{R}^n)^{\mathbb{Z}_+} \to (\mathbb{R}^n)^{\mathbb{Z}_+}$ is defined by $(\Lambda_\tau v)(t) := v(t + \tau)$ for every $t \in \mathbb{Z}_+$ and every $v \in (\mathbb{R}^n)^{\mathbb{Z}_+}$.

2. Preliminaries

We begin with some preliminary definitions and results regarding the following controlled and observed linear difference equation

$$x^+ = Ax + Bu + v, \quad y = Cx, \tag{2.1}$$

where $(A, B, C) \in \mathbb{L} := \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$, $u \in (\mathbb{R}^m)^{\mathbb{Z}_+}$ and $v \in (\mathbb{R}^n)^{\mathbb{Z}_+}$. Here and throughout, $x^+ := \Lambda_1 x$ for $x \in (\mathbb{R}^n)^{\mathbb{Z}_+}$. We denote the transfer function of (A, B, C) by $\mathbf{G}$, that is, $\mathbf{G}(z) = C(zI - A)^{-1}B$. For $K \in \mathbb{C}^{m\times p}$, we define $A^K := A + BKC$ and denote the transfer function of (A^K, B, C) by $\mathbf{G}^K$. It is easily verified that

$$\mathbf{G}^K(z) = \mathbf{G}(z)(I - K\mathbf{G}(z))^{-1}.$$

For $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, we denote the set of stabilising output feedback matrices (over $\mathbb{F}$) for (A, B, C) by

$\mathbb{S}_{\mathbb{F}}(\mathbf{G})$, that is,

$$\mathbb{S}_{\mathbb{F}}(\mathbf{G}) := \left\{K \in \mathbb{F}^{m\times p} : \mathbf{G}^K \in H^\infty_{p\times m}\right\}.$$

Furthermore, for $K \in \mathbb{C}^{m\times p}$ and $r > 0$, we obtain from [29, Lemma 6] that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ if, and only if, $\|\mathbf{G}^K\|_{H^\infty} \le 1/r$.

Application of the feedback law $u = f(y)$ to (2.1), where $f : \mathbb{R}^p \to \mathbb{R}^m$ is a nonlinearity, leads to the closed-loop system

$$x^+ = Ax + Bf(Cx) + v. \tag{2.2}$$

We define the behaviour of (2.2) by

$$\mathcal{B}_f(A, B, C) := \big\{(v, x) \in (\mathbb{R}^n)^{\mathbb{Z}_+} \times (\mathbb{R}^n)^{\mathbb{Z}_+} : (v, x) \text{ satisfies } (2.2)\big\},$$

and note that $(v, x) \in \mathcal{B}_f(A, B, C)$ if, and only if, $(v, x) \in \mathcal{B}_{f-K}(A^K, B, C)$, so-called loop shifting. For ease of notation, from hereon in we shall write $\mathcal{B} = \mathcal{B}_f(A, B, C)$ when no ambiguity shall arise. We note that $\mathcal{B}$ is left-shift invariant, that is,

$$(v, x) \in \mathcal{B} \implies (\Lambda_\sigma v, \Lambda_\sigma x) \in \mathcal{B} \quad \forall\, \sigma \in \mathbb{Z}_+. \tag{2.3}$$

For given $(A, B, C) \in \mathbb{L}$ and $K \in \mathbb{S}_{\mathbb{R}}(\mathbf{G})$, we shall repeatedly make use of the following assumption throughout the rest of this paper:

$$\left.\begin{array}{l}(A, B, C) \text{ is (i) controllable and observable,}\\ \text{or (ii) stabilisable and detectable, and}\\ \qquad \min\limits_{|z|=1} \|\mathbf{G}^K(z)\| < \|\mathbf{G}^K\|_{H^\infty}.\end{array}\right\} \quad \text{(A)}$$

Assumption (A) is the key hypothesis underlying the ISS theory of discrete-time Lur'e systems developed in [29]. It is required for Lemma 2.2, which in turn underpins Theorem 3.2, the main result on semi-global incremental ISS. The last condition in (A) means that $\|\mathbf{G}^K\|$ is not constant on the unit circle and guarantees the existence of suitable solutions to the discrete-time bounded-real Lur'e equations in the absence of controllability and observability, cf. [29, Lemma 3] and [34]. Interestingly, the continuous-time version of the assumption on $\|\mathbf{G}^K\|$ in (A) is not required in the continuous-time setting, see [27, Lemma 2.2] and [29, pp. 1742/1743] for more details.

We conclude the section with two technical results. The first is elementary and so we do not provide a proof.

Lemma 2.1. *Let $h : \mathbb{R}^p \to \mathbb{R}^m$ be continuous, and $W \subseteq \mathbb{R}^p$ be compact. Then $\xi \mapsto \sup_{w\in W} \|h(\xi + w) - h(w)\|$ is a continuous function.*

The second result is an input-to-state stability criterion for

$$x(t+1) = Ax(t) + Bg(t, Cx(t)) + v(t) \quad \forall\, t \in \mathbb{Z}_+, \tag{2.4}$$

a controlled Lur'e system with time-varying nonlinearity.

Lemma 2.2. *Let $(A, B, C) \in \mathbb{L}$, $K \in \mathbb{S}_{\mathbb{R}}(\mathbf{G})$ and $\alpha \in \mathcal{K}_\infty$. If assumption* (A) *holds, then $r := 1/\|\mathbf{G}^K\|_{H^\infty} < \infty$ and there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that, for all $g : \mathbb{Z}_+ \times \mathbb{R}^p \to \mathbb{R}^m$ satisfying, for all $\xi \in \mathbb{R}^p$,*

$$\sup_{t\in\mathbb{Z}_+} \|g(t, \xi) - K\xi\| \le r\|\xi\| - \alpha(\|\xi\|), \tag{2.5}$$

and for every $(v, x) \in (\mathbb{R}^n)^{\mathbb{Z}_+} \times (\mathbb{R}^n)^{\mathbb{Z}_+}$ satisfying (2.4)*,*

$$\|x(t)\| \le \psi(\|x(0)\|, t) + \phi\Big(\sup_{s\in\underline{t-1}} \|v(s)\|\Big) \quad \forall\, t \in \mathbb{N}.$$

Lemma 2.2 extends [29, Theorem 13] to the time-varying case. The proof is a straightforward generalisation of the proof of [29, Theorem 13] and we therefore omit the details.

3. Incremental stability properties

The present section contains our main stability results. The stability notion we consider is so-called semi-globally incrementally input-to-state stable.

Definition 3.1. Let $(A, B, C) \in \mathbb{L}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$. We say that (2.2) is *semi-globally incrementally input-to-state stable* if for any $R > 0$, there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that, for all $(v_i, x_i) \in \mathcal{B}$ with $\|x_i(0)\| + \|v_i\|_{\ell^\infty} \le R$, $i = 1, 2$,

$$\|x_1(t) - x_2(t)\| \le \psi\left(\|x_1(0) - x_2(0)\|, t\right) + \phi\Big(\sup_{s\in\underline{t-1}} \|v_1(s) - v_2(s)\|\Big) \;\forall\, t \in \mathbb{N}. \tag{3.1}$$

Although this stability notion is semi-global, it is suitable for almost all practical applications, as all relevant initial conditions and inputs are likely to have their norm bounded by some $R > 0$. We refer the reader to papers such as [2, 13, 14] for varying notions of global incremental stability.

The following theorem is the main result of the paper.

