Dynamic properties of a class of forced positive higher-order scalar difference equations: persistency, stability and convergence

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Abstract: Persistency, stability and convergence properties are considered for a class of nonlinear, forced, positive, scalar higher-order difference equations. Sufficient conditions for these properties to hold are derived, broadly in terms of the interplay of the linear and nonlinear components of the difference equations. Convergence properties of solutions include to almost periodic functions when subject to asymptotically almost periodic forcing terms. The equations under consideration arise in a number of ecological and biological contexts, with the Allen-Clark population model appearing as a special case. We illustrate our results by several examples from population dynamics.

Keywords: Allen-Clark model, Almost periodic forcing, Difference equation, Persistence, Positive Lur'e system, Stability.

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1 Introduction

We consider the following class of scalar, forced, higher-order initial-value problems:

$$x(t+1) = \sum_{j=0}^{k} \alpha_j x(t-j) + \beta f\Big(u(t), \sum_{j=0}^{k} \gamma_j x(t-j)\Big) + v(t), \quad x(-j) = x^{-j} \in \mathbb{R}, \quad j = 0, \dots, k, \quad t \in \mathbb{N}_0, \tag{1.1}$$

where k is a nonnegative integer. The details of (1.1) are given in Section 2, although we note here that f is a nonlinearity, α_j, β and γ_j are real parameters, and the terms u and v are exogenous forcing terms, and could model control actions or disturbances, depending on the context. We study a suite of relevant dynamical properties of (1.1), namely, boundedness, persistence, stability and convergence, in the situation wherein system (1.1) is positive, and provide sufficient conditions for these properties to hold.

Positive dynamical systems are dynamical systems with the defining property that they leave some positive cone invariant. They are well-studied objects, evidenced by a vast literature with texts including [3, 4, 29, 36, 37]. Their interest is not only mathematical, but also practical as they arise in myriad application areas where state variables are constrained to lie in some positive cone to be meaningful, such as the nonnegative orthant in Euclidean space for necessarily nonnegative quantities.

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One motivation for studying (1.1) is its occurrence in models in theoretical ecology, as the higher-order nature of (1.1) may be interpreted as a delay structure. Time delays are an important feature in this setting, for instance, a known plant survival strategy is to delay the germination of dormant seeds post dispersal [17, Section 1.2]. Similarly, system (1.1) is a forced generalisation of

$$x(t+1) = \alpha x(t) + \beta f(x(t-k)) \quad t \in \mathbb{N}_0 \,, \tag{1.2}$$

known as the Allen-Clark, or Clark, model (after Allen [1] and Clark [10]), see also the bibliographical notes in [7] for other early contributors. In population modelling, (1.2) is a parsimonious extension of uncontrolled standard first-order difference equations (discussed in a number of monographs, such as [9, Chapter 1] or [48, Chapter 2]) to include age-structure, particularly delayed reproductive maturity. There has been much subsequent interest in the role of delay-until-reproductive-maturation in age-structured population models, dating back to at least [40], and further studied and generalised across, for example, [2, 20, 64, 67]. Roughly speaking, these works generalise the Allen-Clark model to include explicit age-structure within a single population model. The model (1.2) is known to admit a unique positive equilibrium under mild assumptions on f, α and β. Stability and attractivity properties of the nonzero equilibrium have been studied in several papers, including [18, 19, 33, 43].

The study of dynamical systems which interact with their wider environment via the inclusion of input (control, forcing) and output (measurement) variables, and their feedback connections, is at the heart of control theory; see, for example [61]. The inclusion of forcing terms in dynamical systems is essential in applied settings. Indeed, forcing terms may represent control actions/interventions, (possibly unwanted or uncertain) variation in the underlying model, and otherwise unmodelled terms which may be significant. When considering the effects of forcing terms, typically one of two perspectives is adopted: their use as controls to establish or maintain desirable dynamic behaviour, or, the robustness of desirable properties of the model with respect to unwanted forcing terms. A strand of control theory associates input and output variables to positive dynamical systems, leading to so-called positive control systems, with recent review paper [49]. The model (1.1) with appropriate nonnegativity assumptions is an instance of a positive control system. Again, in a population dynamics context, the forcing function v in (1.1) may model immigration, and u may capture environmental variation or harvesting (anthropogenic or otherwise).

To connect to another body of literature, equation (1.1) is an instance of a forced (positive, in this case) Lur'e difference equation, or simply Lur'e system, in control-theoretic terminology; see, for example [65]. There are numerous studies of positive Lur'e systems in state-space form, including [5, 15, 16, 22, 23, 24, 25, 58], broadly motivated by their interesting dynamical behaviour and relevance in theoretical ecology, dating back to [50, 63] where 'trichotomies of stability' were established for such systems.

Boundedness of solutions is an innate requirement for positive difference equations motivated by real-world applications of size- or quantity-limited variables. Persistence concepts, broadly referring to the property that certain internal variables (for example, the state or certain linear combinations of state components) are ultimately bounded away from zero, are highly relevant as well. This is particularly the case when $f(w, 0) = 0$ for all w, so that zero is an equilibrium of system (1.1) when unforced (meaning u equal to a constant nominal value, and $v = 0$). Persistence in dynamical systems is a well-established concept with a number of variations including those presented in [21, 22, 23, 24, 55, 59, 66]. The persistence properties we consider are ultimate in that they only apply after some fixed number of time-steps, and are uniform with respect to certain initial conditions and forcing functions as we shall describe. We contend that this concept of persistence is suitable for all practical purposes. Moreover, mild assumptions on the model data ensure that there is a unique positive equilibrium, denoted x_{e}, at a constant forcing pair $(u_{\mathrm{e}}, v_{\mathrm{e}})$. The stability properties we consider relate to x_{e}, and account for the fact that (1.1) is inhomogeneous by appealing to the input-to-state stability (ISS) framework from nonlinear control theory; see [45] for a recent monograph. Roughly speaking, ISS ensures that $\|x(t) - x_{\mathrm{e}}\|$ is bounded in terms of (nonlinear functions of) the difference of the initial state to x_{e}, which decays to 0 over time, as well contributions from $u - u_{\mathrm{e}}$ and $v - v_{\mathrm{e}}$. Here, we provide a range of boundedness, persistence, and stability properties for system (1.1), presented as Proposition 3.3, and Theorems 3.5 and 4.2, respectively.

As a consequence of our stability results we show that, under suitable assumptions, the following convergence property holds: for all constant forcing pairs $(u_{\mathrm{e}}, v_{\mathrm{e}})$, there is a unique constant x_{e} such that, for all u and v converging to u_{e} and v_{e}, respectively, and all non-zero initial conditions, the corresponding solution $x(t)$ of (1.1) converges to x_{e} as $t \to \infty$, see Corollary 4.4 for a precise statement. Finally, the stability properties we derive are sufficiently strong to ensure that (1.1) admits a rather general entrainment-type property (see, for example [34, Chapter 7] for a classical treatment of entrainment), specifically here that for almost periodic forcing terms $v = v_{\mathrm{ap}}$ (in the sense of Bohr): (i) there is a unique almost periodic solution x_{ap} of (1.1), and, (ii) all other solutions x of (1.1) converge to x_{ap}, when subject to v converging to v_{ap}. Moreover, the sets of almost periods of v_{ap} and x_{ap} are closely related, see Theorem 5.1 for a precise statement. The development of the theoretical results is mostly based on a blend of techniques from control theory and positive systems. We highlight that the boundedness and persistence properties are key for our stability arguments. Further, our results apply to general f specified in terms of qualitative and quantitative properties which, in particular, do not necessitate that f is monotone or unimodal. In particular, system (1.1) need not be a monotone control system when unforced, as in [30, 57].

In terms of novelty and contribution, whilst the persistence and stability results from [22, 23] play a pivotal role in establishing the corresponding properties for (1.1), here we are able to formulate conditions directly in terms of the model data in (1.1); moreover, the convergence results in Sections 4 and 5 are new. Paper [24] considers related dynamic properties for a class of positive vector Lur'e systems with unit delay in the nonlinear term and highlights a number of surprising discrepancies as compared to the continuous-time (delay-differential equation) case analysed in [21]. The overlap with the present work is minimal, however. We comment that our inclusion and treatment of forcing terms, and consequent stability and convergence properties, separates our work from much of the literature in this area.

The remainder of the paper is organized as follows. Section 2 contains preliminary material. Our main results appear in Sections 3, 4 and 5, which focus on boundedness and persistence, stability and convergence properties, and the response to almost periodic additive forcing terms, respectively. Four examples are presented in Section 6. In particular, we demonstrate how the results in Sections 3–5 apply to forced versions of the Allen-Clark model (1.2). Section 7 contains a discussion of our results. An ancillary technical theorem appears in the Appendix.

Notation. We set $\mathbb{R}_+ := [0, \infty)$, $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}$ stands for the set of all integers. For $n \in \mathbb{N}$, let $\mathbb{R}^n$ denote the space of column vectors with n real components. We define $\mathbb{R}^n_+$ to be the subset of $\mathbb{R}^n$ consisting of all vectors in $\mathbb{R}^n$ with non-negative components. For $\xi \in \mathbb{R}^n$, we write $\xi \geq 0$ if $\xi \in \mathbb{R}^n_+$, $\xi > 0$ if $\xi \geq 0$ and $\xi \neq 0$, and $\xi \gg 0$ if all components of ξ are positive. If $\xi \gg 0$, then we also say that ξ is strictly positive. Furthermore, let $\xi, \zeta \in \mathbb{R}^n$. If $\xi - \zeta \geq 0$, $\xi - \zeta > 0$ or $\xi - \zeta \gg 0$, then we write $\xi \geq \zeta$, $\xi > \zeta$ or $\xi \gg \zeta$, respectively. Similar conventions apply to real matrices.

We will make use of the following classes of comparison functions:

$$\mathcal{K} := \{\phi : \mathbb{R}_+ \to \mathbb{R}_+ : \phi(0) = 0,\ \phi \text{ is continuous and strictly increasing}\}$$

and $\mathcal{K}_\infty := \{\phi \in \mathcal{K} : \lim_{s\to\infty} \phi(s) = \infty\}$. Furthermore, we denote by $\mathcal{KL}$ the set of all functions $\phi : \mathbb{R}_+ \times \mathbb{N}_0 \to \mathbb{R}_+$ with the following properties: for each fixed $t \in \mathbb{N}_0$, the function $\phi(\,\cdot\,, t)$ is in $\mathcal{K}$, and for each fixed $s \in \mathbb{R}_+$, the function $\phi(s, \cdot\,)$ is non-increasing and $\phi(s, t) \to 0$ as $t \to \infty$. The reader is referred to [35] for more details on comparison functions.

Finally, for a function $y : \mathbb{N}_0 \to \mathbb{R}^n$ and $\theta \in \mathbb{N}_0$, we denote the θ-left translate by y_θ, that is,

$$y_\theta(t) := y(t + \theta) \quad \forall\, t \in \mathbb{N}_0. \tag{1.3}$$

2 A class of nonlinear higher-order difference equations

Consider (1.1) where, throughout, $x^0, \dots, x^{-k} \geq 0$, and α_j, γ_j and β satisfy the following positivity condition.

(P1) $\alpha_j, \gamma_j \geq 0$ for $j = 0, \dots, k$, $\beta > 0$, $\sum_{j=0}^k \gamma_j > 0$ and $\alpha_k + \gamma_k > 0$.

The functions u and v are assumed to take values in non-empty compact sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}_+$, respectively, where V is such that $0 \in V$. It is always assumed that the nonlinearity $f : U \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. For convenience, we set

$$\alpha := \sum_{j=0}^k \alpha_j \quad \text{and} \quad \gamma := \sum_{j=0}^k \gamma_j.$$

Furthermore, we impose the following stability assumption.

(S) The polynomial

$$\mathbf{a}(\zeta) := \zeta^{k+1} - \sum_{j=0}^k \alpha_j \zeta^{k-j} \tag{2.1}$$

is Schur, that is, if $\mathbf{a}(\zeta) = 0$, then $|\zeta| < 1$.

Obviously, (S) is equivalent to the 0-equilibrium of the linear difference equation

$$x(t+1) = \sum_{j=0}^k \alpha_j x(t-j)$$

being asymptotically stable. As a consequence, we have that

$$\alpha = \sum_{j=0}^k \alpha_j < 1. \tag{2.2}$$

The initial-value problem (1.1) has a unique solution $x : \{-k, \dots, -1\} \cup \mathbb{N}_0 \to \mathbb{R}$. As the coefficients and initial data are non-negative, it is clear that the solution of (1.1) has values in $\mathbb{R}_+$. As is usual in control theory, the difference equation in (1.1) is referred to as a Lur'e system (see, for example, [65]), more specifically, (1.1) is an instance of a forced, positive, higher-order Lur'e system in discrete time. It can be thought of as the interconnection of the linear controlled and observed system

$$x(t+1) = \sum_{j=0}^k \alpha_j x(t-j) + \beta w(t) + v(t), \quad y(t) = \sum_{j=0}^k \gamma_j x(t-j) \tag{2.3}$$

and the nonlinearity f via the feedback law $w(t) = f(u(t), y(t))$, see Figure 2.1. Note that in (2.3), v and w are inputs, controls or forcing functions, where w is available for feedback, whereas y is the measurement, observation or output.

Associated with (2.3) is the rational function $\mathbf{G}$ defined by

$$\mathbf{G}(\zeta) := \frac{\beta \sum_{j=0}^k \gamma_j \zeta^{-j}}{\zeta - \sum_{j=0}^k \alpha_j \zeta^{-j}} = \frac{\beta \sum_{j=0}^k \gamma_j \zeta^{k-j}}{\zeta^{k+1} - \sum_{j=0}^k \alpha_j \zeta^{k-j}} = \frac{\beta \mathbf{c}(\zeta)}{\mathbf{a}(\zeta)}, \quad \text{where} \quad \mathbf{c}(\zeta) := \sum_{j=0}^k \gamma_j \zeta^{k-j}, \tag{2.4}$$

and ζ is a complex variable. If $x(-j) = 0$ for $j = 0, \dots, k$ and $v = 0$, then application of the Z-transform $\mathcal{Z}$ to (2.3) yields that

$$(\mathcal{Z}y)(\zeta) = \mathbf{G}(\zeta)(\mathcal{Z}w)(\zeta).$$

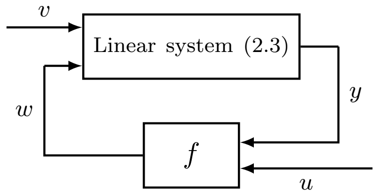

Figure 2.1: Application of the feedback law $w = f(u, y)$ to system (2.3).

The above identity shows that if $x(-j) = 0$ for $j = 0, \dots, k$ (zero initial conditions) and $v = 0$, then the effect of the input w on the output y of system (2.3) is described in the frequency domain by the product of $\mathbf{G}$ and the Z-transform of w. Therefore, $\mathbf{G}$ is called the transfer function of (2.3) with $v = 0$.

