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Persistency and stability of a class of nonlinear forced positive discrete-time systems with delays

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ABSTRACT

Persistence, excitability and stability properties are considered for a class of nonlinear, forced, positive discrete-time systems with delays. As will be illustrated, these equations arise in a number of biological and ecological contexts. Novel sufficient conditions for persistence, excitability and stability are presented. Further, similarities and differences between the delayed equations considered presently and their corresponding undelayed versions are explored, and some striking differences are noted. It is shown that recent results for a corresponding class of positive, nonlinear delay-differential (continuous-time) systems do not carry over to the discrete-time setting. Detailed discussion of three examples from population dynamics is provided.

1. Introduction

We consider boundedness, persistence, excitability and stability properties of the following system of forced, positive, nonlinear difference equations with delay

$$x(t+1) = Ax(t) + bf(u(t), c^\top x(t) + d^\top x(t-1)) + v(t),$$
$$x(0) = x^0,\ x(-1) = x^{-1},\ t \in \mathbb{N}_0. \tag{1.1}$$

Here the matrix A and the vectors b, c, d are nonnegative, f is a nonlinearity mapping into $[0,\infty)$ and x^0, x^{-1} are nonnegative initial vectors. The functions u and v denote external signals which, depending on the context, could play the role of disturbance or control terms. We call u and v forcing functions or inputs and it is assumed that v is nonnegative.

Positive dynamical systems are dynamical systems with the defining property that they leave some positive cone invariant. This defining property captures the natural requirement that the modelled quantities (e.g., concentrations, densities) must take nonnegative values to be physically meaningful. The study of positive systems described by linear dynamic equations is underpinned by the seminal work of Perron and Frobenius in the early 1900s on irreducible and primitive matrices. A range of extensions of these results appear across the literature (see, for example, [1]). Positive dynamical systems are closely related to the concept of monotone dynamical systems [2,3]. Positive and monotone dynamical systems are mathematically interesting owing to their various invariance properties and amenability to comparison arguments, and are known to often admit linear or separable Lyapunov functions (see [4], or, for monotone systems, [5]). Moreover, they are the appropriate modelling framework for numerous applications. The reader can find more information about these systems and their applications in [1,6–9].

Control theory deals with the study of dynamical systems which interact with their wider environment via the inclusion of input (control, forcing) and output (measurement) variables, and their interconnection via feedback. A branch of control theory augments positive and monotone dynamical systems with input and output variables, and gives rise to so-called positive and monotone control systems, see the texts [8,10] or [11,12], respectively. The model (1.1) (with the nonnegativity assumptions stated above) is an example of a positive control system.

The inclusion of forcing terms is essential when seeking to explore the effects of external signals on resulting dynamics, such as the response of a population to, for instance, anthropogenic, environmental or demographic variation. Two perspectives are afforded: first, the robustness of desirable properties of the model with respect to unwanted

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forcing terms, and second, the role of forcing terms (here likely to be interpreted as management actions) in catalysing or ensuring desirable dynamic behaviour, such as population persistence in the context of conservation.

Whilst delays in discrete- and continuous-time dynamical systems play largely the same purpose — to capture some contribution from previous states in the forward evolution of the state — there are significant differences. Namely, delay-differential equations are instances of functional differential equations [13], and to fully describe the time evolution in terms of a state variable requires an infinite-dimensional state space. The same is not true for difference equations: it is always possible to rewrite the original delayed difference equation as an undelayed difference equation with an augmented finite-dimensional state. Consequently, there is the potential to apply persistence and stability results for first-order difference equations [14,15] to delayed systems of the form (1.1). However, rewriting (1.1) in first-order form imposes a certain special structure on the first-order system data which frequently prevents a straightforward application of the results in [14,15]. Here, we develop a bespoke approach to the persistency and stability analysis of (1.1) which takes the said special structure of the first-order version of (1.1) into account.

Delays are an important feature of models arising in mathematical biology and ecology. For instance, many plants grow from seeds, and it is a known plant survival strategy that not all seeds germinate in the year following their dispersal. Seeds which remain dormant underground, so-called seed banks, introduce delays into models. A review of mathematical models including seed banks appears in [16]. Or, in another setting, much attention has been devoted to the role of delay-until-reproductive-maturation in age-structured models, dating back to at least [17], and further studied and generalised across, for example, [18–20].

System (1.1) contains both linear and nonlinear components, and, in control theoretic terminology, is an instance of a forced, positive, delayed discrete-time Lur'e system, see, for example [21]. Lur'e systems of difference equations have gained some recent traction in structured population modelling owing to their ability to represent both density independent and density-dependent vital or transition rates. Indeed, they have been proposed and considered as models in ecology in, for example, [14,15,22–26]. This line of enquiry dates back to the two papers [27,28] where control theoretic tools were used to study so-called "trichotomies of stability". Under mild assumptions, unforced Lur'e systems admit two equilibria, zero and a unique nonzero equilibrium, with the usual interpretation of population absence/extinction and steady state, respectively. For models arising in mathematical biology, boundedness and persistence are fundamental properties. The latter property addresses, roughly speaking, the extent to which the zero state is repelling. Persistence is a well-established concept with a number of variations appearing across, but not limited to, [14,15,29–31]. Moreover, persistence relates to the control theoretic concept of excitability, that is, the use of external inputs to drive the system into a persistency regime.