Theorem 3.2. *Let* $(A, B, C) \in \mathbb{L}$*,* $f : \mathbb{R}^p \to \mathbb{R}^m$ *and* $K \in \mathbb{S}_{\mathbb{R}}(\mathbf{G})$*. If* (A) *holds,* f *satisfies*

$$\|f(\xi + \zeta) - f(\zeta) - K\xi\| < r\|\xi\| \quad \forall\, \xi, \zeta \in \mathbb{R}^p,\ \xi \neq 0, \tag{3.2}$$

where $r := 1/\|\mathbf{G}^K\|_{H^\infty}$*, and there exists* $\eta \in \mathbb{R}^p$ *such that*

$$r\|\xi - \eta\| - \|f(\xi) - f(\eta) - K(\xi - \eta)\| \to \infty \quad \text{as } \|\xi\| \to \infty\,, \tag{3.3}$$

then (2.2) *is semi-globally incrementally input-to-state stable.*

In the next two remarks we provide some commentary on Theorem 3.2, and compare and contrast our present results to those in [5, 13, 14].

Remark 3.3. (i) Under the assumptions of Theorem 3.2, it follows from [5, Lemma 4.2], that for every $\zeta \in \mathbb{R}^p$, there exists $\alpha_\zeta \in \mathcal{K}_\infty$ such that

$$\|f(\xi+\zeta)-f(\zeta)-K\xi\| \leq r\|\xi\|-\alpha_\zeta(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p.$$

In particular, if $f(0) = 0$, then

$$\|f(\xi) - K\xi\| \leq r\|\xi\| - \alpha_0(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p,$$

which we recognise as the main assumption of [29, Theorem 13]. Hence, in this case, the assumptions of Theorem 3.2 guarantee input-to-state stability of (2.2) (see, for example, [19, 29, 31]). The paper [29] does not consider notions of incremental stability.

(ii) Note that condition (3.2) can be written in the form, for all $\xi, \zeta \in \mathbb{R}^p$, $\xi \neq 0$

$$\|\mathbf{G}^K\|_{H^\infty} \frac{\|f(\xi + \zeta) - f(\zeta) - K\xi\|}{\|\xi\|} < 1\,, \tag{3.4}$$

which can be viewed as a small incremental gain condition. However, it is not excluded that there exists $\zeta \in \mathbb{R}^p$ such that the left-hand side of (3.4) converges to 1 as $\|\xi\| \to \infty$ or $\|\xi\| \to 0$, and so, (3.4) is not a small incremental gain condition in the sense of the classical input-output theory of feedback systems [11, 33]. Finally, note that even if (3.3) holds for some $\eta \in \mathbb{R}^p$, then the left-hand side of (3.4) with $\zeta = \eta$ may not be bounded away from 1.

(iii) Although (A) guarantees that $\|\mathbf{G}^K\|_{H^\infty} > 0$, we highlight that, by [13, Remark 3.4], the conclusions of Theorem 3.2 remain valid in the situation wherein $\|\mathbf{G}^K\|_{H^\infty} = 0$ (which is equivalent to $\mathbf{G} = 0$), provided that f is globally Lipschitz. More precisely, if $(A, B, C) \in \mathbb{L}$ is stabilisable and detectable, $\mathbf{G} = 0$ and $f : \mathbb{R}^p \to \mathbb{R}^m$ is globally Lipschitz, then (2.2) is semi-globally incrementally input-to-state stable. ◊

Remark 3.4. The paper [5] considers the CICS property for finite-dimensional, continuous-time Lur'e systems, and does not consider incremental stability notions. The approach in [5] is based on ISS arguments, rather than the semi-global incremental ISS framework adopted here. We note that the assumptions (3.2) and (3.3) on the nonlinear term f are the same as those in [5, Theorem 4.3 (2)] which guarantee the CICS property.

The papers [13, 14] derive incremental ISS properties for large classes of infinite-dimensional discrete-time and continuous-time systems, respectively, under the assumption that there exists $\delta > 0$ such that, for all $\xi, \zeta \in \mathbb{R}^p$

$$\|f(\xi + \zeta) - f(\zeta) - K\xi\| \leq (r - \delta)\|\xi\|\,. \tag{3.5}$$

Obviously, if condition (3.5) is satisfied for some $\delta > 0$, then (3.2) and (3.3) hold, but (3.5) is significantly more restrictive then the combination of (3.2) and (3.3) (see also Example 5.1). Note that if (3.5) is satisfied for some $\delta > 0$, then

$$\|\mathbf{G}^K\|_{H^\infty} \sup_{\substack{\xi,\zeta\in\mathbb{R}^p \\ \xi\neq 0}} \frac{\|f(\xi + \zeta) - f(\zeta) - K\xi\|}{\|\xi\|} < 1,$$

which is a small incremental gain condition in the sense of classical input-output theory [11, 33].

A key difference between the present work and [13, 14] is that Theorem 3.2, via Lemma 2.2 and [29], is underpinned by an ISS Lyapunov theory for Lur'e systems, whilst, in the absence of such a theory in the infinite-dimensional case, [13, 14] is based on small-gain and exponential weighting methods. ◊

Proof of Theorem 3.2. We seek to apply Lemma 2.2. By invoking (3.3) and [5, Statement (1) of Lemma 4.2], we obtain that, for all $\zeta \in \mathbb{R}^p$,

$$r\|\xi-\zeta\|-\|f(\xi)-f(\zeta)-K(\xi-\zeta)\| \to \infty \text{ as } \|\xi\| \to \infty\,.$$

Consequently,

$$r\|\xi\|-\|f(\xi)-f(0)-K\xi\| \to \infty \text{ as } \|\xi\| \to \infty\,. \quad (3.6)$$

By setting $\tilde{f}(\xi) := f(\xi) - f(0)$ for all $\xi \in \mathbb{R}^p$, it is clear from (3.2) and (3.6) that $\tilde{f} - K : \mathbb{R}^p \to \mathbb{R}^m$ is continuous, $\|\tilde{f}(\xi) - K\xi\| < r\|\xi\|$ for all $\xi \in \mathbb{R}^p\backslash\{0\}$, and $r\|\xi\| - \|\tilde{f}(\xi) - K\xi\| \to \infty$ as $\|\xi\| \to \infty$. Hence, $\tilde{f} - K$ satisfies the hypotheses of [5, Statement 2 of Lemma 4.2] and so, there exists $\alpha_0 \in \mathcal{K}_\infty$ such that

$$\|\tilde{f}(\xi) - K\xi\| \le r\|\xi\| - \alpha_0(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p\,. \quad (3.7)$$

We shall use (3.7) to establish the existence of $\alpha_1 \in \mathcal{K}_\infty$ and $s_1 > 0$ such that

$$\|f(\xi) - K\xi\| \le r\|\xi\| - \alpha_1(\|\xi\|) \qquad \forall\, \xi \in \mathbb{R}^p \;\text{ s.t. } \|\xi\| \ge s_1\,. \quad (3.8)$$

For which purpose, we note that, for all $\xi \in \mathbb{R}^p$,

$$\begin{aligned}\|f(\xi) - K\xi\| &\le \|\tilde{f}(\xi) - K\xi\| + \|f(0)\| \\ &\le r\|\xi\| - \alpha_0(\|\xi\|) + \|f(0)\|\,,\end{aligned}$$

where we have used (3.7). Defining $\tilde{\alpha} \in \mathcal{K}_\infty$ by

$$\tilde{\alpha}(s) := \begin{cases}\alpha_0(s), & \text{if } s \ge 1,\\ \alpha_0(1)s, & \text{if } 0 \le s < 1,\end{cases}$$

we obtain that, for all $\xi \in \mathbb{R}^p$ such that $\|\xi\| \ge 1$,

$$\|f(\xi) - K\xi\| \le r\|\xi\| - \tilde{\alpha}(\|\xi\|) + \|f(0)\|.$$

Therefore, if we let $s_1 \ge 1$ be such that $\tilde{\alpha}(s) > \|f(0)\|$ for all $s \ge s_1$, and define $\alpha_1 \in \mathcal{K}_\infty$ by

$$\alpha_1(s) := \begin{cases}\tilde{\alpha}(s) - \|f(0)\|, & \text{if } s \ge s_1\\ (\tilde{\alpha}(s_1) - \|f(0)\|)s/s_1, & \text{if } 0 \le s < s_1,\end{cases}$$

we obtain that (3.8) holds.