Assuming that (S) holds, we set

$$\|\mathbf{G}\|_{H^\infty} := \sup_{|\zeta| \geq 1} |\mathbf{G}(\zeta)| = \sup_{|\zeta|=1} |\mathbf{G}(\zeta)| < \infty,$$

where H^∞ refers to the space of all bounded holomorphic functions defined on the complement of the closed unit disc. If $x(-j) = 0$ for $j = 0, \dots, k$ and $v = 0$ in (2.3), then the associated output $y = y_w$ depends only on w, and

$$\sup\{\|y_w\|_{\ell^2} : \|w\|_{\ell^2} = 1\} = \|\mathbf{G}\|_{H^\infty}, \quad \text{where} \quad \|w\|_{\ell^2} := \sqrt{\sum_{t=0}^{\infty} |w(t)|^2}.$$

The above identity provides an appealing interpretation of $\|\mathbf{G}\|_{H^\infty}$ in time-domain terms.

For $\zeta \in \mathbb{C}$ such that $|\zeta| = 1$, we have

$$\left| \beta \sum_{j=0}^{k} \gamma_j \zeta^{k-j} \right| \leq \beta\gamma \quad \text{and} \quad |\mathbf{a}(\zeta)| \geq 1 - \alpha > 0,$$

where the last inequality follows from (S) and (2.2). Consequently,

$$\mathbf{G}(1) \leq \|\mathbf{G}\|_{H^\infty} \leq \frac{\beta\gamma}{1-\alpha} = \mathbf{G}(1),$$

showing that

$$\|\mathbf{G}\|_{H^\infty} = \mathbf{G}(1).$$

We define

$$p := \frac{1}{\mathbf{G}(1)} = \frac{1}{\|\mathbf{G}\|_{H^\infty}}, \qquad \text{where} \quad p := \infty \quad \text{if } \mathbf{G}(1) = \|\mathbf{G}\|_{H^\infty} = 0.$$

Note that $\mathbf{G}(1) > 0$ if (P1) is satisfied. Setting $\tilde{x}(t) := \big(x(t), \dots, x(t-k)\big)^\top$ and defining $b, c \in \mathbb{R}^{k+1}$ and $A \in \mathbb{R}^{(k+1)\times(k+1)}$ by

$$b := \begin{pmatrix} \beta \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad c := \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_k \end{pmatrix}, \quad A := \begin{pmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{k-1} & \alpha_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{2.5}$$

equation (1.1) can be expressed in the form

$$\tilde{x}(t+1) = A\tilde{x}(t) + bf(u(t), c^\top \tilde{x}(t)) + \tilde{v}(t), \;\; t \in \mathbb{N}_0, \quad \tilde{x}(0) = (x^0, \dots, x^{-k})^\top =: \tilde{x}^0, \tag{2.6}$$

where $\tilde{v}(t) := (v(t), 0, \dots, 0)^\top \in \mathbb{R}^{k+1}$. We note that $b > 0$, $c > 0$, and A is a companion matrix, see, for example, [31, Section 3.3 and Problem 3.3 P11]. As $\det(\zeta I - A) = \mathbf{a}(\zeta)$, all eigenvalues of A have modulus smaller than 1, that is, the matrix A is asymptotically stable. It is straightforward to show that

$$\mathbf{G}(\zeta) = c^\top (\zeta I - A)^{-1} b.$$

Throughout, let $\mathcal{I}$ be the set of positive integers defined by

$$\mathcal{I} := \big\{ i \in \{1, \dots, k+1\} : \alpha_{i-1} + \gamma_{i-1} > 0 \big\} = \big\{ i \in \{1, \dots, k+1\} : \alpha_{i-1} + \beta\gamma_{i-1} > 0 \big\}.$$

If (P1) is satisfied, then $k+1 \in \mathcal{I}$. The greatest common divisor of the elements of $\mathcal{I}$ is denoted by $\gcd \mathcal{I}$. In the following proposition we explore certain positivity properties of the linear part of system (2.6) which are essential for the developments in Sections 3–5.

Proposition 2.1. *Assume that* (P1) *holds and let A, b and c as in* (2.5)*. The following statements hold.*

(1) *The matrix $A + bc^\top$ is primitive if, and only if,* $\gcd \mathcal{I} = 1$.

(2) *There exists $\tau \in \mathbb{N}_0$ such that $c^\top (A + bc^\top)^\tau \gg 0$ if, and only if,* $\gcd \mathcal{I} = 1$.

(3) *If* $\gcd \mathcal{I} = 1$*, then $(A + bc^\top)^{(2+l)k-l} \gg 0$, where $l := \min\{j : \alpha_j + \gamma_j \neq 0\}$.*

(4) *If $\alpha_0 + \gamma_0 > 0$, then $c^\top (A + bc^\top)^{k+m} \gg 0$, where $m := \min\{j : \gamma_j \neq 0\}$.*

We remark that for the ultimate c-persistency result in Section 3, see Theorem 3.5, the strict positivity of $c^\top (A+bc^\top)^\tau$ for some $\tau \in \mathbb{N}_0$ plays a key role: the smallest τ such that $c^\top (A+bc^\top)^\tau \gg 0$ is the time at which c-persistency 'kicks in'. Trivially, if $(A + bc^\top)^\tau \gg 0$, then $c^\top (A + bc^\top)^\tau \gg 0$. The converse implication is obviously not true. In particular, $(A+bc^\top)^\tau$ may not be strictly positive for the minimal τ such that $c^\top (A + bc^\top)^\tau \gg 0$.

The proof of the implication $c^\top (A+bc^\top)^\tau \gg 0 \;\Rightarrow\; \gcd \mathcal{I} = 1$ claimed in statement (2) is facilitated by the following lemma.

Lemma 2.2. *Let $r := (r_0, r_1, \dots, r_k)^\top \in \mathbb{R}_+^{k+1}$ and let $e_1, \dots, e_{k+1}$ be the canonical basis of $\mathbb{R}^{k+1}$. Assume that* (P1) *holds, $d := \gcd \mathcal{I} > 1$ and there exists $i_0 \in \{0, \dots, d-1\}$ such that $r^\top e_i = r_{i-1} = 0$ for all $i \in \{1, \dots, k+1\}$ such that $i \not\equiv i_0 \mod d$. Then $r^\top (A + bc^\top) e_i = 0$ for all $i \in \{1, \dots, k+1\}$ such that $i \not\equiv (i_0 - 1) \mod d$.*

Proof. Note that

$$r^\top (A+bc^\top) = \big(r_0(\alpha_0+\beta\gamma_0)+r_1, r_0(\alpha_1+\beta\gamma_1)+r_2, \dots, r_0(\alpha_{k-1}+\beta\gamma_{k-1})+r_k, r_0(\alpha_k+\beta\gamma_k) \big). \quad (2.7)$$

We distinguish between the cases $r_0 > 0$ and $r_0 = 0$.

Case 1: $r_0 > 0$. In this case, it follows from the hypothesis on r that $1 \equiv i_0 \mod d$, implying that $i_0 = 1$. As $\alpha_k + \beta\gamma_k > 0$, we have that $k+1 \in \mathcal{I}$, and thus, $k+1 \equiv 0 \mod d$. Since $i_0 = 1$, the condition $k+1 \not\equiv (i_0 - 1) \mod d$ is not satisfied (we note in passing, that nevertheless, $r^\top (A + bc^\top) e_{k+1} = 0$). Let $i \in \{1, \dots, k\}$ be such that $i \not\equiv (i_0 - 1) \mod d$, or equivalently

$$i + 1 \not\equiv 1 \mod d. \quad (2.8)$$

Invoking again the hypothesis on r, we have that $r_i = 0$. Furthermore, (2.8) is equivalent to $i \not\equiv 0 \mod d$, showing that $i \notin \mathcal{I}$, whence $\alpha_{i-1} + \beta\gamma_{i-1} = 0$. It now follows from (2.7) that

$$r^\top (A + bc^\top) e_i = r_0(\alpha_{i-1} + \beta\gamma_{i-1}) + r_i = 0.$$

Case 2: $r_0 = 0$. Invoking (2.7), we see that $r^T (A + bc^\top) e_{k+1} = r_0(\alpha_k + \beta\gamma_k) = 0$. Let now $i \in \{1, \dots, k\}$ be such that $i + 1 \not\equiv i_0 \mod d$. Then, by hypothesis, $r_i = 0$, and thus,

$$r^\top (A + bc^\top) e_i = r_0(\alpha_{i-1} + \beta\gamma_{i-1}) + r_i = 0,$$

completing the proof. □

We continue with the proof of Proposition 2.1.

Proof of Proposition 2.1. (1) Note that A and $A+bc^\top$, as companion matrices, have the structure of a Leslie matrix familiar from stage-structured population models. Thus, irreducibility and primitivity results for Leslie matrices (see, for example [62]) can be applied to $A+bc^\top$. As $\alpha_k+\gamma_k>0$, [62, Theorem 6] yields that primitivity of $A+bc^\top$ is equivalent to $\gcd \mathcal{I}=1$.

(2) If $\gcd\mathcal{I}=1$, then, by statement (1), $A+bc^\top$ is primitive, whence $c^\top(A+bc^\top)^\tau \gg 0$ for some $\tau\in\mathbb{N}_0$.

We prove the converse by contraposition. To this end assume that $\gcd\mathcal{I}=d>1$ and define $c_\tau = \big(c_{\tau,0},\dots,c_{\tau,k}\big)^\top \in \mathbb{R}^{k+1}_+$ by $c_\tau^\top := c^\top(A+bc^\top)^\tau$ for every $\tau\in\mathbb{N}_0$. It is sufficient to show that, for every $\tau\in\mathbb{N}_0$, there exists $i_\tau\in\{0,\dots,d-1\}$ such that $c_{\tau,i-1}=0$ for all $i\in\{1,\dots,k+1\}$ satisfying $i\not\equiv i_\tau \mod d$. We do this by induction on τ. We have that $c_{0,i-1}=\gamma_{i-1}=0$ for all $i\in\{1,\dots,k+1\}$ such that $i\not\equiv 0 \mod d$ and the claim holds for $\tau=0$ with $i_0:=0$. Let now $\tau\in\mathbb{N}_0$ and assume that there exists $i_\tau\in\{0,\dots,d-1\}$ such that $c_{\tau,i-1}=0$ for all $i\in\{1,\dots,k+1\}$ satisfying $i\not\equiv i_\tau \mod d$. Then

$$c_{\tau+1,i-1}=c_{\tau+1}^\top e_i = c_\tau^\top(A+bc^\top)e_i \quad \forall i\in\{1,\dots,k+1\}.$$

Setting

$$i_{\tau+1} := \begin{cases} i_\tau-1, & \text{if } i_\tau\neq 0\\ d-1, & \text{if } i_\tau=0,\end{cases}$$

we have that $i_{\tau+1}\in\{0,\dots,d-1\}$, and an application of Lemma 2.2 with $r=c_\tau$ shows that $c_{\tau+1,i-1}=0$ for all $i\in\{1,\dots,k+1\}$ such that $i\not\equiv i_{\tau+1} \mod d$, completing the proof.

(3) Assume that $\gcd\mathcal{I}=1$. By statement (1), $A+bc^\top$ is primitive. It follows from [31, Theorem 8.5.7] that $(A+bc^\top)^{(k+1)+\ell(k-1)}\gg 0$, where ℓ is the length of the shortest cycle in the directed graph associated with $A+bc^\top$. Exploiting the companion matrix structure of $A+bc^\top$, it is straightforward to show that $\ell=l+1$. Consequently, $(k+1)+\ell(k-1)=(2+l)k-l$, and so, $(A+bc^\top)^{(2+l)k-l}\gg 0$, establishing the claim.

(4) As $\alpha_0+\gamma_0>0$, we have that $1\in\mathcal{I}$, implying that $\gcd\mathcal{I}=1$. Hence, by statement (3) with $l=0$, we obtain that $c^\top(A+bc^\top)^{2k}\gg 0$. To show that $c^\top(A+bc^\top)^{k+m}\gg 0$, we proceed in two steps.

Step 1. Set $\hat c:=(\gamma_0,0,\dots,0,\gamma_k)^\top$, $\hat c_m:=(0,\dots,0,\gamma_m,0,\dots,0)^\top$ (where γ_m is in position $m+1$), and

$$\hat A := \begin{pmatrix} \alpha_0 & 0 & \cdots & 0 & \alpha_k\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{pmatrix} \in \mathbb{R}^{(k+1)\times(k+1)}.$$

We claim that

$$\hat c_m^\top(\hat A+b\hat c^\top)^{k+m}\gg 0. \tag{2.9}$$

Noting that

$$\hat A+b\hat c^\top = \begin{pmatrix} \alpha_0+\beta\gamma_0 & 0 & \cdots & 0 & \alpha_k+\beta\gamma_k\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{pmatrix} = \begin{pmatrix} + & 0 & \cdots & 0 & +\\ + & 0 & \cdots & 0 & 0\\ 0 & + & \cdots & 0 & 0\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & \cdots & + & 0\end{pmatrix} \in \mathbb{R}^{(k+1)\times(k+1)},$$

where here, and in the following, $+$ denotes a positive entry, the value of which is immaterial. It is routine to check that, for every $j=0,\dots,m$,

$$\hat c_m^\top(\hat A+b\hat c^\top)^j \geq (0,\dots,0,+,0,\dots,0), \quad \text{where } + \text{ is in position } m+1-j.$$

Furthermore, for every $j = 1, \dots, k$, we have that

$$\hat{c}_m^\top (\hat{A} + b\hat{c}^\top)^{m+j} \geq (+, 0, \dots, 0, +, \dots, +), \quad \text{where } + \text{ is in the positions 1 and } k+1-i \text{ for all } i = 0, \dots, j-1.$$

In particular,

$$\hat{c}_m^\top (\hat{A} + b\hat{c}^\top)^{k+m} = (+, +, \dots, +) \gg 0,$$

establishing (2.9).

Step 2. As $c \geq \hat{c}, \hat{c}_m$ and $A \geq \hat{A}$, we have that $A + bc^\top \geq \hat{A} + b\hat{c}^\top$. Consequently,

$$c^\top (A + bc^\top)^j \geq \hat{c}_m^\top (\hat{A} + b\hat{c}^\top)^j \quad \forall\, j \in \mathbb{N}_0,$$

and it follows from (2.9) that $c^\top (A + bc^\top)^{k+m} \gg 0$. ☐

3 Boundedness and persistence

In this section, we explore boundedness and persistence properties of the Lur'e system (1.1). For which purpose, we require the following, not very restrictive, assumption on the linear part of system (1.1).

(L) At least one of the following two conditions holds:

$$\text{(i)} \quad \min_{|\zeta|=1} |\mathbf{G}(\zeta)| < \|\mathbf{G}\|_{H^\infty}, \qquad \text{(ii)} \quad \mathbf{a} \text{ and } \mathbf{c} \text{ are coprime},$$

where the polynomials $\mathbf{a}$ and $\mathbf{c}$ are defined in (2.1) and (2.4), respectively.