Next, fix $R > 0$ and combine (3.8) with [29, Corollary 17] to obtain the existence of a constant $\rho > 0$ such that, for all $(v, x) \in \mathcal{B}$ with $\|x(0)\| + \|v\|_{\ell^\infty} \le R$,

$$\|Cx(t)\| \le \rho \quad \forall\, t \in \mathbb{Z}_+\,. \quad (3.9)$$

We set $W := \{\xi \in \mathbb{R}^p : \|\xi\| \le \rho\} \subseteq \mathbb{R}^p$ and claim that

$$r\|\xi\| - \sup_{w\in W} \|f(\xi + w) - f(w) - K\xi\| \to \infty \qquad \text{as } \|\xi\| \to \infty\,. \quad (3.10)$$

To avoid interruption of the argument, we relegate the validation of (3.10) to the end of the proof.

Invoking (3.2), the continuity of f and compactness of W, we conclude that, for all $\xi \in \mathbb{R}^p\backslash\{0\}$,

$$\sup_{w\in W} \|f(\xi + w) - f(w) - K\xi\| < r\|\xi\|\,. \quad (3.11)$$

By Lemma 2.1, the function $\xi \mapsto \sup_{w\in W} \|f(\xi + w) - f(w) - K\xi\|$ is continuous which, in conjunction with (3.10) and (3.11) and an application of [5, Statement (2) of Lemma 4.2], shows that there exists $\alpha \in \mathcal{K}_\infty$ such that, for all $\xi \in \mathbb{R}^p$

$$\sup_{w\in W} \|f(\xi + w) - f(w) - K\xi\| \le r\|\xi\| - \alpha(\|\xi\|)\,. \quad (3.12)$$

Let $(v_i, x_i) \in \mathcal{B}$ with $\|x_i(0)\| + \|v_i\|_{\ell^\infty} \le R$, $i = 1, 2$, and define $g : \mathbb{Z}_+ \times \mathbb{R}^p \to \mathbb{R}^m$ by

$$g(t, \xi) := f\left(\xi + Cx_2(t)\right) - f\left(Cx_2(t)\right) \qquad \forall\, (t, \xi) \in \mathbb{Z}_+ \times \mathbb{R}^p\,.$$

Note that $x := x_1 - x_2$ and $v := v_1 - v_2$ satisfy (2.4). Moreover, from (3.9) and (3.12), we see that (2.5) holds. We now invoke Lemma 2.2 to obtain the existence of $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that

$$\|x(t)\| \le \psi(\|x(0)\|, t) + \phi\Big(\sup_{s\in\underline{t-1}} \|v(s)\|\Big) \quad \forall\, t \in \mathbb{N}\,.$$

Since ψ and ϕ depend only on (A, B, C) and α (and not on (v_1, x_1) or (v_2, x_2)), we see that this gives semi-global incremental input-to-state stability.

All that is left to prove is (3.10). Since f is continuous, and W is compact, it follows that for each $\xi \in \mathbb{R}^p$, there exists $w_\xi \in W$ such that

$$\begin{aligned}&\sup_{w\in W} \|f(\xi + w) - f(w) - K\xi\| \\ &= \|f(\xi + w_\xi) - f(w_\xi) - K\xi\|\,.\end{aligned} \quad (3.13)$$

In particular, $\|w_\xi\| \le \rho$ for all $\xi \in \mathbb{R}^p$. Thus, we estimate, by use of (3.7), that

$$\begin{aligned}&\|f(\xi + w_\xi) - f(w_\xi) - K\xi\| \\ &\le \|f(\xi + w_\xi) - f(0) - K(\xi + w_\xi)\| + \|f(0)\| \\ &\quad + \|Kw_\xi\| + \|f(w_\xi)\| \\ &\le r\|\xi\| + r\|w_\xi\| - \alpha_0(\|\xi + w_\xi\|) + \|f(0)\| \\ &\quad + \|K\|\|w_\xi\| + \|f(w_\xi)\|\,.\end{aligned} \quad (3.14)$$

By the reverse triangle inequality, we have that

$$\alpha_0(\|\xi + w_\xi\|) \ge \alpha_0(\|\xi\| - \|w_\xi\|) \qquad \forall\, \xi \in \mathbb{R}^p,\ \|\xi\| \ge \|w_\xi\|\,. \quad (3.15)$$

Hence, combining (3.13)–(3.15) gives

$$\begin{aligned}
&r\|\xi\| - \sup_{w\in W}\|f(\xi+w) - f(w) - K\xi\| \\
&\geq -r\|w_\xi\| + \alpha_0(\|\xi\| - \|w_\xi\|) - \|f(0)\| \\
&\quad - \|K\|\|w_\xi\| - \|f(w_\xi)\| \\
&\to \infty \quad \text{as } \|\xi\| \to \infty,
\end{aligned}$$

establishing (3.10). □

Remark 3.5. The above and subsequent results can be generalised to systems of the form

$$x^+ = Ax + Bu + v, \quad y = Cx, \quad u = f(y+w),$$

where $w \in (\mathbb{R}^p)^{\mathbb{Z}_+}$ is an output disturbance. This more general system can be analysed by using methods similar to those employed here, and so, in the interest of brevity, we do not give formal statements and instead refer the reader to the forthcoming thesis [12]. ◊

We conclude this section with a corollary of Theorem 3.2. The result presents sufficient conditions, reminiscent of the well-known circle-criterion (see, for example, [23]), which guarantee that (2.2) is semi-globally incrementally input-to-state stable. Before giving this result, we recall that a $\mathbb{C}^{m\times m}$-valued rational function $\mathbf{H}$ is *positive real* if $\mathbf{H}(z) + \mathbf{H}(z)^*$ is positive semi-definite for every $z \in \mathbb{E}$ which is not a pole of $\mathbf{H}$.

Corollary 3.6. *Let $(A, B, C) \in \mathbb{L}$, $f : \mathbb{R}^p \to \mathbb{R}^m$ and $K_1, K_2 \in \mathbb{R}^{m\times p}$. Assume that $\mathbf{H} := (I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1}$ is positive real and that either (A, B, C) is* (i) *controllable and observable, or* (ii) *stabilisable and detectable and there exists $z \in \mathbb{C}$ such that $|z| = 1$ and $\mathbf{H}(z) + \mathbf{H}(z)^*$ is positive definite. If, in addition,*

$$\langle f(\xi+\zeta) - f(\zeta) - K_1\xi, f(\xi+\zeta) - f(\zeta) - K_2\xi\rangle < 0 \quad \forall\, \xi, \zeta \in \mathbb{R}^p,\ \xi \neq 0, \tag{3.16}$$

and there exists $\eta \in \mathbb{R}^p$ such that

$$\frac{1}{\|\xi\|}\langle f_\eta(\xi) - K_1\xi, f_\eta(\xi) - K_2\xi\rangle \to -\infty \quad \text{as } \|\xi\| \to \infty,$$

where $f_\eta(\xi) := f(\xi + \eta) - f(\eta)$, then (2.2) *is semi-globally incrementally input-to-state stable.*

The following proof uses ideas and methods from the proofs of [5, Corollary 4.15] and [29, Corollary 11].