The above coprimeness condition can be characterized in terms of the linear observed system

$$x(t+1) = Ax(t), \quad y(t) = c^\top x(t), \tag{3.1}$$

where A and c are given by (2.5). Recall that system (3.1) (or the pair $(c^\top, A)$) is said to be *observable* if the following implication holds:

$$\big(c^\top A^t x^0 = 0 \ \ \forall\, t \in \mathbb{N}_0\big) \ \Rightarrow \ \big(x^0 = 0\big).$$

It is well known that (3.1) is observable if, and only if, the so-called *observability matrix*

$$\mathcal{O}(c^\top, A) := \begin{pmatrix} c^\top \\ c^\top A \\ \vdots \\ c^\top A^k \end{pmatrix} \in \mathbb{R}^{(k+1)\times(k+1)}$$

is invertible, see, for example, [38, Corollary 18.2] or [52, Theorem 25.12].

Lemma 3.1. *The polynomials* $\mathbf{a}$ *and* $\mathbf{c}$ *are coprime if, and only if, the pair* $(c^\top, A)$ *is observable.*

Proof. Invoking the so-called Hautus criterion for observability (see [42, Theorem 3.21] or [52, Theorem 13.15]), we need to show the equivalence of the coprimeness of $\mathbf{a}$ and $\mathbf{c}$ and the following full rank condition

$$\operatorname{rank} \begin{pmatrix} \zeta I - A \\ c^\top \end{pmatrix} = k + 1 \quad \forall\, \zeta \in \mathbb{C}. \tag{3.2}$$

We prove the contrapositive, that is, we show that the existence of a common root of $\mathbf{a}$ and $\mathbf{c}$ is equivalent to the failure of the rank condition (3.2). We start by assuming that (3.2) does not hold. Then there exists $\lambda \in \mathbb{C}$ such that

$$\operatorname{rank}\begin{pmatrix}\lambda I - A \\ c^\top\end{pmatrix} < k+1.$$

Consequently, for suitable, $a := (a_0, \ldots, a_k)^\top \in \mathbb{C}^{k+1}$, $a \neq 0$,

$$(\lambda I - A)a = 0 \quad \text{and} \quad c^\top a = 0. \tag{3.3}$$

Thus, by the first of the above two identities,

$$a_0 = \lambda a_1, \;\; a_1 = \lambda a_2, \;\; \ldots \;\; , a_{k-1} = \lambda a_k,$$

and so, $a_k \neq 0$ and $a_j = \lambda^{k-j} a_k$ for $j = 0, \ldots, k$. Using the first identity in (3.3) once more, we obtain

$$\mathbf{a}(\lambda) a_k = \Big(\lambda^{k+1} - \sum_{j=0}^{k} \alpha_j \lambda^{k-j}\Big) a_k = 0.$$

Hence, $\mathbf{a}(\lambda) = 0$. The second identity in (3.3) yields

$$\mathbf{c}(\lambda) a_k = \Big(\sum_{j=0}^{k} \gamma_j \lambda^{k-j}\Big) a_k = \sum_{j=0}^{k} \gamma_j a_j = c^\top a = 0,$$

showing that $\mathbf{c}(\lambda) = 0$. We conclude that $\mathbf{a}$ and $\mathbf{c}$ are not coprime.

Conversely, assume that $\mathbf{a}$ and $\mathbf{c}$ are not coprime. Then $\mathbf{a}$ and $\mathbf{c}$ have a common root λ. Setting $a := a_k\big(\lambda^k, \lambda^{k-1}, \ldots, 1\big)^\top$ for arbitrary $a_k \neq 0$, the above steps can be reversed to arrive at (3.3) which in turn implies that (3.2) does not hold. □

Remark 3.2. It follows from Lemma 3.1 and basic linear control theory (see, for example, [38, 52]) that, if the stability assumption (S) holds, then (L) is equivalent to [22, Assumption (A4)] and (L) is also equivalent to [53, Assumption (A)]. These equivalences allow us to apply certain results in [22, 53] in the current setting. In this paper, we prefer (L) to [22, Assumption (A4)] because, in contrast to the latter, (L) is formulated more directly in terms of the coefficients appearing in the higher-order system (1.1) and avoids control theoretic concepts. ◊

We introduce the following assumptions on the nonlinearity f.

(N1) $f(w,z) > 0$ for all $w \in U$ and $z > 0$ and

$$\lim_{z\to\infty} \big(pz - \max_{w\in U} f(w,z)\big) = \infty\,.$$

(N2) (N1) holds, $p < \infty$ and

$$\liminf_{z\downarrow 0} \Big(\min_{w\in U} \frac{f(w,z)}{z}\Big) > p\,. \tag{3.4}$$

A sufficient (but not necessary) condition for (N1) to hold is given by

(N1′) $f(w,z) > 0$ for all $w \in U$ and $z > 0$ and

$$\limsup_{z\to\infty} \Big(\max_{w\in U} \frac{f(w,z)}{z}\Big) < p. \tag{3.5}$$

Certain versions of assumptions (N1) and (N2) were employed in [22] (in a somewhat different setting), and, with (N1) replaced by (N1′), they also appear in [21, 23, 63]. The interested reader can find a biological interpretation of (3.4) and (3.5) in [23, Remark 4.2].

The following proposition provides a sufficient condition for the solutions of (1.1) to be bounded.

Proposition 3.3. *Assume that* (P1), (S), (L) *and* (N1) *hold, and let* $\Gamma \subset \mathbb{R}_+^{k+1}$ *be compact. Then there exists* $\rho > 0$ *such that the solution* x *of* (1.1) *satisfies*

$$|x(t)| \le \rho \quad \forall\, t \in \mathbb{N}_0$$

for all $u : \mathbb{N}_0 \to U$*,* $v : \mathbb{N}_0 \to V$ *and all initial conditions* $(x^0, x^{-1} \dots, x^{-k})^\top \in \Gamma$*.*

Proof. Define a set-valued function F by $F(z) := \{f(w,z) : w \in U\}$ for all $z \in \mathbb{R}_+$ and note that if x is a solution of (1.1), then $\tilde{x}$ given by $\tilde{x}(t) := \big(x(t), x(t-1), \dots, x(t-k)\big)^\top$ satisfies the difference inclusion

$$\tilde{x}(t+1) - A\tilde{x}(t) - \tilde{v}(t) \in bF(c^\top \tilde{x}(t)) \quad \forall\, t \in \mathbb{N}_0, \tag{3.6}$$

where $\tilde{v}(t) := \big(v(t), 0, \dots, 0\big)^\top$. Invoking (N1), we conclude that there exists $z_0 \ge 0$ and $\theta \in \mathcal{K}_\infty$ such that

$$\max F(z) = \max_{w \in U} f(w,z) \le pz - \theta(z) \quad \forall\, z \ge z_0.$$

Therefore, invoking Remark 3.2, it follows from the inclusion version of [53, Corollary 17][1] that, for compact $\Gamma \subset \mathbb{R}_+^{k+1}$, there exists $\rho > 0$ such that the solution $\tilde{x}$ of (3.6) satisfies

$$\|\tilde{x}(t)\| \le \rho \quad \forall\, t \in \mathbb{N}_0$$

for all initial conditions $\tilde{x}(0) \in \Gamma$ and all $v : \mathbb{N}_0 \to V$, establishing the claim. □

Next we introduce a persistency concept which will play a key role in this paper.

Definition 3.4. Let $d = (d_0, \dots, d_k)^\top \in \mathbb{R}^{k+1}$. We say that (1.1) is *ultimately semi-globally d-persistent* if, for every compact subset $\Gamma \subset \mathbb{R}_+^{k+1}$, $0 \notin \Gamma$, there exist $\tau \in \mathbb{N}_0$ and $\eta > 0$ such that the solution x of (1.1) satisfies

$$\sum_{j=0}^{k} d_j x(t + \tau - j) \ge \eta \quad \forall\, t \in \mathbb{N}_0 \tag{3.7}$$

for all $u : \mathbb{N}_0 \to U$, $v : \mathbb{N}_0 \to V$ and all initial conditions $(x^0, x^{-1} \dots, x^{-k})^\top \in \Gamma$. ◊

If (1.1) is ultimately semi-globally d-persistent for some $d \gg 0$, then (1.1) is ultimately semi-globally d-persistent for every $d \gg 0$, and we simply say that (1.1) is ultimately semi-globally persistent. In particular, if (1.1) is ultimately semi-globally persistent, then, for every compact subset $\Gamma \subset \mathbb{R}_+^{k+1}$, $0 \notin \Gamma$, there exist $\tau \in \mathbb{N}_0$ and $\eta > 0$ such that

$$\|\big(x(t+\tau), x(t+\tau-1), \dots, x(t+\tau-k)\big)^\top\|_1 \ge \eta \quad \forall\, t \in \mathbb{N}_0, \tag{3.8}$$

where $\|\cdot\|_1$ denotes the 1-norm on $\mathbb{R}^{k+1}$. Obviously, ultimate semi-global d-persistency for some $d \ge 0$ implies ultimate semi-global persistency, but the converse is not true in general. Furthermore, if (3.7) or (3.8) hold for $\tau = 0$, then we drop the word 'ultimately' and say that (1.1) is semi-globally d-persistent or semi-globally persistent, repectively.

It is clear that the persistency concept in Definition 3.4 depends on the compact sets U and V. However, as it is assumed that, in a given context, U and V are fixed, and in order to avoid lengthy and

[1]Whilst [53, Corollary 17] applies to difference equations, an inspection of the proof of [53, Corollary 17] shows that it extends in straightforward way to set-valued nonlinearities.

awkward terminology, we do not make the dependency on U and V explicit. Finally, it is straightforward to show that if $z \to f(w,z)$ is non-decreasing for every $w \in U$, then (1.1) is ultimately semi-globally d-persistent if, for every compact subset $\Gamma \subset \mathbb{R}_+^{k+1}$, $0 \notin \Gamma$, there exist $\tau \in \mathbb{N}_0$ and $\eta > 0$ such that the solution x of (1.1) satisfies (3.7) for all initial conditions $(x^0, x^{-1} \dots, x^{-k})^\top \in \Gamma$, all $u : \mathbb{N}_0 \to U$ and $v(t) \equiv 0$.

To investigate persistency properties of (1.1), we shall apply ideas from [23] for undelayed difference equations to the augmented system (2.6). To this end, we introduce the following condition.

(P2) (P1) holds and $\gcd \mathcal{I} = 1$.

By Proposition 2.1, if (P2) is satisfied, then there exists $\tau \in \mathbb{N}_0$ such that $c^\top(A + bc^\top)^\tau \gg 0$.

Theorem 3.5. *Assume that* (S), (L), (N2) *and* (P2) *hold. Then system* (1.1) *is semi-globally persistent and ultimately semi-globally c-persistent, where c is given by* (2.5)*. In particular, if* $\tau \in \mathbb{N}_0$ *is such that* $c^\top(A+bc^\top)^\tau \gg 0$*, then, for every compact subset* $\Gamma \subset \mathbb{R}_+^{k+1}$ *not containing* 0*, there exists* $\eta > 0$ *such that the solution* x *of* (1.1) *satisfies*

$$\sum_{j=0}^{k} \gamma_j x(t + \tau - j) \geq \eta \quad \forall\, t \in \mathbb{N}_0$$

for all $u : \mathbb{N}_0 \to U$*,* $v : \mathbb{N}_0 \to V$ *and all initial conditions* $(x^0, x^{-1} \dots, x^{-k})^\top \in \Gamma$*.*

Proof. Let $\Gamma \subset \mathbb{R}_+^{k+1}$ be compact and such that $0 \notin \Gamma$. It follows from Proposition 3.3 that there exists $\rho > 0$ such that the solution x of (1.1) satisfies $\sup_{t \in \mathbb{N}_0} |x(t)| \leq \rho$ for all $u : \mathbb{N}_0 \to U$, $v : \mathbb{N}_0 \to V$ and all initial conditions $(x^0, x^{-1} \dots, x^{-k})^\top \in \Gamma$. Moreover, by Proposition 2.1, assumption (P2) implies that $c^\top(A + bc^\top)^\tau \gg 0$ for some $\tau \in \mathbb{N}_0$. Therefore, the arguments in the proof of [23, Theorem 4.4][2] can be applied to establish that the augmented system (2.6) is semi-globally persistent and ultimately semi-globally c-persistent in the sense of [23], implying that (1.1) is semi-globally persistent and ultimately semi-globally c-persistent in the above sense. $\square$

4 Stability and convergence

In the following, under suitable assumptions on the nonlinearity f, we are going to explore certain stability and convergence properties of the forced system (1.1). To this end, we introduce, for each $w \in U$, the function

$$F_w : \mathbb{R}_+ \to \mathbb{R},\ z \mapsto z - \mathbf{G}(1) f(w,z) = z - \frac{\beta\gamma}{1-\alpha} f(w,z)$$

and investigate some of its properties. Of course, for F_w to be meaningful, 1 should not be a pole of $\mathbf{G}$. The latter is guaranteed by hypothesis (S).

Lemma 4.1. *Assume that* (P1), (S) *and* (N2) *are satisfied. For every* $w \in U$*, the following statements hold.*

(1) *There exists a unique* $z_w \in F_w^{-1}(0)$ *such that* $z_w > 0$ *and* $F_w(z) < 0$ *for all* $z \in (0, z_w)$*.*

(2) $\mathbb{R}_+ \subset F_w([z_w, \infty))$*, or, equivalently,* $F_w^{-1}(z) \cap [z_w, \infty) \neq \emptyset$ *for all* $z \in \mathbb{R}_+$*.*

[2] Here it is important to recall that (N1) forms part of the condition (N2). Whilst in [23, Theorem 4.4] it is imposed that (N1′) holds (an assumption more restrictive than (N1)), in the proof of [23, Theorem 4.4] condition (3.5) is used to establish uniform boundedness of the state trajectories generated by initial conditions in Γ and forcing functions $u : \mathbb{N}_0 \to U$ and $v : \mathbb{N}_0 \to V$. As we have seen, in the current setting, this uniform boundedness property is guaranteed by Proposition 3.3 which only assumes that (N1) holds.

(3) $F_w^{-1}(0)\backslash\{0\} \subset [z_w, \infty)$ *and* $F_w^{-1}(z) \subset (z_w, \infty)$ *for all* $z > 0$.

(4) *Let* $\xi \in V$ *and* $z_{w,\xi} \in F_w^{-1}(\gamma\xi/(1-\alpha))$. *Then* $x_{w,\xi} := z_{w,\xi}/\gamma$ *satisfies*

$$x_{w,\xi} = \alpha x_{w,\xi} + \beta f(w, \gamma x_{w,\xi}) + \xi. \tag{4.1}$$

Furthermore, $pz_{w,\xi} - \xi/\beta = f(w, z_{w,\xi})$.

(5) *Let* $\xi \in V$ *and* $z_{w,\xi} \in F_w^{-1}(\gamma\xi/(1-\alpha))\backslash\{0\}$. *If*

$$|f(w,z) - f(w, z_{w,\xi})| < p|z - z_{w,\xi}| \quad \forall\, z > 0,\ z \neq z_{w,\xi} \tag{4.2}$$

then $F_w^{-1}(\gamma\xi/(1-\alpha))\backslash\{0\} = \{z_{w,\xi}\}$. *In particular, if* (4.2) *holds with* $\xi = 0$, *then* $z_{w,0} = z_w$.

Proof. Let $w \in U$. As $p := 1/\mathbf{G}(1)$, the function F_w can be expressed as

$$F_w(z) = \frac{pz - f(w,z)}{p} \quad \forall\, z \in \mathbb{R}_+. \tag{4.3}$$

(1) It follows from (4.3) and (N2) that there exist $z_- > 0$ and $z_+ > z_-$ such that $F_w(z) < 0$ for all $z \in (0, z_-)$ and $F_w(z) > 0$ for all $z \in (z_+, \infty)$. The intermediate-value theorem for continuous functions guarantees the existence of $z_w > 0$ with the stated properties.

(2) By (4.3) and (N2), $F_w(z) \to \infty$ as $z \to \infty$. As $F_w(z_w) = 0$, it follows from the intermediate-value theorem for continuous functions that $\mathbb{R}_+ \subset F_w([z_w, \infty))$.

(3) By statement (1), $F_w(z) < 0$ for all $z \in (0, z_w)$. Hence, $F_w^{-1}(0)\backslash\{0\} \subset [z_w, \infty)$ and $F_w^{-1}(z) \subset (z_w, \infty)$ for all $z > 0$.

(4) Let $\xi \in V$ and $z_{w,\xi} \in F_w^{-1}(\gamma\xi/(1-\alpha))$. Then

$$z_{w,\xi} - \frac{\beta\gamma}{1-\alpha} f(w, z_{w,\xi}) = \frac{\gamma\xi}{1-\alpha}, \tag{4.4}$$

and thus,

$$(1-\alpha)x_{w,\xi} - \beta f(w, z_{w,\xi}) = \xi,$$

from which (4.1) follows. Moreover, multiplying (4.4) by $p = (1-\alpha)/(\beta\gamma)$ leads to $pz_{w,\xi} - \xi/\beta = f(w, z_{w,\xi})$.

(5) Let $\xi \in V$ and $z_{w,\xi}, y \in F_w^{-1}(\gamma\xi/(1-\alpha))\backslash\{0\}$. By statement (4),

$$f(w, z_{w,\xi}) = pz_{w,\xi} - \xi/\beta \quad \text{and} \quad f(w,y) = py - \xi/\beta,$$

and so, $|f(w, z_{w,\xi}) - f(w,y)| = p|z_{w,\xi} - y|$. It follows from (4.2) that $y = z_{w,\xi}$. □

In the following, for given $u_\mathrm{e} \in U$ and $v_\mathrm{e} \in V$, we shall identify equilibria of the difference equation

$$x(t+1) = \sum_{j=0}^{k} \alpha_j x(t-j) + \beta f\big(u_\mathrm{e}, \sum_{j=0}^{k} \gamma_j x(t-j)\big) + v_\mathrm{e}, \tag{4.5}$$

that is, of system (1.1) with $u(t) \equiv u_\mathrm{e}$ and $v(t) \equiv v_\mathrm{e}$, and investigate the stability properties of these equilibria. As (1.1) is a forced system, this will require results from the so-called input-to-state stability theory of nonlinear control theory which provides an extension of Lyapunov theory to forced systems [14, 45, 60].

It follows from statement (4) of Lemma 4.1 that, for $z_\mathrm{e}(u_\mathrm{e}, v_\mathrm{e}) \in F_{u_\mathrm{e}}^{-1}\big(\gamma v_\mathrm{e}/(1-\alpha)\big)$ and $x_\mathrm{e}(u_\mathrm{e}, v_\mathrm{e}) := z_\mathrm{e}(u_\mathrm{e}, v_\mathrm{e})/\gamma$, we have that

$$x_\mathrm{e}(u_\mathrm{e}, v_\mathrm{e}) = \alpha x_\mathrm{e}(u_\mathrm{e}, v_\mathrm{e}) + \beta f\big(u_\mathrm{e}, \gamma x_\mathrm{e}(u_\mathrm{e}, v_\mathrm{e}\big) + v_\mathrm{e} \quad \text{and} \quad pz_\mathrm{e}(u_\mathrm{e}, v_\mathrm{e}) - v_\mathrm{e}/\beta = f(u_\mathrm{e}, z_\mathrm{e}(u_\mathrm{e}, v_\mathrm{e})). \tag{4.6}$$

In particular, $x_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}})$ is an equilibrium of system (4.5). Moreover, by statement (5) of Lemma 4.1, if

$$|f(u_{\mathrm{e}}, z) - f(u_{\mathrm{e}}, z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}}))| < p|z - z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}})| \quad \forall\, z > 0,\ z \neq z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}}), \tag{4.7}$$

then $x_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}})$ is the unique positive equilibrium of (4.5).

The following hypothesis (cf. [23, hypothesis (N3)]) will play a key role in the context of the stability and convergence theory to be developed.

(N3) Hypothesis (N2) and inequality (4.7) hold.

The inequality (4.7) is a so-called sector condition. Appealing to (4.6), we see that the graphical interpretation of (4.7) is as follows: the graph of $z \mapsto f(z, u_{\mathrm{e}})$ is strictly 'sandwiched' between the straight lines $z \mapsto pz - v_{\mathrm{e}}/\beta$ and $z \mapsto -pz + 2z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}}) - v_{\mathrm{e}}/\beta$, with the three graphs intersecting at the point $(z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}}), pz_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}}) - v_{\mathrm{e}}/\beta)$. A number of sufficient conditions on the nonlinearity f for (N3) to hold and classes of examples satisfying (N3) can be found in [21, 23, 22], see, for example, [22, Lemma 5.4 and Table 5.1].

Statement (1) of the following theorem, provides a stability result which is very much in the spirit of the input-to-state stability from nonlinear control theory, see the survey articles [14, 60], the book section [42, Section 5.8] and the recent monograph [45]. Statement (2) is reminiscent of control theoretic convergent-input convergent-state results [6].

Theorem 4.2. *Let $u_{\mathrm{e}} \in U$, $v_{\mathrm{e}} \in V$ and $z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}}) \in F_{u_{\mathrm{e}}}^{-1}(\gamma v_{\mathrm{e}}/(1-\alpha)) \backslash \{0\}$ and set $x_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}}) := z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}})/\gamma$. If* (P1), (S), (L) *and* (N3) *hold and* (1.1) *is ultimately semi-globally c-persistent, then the following statements hold.*

(1) *For every compact set $\Gamma \subset \mathbb{R}_+^{k+1}$ such that $0 \notin \Gamma$, there exist $\psi \in \mathcal{KL}$, $\phi \in \mathcal{K}$ and $r > 0$ such that, for all initial conditions $(x^0, x^{-1}, \ldots, x^{-k})^\top \in \Gamma$, all $u\colon \mathbb{N}_0 \to U$ and all $v\colon \mathbb{N}_0 \to V$, the solution x of* (1.1) *satisfies*

$$|x(t) - x_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}})| \leq \psi\Big(\sum_{j=0}^{k} |x^{-j} - x_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}})|, t\Big) + \phi\big(\|v - v_{\mathrm{e}}\|_{\ell^\infty(0,t)} + \|\beta_r \circ u\|_{\ell^\infty(0,t)}\big) \quad \forall\, t \in \mathbb{N}_0, \tag{4.8}$$

where $\|v - v_{\mathrm{e}}\|_{\ell^\infty(0,t)} := \max\{\|v(s) - v_{\mathrm{e}}\| : s = 0, 1, \ldots, t\}$ and

$$\beta_r(w) := \max_{0 \leq z \leq r} |f(u_{\mathrm{e}}, z) - f(w, z)| \quad \forall\, w \in U. \tag{4.9}$$

(2) *For all $(x^0, x^{-1}, \ldots, x^{-k})^\top \in \mathbb{R}_+^{k+1}$, all $u\colon \mathbb{N}_0 \to U$, all $v\colon \mathbb{N}_0 \to V$ such that $\sum_{j=0}^{k} x^{-j} + \|v\|_{\ell^\infty} > 0$, $u(t) \to u_{\mathrm{e}}$ and $v(t) \to v_{\mathrm{e}}$ as $t \to \infty$, the solution x of* (1.1) *has the convergence property $x(t) \to x_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}})$ as $t \to \infty$.*

We remark that if (N1) is replaced by the stronger condition (N1′) and the additional assumption

$$\limsup_{z \to z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}})} \frac{\big|f(u_{\mathrm{e}}, z) - f(u_{\mathrm{e}}, z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}}))\big|}{\big|z - z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}})\big|} < p \tag{4.10}$$

is satisfied, then ψ and ϕ are of the form $\psi(s, t) = \lambda \kappa^t s$ and $\phi(s) = \nu s$, where $\lambda, \nu > 0$ and $\kappa \in (0, 1)$, as follows from arguments similar to those used in the proof of [23, Theorem 5.2]. The condition (4.10) means that the graph of $z \mapsto f(u_{\mathrm{e}}, z)$ is not tangential to the lines $z \mapsto pz - v_{\mathrm{e}}/\beta$ and $z \mapsto -pz + 2z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}}) - v_{\mathrm{e}}/\beta$ at $z = z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}})$.

If, in Theorem 4.2, the nonlinearity f is globally Lipschitz in its first variable, that is, there exists $\lambda > 0$ such that

$$|f(w_1, z) - f(w_2, z)| \leq \lambda \|w_1 - w_2\| \quad \forall\, z \in \mathbb{R}_+,\ \forall\, w_1, w_2 \in U, \tag{4.11}$$

then the constant r becomes redundant and (4.8) simplifies to

$$|x(t)-x_{\mathrm{e}}(u_{\mathrm{e}},v_{\mathrm{e}})| \le \psi\Big(\sum_{j=0}^{k}|x^{-j}-x_{\mathrm{e}}(u_{\mathrm{e}},v_{\mathrm{e}})|,t\Big)+\phi\big(\|v-v_{\mathrm{e}}\|_{\ell^\infty(0,t)}+\lambda\|u_{\mathrm{e}}-u\|_{\ell^\infty(0,t)}\big) \quad \forall\, t\in\mathbb{N}_0.$$

The global Lipschitz property (4.11) is, for example, satisfied for the following nonlinearities.

(a) Beverton-Holt nonlinearity:

$$f(w,z):=\frac{a_1 z}{a_2+wz}, \quad z\ge 0,\ w\in U:=[u_0,u_1], \quad \text{where } a_1>0,\ a_2>0 \text{ and } 0<u_0<u_1. \tag{4.12}$$

(b) Ricker nonlinearity [51]:

$$f(w,z):=z\mathrm{e}^{-\rho wz}, \quad z\ge 0,\ w\in U:=[u_0,u_1], \quad \text{where } \rho>0 \text{ and } 0<u_0<u_1.$$

Proof of Theorem 4.2. Throughout the proof, we shall write z_{e} and x_{e} for $z_{\mathrm{e}}(u_{\mathrm{e}},v_{\mathrm{e}})$ and $x_{\mathrm{e}}(u_{\mathrm{e}},v_{\mathrm{e}})$, respectively, and make use of the notation for left translates defined in (1.3).

(1) It follows from Proposition 3.3 and Theorem 3.5 that there exist $r>0$, a compact set $\hat\Gamma\subset\mathbb{R}^{k+1}$, $\tau\in\mathbb{N}_0$ and $\eta\in(0,z_{\mathrm{e}})$ such that, for all $(x^0,x^{-1},\dots,x^{-k})^\top\in\Gamma$, all $u\colon\mathbb{N}_0\to U$ and all $v\colon\mathbb{N}_0\to V$, the solution x of (1.1) satisfies

$$\sum_{j=0}^{k}\gamma_j x(t-j)\le r, \quad (x(t)-x_{\mathrm{e}},\dots,x(t-k)-x_{\mathrm{e}})^\top\in\hat\Gamma \quad\text{and}\quad \sum_{j=0}^{k}\gamma_j x_\tau(t-j)\ge\eta \quad \forall\, t\in\mathbb{N}_0.$$

Defining

$$\hat f:\mathbb{R}\to\mathbb{R},\quad z\mapsto\begin{cases} f(u_{\mathrm{e}},z+z_{\mathrm{e}})-f(u_{\mathrm{e}},z_{\mathrm{e}}), & z\ge -z_{\mathrm{e}}+\eta\\ f(u_{\mathrm{e}},\eta)-f(u_{\mathrm{e}},z_{\mathrm{e}}), & z<-z_{\mathrm{e}}+\eta\end{cases} \tag{4.13}$$

and invoking (4.6), we see that x satisfies

$$x_\tau(t+1)-x_{\mathrm{e}}=\sum_{j=0}^{k}\alpha_j(x_\tau(t-j)-x_{\mathrm{e}})+\beta\hat f\Big(\sum_{j=0}^{k}\gamma_j(x_\tau(t-j)-x_{\mathrm{e}})\Big)+v_\tau(t)-v_{\mathrm{e}}+q_\tau(t) \quad \forall\, t\in\mathbb{N}_0, \tag{4.14}$$

where

$$q(t):=f\Big(u(t),\sum_{j=0}^{k}\gamma_j x(t-j)\Big)-f\Big(u_{\mathrm{e}},\sum_{j=0}^{k}\gamma_j x(t-j)\Big). \tag{4.15}$$

We note that

$$|q(t)|\le\beta_r(u(t))=(\beta_r\circ u)(t) \quad \forall\, t\in\mathbb{N}_0. \tag{4.16}$$

Noting that

$$|\hat f(z)|<p|z| \quad \forall\, z\in\mathbb{R},\ z\ne 0 \quad\text{and}\quad \big(p|z|-|\hat f(z)|\big)\to\infty \quad\text{as } |z|\to\infty,$$

we conclude that there exists $\rho\in\mathcal{K}_\infty$ such that

$$|\hat f(z)|\le p|z|-\rho(|z|) \quad \forall\, z\in\mathbb{R}. \tag{4.17}$$

An application of Theorem A.1 in the context of (4.14), with initial-value set given by $\hat\Gamma$, shows that there exist $\hat\psi\in\mathcal{KL}$ and $\hat\phi\in\mathcal{K}$ such that

$$|x_\tau(t)-x_{\mathrm{e}}|\le\hat\psi\Big(\sum_{j=0}^{k}|x_\tau(-j)-x_{\mathrm{e}}|,t\Big)+\hat\phi\big(\|v_\tau-v_{\mathrm{e}}\|_{\ell^\infty(0,t)}+\|q_\tau\|_{\ell^\infty(0,t)}\big) \quad \forall\, t\in\mathbb{N}_0. \tag{4.18}$$

As every solution x of (1.1) satisfies

$$x(t+1) - x_\mathrm{e} = \sum_{j=0}^{k} \alpha_j (x(t-j) - x_\mathrm{e}) + \beta \tilde{f}\Big(\sum_{j=0}^{k} \gamma_j (x(t-j) - x_\mathrm{e})\Big) + v(t) - v_\mathrm{e} + q(t) \quad \forall\, t \in \mathbb{N}_0,$$

where

$$\tilde{f} : \mathbb{R} \to \mathbb{R}, \quad z \mapsto \begin{cases} f(u_\mathrm{e}, z + z_\mathrm{e}) - f(u_\mathrm{e}, z_\mathrm{e}), & z \geq -z_\mathrm{e} \\ f(u_\mathrm{e}, 0) - f(u_\mathrm{e}, z_\mathrm{e}), & z < -z_\mathrm{e}, \end{cases}$$

it follows from the linear boundedness of $\tilde{f}$ [3] that there exist $\kappa, \lambda > 0$ such that

$$|x(t) - x_\mathrm{e}| \leq \kappa \sum_{j=0}^{k} |x(-j) - x_\mathrm{e}| + \lambda\big(\|v - v_\mathrm{e}\|_{\ell^\infty(0,t)} + \|q\|_{\ell^\infty(0,t)}\big) \quad \forall\, t \in \{0, 1 \ldots, \tau\}.$$

Combining this with (4.18), we conclude that there exist $\psi \in \mathcal{KL}$, $\phi \in \mathcal{K}$ and $r > 0$ such that, for all initial conditions $(x^0, x^{-1}, \ldots, x^{-k})^\top \in \Gamma$, all $u\colon \mathbb{N}_0 \to U$ and all $v\colon \mathbb{N}_0 \to V$, the solution x of (1.1) satisfies

$$|x(t) - x_\mathrm{e}, v_\mathrm{e})| \leq \psi\Big(\sum_{j=0}^{k} |x^{-j} - x_\mathrm{e}|, t\Big) + \phi\big(\|v - v_\mathrm{e}\|_{\ell^\infty(0,t)} + \|q\|_{\ell^\infty(0,t)}\big) \quad \forall\, t \in \mathbb{N}_0.$$

This, together with (4.16), yields (4.8).