Proof of Corollary 3.6. We begin by defining

$$L := \frac{1}{2}(K_1 - K_2) \quad \text{and} \quad M := \frac{1}{2}(K_1 + K_2).$$

We then obtain that, for all $\xi, \zeta \in \mathbb{R}^p$,

$$\begin{aligned}
&\langle f(\xi+\zeta) - f(\zeta) - K_1\xi, f(\xi+\zeta) - f(\zeta) - K_2\xi\rangle \\
&= \|f(\xi+\zeta) - f(\zeta) - M\xi\|^2 - \|L\xi\|^2.
\end{aligned}$$

By combining this with (3.16), it can easily be seen that $\ker L = \{0\}$. Subsequently, L^TL is invertible and we define $\tilde{L} := (L^TL)^{-1}L^T$, which is a left-inverse of L. Now, we highlight that $(v, x) \in \mathcal{B}$ if, and only if, (v, x) satisfies

$$x^+ = A^{K_1}x + Bg(LCx) + v, \tag{3.17}$$

where $g : \mathbb{R}^m \to \mathbb{R}^m$ is defined by $g(\xi) := f(\tilde{L}\xi) - K_1\tilde{L}\xi$ for all $\xi \in \mathbb{R}^m$. To complete the proof, it is therefore sufficient to show that (3.17) is semi-globally incrementally input-to-state stable. To do so, we shall show that the assumptions of Theorem 3.2 are satisfied in the context of (3.17). To this end, we set $K = -L\tilde{L}$ and define $\mathbf{F}$ to be the transfer function of (A^{K_1}, B, LC), the linear system underlying the Lur'e system (3.17), that is,

$$\mathbf{F}(z) := LC(zI - A^{K_1})^{-1}B = L\mathbf{G}^{K_1}(z).$$

By invoking arguments similar to those used in the proof of [29, Corollary 11], it can be shown that $1/\|\mathbf{F}^K\|_{H^\infty} \geq 1$ where $\mathbf{F}^K = \mathbf{F}(I - K\mathbf{F})^{-1}$ and that (A) holds in the context of (3.17). Furthermore, it is also easy to show, again by following similar arguments to those seen in proof of [29, Corollary 11], that

$$\|g(\xi+\zeta) - g(\zeta) - K\xi\| < \|\xi\| \quad \forall\, \xi, \zeta \in \mathbb{R}^m,\ \xi \neq 0,$$

and that there exists $\beta \in \mathcal{K}_\infty$ such that

$$\|g(\xi+L\eta) - g(L\eta) - K\xi\| \leq \|\xi\| - \beta(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^m.$$

We therefore may invoke Theorem 3.2 to deduce that (3.17), and hence (2.2), is semi-globally incrementally input-to-state stable. □

4. Lur'e systems subject to almost periodic forcing

In this section, we use Theorem 3.2 to investigate the response of discrete-time Lur'e systems to asymptotically almost periodic forcing. In particular, we will show that state trajectories corresponding to asymptotically almost periodic inputs are asymptotically almost periodic.

We begin by presenting some relevant background material. Let $Z = \mathbb{Z}$ or $\mathbb{Z}_+$. A set $S \subseteq Z$ is called *relatively dense* (in Z) if there exists $L \in \mathbb{N}$ such that

$$\{a, \dots, a+L\} \cap S \neq \emptyset \quad \forall\, a \in Z.$$

For $\varepsilon > 0$, we say that $\tau \in Z$ is an *ε-period* of $v \in (\mathbb{R}^n)^Z$ if

$$\|v(t) - v(t+\tau)\| \leq \varepsilon \quad \forall\, t \in Z.$$

We denote by $P(v, \varepsilon) \subseteq Z$ the set of ε-periods of v and we say that $v \in (\mathbb{R}^n)^Z$ is *almost periodic* if $P(v, \varepsilon)$ is relatively dense in Z for every $\varepsilon > 0$. We denote the set of almost periodic functions $v \in (\mathbb{R}^n)^Z$ by $AP(Z, \mathbb{R}^n)$ and note that $AP(Z, \mathbb{R}^n)$ is a closed linear subspace of $\ell^\infty(Z, \mathbb{R}^n)$. Trivially, a periodic function is almost periodic. An example of a function which is almost periodic but not periodic, is $v \in (\mathbb{R}^n)^Z$ defined by

$$v(t) := \sin(\pi\sqrt{2}t) \quad \forall\, t \in Z\,.$$

The straightforward proof of the following lemma is left to the reader.

Lemma 4.1. *If $v^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^n)$, then, for every $T \in \mathbb{Z}_+$*

$$\sup_{t \in \mathbb{Z}_+,\, t \geq T} \|v^{\mathrm{ap}}(t)\| = \|v^{\mathrm{ap}}\|_{\ell^\infty}.$$

Furthermore, if $v^{\mathrm{ap}} \in AP(\mathbb{Z}, \mathbb{R}^n)$, then, for every $T \in \mathbb{Z}$,

$$\sup_{t \in \mathbb{Z},\, t \geq T} \|v^{\mathrm{ap}}(t)\| = \|v^{\mathrm{ap}}\|_{\ell^\infty}$$
$$\text{and} \quad \sup_{t \in \mathbb{Z},\, t \leq T} \|v^{\mathrm{ap}}(t)\| = \|v^{\mathrm{ap}}\|_{\ell^\infty}.$$

A consequence of Lemma 4.1 is that almost periodic functions are completely determined by their "infinite tails": if $v, w \in AP(\mathbb{Z}_+, \mathbb{R}^n)$ and there exists $\tau \in \mathbb{Z}_+$ such that $v(t) = w(t)$ for all $t \in \overline{\tau}$, then $v = w$; similarly, if $v, w \in AP(\mathbb{Z}, \mathbb{R}^n)$ and there exists $\tau \in \mathbb{Z}$ such that $v(t) = w(t)$ for all $t \in \overline{\tau}$, or for all $t \in -\overline{\tau}$, then $v = w$.

We say that a function $v \in (\mathbb{R}^n)^{\mathbb{Z}_+}$ is *asymptotically almost periodic* if it is of the form $v = v^{\mathrm{ap}} + w$ with $v^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^n)$ and $w \in c_0(\mathbb{Z}_+, \mathbb{R}^n)$, where $c_0(\mathbb{Z}_+, \mathbb{R}^n)$ is the space of functions $u \in (\mathbb{R}^n)^{\mathbb{Z}_+}$ such that $\lim_{t\to\infty} u(t) = 0$. The space of all asymptotically almost periodic functions $v \in (\mathbb{R}^n)^{\mathbb{Z}_+}$ is denoted by $AAP(\mathbb{Z}_+, \mathbb{R}^n)$, that is,

$$AAP(\mathbb{Z}_+, \mathbb{R}^n) = AP(\mathbb{Z}_+, \mathbb{R}^n) + c_0(\mathbb{Z}_+, \mathbb{R}^n).$$

Noting that, by Lemma 4.1,

$$\|v + w\|_{\ell^\infty} \geq \|v\|_{\ell^\infty} \quad \forall\, v \in AP(\mathbb{Z}_+, \mathbb{R}^n),$$
$$\forall\, w \in c_0(\mathbb{Z}_+, \mathbb{R}^n),$$

it is easy to see that $AAP(\mathbb{Z}_+, \mathbb{R}^n)$ is a closed subspace of $\ell^\infty(\mathbb{Z}_+, \mathbb{R}^n)$.