(2) Let $(x^0, x^{-1}, \ldots, x^{-k})^\top \in \mathbb{R}_+^{k+1}$, $u\colon \mathbb{N}_0 \to U$ and $v\colon \mathbb{N}_0 \to V$ be such that $\sum_{j=0}^{k} x^{-j} + \|v\|_{\ell^\infty} > 0$, $u(t) \to u_\mathrm{e}$ and $v(t) \to v_\mathrm{e}$ as $t \to \infty$, and let x be the corresponding solution of (1.1). We claim that there exist $\tau \in \mathbb{N}_0$ and $\eta > 0$ such that

$$\sum_{j=0}^{k} \gamma_j x_\tau(t-j) \geq \eta \quad \forall\, t \in \mathbb{N}_0. \tag{4.19}$$

If $\sum_{j=0}^{k} x^{-j} > 0$, then this is an immediate consequence of Theorem 3.5. Let us now consider the case wherein $\sum_{j=0}^{k} x^{-j} = 0$. Then $\|v\|_{\ell^\infty} > 0$, and so there exists $\sigma \in \mathbb{N}_0$ such that $x(\sigma) > 0$. As x_σ solves (1.1) with u and v replaced by u_σ and v_σ, respectively, and $\big(x_\sigma(0), x_\sigma(-1), \ldots, x_\sigma(-k)\big)^\top \neq 0$, Theorem 3.5 guarantees the existence of $\tilde{\sigma} \in \mathbb{N}_0$ and $\eta > 0$ such that

$$\sum_{j=0}^{k} \gamma_j x_\sigma(t + \tilde{\sigma} - j) \geq \eta \quad \forall\, t \in \mathbb{N}_0.$$

Consequently, (4.19) holds with $\tau = \sigma + \tilde{\sigma}$.

Combining (4.19) with the fact that x is bounded (as follows from Proposition 3.3), we see that

$$X := \mathrm{closure}\{(x_\tau(t), x_\tau(t-1), \ldots, x_\tau(t-k))^\top : t \in \mathbb{N}_0\} \subset \mathbb{R}_+^{k+1}$$

is compact and $0 \notin X$. Invoking statement (1) with $\Gamma = X$, shows that there exist $\psi \in \mathcal{KL}$, $\phi \in \mathcal{K}$ and $r > 0$ such that, for every $\theta \geq \tau$,

$$|x_\theta(t) - x_\mathrm{e}| \leq \psi\Big(\sum_{j=0}^{k} |x_\theta(-j) - x_\mathrm{e}|, t\Big) + \phi\big(\|v_\theta - v_\mathrm{e}\|_{\ell^\infty(0,t)} + \|\beta_r \circ u_\theta\|_{\ell^\infty(0,t)}\big) \quad \forall\, t \in \mathbb{N}_0, \tag{4.20}$$

[3] Whilst the definition of $\tilde{f}$ is identical to that of $\hat{f}$ with $\eta = 0$, we remark that in the case wherein $v_\mathrm{e} = 0$ and $f(u_\mathrm{e}, 0) = 0$, we have that $f(u_\mathrm{e}, z_\mathrm{e}) = pz_\mathrm{e}$, and thus $|\tilde{f}(u_\mathrm{e}, -z_\mathrm{e})| = p| - z_\mathrm{e}|$. Consequently, there does not exist $\rho \in \mathcal{K}_\infty$ such that (4.17) holds with $\hat{f}$ replaced by $\tilde{f}$.

where β_r is defined by (4.9). Let $\varepsilon > 0$ be given. As $u(t) \to u_{\mathrm{e}}$ and $v(t) \to v_{\mathrm{e}}$ as $t \to \infty$, there exists $\theta_0 \geq \tau$ such that

$$\phi\big(\|v_{\theta_0} - v_{\mathrm{e}}\|_{\ell^\infty(0,t)} + \|\beta_r \circ u_{\theta_0}\|_{\ell^\infty(0,t)}\big) \leq \frac{\varepsilon}{2} \quad \forall\, t \in \mathbb{N}_0.$$

Finally, choosing $\theta_1 \in \mathbb{N}_0$ such that

$$\psi\Big(\sum_{j=0}^{k} |x_{\theta_0}(-j) - x_{\mathrm{e}}|, t\Big) \leq \frac{\varepsilon}{2} \quad \forall\, t \geq \theta_1,$$

it follows from (4.20) (with $\theta = \theta_0$) that $|x_{\theta_0}(t) - x_{\mathrm{e}}| \leq \varepsilon$ for all $t \geq \theta_1$. Hence, $|x(t) - x_{\mathrm{e}}| \leq \varepsilon$ for all $t \geq \theta_0 + \theta_1$, completing the proof. □

The following corollary is an immediate consequence of Theorems 3.5 and 4.2.

Corollary 4.3. *Let $u^{\mathrm{e}} \in U$, $v^{\mathrm{e}} \in V$ and $z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}}) \in F_{u_{\mathrm{e}}}^{-1}(\gamma v_{\mathrm{e}}/(1-\alpha))$ and set $x_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}}) := z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}})/\gamma$. If* (S), (L), (P2) *and* (N3) *are satisfied, then statements* (1) *and* (2) *of* Theorem 4.2 *hold.*

The final result in this section provides a sufficient condition for the convergence property in statement (2) of Theorem 4.2 to hold for every $v_{\mathrm{e}} \in V$.

Corollary 4.4. *Assume that* (S), (L), (P2) *and* (N2) *are satisfied and let $u_{\mathrm{e}} \in U$ be fixed, but arbitrary. Then there exists $z_{u_{\mathrm{e}}} := z_{\mathrm{e}}(u_{\mathrm{e}}, 0) \in F_{u_{\mathrm{e}}}^{-1}(0)$ such that $z_{u_{\mathrm{e}}} > 0$ and $(0, z_{u_{\mathrm{e}}}) \cap F_{u_{\mathrm{e}}}^{-1}(0) = \emptyset$. If*

$$\left|\frac{f(u_{\mathrm{e}}, z) - f(u_{\mathrm{e}}, \xi)}{z - \xi}\right| < p \quad \forall\, (z, \xi) \in (0, \infty) \times [z_{u_{\mathrm{e}}}, \infty),\ z \neq \xi, \tag{4.21}$$

then, for all $(x^0, x^{-1}, \ldots, x^{-k})^\top \in \mathbb{R}_+^{k+1}$, all $v_{\mathrm{e}} \in V$, all $u\colon \mathbb{N}_0 \to U$, all $v\colon \mathbb{N}_0 \to V$ such that $\sum_{j=0}^{k} x^{-j} + \|v\|_{\ell^\infty} > 0$, $u(t) \to u_{\mathrm{e}}$ and $v(t) \to v_{\mathrm{e}}$ as $t \to \infty$, the solution x of (1.1) *has the convergence property $x(t) \to x_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}})$ as $t \to \infty$, where $x_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}}) = z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}})/\gamma$ and $\{z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}})\} = F_{u_{\mathrm{e}}}^{-1}(\gamma v_{\mathrm{e}}/(1-\alpha))$.*

Proof. The existence of $z_{u_{\mathrm{e}}} \in F_{u_{\mathrm{e}}}^{-1}(0)$ such that $z_{u_{\mathrm{e}}} > 0$ and $(0, z_{u_{\mathrm{e}}}) \cap F_{u_{\mathrm{e}}}^{-1}(0) = \emptyset$ is a consequence of statement (1) of Lemma 4.1.

To prove the convergence property, assume that (4.21) holds and let $v_{\mathrm{e}} \in V$. By statement (2) of Lemma 4.1,

$$F_{u_{\mathrm{e}}}^{-1}\big(\gamma v_{\mathrm{e}}/(1-\alpha)\big) \cap [z_{u_{\mathrm{e}}}, \infty) \neq \emptyset.$$

For $z_{\mathrm{e}}(u_{\mathrm{e}}, v_{\mathrm{e}}) \in F_{u_{\mathrm{e}}}^{-1}\big(\gamma v_{\mathrm{e}}/(1-\alpha)\big) \cap [z_{u_{\mathrm{e}}}, \infty)$, condition (4.21) guarantees that (4.7), and hence, (N3) is satisfied. The claim now follows from statement (2) of Theorem 4.2. □

In the following, we identify classes of nonlinearities which satisfy the relevant assumptions in Corollary 4.4. The next two lemmas are straightforward consequences of [6, Lemma 6.8] and [6, Lemma 6.9], respectively.

Lemma 4.5. *Assume that $f : U \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, where $U \subset \mathbb{R}^n$ is compact, and let $p > 0$. Let $u_{\mathrm{e}} \in U$ and assume that $z \mapsto f(u_{\mathrm{e}}, z)$ is continuously differentiable, $f(u_{\mathrm{e}}, 0) = 0$, $f'(u_{\mathrm{e}}, z) \geq 0$ for all $z > 0$, $f'(u_{\mathrm{e}}, 0) > p$, $z \mapsto f'(u_{\mathrm{e}}, z)$ is non-increasing and $\lim_{z\to\infty} f'(u_{\mathrm{e}}, z) < p$, where f' denotes the derivative of f with respect to the second argument z. Then there exists $z_{u_{\mathrm{e}}} > 0$ such that $f(u_{\mathrm{e}}, z_{u_{\mathrm{e}}}) = p z_{u_{\mathrm{e}}}$ and* (4.21) *holds.*

As a specific example, consider the Beverton-Holt nonlinearity given by (4.12): for $p > 0$ and $u_{\mathrm{e}} \in U = [u_0, u_1]$, the conditions in Lemma 4.5 are satisfied, provided that $p < a_1/a_2$. The latter condition is also sufficient for (N2) to hold for the Beverton-Holt nonlinearity.

The next lemma considers a class of Ricker nonlinearities.

Lemma 4.6. *Let $U := [u_0, u_1]$, where $0 < u_0 < u_1$ and let $f : U \times \mathbb{R}_+ \to \mathbb{R}_+$ be given by*

$$f(w, z) := z\mathrm{e}^{-\rho w z} \quad \forall\, (w, z) \in U \times \mathbb{R}_+,$$

where ρ is a positive parameter. Assume that

$$\mathrm{e}^{-2} \le p < 1. \tag{4.22}$$

Then (N2) *is satisfied, and, for given $u_\mathrm{e} \in U$, the number $z_{u_\mathrm{e}} := -(\ln p)/(\rho u_\mathrm{e}) > 0$ is the unique positive solution of the equation $f(u_\mathrm{e}, z) = pz$ and* (4.21) *holds.*

A further class of nonlinearities satisfying (4.21) is provided in Lemma 5.3, at the end of the next section.

5 Response to almost periodic additive forcing

In this section, we will investigate the response of system (1.1) to non-negative almost periodic additive forcing functions v under the assumption that $u(t) \to u_\mathrm{e}$ as $t \to \infty$. The theory of almost periodic functions defined on $\mathbb{Z}$ parallels that of functions defined on $\mathbb{R}$ (see, for example, [11] as a general reference on almost periodicity). The basic theory of almost periodic functions defined on $\mathbb{Z}$ was developed in [54], and further details can be found, for example, in [11, Section I.6] and [28, Appendix B].

We begin by presenting some relevant background material on almost periodic functions defined on the discrete-time domain $\mathbb{Z}$. A set $Z \subset \mathbb{Z}$ is called *relatively dense* (in $\mathbb{Z}$) if there exists $s \in \mathbb{N}$ such that

$$\{t, \dots, t+s\} \cap Z \neq \emptyset \quad \forall\, t \in \mathbb{Z}.$$

For $\varepsilon > 0$, we say that $t_0 \in \mathbb{Z}$ is an *ε-period* of $w : \mathbb{Z} \to \mathbb{R}^m$ if $\|w(t) - w(t + t_0)\| \le \varepsilon$ for all $t \in \mathbb{Z}$. We denote by $P(w, \varepsilon) \subset \mathbb{Z}$ the set of ε-periods of w and we say that $w : \mathbb{Z} \to \mathbb{R}^m$ is *almost periodic* if $P(w, \varepsilon)$ is relatively dense in $\mathbb{Z}$ for every $\varepsilon > 0$. We denote the set of almost periodic functions $w : \mathbb{Z} \to W \subset \mathbb{R}^m$ by $AP(\mathbb{Z}, W)$. The functions in $AP(\mathbb{Z}, W)$ are bounded, and, if W is a linear subspace of $\mathbb{R}^m$, then $AP(\mathbb{Z}, W)$ is a closed subspace of $\ell^\infty(\mathbb{Z}, W)$. It is convenient to set $AP(\mathbb{Z}) := AP(\mathbb{Z}, \mathbb{R})$. Trivially, a periodic function is almost periodic. An example of a function which is almost periodic, but not periodic, is $w : \mathbb{Z} \to \mathbb{R}$ defined by $w(t) := \sin(\pi\sqrt{2}\, t)$ for $t \in \mathbb{Z}$.

The theorem below is the main result of this section.

Theorem 5.1. *Assume that* (S), (P2), (L) *and* (N2) *are satisfied, let $u_\mathrm{e} \in U$ be fixed and let $v^{\mathrm{ap}} \in AP(\mathbb{Z}, V)$. If*

$$\left| \frac{f(u_\mathrm{e}, z) - f(u_\mathrm{e}, \xi)}{z - \xi} \right| < p \quad \forall\, (z, \xi) \in (0, \infty) \times (0, \infty),\ z \neq \xi, \tag{5.1}$$

then the following statements hold.

(1) *There exists $x^{\mathrm{ap}} \in AP(\mathbb{Z}, \mathbb{R}_+)$ satisfying the bilateral equation*

$$x(t+1) = \sum_{j=0}^{k} \alpha_j x(t-j) + \beta f\Big(u_\mathrm{e}, \sum_{j=0}^{k} \gamma_j x(t-j)\Big) + v^{\mathrm{ap}}(t) \quad \forall\, t \in \mathbb{Z}, \tag{5.2}$$

and x^{ap} is the unique bounded (on $\mathbb{Z}$) solution of (5.2)*. Furthermore,*

$$\inf_{t \in \mathbb{Z}} \Big(\sum_{j=0}^{k} \gamma_j x^{\mathrm{ap}}(t-j) \Big) > 0, \tag{5.3}$$

and, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $P(v^{\mathrm{ap}}, \delta) \subset P(x^{\mathrm{ap}}, \varepsilon)$. In particular, if v^{ap} is t_0-periodic for some $t_0 \in \mathbb{N}$, then x^{ap} is t_0-periodic.