As an immediate consequence of Lemma 4.1, we obtain the following result.

Lemma 4.2. *The following statements hold.*

(i) $AP(\mathbb{Z}_+, \mathbb{R}^n) \cap c_0(\mathbb{Z}_+, \mathbb{R}^n) = \{0\}$.

(ii) *If $v \in AAP(\mathbb{Z}_+, \mathbb{R}^n)$, then the decomposition $v = v^{\mathrm{ap}} + w$, where $v^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^n)$ and $w \in c_0(\mathbb{Z}_+, \mathbb{R}^n)$, is unique.*

It is well-known that $v \in (\mathbb{R}^n)^{\mathbb{Z}}$ is almost periodic if, and only if, the set of translates $\{\Lambda_\tau v : \tau \in \mathbb{Z}\}$ is relatively compact in $\ell^\infty(\mathbb{Z}, \mathbb{R}^n)$. Since, for any $v \in c_0(\mathbb{Z}_+, \mathbb{R}^n)$, the set of translates $\{\Lambda_\tau v : \tau \in \mathbb{Z}_+\}$ is relatively compact in $\ell^\infty(\mathbb{Z}_+, \mathbb{R}^n)$, it is clear that the above characterisation of almost periodicity on $\mathbb{Z}$ is not valid for functions in $(\mathbb{R}^n)^{\mathbb{Z}_+}$. Interestingly, the elements of $AAP(\mathbb{Z}_+, \mathbb{R}^n)$ are precisely the functions v in $(\mathbb{R}^n)^{\mathbb{Z}_+}$ for which the set $\{\Lambda_\tau v : \tau \in \mathbb{Z}_+\}$ is relatively compact in $\ell^\infty(\mathbb{Z}_+, \mathbb{R}^n)$, see [10]. For more information on and further characterisations of almost periodicity, we refer the reader to the literature, see, for example, [6, 7, 15, 30].

There exists a close relationship between the spaces $AP(\mathbb{Z}_+, \mathbb{R}^n)$ and $AP(\mathbb{Z}, \mathbb{R}^n)$ which we now briefly explain. Following an idea in [4, Remark on p. 318], for every $v \in AP(\mathbb{Z}_+, \mathbb{R}^n)$, we define a function $v_{\mathrm{e}} \in (\mathbb{R}^n)^{\mathbb{Z}}$ by

$$v_{\mathrm{e}}(t) = \lim_{k\to\infty} v(t + \tau_k) \quad \forall\, t \in \mathbb{Z},$$

where $\tau_k \in P(v, 1/k)$ for each $k \in \mathbb{N}$ and $\tau_k \to \infty$ as $k \to \infty$. For given $t \in \mathbb{Z}$, we have

$$\begin{aligned}\|v(t+\tau_k) - v(t+\tau_l)\| &\leq \|v(t+\tau_k) - v(t+\tau_k+\tau_l)\| \\ &\quad + \|v(t+\tau_k+\tau_l) - v(t+\tau_l)\| \\ &\leq \frac{1}{l} + \frac{1}{k},\end{aligned}$$

for all $k, l \in \mathbb{N}$ sufficiently large, and so $(v(t+\tau_k))_k$ is a Cauchy sequence. Hence $v_{\rm e}(t)$ is well-defined for each $t \in \mathbb{Z}$. It is clear that $v_{\rm e}(t) = v(t)$ for all $t \in \mathbb{Z}_+$, that is, $v_{\rm e}$ extends v to $\mathbb{Z}$. Furthermore, it is not difficult to show that $v_{\rm e} \in AP(\mathbb{Z}, \mathbb{R}^n)$ and there is no other function in $AP(\mathbb{Z}, \mathbb{R}^n)$ which extends v to $\mathbb{Z}$. Moreover, Lemma 4.1 guarantees that

$$\sup_{t\in\mathbb{Z}} \|v_{\rm e}(t)\| = \sup_{t\in\mathbb{Z}_+} \|v(t)\|.$$

It is now clear that the map $AP(\mathbb{Z}_+, \mathbb{R}^n) \to AP(\mathbb{Z}, \mathbb{R}^n)$, $v \mapsto v_{\rm e}$ is an isometric isomorphism.

The next result is our main result of this section.

Theorem 4.3. *Imposing the assumptions of Theorem 3.2, let $v^{\rm ap} \in AP(\mathbb{Z}_+, \mathbb{R}^n)$. The following statements hold.*

(i) *There exists a unique $x^{\rm ap} \in AP(\mathbb{Z}_+, \mathbb{R}^n)$ such that $(v^{\rm ap}, x^{\rm ap}) \in \mathcal{B}$ and, for all $(v, x) \in \mathcal{B}$ such that $v - v^{\rm ap} \in c_0(\mathbb{Z}_+, \mathbb{R}^n)$,*

$$\lim_{t\to\infty} \|x(t) - x^{\rm ap}(t)\| = 0\,. \qquad (4.1)$$

Furthermore, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $P(v^{\rm ap}, \delta) \subseteq P(x^{\rm ap}, \varepsilon)$. In particular, if $v^{\rm ap}$ is τ-periodic, then $x^{\rm ap}$ is τ-periodic.

(ii) *The almost periodic extension $x^{\rm ap}_{\rm e}$ of $x^{\rm ap}$ to $\mathbb{Z}$ is the unique bounded solution on $\mathbb{Z}$ of*

$$z^+ = Az + Bf(Cz) + v^{\rm ap}_{\rm e}\,. \qquad (4.2)$$

Remark 4.4. (i) Theorem 4.3 shows that, under the assumptions of Theorem 3.2, asymptotically almost periodic inputs generate asymptotically almost periodic state trajectories.

(ii) Statements (i) and (ii) remain valid if, in Theorem 4.3, the assumptions are replaced by the hyptheses of Corollary 3.6.

(iii) Under the assumptions of Theorem 3.2, it is straightforward to show, by combining Theorem 3.2, Lemma 4.1 and Theorem 4.3, that for all $R > 0$, there exists $\phi \in \mathcal{K}$ such that, for all $(v_i, x_i) \in \mathcal{B}$ with $v_i \in AAP(\mathbb{Z}_+, \mathbb{R}^n)$ and $\|x_i(0)\| + \|v_i\|_{\ell^\infty} \leq R$, $i = 1, 2$,

$$\|x_1^{\rm ap} - x_2^{\rm ap}\|_{\ell^\infty} \leq \phi(\|v_1^{\rm ap} - v_2^{\rm ap}\|_{\ell^\infty})\,,$$

where $v_i^{\rm ap}$ and $x_i^{\rm ap}$ are the almost periodic parts of v_i and x_i, respectively, $i = 1, 2$. ◊

The proof of Theorem 4.3 is facilitated by a technical lemma, which we state and prove first. For a function $\Phi : \mathbb{R}^p \to \mathbb{R}^p$ and $U \subseteq \mathbb{R}^p$, we let $\Phi^{-1}(U)$ denote the preimage of U under Φ, and denote its cardinality by $\#\Phi^{-1}(U)$. For $\xi \in \mathbb{R}^p$, we set $\Phi^{-1}(\xi) := \Phi^{-1}(\{\xi\})$. Finally, we let θ denote the constant function on $\mathbb{Z}_+$ with value one.

Lemma 4.5. *Imposing the assumptions of Theorem 3.2, define $F_K : \mathbb{R}^p \to \mathbb{R}^p$ by*

$$F_K(\xi) := \xi - \mathbf{G}^K(1)(f(\xi) - K\xi) \quad \forall\, \xi \in \mathbb{R}^p\,.$$

The following statements hold.

(i) *F_K is surjective and $\#F_K^{-1}(\xi) = 1$ for all $\xi \in \operatorname{im} C$.*

(ii) *For all $v^\infty \in \mathbb{R}^n$, by letting $y^\infty \in F_K^{-1}(C(I - A^K)^{-1}v^\infty)$ and setting*

$$x^\infty := (I - A^K)^{-1}\big(B(f(y^\infty) - Ky^\infty) + v^\infty\big)\,,$$

we have that $y^\infty = Cx^\infty$ and $(v^\infty\theta, x^\infty\theta) \in \mathcal{B}$.

Proof. Statement (i) is a discrete-time analogue of [5, Proposition 4.1]. The proof carries over to the discrete-time case and is therefore omitted. For statement (ii), fix $v^\infty \in \mathbb{R}^n$ and let $x^\infty \in \mathbb{R}^n$ be as in statement (ii) where, by statement (i), y^∞ is the unique element in the singleton $F_K^{-1}(C(I - A^K)^{-1}v^\infty)$. We then see that

$$\begin{aligned} y^\infty &= F_K(y^\infty) + \mathbf{G}^K(1)(f(y^\infty) - Ky^\infty) \\ &= C(I - A^K)^{-1}(B(f(y^\infty) - Ky^\infty) + v^\infty) \\ &= Cx^\infty, \end{aligned}$$

and thus,

$$\begin{aligned} x^\infty &= A^K x^\infty + B(f(y^\infty) - Ky^\infty) + v^\infty \\ &= Ax^\infty + Bf(Cx^\infty) + v^\infty. \end{aligned}$$

Consequently, $(v^\infty\theta, x^\infty\theta) \in \mathcal{B}$, establishing statement (ii). □

Proof of Theorem 4.3. We begin by proving statement (i). For which purpose, we use Lemma 4.5 to yield that, for every $v^* \in \mathbb{R}^n$, there exists (a unique) $x^* \in \mathbb{R}^n$ such that $(v^*\theta, x^*\theta) \in \mathcal{B}$. Fix such a pair (v^*, x^*). Let $(v^{\rm ap}, x) \in \mathcal{B}$ and note that, since $v^{\rm ap}$ is almost periodic, $v^{\rm ap}$ is bounded. Thus there exists $R_1 > 0$ such that

$$\|x(0)\| + \|v^{\rm ap}\|_{\ell^\infty}\,, \|x^*\| + \|v^*\| \leq R_1. \qquad (4.3)$$

Hence, by Theorem 3.2, there exist $\psi_1 \in \mathcal{KL}$ and $\phi_1 \in \mathcal{K}$ (dependent on R_1) such that

$$\|x(t) - x^*\| \leq \psi_1 (\|x(0) - x^*\|, t) + \phi_1\Big(\sup_{s \in \underline{t}} \|v^{\rm ap}(s) - v^*\| \Big) \quad \forall\, t \in \mathbb{N}\,.$$

Combining this with (4.3), we conclude that x is bounded. Hence, there exists $R > 0$ such that

$$\|x\|_{\ell^\infty} + \|v^{\rm ap}\|_{\ell^\infty} \leq R\,.$$

Since $v^{\rm ap}$ is almost periodic, there exists $\tau_k \in P(v^{\rm ap}, 1/k)$ for every $k \in \mathbb{N}$ such that $\tau_k \to \infty$ as $k \to \infty$. Inspired by an argument from the proof of [2, Proposition 4.4], we claim that $(\Lambda_{\tau_k} x)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\ell^\infty(\mathbb{Z}_+, \mathbb{R}^n)$. To show this, we first invoke Theorem 3.2 to obtain $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ (dependent on R) such that, for all $(v_1, x_1), (v_2, x_2) \in \mathcal{B}$, (3.1) holds, provided that $\|x_i(0)\| + \|v_i\|_{\ell^\infty} \leq R$ for $i = 1, 2$. Subsequently, we let $k, l \in \mathbb{N}$ be sufficiently large so that

$$\psi(2R, \tau_k), \psi(2R, \tau_l) \leq \frac{\varepsilon}{2} \quad \text{and} \quad \phi\left(\frac{1}{k} + \frac{1}{l}\right) \leq \frac{\varepsilon}{2},$$

and, without loss of generality, assume that $\tau_l \geq \tau_k$. Then, for all $t \in \mathbb{Z}_+$,

$$\begin{aligned}
&\sup_{s \in \underline{\tau_k}} \|v^{\rm ap}(s+t) - v^{\rm ap}(s+t+\tau_l - \tau_k)\| \\
\leq &\sup_{s \in \underline{\tau_k}} \|v^{\rm ap}(s+t) - v^{\rm ap}(s+t+\tau_l)\| \\
&\quad + \sup_{s \in \underline{\tau_k}} \|v^{\rm ap}(s+t+\tau_l) - v^{\rm ap}(s+t+\tau_l - \tau_k)\| \\
\leq &\frac{1}{l} + \frac{1}{k}.
\end{aligned}$$

Hence,

$$\phi\Big(\sup_{s \in \underline{\tau_k}} \|v^{\rm ap}(s+t) - v^{\rm ap}(s+t+\tau_l - \tau_k)\| \Big) \leq \frac{\varepsilon}{2},$$

which, when combined with (2.3) and (3.1), implies

$$\begin{aligned}
&\|(\Lambda_{\tau_k} x)(t) - (\Lambda_{\tau_l} x)(t)\| \\
&\qquad = \|(\Lambda_t x)(\tau_k) - (\Lambda_{t+\tau_l-\tau_k} x)(\tau_k)\| \\
&\qquad \leq \psi(\|x(t) - x(t+\tau_l - \tau_k)\|, \tau_k) + \frac{\varepsilon}{2} \\
&\qquad \leq \psi(2R, \tau_k) + \frac{\varepsilon}{2} \leq \varepsilon\,.
\end{aligned}$$

Whence, we have shown that $(\Lambda_{\tau_k} x)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\ell^\infty(\mathbb{Z}_+, \mathbb{R}^n)$ and so converges to a function $x^{\rm ap} \in \ell^\infty(\mathbb{Z}_+, \mathbb{R}^n)$. To show that $(v^{\rm ap}, x^{\rm ap}) \in \mathcal{B}$, we note that, for all $k \in \mathbb{Z}_+$ and $t \in \mathbb{Z}_+$, by (2.3)

$$\begin{aligned}
(\Lambda_{\tau_k} x)(t+1) = A(\Lambda_{\tau_k} x)(t) + Bf(C(\Lambda_{\tau_k} x)(t)) \\
+ (\Lambda_{\tau_k} v^{\rm ap})(t).
\end{aligned}$$

Since f is continuous, by using that $\tau_k \in P(v^{\rm ap}, 1/k)$ for all $k \in \mathbb{N}$ and taking the limit as $k \to \infty$, we see that $(v^{\rm ap}, x^{\rm ap}) \in \mathcal{B}$.