(2) *Let u and v be functions from $\mathbb{N}_0$ to U and V respectively, and let $x : \{-k,\dots,-1\}\cup\mathbb{N}_0$ be a solution of the initial-value problem* (1.1). *If* $\lim_{t\to\infty} u(t) = u^{\mathrm{e}}$ *and* $\lim_{t\to\infty}\big(v(t)-v^{\mathrm{ap}}(t)\big) = 0$, *then* $\lim_{t\to\infty}\big(x(t) - x^{\mathrm{ap}}(t)\big) = 0$ *and*

$$\liminf_{t\to\infty}\Big(\sum_{j=0}^{k}\gamma_j x(t-j)\Big) = \inf_{t\in\mathbb{Z}}\Big(\sum_{j=0}^{k}\gamma_j x^{\mathrm{ap}}(t-j)\Big) > 0. \tag{5.4}$$

We remark that, by (N2), there exists $z_{\mathrm{e}} > 0$ such that $f(u_{\mathrm{e}}, z) > pz$ for all $z \in (0, z_{\mathrm{e}})$ and $f(u_{\mathrm{e}}, z_{\mathrm{e}}) = pz_{\mathrm{e}}$. On the other hand, (5.1) implies $|f(u_{\mathrm{e}}, 0) - f(u_{\mathrm{e}}, z)| \le pz$ for all $z \ge 0$. Consequently, if (N2) and (5.1) hold, then $f(u_{\mathrm{e}}, 0) > 0$. Note the difference between conditions (4.21) and (5.1): the range of ξ for which the inequality is required to hold is $[z_{\mathrm{e}}, \infty)$ in (4.21) as compared to $(0, \infty)$ in (5.1).

Almost periodicity can also be defined for functions with domain $\mathbb{N}_0$ by simply replacing $\mathbb{Z}$ with $\mathbb{N}_0$ in the above definitions of relative denseness, ε-period and almost periodicity. Letting $AP(\mathbb{N}_0,\mathbb{R}^m)$ denote space of almost periodic functions $\mathbb{N}_0 \to \mathbb{R}^m$, then, as explained in [26], the restriction map $AP(\mathbb{Z},\mathbb{R}^m) \to AP(\mathbb{N}_0,\mathbb{R}^m)$, $w \mapsto w|_{\mathbb{N}_0}$ is bijective. In particular, in Theorem 5.1, we could let v^{ap} be an almost periodic function defined on $\mathbb{N}_0$, provided that, in (5.2), v^{ap} is replaced by its unique bilateral extension to $\mathbb{Z}$.

The following simple lemma will be used in the proof of Theorem 5.1.

Lemma 5.2. *Let $w \in AP(\mathbb{Z},\mathbb{R}^m)$. If $\lim_{t\to\infty}\inf\{\|w(t)-\xi\| : \xi \in \mathbb{R}^m_+\} = 0$, then $w(t) \in \mathbb{R}^m_+$ for all $t \in \mathbb{Z}$.*

The proof of the contrapositive statement is straightforward, using only the definition of almost periodicity. For the sake of brevity, we leave the details to the reader.

Proof of Theorem 5.1. The key idea is to apply [26, Theorem 4.3]. To this end, we need to rewrite the higher-order system in first-order form and, in a second step, 'transform' the nonlinearity in a suitable way, as the theory in [26] is developed for general (not necessarily non-negative) state-space systems.

Let u and v be functions from $\mathbb{N}_0$ to U and V such that $u(t) \to u_{\mathrm{e}}$ and $v(t) - v^{\mathrm{ap}}(t) \to 0$ as $t \to \infty$, and let x be the solution of the initial-value problem (1.1). By (N2) and (5.1) there exists a unique positive solution z_{e} to the equation $pz = f(u_{\mathrm{e}}, z)$. We claim that there exist $\sigma \in \mathbb{N}_0$ and $\eta \in (0, z_{\mathrm{e}})$ such that

$$c^\top \tilde{x}(t+\sigma) = \sum_{j=0}^{k}\gamma_j x(t+\sigma-j) \ge \eta \quad \forall\, t \in \mathbb{N}_0, \tag{5.5}$$

where $\tilde{x}(t) := \big(x(t), x(t-1), \dots, x(t-k)\big)^\top$. If $\tilde{x}(0) = \big(x(-k),\dots,x(0)\big)^\top \ne 0$, this is an immediate consequence of Theorem 3.5. Let us now consider the case wherein $\tilde{x}(0) = \big(x(-k),\dots,x(0)\big)^\top = 0$. As has been already pointed out, it follows from (N2) and (5.1) that $f(u_{\mathrm{e}}, 0) > 0$. As f is continuous and $u(t) \to u_{\mathrm{e}}$ as $t \to \infty$, it follows that $x(t) \not\equiv 0$, whence $x(\theta) > 0$ for some $\theta \in \mathbb{N}_0$. It now follows as in the proof of statement (2) of Theorem 4.2 that (5.5) holds with suitable $\sigma \in \mathbb{N}_0$ and $\eta \in (0, z_{\mathrm{e}})$.

Proposition 3.3 guarantees that x is bounded. Consequently, as f is continuous (and hence uniformly continuous on compact sets), and $q(t)$ defined in (4.15) converges to 0 at $t \to \infty$:

$$\lim_{t\to\infty} q(t) = 0.$$

We note that x satisfies

$$x(t+1) = \sum_{j=0}^{k}\alpha_j x(t-j) + \beta f\Big(u_{\mathrm{e}}, \sum_{j=0}^{k}\gamma_j x(t-j)\Big) + w(t) \quad \forall\, t \in \mathbb{N}_0,$$

where $w(t) := v(t) + q(t)$, or, equivalently,

$$\tilde{x}(t+1) = A\tilde{x}(t) + bf\big(u_{\rm e}, c^\top \tilde{x}(t)\big) + \tilde{w}(t) \quad \forall t \in \mathbb{N}_0,$$

where $\tilde{w}(t) := \big(w(t), 0, \ldots, 0\big)^\top$ for all $t \in \mathbb{N}_0$. Setting $\tilde{v}^{\rm ap}(t) := (v^{\rm ap}(t), 0, \ldots, 0)^\top$ for all $t \in \mathbb{Z}$, we have that

$$\lim_{t\to\infty} \big(\tilde{w}(t) - \tilde{v}^{\rm ap}(t)\big) = 0. \tag{5.6}$$

Let $\hat{f}$ be as in (4.13) and note that, by (5.1) and (N2),

$$\left|\frac{\hat{f}(z) - \hat{f}(\xi)}{z - \xi}\right| < p \quad \forall\,(z,\xi) \in \mathbb{R}\times\mathbb{R}\ z \neq \xi \qquad \text{and} \qquad \lim_{|z|\to\infty} \big(p|z| - |\hat{f}(z)|\big) = \infty. \tag{5.7}$$

For an arbitrary function $\hat{w} : \mathbb{T} \to \mathbb{R}^{k+1}$, where $\mathbb{T} = \mathbb{N}_0$ or $\mathbb{T} = \mathbb{Z}$, consider the system

$$y(t+1) = Ay(t) + b\hat{f}\big(c^\top y(t)\big) + \hat{w}(t) \quad \forall\, t \in \mathbb{T}. \tag{5.8}$$

It follows from (S), (L) and (5.7) that system (5.8) satisfies the hypotheses of [26, Theorem 4.3], and therefore, [26, Theorem 4.3] guarantees that

(a) if, in (5.8), $\mathbb{T} = \mathbb{Z}$ and $\hat{w} = \tilde{v}^{\rm ap}$, then there exists a unique bounded solution $y^{\rm ap} : \mathbb{Z} \to \mathbb{R}^{k+1}$ of (5.8), $y^{\rm ap} \in AP(\mathbb{Z}, \mathbb{R}^{k+1})$, and, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $P(\tilde{v}^{\rm ap}, \delta) \subset P(y^{\rm ap}, \varepsilon)$;

(b) if, in (5.8), $\mathbb{T} = \mathbb{N}_0$ and $\hat{w} = \tilde{w}$, then, since $\tilde{w}(t) - \tilde{v}^{\rm ap}(t) \to 0$ as $t \to \infty$ by (5.6), $y(t) - y^{\rm ap}(t) \to 0$ as $t \to \infty$ for every solution y of (5.8).

Using the notation for left translates defined in (1.3), we have that $\tilde{v}^{\rm ap}_\sigma$ and $y^{\rm ap}_\sigma$ are almost periodic, and it is clear that statements (a) and (b) remain valid when $\tilde{v}^{\rm ap}$, $y^{\rm ap}$ and $\tilde{w}$ are replaced by $\tilde{v}^{\rm ap}_\sigma$, $y^{\rm ap}_\sigma$ and $\tilde{w}_\sigma$, respectively. Therefore, as $y^x : \mathbb{N}_0 \to \mathbb{R}^{k+1}$ defined by $y^x(t) := \tilde{x}(t+\sigma) - \tilde{x}_{\rm e}$, where

$$\tilde{x}_{\rm e} := \big(x_{\rm e}, \ldots, x_{\rm e}\big)^\top = \big(z_{\rm e}/\gamma, \ldots, z_{\rm e}/\gamma\big)^\top \in (0,\infty)^{k+1},$$

satisfies (5.8) with $\mathbb{T} = \mathbb{N}_0$ and $\hat{w} = \tilde{w}_\sigma$, we have that $y^x(t) - y^{\rm ap}_\sigma(t) \to 0$ as $t \to \infty$. Consequently,

$$\tilde{x}(t+\sigma) - \big(y^{\rm ap}_\sigma(t) + \tilde{x}_{\rm e}\big) \to 0 \quad \text{as } t \to \infty. \tag{5.9}$$

As $\tilde{x}(t) \geq 0$ for all $t \in \mathbb{N}_0$ and the function $t \mapsto y^{\rm ap}_\sigma(t) + \tilde{x}_{\rm e}$ is almost periodic, it follows from Lemma 5.2 that $y^{\rm ap}_\sigma(t) + \tilde{x}_{\rm e} \geq 0$ for all $t \in \mathbb{Z}$. Defining $\tilde{x}^{\rm ap} \in AP(\mathbb{Z}, \mathbb{R}^{k+1}_+)$ by

$$\tilde{x}^{\rm ap}(t) := y^{\rm ap}_\sigma(t-\sigma) + \tilde{x}_{\rm e} = y^{\rm ap}(t) + \tilde{x}_{\rm e} \quad \forall\, t \in Z,$$

we have that

$$\tilde{x}^{\rm ap}(t+1) = A\tilde{x}^{\rm ap}(t) + bf\big(u_{\rm e}, c^\top \tilde{x}^{\rm ap}(t)\big) + \tilde{v}^{\rm ap}(t) \quad \forall\, t \in \mathbb{Z}.$$

Furthermore, by (a)

$$P(\tilde{v}^{\rm ap}, \delta) \subset P(y^{\rm ap}, \varepsilon) = P(\tilde{x}^{\rm ap}, \varepsilon).$$

Invoking (5.9) yields

$$\tilde{x}(t) - \tilde{x}^{\rm ap}(t) = \tilde{x}(t) - \big(y^{\rm ap}_\sigma(t-\sigma) + \tilde{x}_{\rm e}\big) \to 0 \quad \text{as } t \to \infty. \tag{5.10}$$

Denoting the first component of $\tilde{x}^{\rm ap}(t)$ by $x^{\rm ap}(t)$, it is clear that $x^{\rm ap}$ is almost periodic, $x^{\rm ap}$ satisfies (5.2), and, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $P(v^{\rm ap}, \delta) \subset P(x^{\rm ap}, \varepsilon)$. Also, by (5.10),

$$x(t) - x^{\rm ap}(t) \to 0 \quad \text{as } t \to \infty. \tag{5.11}$$

Combining (5.11) with (5.5), the almost periodicity of the function $t \mapsto c^\top \tilde{x}^{\mathrm{ap}}(t) = \sum_{j=0}^{k} \gamma_j x^{\mathrm{ap}}(t - j)$ and Lemma 5.2 shows that (5.3) holds. Finally, by almost periodicity of $c^\top \tilde{x}^{\mathrm{ap}}$, we have that $\inf_{t\in\mathbb{Z}} c^\top \tilde{x}^{\mathrm{ap}}(t) = \liminf_{t\to\infty} c^\top \tilde{x}^{\mathrm{ap}}(t)$, and thus, by (5.10) and (5.3),

$$\liminf_{t\to\infty} c^\top \tilde{x}(t) = \inf_{t\in\mathbb{Z}} c^\top \tilde{x}^{\mathrm{ap}}(t) > 0$$

establishing (5.4) and completing the proof. □

We close this section by specifying a class of nonlinearities which satisfy the condition (5.1) in Theorem 5.1. The lemma below follows from a straightforward application of the mean-value theorem for differentiation.

Lemma 5.3. *Let $U \subset \mathbb{R}^n$ be compact, $u_{\mathrm{e}} \in U$ and $p > 0$. Assume that $z \mapsto f(u_{\mathrm{e}}, z)$ is continuously differentiable, $f(u_{\mathrm{e}}, 0) > 0$ and*

$$\sup_{z\geq 0} |f'(u_{\mathrm{e}}, z)| < p, \tag{5.12}$$

where f' denotes the derivative of f with respect to the second argument z. Then there exists a unique $z_{\mathrm{e}} > 0$ such that $f(u_{\mathrm{e}}, z_{\mathrm{e}}) = pz_{\mathrm{e}}$ and (5.1) holds.

Note that if (5.12) holds, but $f(u_{\mathrm{e}}, 0) = 0$, then $f(u_{\mathrm{e}}, z) < pz$ for all $z > 0$, implying that there does not exist $z_{\mathrm{e}} > 0$ such that $f(u_{\mathrm{e}}, z_{\mathrm{e}}) = pz_{\mathrm{e}}$.

Finally, we note that if f satisfies the assumptions of Lemma 5.3, then condition (4.21) in Corollary 4.4 holds.

6 Examples

To illustrate the results in Sections 3–5, we discuss four examples.

Example 6.1 (The forced Allen-Clark model). Here we consider the nonlinear, scalar, higher-order (or delayed) difference equation

$$x(t+1) = \alpha x(t) + \beta f(u(t), x(t-k)) + v(t) \quad t \in \mathbb{N}_0\,, \tag{6.1}$$

where $\alpha \geq 0$, $\beta > 0$ and $k \in \mathbb{N}_0$ are constants, and $f = f(w, z)$ is a nonlinearity. Obviously, (6.1) is itself a forced (or controlled) version of the Allen-Clark model (1.2), both of which are special cases of the general higher-order difference equation (1.1) with

$$\alpha_0 = \alpha, \quad \alpha_1 = \ldots = \alpha_k = 0, \quad \gamma_k = 1, \quad \gamma_0 = \ldots = \gamma_{k-1} = 0\,.$$

Equation (6.1) will be referred to as the forced Allen-Clark model. We will see below that the theoretical results of Sections 3-5 apply if $\alpha > 0$, but they do not apply if $\alpha = 0$. As has been already indicated, the term v facilitates modelling immigration into a population, and the term u may capture environmental variation or harvesting, either anthropogenic or otherwise. For example, the model [22, equation (6.7)] is of the form (6.1) and expresses the juvenile-only harvesting situation of a population presented in [68], and further studied in [41], with harvesting rates are assumed constant in [41, 68].