To show that $x^{\rm ap} \in AP(\mathbb{Z}_+, \mathbb{R}^n)$, we fix $\varepsilon > 0$ and note that, since $\phi(0) = 0$ and ϕ is continuous, there exists $\delta > 0$ such that

$$\phi(s) \leq \varepsilon \quad \forall\, s \in [0, \delta]. \tag{4.4}$$

Let $\tau \in P(v^{\rm ap}, \delta)$. Then, by combining (2.3) with (3.1) and (4.4), we see that, for all $t \in \mathbb{Z}_+$ and all $k \in \mathbb{N}$,

$$\begin{aligned}
&\|(\Lambda_{\tau_k} x)(t) - (\Lambda_{\tau_k} x)(t+\tau)\| \\
&\qquad = \|(\Lambda_t x)(\tau_k) - (\Lambda_{t+\tau} x)(\tau_k)\| \\
&\qquad \leq \psi(\|x(t) - x(t+\tau)\|, \tau_k) + \varepsilon.
\end{aligned} \tag{4.5}$$

Since $(\tau_k)_{k\in\mathbb{N}}$ converges to ∞ as $k \to \infty$, (4.5) yields that

$$\|x^{\rm ap}(t) - x^{\rm ap}(t+\tau)\| \leq \varepsilon \quad \forall\, t \in \mathbb{Z}_+\,,$$

showing that $\tau \in P(x^{\rm ap}, \varepsilon)$. It follows that $P(v^{\rm ap}, \delta) \subseteq P(x^{\rm ap}, \varepsilon)$. Since $v^{\rm ap}$ is almost periodic, it follows that $P(x^{\rm ap}, \varepsilon)$ is relatively dense in $\mathbb{Z}_+$, showing that $x^{\rm ap} \in AP(\mathbb{Z}_+, \mathbb{R}^n)$.

In order to establish (4.1), let $(v, x) \in \mathcal{B}$ such that $v - v^{\rm ap} \in c_0(\mathbb{Z}_+, \mathbb{R}^n)$. Obviously, v is bounded, and so an application of Theorem 3.2 yields the existence of $\tilde{R} > 0$ such that

$$\|x\|_{\ell^\infty} + \|v\|_{\ell^\infty}\,, \|x^{\rm ap}\|_{\ell^\infty} + \|v^{\rm ap}\|_{\ell^\infty} \leq \tilde{R}\,.$$

Furthermore, Theorem 3.2 guarantees the existence of $\tilde{\psi} \in \mathcal{KL}$ and $\tilde{\phi} \in \mathcal{K}$ such that, for all $(v_i, x_i) \in \mathcal{B}$ with $\|x_i(0)\| + \|v_i\|_{\ell^\infty} \leq \tilde{R}$, $i = 1, 2$, (3.1) holds with ψ and ϕ replaced by $\tilde{\psi}$ and $\tilde{\phi}$, respectively. In particular, setting $w := v - v^{\rm ap}$ and using (2.3), we see that, for all $t \in \mathbb{N}$,

$$\begin{aligned}
&\|x(t) - x^{\rm ap}(t)\| \\
&= \|(\Lambda_{\lfloor t/2 \rfloor} x)(\lceil t/2 \rceil) - (\Lambda_{\lfloor t/2 \rfloor} x^{\rm ap})(\lceil t/2 \rceil)\| \\
&\leq \tilde{\psi}\left(\|x(\lfloor t/2 \rfloor) - x^{\rm ap}(\lfloor t/2 \rfloor)\|\,, \lceil t/2 \rceil\right) \\
&\qquad + \tilde{\phi}\Big(\sup_{s \in \underline{\lceil t/2 \rceil}} \left\|(\Lambda_{\lfloor t/2 \rfloor} w)(s)\right\| \Big)\,.
\end{aligned}$$

By applying (3.1) to the term $\|x(\lfloor t/2 \rfloor) - x^{\mathrm{ap}}(\lfloor t/2 \rfloor)\|$ in the above inequality, we deduce that, for all $t \in \mathbb{N}$,

$$\|x(t) - x^{\mathrm{ap}}(t)\| \leq \tilde{\psi}\left(\tilde{\psi}(\|x(0) - x^{\mathrm{ap}}(0)\|, \lfloor t/2 \rfloor) + \tilde{\phi}\Big(\sup_{s \in \underline{\lfloor t/2 \rfloor}} \|w(s)\|\Big), \lceil t/2 \rceil\right) + \tilde{\phi}\Big(\sup_{s \in \underline{\lceil t/2 \rceil}} \big\|(\Lambda_{\lfloor t/2 \rfloor} w)(s)\big\|\Big).$$

Finally, the right hand side of the above inequality converges to 0 as $t \to \infty$, showing that (4.1) holds. It is easily seen, by a combination of (4.1) with Lemma 4.2, that x^{ap} is the unique almost periodic function in $(\mathbb{R}^n)^{\mathbb{Z}_+}$ satisfying (4.1) and such that $(v^{\mathrm{ap}}, x^{\mathrm{ap}}) \in \mathcal{B}$, completing the proof of statement (i).

We proceed to prove statement (ii). To this end, note that the almost periodic extension $x^{\mathrm{ap}}_{\mathrm{e}}$ of x^{ap} to $\mathbb{Z}$ is bounded. We shall now show that $x^{\mathrm{ap}}_{\mathrm{e}}$ satisfies (4.2). To this end, note that since $\Lambda_1 x^{\mathrm{ap}}_{\mathrm{e}}$ is the almost periodic extension of $\Lambda_1 x^{\mathrm{ap}}$, we hence see that $\Lambda_1 x^{\mathrm{ap}}_{\mathrm{e}}$ is also the almost periodic extension of $Ax^{\mathrm{ap}} + Bf(Cx^{\mathrm{ap}}) + v^{\mathrm{ap}}$. Moreover, since $x^{\mathrm{ap}}_{\mathrm{e}}$ is almost periodic and f is (globally) Lipschitz, $f(Cx^{\mathrm{ap}}_{\mathrm{e}})$ is almost periodic and, consequently, $Ax^{\mathrm{ap}}_{\mathrm{e}} + Bf(Cx^{\mathrm{ap}}_{\mathrm{e}}) + v^{\mathrm{ap}}_{\mathrm{e}}$ is also an almost periodic extension of $Ax^{\mathrm{ap}} + Bf(Cx^{\mathrm{ap}}) + v^{\mathrm{ap}}$. Hence, by the uniqueness of almost periodic extensions,

$$\Lambda_1 x^{\mathrm{ap}}_{\mathrm{e}} = Ax^{\mathrm{ap}}_{\mathrm{e}} + Bf(Cx^{\mathrm{ap}}_{\mathrm{e}}) + v^{\mathrm{ap}}_{\mathrm{e}},$$

that is, $x^{\mathrm{ap}}_{\mathrm{e}}$ is a solution of (4.2).

To show that $x^{\mathrm{ap}}_{\mathrm{e}}$ is the unique bounded solution of (4.2) on $\mathbb{Z}$, let $\tilde{x} \in (\mathbb{R}^n)^{\mathbb{Z}}$ be another bounded solution of (4.2). Let $R > 0$ be such that

$$\|x^{\mathrm{ap}}_{\mathrm{e}}\|_{\ell^\infty} + \|\tilde{x}\|_{\ell^\infty} + \|v^{\mathrm{ap}}_{\mathrm{e}}\|_{\ell^\infty} \leq R,$$

and apply Theorem 3.2 to obtain the existence of $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that (3.1) holds for all $(v_i, x_i) \in \mathcal{B}$ with $\|x_i(0)\| + \|v_i\|_{\ell^\infty} \leq R$, $i = 1, 2$. Let $\varepsilon > 0$ and $t \in \mathbb{Z}$ and choose $\tau \in \mathbb{Z}$ such that $\tau \leq t$ and

$$\psi(2R, t - \tau) \leq \varepsilon.$$

Now, since the restrictions of $(\Lambda_\tau v^{\mathrm{ap}}_{\mathrm{e}}, \Lambda_\tau x^{\mathrm{ap}}_{\mathrm{e}})$ and $(\Lambda_\tau v^{\mathrm{ap}}_{\mathrm{e}}, \Lambda_\tau \tilde{x})$ to $\mathbb{Z}_+$ are in $\mathcal{B}$ and satisfy $\|(\Lambda_\tau x^{\mathrm{ap}}_{\mathrm{e}})(0)\| + \|(\Lambda_\tau \tilde{x})(0)\| + \|\Lambda_\tau v^{\mathrm{ap}}_{\mathrm{e}}\|_{\ell^\infty} \leq R$, (3.1) guarantees that

$$\begin{aligned}\|x^{\mathrm{ap}}_{\mathrm{e}}(t) - \tilde{x}(t)\| &= \|(\Lambda_\tau x^{\mathrm{ap}}_{\mathrm{e}})(t - \tau) + (\Lambda_\tau \tilde{x})(t - \tau)\| \\ &\leq \psi(\|x^{\mathrm{ap}}_{\mathrm{e}}(\tau) - \tilde{x}(\tau)\|, t - \tau) \\ &\leq \psi(2R, t - \tau) \leq \varepsilon.\end{aligned}$$