In accordance with Section 2, the rational function $\mathbf{G}$, the number p and the polynomials $\mathbf{a}$ and $\mathbf{c}$ associated with (6.1) are given by

$$\mathbf{G}(\zeta) = \frac{\beta}{\zeta^k(\zeta - \alpha)}, \quad p := \frac{1}{\mathbf{G}(1)} = \frac{1-\alpha}{\beta} \in (0, \infty), \quad \mathbf{a}(\zeta) = \zeta^k(\zeta - \alpha), \quad \mathbf{c}(\zeta) \equiv 1. \tag{6.2}$$

We investigate under which conditions the key hypotheses of this paper are satisfied. It is clear that (L) holds because the comprimeness condition (ii) in (L) is trivially satisfied. The linear stability condition (S) holds if, and only if, $\alpha < 1$. Furthermore, if $\alpha > 0$, then $\mathcal{I} = \{1, k+1\}$, implying that (P2) is satisfied. We note that if $\alpha = 0$, then $\mathcal{I} = \{k+1\}$, and so, (P2) fails to hold whenever $k \neq 0$. Throughout the rest of this example, it will be assumed that $\alpha \in (0, 1)$.

Hypotheses (N1)–(N3), as well as the inequality (4.21), depend on the nonlinearity f, and its relation to the positive parameter p, and require f to enjoy certain qualitative properties. As a specific example, we consider (6.1) with

$$f(w, z) = z\mathrm{e}^{-\kappa w z} \quad z \geq 0,\ w \in U, \quad \text{where } U \subset (0, \infty) \text{ is compact,}$$

that is,

$$x(t+1) = \alpha x(t) + \beta x(t-k)\mathrm{e}^{-\kappa u(t) x(t-k)} + v(t) \quad t \in \mathbb{N}_0\,. \tag{6.3}$$

We remark that (6.3) is a forced version of the model [56, equation (6)] for biomass of mature fish. Table 6.1 relates the notation used presently to that in [56, equation (6)]. The cases of $\alpha = 0$ or $\alpha > 0$ correspond to semelparous and iteroparous species, respectively, although recall that we consider $\alpha > 0$ here only.

Symbol	**Symbol in [56]**	**Interpretation**
$x(t)$	$B(t)$	biomass of mature fish in the population at time t
α	e^{-Z}	$Z = M_p + F > 0$ is the overall instantaneous mortality rate, and is the sum of natural mortality M_p and fishing mortality F
β	α	maximum per capita reproduction rate (at low population abundance)
κ	β	density-dependent mortality near equilibrium abundance parameter

Table 6.1: Comparison of notation used in model (6.3).

In the idealised situation in which reproduction in (6.1) is density-independent, meaning $f(w, z) = f(z) = z$, the quantity $\beta/(1-\alpha) = 1/p = \mathbf{G}(1)$ is readily shown to equal the *inherent net reproductive number* [13, pp. 7–9, Definition 1] of the linear model (6.3). Adapting the conclusion on [13, p. 9], $\beta/(1-\alpha)$ equals the expected amount of biomass produced, per unit of biomass, over the course of its lifetime. Therefore, the existence of a non-trivial equilibrium of the density-dependent model (6.3) requires that

$$\frac{\beta}{1-\alpha} > 1 \iff p < 1.$$

In the following numerical simulations we fix the model data

$$\left.\begin{aligned} &k = 2, \quad \alpha = 0.1, \quad \beta = 6, \quad \kappa = 1.5, \quad x^0 = 1, \quad x^{-1} = 1.5, \quad x^{-2} = 0, \\ &U = [0.9, 1.1], \quad u_\mathrm{e} = 1, \quad V = [0, 10]\,. \end{aligned}\right\} \tag{6.4}$$

Then $p = 0.15 < 1$, and as already discussed, (P2), (S) and (L) are satisfied. A graph of the nonlinearity $f(u_\mathrm{e}, \cdot) = f(1, \cdot)$ is shown in Figure 6.1a, along with the straight lines $z \mapsto pz$ and $z \mapsto -pz + 2z_\mathrm{e}(1, 0)$ determining the sector condition (4.7). Furthermore, as $\mathrm{e}^{-2} < p = 0.15 < 1$, Lemma 4.6 yields that (N2) and (4.21) hold with model data as in (6.4), whence the hypotheses of Theorem 3.5, Theorem 4.2 and Corollary 4.4, our main persistence, stability and convergence results, are satisfied. In the context of (6.3), the key condition $z_\mathrm{e} = z_\mathrm{e}(u_\mathrm{e}, v_\mathrm{e}) \in F_{u_\mathrm{e}}^{-1}\big(\gamma v_\mathrm{e}/(1-\alpha)\big)\backslash\{0\}$ from Section 4 becomes

$$z_\mathrm{e} - \frac{\beta}{1-\alpha} z_\mathrm{e}\mathrm{e}^{-\kappa u_\mathrm{e} z_\mathrm{e}} = \frac{v_\mathrm{e}}{1-\alpha}. \tag{6.5}$$

As $\gamma = 1$, we have that $x_\mathrm{e} = x_\mathrm{e}(u_\mathrm{e}, v_\mathrm{e}) = z_\mathrm{e}(u_\mathrm{e}, v_\mathrm{e}) = z_\mathrm{e}$.

To illustrate statement (1) of Theorem 4.2, let $v_\mathrm{e} = 0$ and consider

$$u(t) = 1 + \theta \sin(t/4) \quad \text{and} \quad v(t) = \theta r(t) \quad t \in \mathbb{N}_0\,, \tag{6.6}$$

where $r(t)$ is equal to a uniform (pseudo)random number in $[0,1]$ for each $t \in \mathbb{N}_0$, and $\theta \in [0,0.1]$ is an amplitude parameter. Figures 6.1b, 6.1c and 6.1d shows plots of $x(t;x^0,x^{-1},x^{-2},1,v)$, $x(t;x^0,x^{-1},x^{-2},u,0)$ and $x(t;x^0,x^{-1},x^{-2},u,v)$ [4] against t, respectively, in each case for $\theta = 0.05$ and $\theta = 0.1$. For comparison, in each plot a graph of the unforced solution $x(t;x^0,x^{-1},x^{-2},1,0)$ is displayed and is seen to converge to $x_{\mathrm{e}}(1,0)$ as t increases. As predicted by the estimate (4.8) in Theorem 4.2, the deviation of $x(t)$ from $x_{\mathrm{e}}(1,0)$ decreases as θ decreases. Observe that despite v taking only nonnegative values, the values of $x(t;x^0,x^{-1},x^{-2},1,v)$ are occasionally smaller than those of the unforced solution, a consequence of the non-monotonicity of the Ricker nonlinearity, capturing so-called overcompensatory recruitment.

To illustrate Corollary 4.4, we consider the convergent additive forcing functions

$$v_1(t) = v_{\mathrm{e},1}(1 + (-0.9)^t), \quad v_2(t) = v_{\mathrm{e},2} + t\mathrm{e}^{-t}, \quad v_3(t) = v_{\mathrm{e},3} \quad t \in \mathbb{N}_0\,, \tag{6.7}$$

with $v_{\mathrm{e},i} = 0.5i$ for $i = 1,2,3$. Figure 6.1e shows graphs of $x(t;x^0,x^{-1},x^{-2},1,v_i)$ against t, for $i \in \{1,2,3\}$. In each case, convergence $x(t) \to x_{\mathrm{e}}(1,v_{\mathrm{e},i})$ as $t \to \infty$ is observed, in accordance with Corollary 4.4. The limits $x_{\mathrm{e}}(1,v_{\mathrm{e},i})$ agree with the solutions of (6.5), obtained numerically by using the MATLAB command `fsolve`, here giving

$$x_{\mathrm{e},1} = 1.5586, \quad x_{\mathrm{e},2} = 1.8674, \quad x_{\mathrm{e},3} = 2.2049\,.$$

System (6.3) is known to admit oscillatory solutions, even in the unforced case, if condition (4.22) fails. This occurs, for instance, when β in (6.4) is replaced by $\beta = 12$, in which case $\mathrm{e}^{-2} > p = 0.075$. However, system (6.3) still satisfies the hypotheses of Theorem 3.5 and exhibits the ultimate semi-global persistence property of Definition 3.4. As a numerical illustration, Figure 6.1f plots in grayscale 40 solutions of (6.3) with (pseudo)random initial conditions such that

$$x^0 \in [0.1, 5] \quad \text{and} \quad x^{-1} = x^{-2} = 0, \tag{6.8}$$

and u and v as in (6.6) with $\theta = 0.1$. The inset shows a plot of $x(t)$ against t for $0 \le t \le 5$ with a logarithmic scale on the vertical axis. We comment that the purpose of Figure 6.1f is not to follow individual solutions, but rather to visualise a system-level property. The simulations shown in Figure 6.1f are in accordance with Theorem 3.5: indeed, $\tau = 4$ is the minimal $\tau \in \mathbb{N}_0$ such that $c^\top(A + bc^\top)^\tau \gg 0$, and thus, Theorem 3.5 guarantees the existence of a number $\eta > 0$ such that, for all initial conditions satisfying (6.8) and all forcing functions u and v with values in U and V, respectively, the corresponding solution x satisfies $x(t+2) \ge \eta$ for all $t \in \mathbb{N}_0$. $\Diamond$

Example 6.2 (A plant population model with seed bank). Many plants grow from seeds, and it is a known plant survival strategy that not all seeds germinate in the year following their dispersal. Seeds which remain dormant underground comprise what is often termed the *seed bank*. Thus, mathematical models for seed banks inherently contain delays (and hence, in discrete-time, higher-order terms). A nice review of mathematical models of plant species with seed banks appears in [46], and a comprehensive construction of a model for single local plant populations with linear growth appears in [17, Section 1.2, p.8]. Here, we show how certain plant models inspired by those in [17] and [46] are of the form (1.1). We assume that seeds may survive $k+1$ years in the seed bank, and that older seeds are not viable. In particular, the present framework allows for any fixed dormancy period. Similar to [46, Section 2], we let $s(t)$ and $a(t)$ denote the number of germinating seeds and adult plants of generation t, respectively, which are assumed to satisfy the following difference equations

$$s(t+1) = \sum_{j=0}^{k} \gamma_j a(t-j), \quad a(t+1) = f(s(t+1)), \quad t \in \mathbb{N}_0. \tag{6.9}$$

[4] Here and in Example 6.4, this notation is used to emphasize the dependence of solutions on the initial conditions and forcing functions, and to distinguish notationally between several solutions.

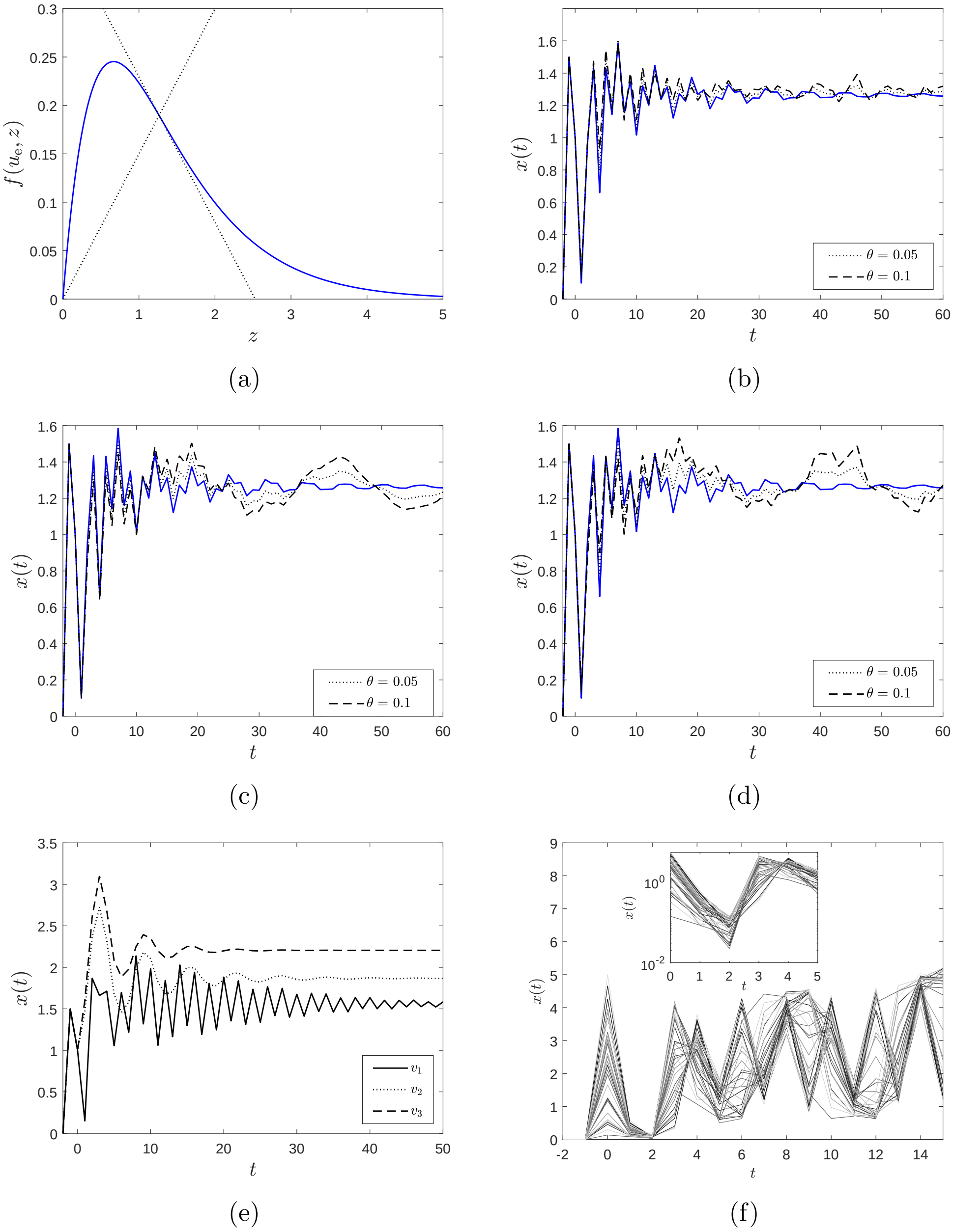

Figure 6.1: Simulations of the forced Allen-Clark model (6.3) with model data (6.4) and (6.6). In panels (b), (c) and (d) the blue curves show the solution of the unforced model ($u(t) \equiv 1$ and $v(t) \equiv 0$). Panel (e) displays graphs of the solutions corresponding to the forcing functions given in (6.7). Panel (f) contains simulations with $\beta = 12$ and forcing term (6.6) with $\theta = 0.1$.

Here $\gamma_j \geq 0$ are constants which capture the combination of the survival of seeds, the fraction that delay their germination, and the number of seeds produced per plant. Consequently, the first equation determines the number of new seedlings, and the second equation models the density-dependent growth of germinating seeds into adult plants over the course of a season, captured by the nonlinearity $f : \mathbb{R}_+ \to \mathbb{R}_+$. Eliminating $s(t+1)$ from (6.9), we obtain

$$a(t+1) = f\Big(\sum_{j=0}^{k} \gamma_j a(t-j)\Big) \quad t \in \mathbb{N}_0 \,,$$

which is of the form (1.1) with $\beta = 1$ and $\alpha_j = 0$ for every $j \in \{0, 1, \ldots k\}$. In particular, the above model fits the scope of the current work. Hypothesis (P1) holds, provided that $\gamma_k > 0$, which we shall assume, and hypothesis (S) is always satisfied. In this example

$$\mathbf{a}(\zeta) := \zeta^{k+1} \quad \text{and} \quad \mathbf{c}(\zeta) := \gamma_0\zeta^k + \gamma_1\zeta^{k-1} + \cdots + \gamma_k\,,$$

which are coprime as $\gamma_k > 0$, implying that hypothesis (L) is satisfied. Furthermore, as $\mathcal{I} = \{i \in \{1,\ldots,k+1\} : \gamma_{i-1} > 0\}$, a sufficient condition for (P2) to hold is the existence of an integer $i \in \{1,\ldots,k+1\}$ such that $\gamma_{i-1} > 0$ and i and $k+1$ are coprime (for example, if $\gamma_0 > 0$). As usual, properties (N1)–(N3) depend on the nonlinearity f and its interplay with the positive parameter $p = 1/\mathbf{G}(1) = 1/(\sum_{j=0}^k \gamma_j)$. In closing, we comment that the inclusion of forcing terms in model (6.9) seems very natural, and the main results of Sections 3–5 would apply in this setting. ♢

Example 6.3 (Delay independent stability)*.* Here we demonstrate that there exist scenarios in which the Allen-Clark model (6.1) has delay-independent global stability properties. To this end, consider (6.1) and assume that $\alpha \in (0,1)$, $\beta > 0$ and $f : U \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. Moreover, fix $u_{\mathrm{e}} \in U$ and let $v_{\mathrm{e}} = 0$. As shown in Example 6.1, the constant p is given by $p = (1-\alpha)/\beta$ (not depending on k) and hypotheses (S), (P2) and (L) are satisfied for every $k \in \mathbb{N}_0$. Consequently, an application of Corollary 4.3 shows that if f satisfies (N3) (a condition which is independent of k as p does not depend on k), then, for every $k \in \mathbb{N}_0$, the unique positive number $x_{\mathrm{e}} := x_{\mathrm{e}}(u_{\mathrm{e}}, 0)$ satisfying $px_{\mathrm{e}} = f(u_{\mathrm{e}}, x_{\mathrm{e}})$ is an equilibrium of (6.1) with $v(t) \equiv 0$, and furthermore, given an arbitrary compact set $\Gamma \subset \mathbb{R}_+^{k+1}\backslash\{0\}$, there exists $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that, for all initial conditions $(x^0,\ldots,x^{-k})^\top \in \Gamma$, all $u : \mathbb{N}_0 \to U$ and all $V : \mathbb{N}_0 \to V$ the solution x of (6.1) satisfies (4.8) (with $v_{\mathrm{e}} = 0$). In particular, if $u(t) \equiv u_{\mathrm{e}}$ and $v(t) \equiv 0$, then

$$|x(t) - x_{\mathrm{e}}| \le \psi\Big(\sum_{j=0}^{k} |x^{-j} - x_{\mathrm{e}}|, t\Big) \quad \forall\, t \in \mathbb{N}_0,$$

showing that x_{e} is a globally asymptotically stable equilibrium of the unforced Allen-Clark model (here considered on the domain $\mathbb{R}_+^{k+1}\backslash\{0\}$). We conclude that, in the context of (6.1), condition (N3) guarantees delay-independent global asymptotic stability.

These findings contrast with those of [43] which show that conditions under which the positive equilibrium x_{e} of (1.2) is globally asymptotically stablefor in the undelayed $k = 0$ case, are not sufficient for global asymptotic stabilitywhen $k \ge 3$. In other words, global asymptotic stability in these settings depends on the delay. ♢

Example 6.4 (Blood cell model)*.* The Allen-Clark model (1.2) also arises as a discretisation of the delay-differential equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-\delta)) \quad t \in \mathbb{R}_+ \quad \text{where } \delta, \mu > 0, \tag{6.10}$$

see [39]. We remark that (6.10) is sometimes referred to as Nicholson or Mackey-Glass equation [27, 44]. In the special case wherein $f(z) = \mathrm{e}^{-\kappa z}$ for all $z \ge 0$, (6.10) is the so-called (reduced) Lasota-Wazewska haematology model [47, Equation (5.24)]. Here we consider the following forced discrete-time version of this model, namely

$$x(t+1) = \alpha x(t) + \beta f(u(t), x(t-k)) + v(t) \quad t \in \mathbb{N}_0\,, \tag{6.11}$$

with $k \in \mathbb{N}_0$, $\alpha \in (0,1)$, $\beta, \kappa > 0$, u and v taking values in non-empty compact sets $U \subset (0,\infty)$ and $V \subset \mathbb{R}_+$, respectively, and $f(w,z) = \mathrm{e}^{-\kappa wz}$ for all $(w,z) \in U \times \mathbb{R}_+$. The forcing terms u and v could, for example, model the intake of drugs affecting blood cell production and transfusions, respectively. In the unforced case ($u(t) \equiv 1$ and $v(t) \equiv 0$), (6.11) has been studied, for example, in [32], whilst in [8], equation (6.11) is analysed in the special case wherein $k = 0$, $u(t) \equiv 1$ and $v(t) \equiv \text{const} \ge 0$.

Since (6.11) is a forced Allen-Clark model, conditions (S), (L) and (P2) hold, and p associated with (6.11) is given by (6.2), see Example 6.1. It is clear that (N2) is satisfied. Furthermore, for any $u_{\mathrm{e}} \in U$, we have that

$$f(u_{\mathrm{e}}, 0) = 1 > 0 \quad \text{and} \quad \sup_{z \geq 0} |f'(u_{\mathrm{e}}, z)| = u_{\mathrm{e}} \kappa \,,$$

and thus, Lemma 5.3 guarantees that (5.1) is satisfied whenever

$$0 < u_{\mathrm{e}} < \frac{1-\alpha}{\beta \kappa} \,. \tag{6.12}$$

As a consequence, if the inequality (6.12) holds, then Theorem 5.1 is applicable.

For numerical simulations, we fix the model data

$$k = 1, \quad \alpha = 0.3, \quad \beta = 0.6, \quad \kappa = 1.5, \quad U = [0.05, 4], \quad u_{\mathrm{e}} = 1, \quad V = [0, 4] \,, \tag{6.13}$$

in which case inequality (6.12) is satisfied. Therefore, Theorem 5.1 guarantees that the response of the system is asymptotically (almost) periodic if $u(t) \to u_{\mathrm{e}}$ and $v : \mathbb{N}_0 \to V$ is (almost) periodic. Let v be given by

$$v(t) = 0.5 + 0.2\big(\cos(0.4\pi t) + 0.5 \sin(0.3\pi t)\big) \quad \forall\, t \in \mathbb{N}_0 \,, \tag{6.14}$$

a periodic function with period 20. The graphs in Figure 6.2 show the solutions of (6.11) corresponding to v as in (6.14) and u, x^0 and x^{-1} as specified below:

$$\mathrm{I}: \quad u = u_{\mathrm{e}}, \quad x^0 = 1, \quad x^{-1} = 1 \tag{6.15a}$$

$$\mathrm{II}: \quad u = u_*, \quad x^0 = 0.75, \quad x^{-1} = 0 \tag{6.15b}$$

$$\mathrm{III}: \quad u = 2u_{\mathrm{e}} - u_*, \quad x^0 = 0, \quad x^{-1} = 1.5, \tag{6.15c}$$

where $u_*(t) := u_{\mathrm{e}}(1 + (-0.95)^t)$ for all $t \in \mathbb{N}_0$. We observe that, in accordance with Theorem 5.1, each solution tends asymptotically to the same periodic function.

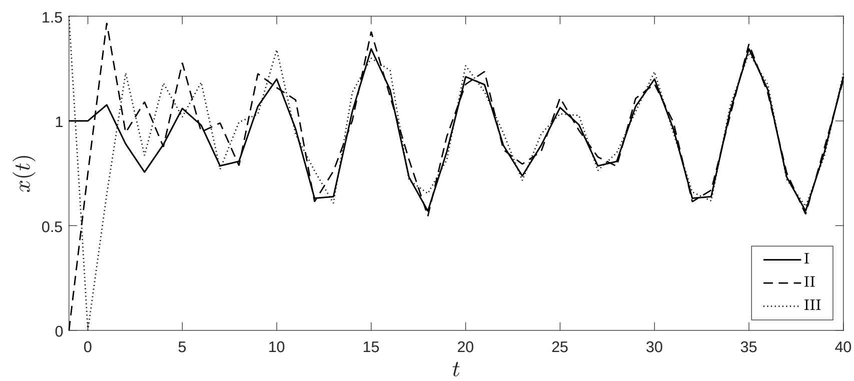

Figure 6.2: Simulations of the forced Allen-Clark model (6.11) with model data (6.13) and forcing functions and initial conditions given by (6.14) and (6.15).

$\Diamond$

7 Discussion

The dynamic properties of boundedness, persistency, stability and convergence have been considered for the class of nonlinear, positive, scalar, higher-order difference equations (1.1). Sufficient conditions for these properties have been provided across our main results of Proposition 3.3, Theorems 3.5, 4.2 and 5.1, and Corollary 4.4. Persistency plays a pivotal role throughout and the key hypothesis (P2) is an easily checkable condition which, together with (S), (L) and (N2), guarantees that (1.1) is persistent. Our work traces its inspiration to [63], complements related work of the authors [21]–[24], and enhances aspects of these papers.

The application of our results to a range of models arising in theoretical biology and ecology — the Allen-Clark model in population dynamics, plant models with seed banks, and haematology models — has been presented across Examples 6.1, 6.2 and 6.4, respectively. For the Allen-Clark model (1.2), our results, when applicable, ensure delay-independent stability as described in Example 6.3. A distinguishing feature, as compared to the literature, is the inclusion of forcing terms. The stability and convergence results in Sections 4 and 5, based on control-theoretic input-to-state stability ideas and which go beyond standard Lyapunov theory and apply to the forced system (1.1), are not restricted to the analysis of the stability properties of the equilibrium of the unforced version of (1.1).

In terms of open problems, we have made essential use of the sector condition (N3) to ensure stability. Careful analyses such as [25] have identified classes of unforced higher-order scalar difference equations, which are special cases of (1.1) and where (N3) is violated, yet global asymptotic stability of a non-zero equilibrium is still ensured. The work [25] exploits that the dynamics of a certain higher-order difference equation may be dominated by the dynamics of a first-order difference equation with a positive global attractor. The extent to which these methods, or the use of envelopments by linear fractional functions [12], may be applicable to forced problems are interesting topics for future research.

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The authors report there are no competing interests to declare.

Data availability statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

A Appendix

Consider the following higher-order Lur'e difference equation, which is of the same form as (1.1), but without any non-negativity constraints and the nonlinearity is independent of any external forcing:

$$x(t+1) = \sum_{j=0}^{k} \alpha_j x(t-j) + \beta h\Big(\sum_{j=0}^{k} \gamma_j x(t-j)\Big) + v(t), \quad x(-j) = x^{-j} \in \mathbb{R}, \quad j = 0, \ldots, k, \quad t \in \mathbb{N}_0, \tag{A.1}$$

where $k \in \mathbb{N}_0$, $\alpha_j, \gamma_j, \beta \in \mathbb{R}$ for $j = 0, \ldots, k$, $\beta \neq 0$, $\sum_{j=0}^{k} |\gamma_j| > 0$ and $|\alpha_k| + |\gamma_k| > 0$. The nonlinearity $h : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous and the function v takes values in $\mathbb{R}$. As in Section 2, set $\mathbf{a}(\zeta) := \zeta^{k+1} - \sum_{j=0}^{k} \alpha_j \zeta^{k-j}$, $\mathbf{c}(\zeta) := \sum_{j=0}^{k} \gamma_j \zeta^{k-j}$ and

$$\mathbf{G}(\zeta) := \frac{\beta \sum_{j=0}^{k} \gamma_j \zeta^{k-j}}{\zeta^{k+1} - \sum_{j=0}^{k} \alpha_j \zeta^{k-j}} = \frac{\beta \mathbf{c}(\zeta)}{\mathbf{a}(\zeta)}, \quad \text{where } \zeta \in \mathbb{C}.$$

Let $A \in \mathbb{R}^{(k+1)\times(k+1)}$ and $b, c \in \mathbb{R}^{k+1}$ be given by (2.5) and set $\tilde{v}(t) := (v(t), 0, \ldots, 0)^\top$. Then equation (A.1) can be expressed in the form

$$\tilde{x}(t+1) = Ax(t) + bh(c^\top \tilde{x}(t)) + \tilde{v}(t), \quad \tilde{x}(0) = (x^0, \ldots, x^{-k})^\top, \; t \in \mathbb{N}_0. \tag{A.2}$$

Furthermore, $\mathbf{G}(\zeta) = c^\top(\zeta I - A)^{-1}b$.

The following theorem is a special case of [53, Theorem 13]

Theorem A.1. *Assume that* (S) *and* (L) *hold. If there there exists* $\rho \in \mathcal{K}_\infty$ *such that*

$$|h(z)| \leq (1/\|\mathbf{G}\|_{H^\infty})|z| - \rho(|z|) \quad \forall\, z \in \mathbb{R},$$

then there exist $\psi \in \mathcal{KL}$ *and* $\phi \in \mathcal{K}$ *such that, for all* $(x^0, x^{-1}, \ldots, x^{-k})^\top \in \mathbb{R}^{k+1}$ *and all* $v : \mathbb{N}_0 \to \mathbb{R}$, *the solution* x *of* (A.1) *satisfies*

$$|x(t)| \leq \psi\Big(\sum_{j=0}^{k} |x^{-j}|, t\Big) + \phi\big(\|v\|_{\ell^\infty(0,t)}\big) \quad \forall\, t \in \mathbb{N}_0.$$

Proof. Consider the first-order (or state-space) formulation (A.2) of the higher-order system (A.1) and note that, by the special structure of A and b, the linear system

$$x(t+1) = Ax(t) + bu(t), \quad y(t) = c^\top x(t)$$

is controllable. Trivially, by the linear stability assumption (S), this system is also stabilizable and detectable. Consequently, invoking (L) and Lemma 3.1, we conclude that [53, Assumption A] is satisfied. Exploiting the linear stability assumption (S) once more, it follows that [53, Theorem 13] applies to (A.2) (with, in the notation of [53], linear stabilizing feedback $K = 0$), establishing the claim. $\square$

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