Since ε was arbitrary, we see that $x^{\mathrm{ap}}_{\mathrm{e}}(t) = \tilde{x}(t)$ and, since t was also arbitrary, it follows that $x^{\mathrm{ap}}_{\mathrm{e}} = \tilde{x}$, completing the proof. □

From Lemma 4.5 and Theorem 4.3 we obtain the following corollary which states that, under the assumptions of Theorem 3.2, the Lur'e system (2.2) has the CICS property.

Corollary 4.6. *Imposing the assumptions of Theorem 3.2, let $v^\infty \in \mathbb{R}^n$ be given, and let x^∞ be as in statement (ii) of Lemma 4.5. Then, for all $(v, x) \in \mathcal{B}$ with $\lim_{t\to\infty} v(t) = v^\infty$, it follows that $\lim_{t\to\infty} x(t) = x^\infty$.*

Remark 4.7. (i) Corollary 4.6 is a discrete-time version of continuous-time results in [5]. The incremental ISS methodology used here to obtain Corollary 4.6 is quite different to that invoked in [5].

(ii) It may appear that x^∞ and y^∞ in Lemma 4.5 depend on the choice of K. But this is not the case, as follows from Corollary 4.6, or from a purely "algebraic" argument based on condition (3.2). ◊

Under the assumptions of Theorem 3.2, consider the (non-autonomous) system

$$z(t + 1) = g(t, z(t)), \tag{4.6}$$

where $g(t, \xi) := A\xi + Bf(C\xi) + v(t)$ for all $(t, \xi) \in \mathbb{Z} \times \mathbb{R}^n$ and $v \in AP(\mathbb{Z}, \mathbb{R}^n)$. For $(t_0, x^0) \in \mathbb{Z} \times \mathbb{R}^n$, we denote the solution to (4.6) with initial state x^0 at time t_0 by $x(\cdot\,; t_0, x^0)$. Theorem 4.3 yields the existence of a unique bounded solution $x_{\mathrm{b}} \in (\mathbb{R}^n)^{\mathbb{Z}}$ of (4.6). If we now apply Theorem 3.2 and use methods similar to those employed in the proof of statement (ii) of Theorem 4.3, we obtain that for all $R > 0$, there exists $\psi \in \mathcal{KL}$ such that, for all $(t_0, x^0) \in \mathbb{Z} \times \mathbb{R}^n$ with $\|x^0\| \leq R$,

$$\|x(t; t_0, x^0) - x_{\mathrm{b}}(t)\| \leq \psi(\|x^0 - x_{\mathrm{b}}(t_0)\|, t - t_0) \quad \forall\, t \in \mathbb{Z},\ t \geq t_0.$$

This shows that (4.6) satisfies a semi-global version of the definition of a uniformly convergent system given in [25].

We conclude this section with a brief comparison of Theorem 4.3 to related results in the literature. The most relevant results in this context are [26, Theorem 1] and [35, Theorem 1], both of which consider continuous-time systems and are restricted to 'scalar' nonlinearities, that is, $m = p = 1$. A state-space and Lyapunov approach is used in [35], whilst the analysis in [26] is based on input-output methods. A careful inspection of the assumptions imposed in [26, Theorem 1] and [35, Theorem 1] shows that, in each case, they are equivalent to the existence of a $\delta > 0$ such that

$$|f(\xi+\zeta)-f(\zeta)-k\xi| \leq (r-\delta)|\xi| \quad \forall\, \xi,\zeta \in \mathbb{R}, \quad (4.7)$$

where $r = 1/\|\mathbf{G}^k\|_{H^\infty}$ and k is a suitable stabilizing scalar real gain. Note that (4.7) is simply the scalar version of (3.5) and we conclude that the assumptions required in [26, Theorem 1] and [35, Theorem 1] are considerable more restrictive than those imposed in Corollary 4.3, cf. statements (iii) and (iv) of Remark 3.3. Finally, we note that neither [26] nor [35] addresses the relationship between the almost periods of the almost peridic forcing v^{ap} and its corresponding almost periodic state trajectory x^{ap}.

5. An example

We consider a simple system for which the hypotheses of Theorem 3.2 hold, but not the (stronger) assumptions of [13, Theorem 3.2]. This latter result guarantees exponential incremental input-to-state stability of (2.2).

Example 5.1. Consider the Lur'e system (2.2) with

$$A := \frac{1}{2}\begin{pmatrix} 2 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and $C = (1,0,0)$, and $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(\xi) := \frac{1}{2}\,\mathrm{sign}(\xi)\ln(1+|\xi|) \quad \forall\, \xi \in \mathbb{R}.$$

It is easy to check that A is Schur, that

$$\mathbf{G}(z) = \frac{1/4}{(z-1/2)^3},$$

and further, that

$$|\mathbf{G}(z)| \leq 2 = |\mathbf{G}(1)| \quad \forall\, z \in \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}.$$

We hence deduce that

$$\|\mathbf{G}\|_{H^\infty} = |\mathbf{G}(1)| = 2.$$

Moreover, it can easily be verified that (A) holds. Since $f'(0) = 1/2$, it is also clear that there does not exist $\delta > 0$ such that

$$|f(\xi+\zeta) - f(\zeta)| \leq (1/2-\delta)|\xi| \quad \forall\, \xi,\zeta \in \mathbb{R}, \quad (5.1)$$

that is, f does not satisfy the assumptions of [13, Theorem 3.2]. However, since f is continuously differentiable with

$$f'(0) = \frac{1}{2} \quad \text{and} \quad f'(\xi) \in (0,1/2) \quad \forall\, \xi \in \mathbb{R}\backslash\{0\},$$

[5, Lemma 4.9] yields that (3.2) holds with $K = 0$ and $r = 1/2$. Finally,

$$\frac{1}{2}|\xi| - |f(\xi)| \to \infty \text{ as } |\xi| \to \infty$$

and so (3.3) is satisfied with $K = 0$, $r = 1/2$ and $\eta = 0$. Therefore, Theorem 3.2 gives that (2.2) is semi-globally incrementally input-to-state stable. Moreover, Lemma 4.5 yields that (2.2) has the CICS property and, from Theorem 4.3, we obtain that asymptotically almost periodic inputs generate asymptotically almost periodic state trajectories.

We comment that the nonlinearity f is a modification of that found in [5, Example 4.12], and does not satisfy (5.1) for $\zeta = 0$ as $\xi \to 0$. It is easy to find other examples of functions which instead do not satisfy (5.1) at infinity, that is, there exists $\zeta \in \mathbb{R}$,

$$\frac{1}{|\xi|}|f(\xi+\zeta) - f(\zeta)| \to \frac{1}{2} \text{ as } |\xi| \to \infty,$$

but do satisfy (3.2) and (3.3). $\Diamond$

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