Here, we provide a range of boundedness, persistence, excitability and stability properties for the Lur'e system (1.1) which are important and interesting, both mathematically and from an applications perspective, as outlined above. Investigating these properties, and developing sufficient conditions under which they occur, comprise the main objectives of the present work. Whilst the terminology is broadly intuitive, we introduce each concept precisely to make the work self-contained. We note here that the persistence properties we consider are ultimate in that they only apply after some fixed number of time-steps, and uniform with respect to certain compact sets of initial conditions. We contend that this concept of persistence is suitable for all practical purposes. Our main boundedness and persistence results are Proposition 3.1 and Theorem 3.12, respectively. Theorem 4.1 is the main stability result, from which a number of corollaries are derived which apply to convergent forcing terms. Our results are rigorous and, by way of our analysis, we establish sufficient conditions for boundedness, persistence, excitability or stability by identifying conditions on: (i) the "linear terms" appearing in (1.1) (that is, the data A, b, c, d), (ii) the classes of nonlinearity f considered, and (importantly), (iii) the interplay between (i) and (ii). This approach is typical when studying Lur'e systems. Mathematically, our argumentation is underpinned by a blend of positive system (comparison) and control theoretic techniques.

Our results build on, refine and further develop our earlier works [14,15] which consider boundedness, persistence and stability properties of variations of model (1.1) above, but without delay. The main novelty of our work, particularly compared to [14,15], is twofold: the inclusion of the delay term in (1.1), and the development of sufficient conditions for a certain excitability property (introduced in this paper). To give more details, we demonstrate that there is some overlap between the boundedness and stability properties of (1.1) and the undelayed case, yet there are considerable differences when it comes to persistence. Roughly speaking, the inclusion of delays can make persistence "harder" than in the undelayed case: a potential intuitive explanation being that the linear and nonlinear terms in (1.1), both of which are nonnegative and so contribute positively towards persistence, now need not be acting synchronously. Interestingly, we show that the discrete-time analogue of assumptions which are sufficient for persistence in a system of delayed Lur'e differential equations, the focus of [32], are *not* sufficient for comparable persistence notions in the context of the discrete-time system (1.1). In other words, there are some discrepancies between persistence in the discrete- and continuous-time cases.

The present work is organised as follows. Section 2 contains preliminary material. Our main results appear across Sections 3 and 4, dedicated to boundedness, excitability, persistence properties, and stability results, respectively. Three examples from population dynamics are presented in Section 5. One typical application is when (1.1) represents a stage-structured population, where the nonlinear term $bf(c^\top x(t) + d^\top x(t-1))$ captures recruitment into the population, the delay term $d^\top x(t-1)$ modelling delays in recruitment. This is the situation considered in Example 5.1. Moreover, in Examples 5.2 and 5.3, we consider populations spatially structured in several patches and study how modifying the dispersal rate between these patches affects the persistence and the asymptotic size of the population. Interestingly, we find that there are only two possible response scenarios of the population size to an increase of dispersal for these models. This contrasts with the four scenarios observed recently in [33–35] for other simple continuous- and discrete-time dispersal models. Finally, a summary appears in Section 6.

Notation. We set $\mathbb{R}_+ := [0,\infty)$, $\mathbb{N} := \{1,2,\dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, let $\mathbb{R}^n$ denote the space of column vectors with n real components. We define $\mathbb{R}^n_+$ to be the subset of $\mathbb{R}^n$ consisting of all vectors in $\mathbb{R}^n$ with non-negative components. For $\xi \in \mathbb{R}^n$, we write $\xi \geq 0$ if $\xi \in \mathbb{R}^n_+$, $\xi > 0$ if $\xi \geq 0$ and $\xi \neq 0$, and $\xi \gg 0$ if all components of ξ are positive. If $\xi \gg 0$, then we also say that ξ is *strictly positive*. Furthermore, let $\xi, \zeta \in \mathbb{R}^n$. If $\xi - \zeta \geq 0$, $\xi - \zeta > 0$ or $\xi - \zeta \gg 0$, then we write $\xi \geq \zeta$, $\xi > \zeta$ or $\xi \gg \zeta$, respectively. Similar conventions apply to real matrices. For vectors $\xi \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}^m$, we set

$$[\![\xi, \zeta]\!] := \begin{pmatrix} \xi \\ \zeta \end{pmatrix} \in \mathbb{R}^{n+m}.$$

For a square matrix M (with real or complex entries), the spectrum of M is denoted by $\mathrm{spec}(M)$. A square matrix is called *asymptotically stable* if its spectral radius is less than one. Finally, for a subset $S \subset \mathbb{R}^n$, let ∂S denote the boundary of S.

2. Preliminaries

Consider the system of positive, nonlinear, forced, delayed difference equations (1.1). Here $A \in \mathbb{R}^{n\times n}_+$, $b, c, d \in \mathbb{R}^n_+$ and $x^0, x^{-1} \in \mathbb{R}^n_+$. It will be assumed throughout that

$$b \neq 0, \quad d \neq 0, \quad A \text{ is asymptotically stable.}$$

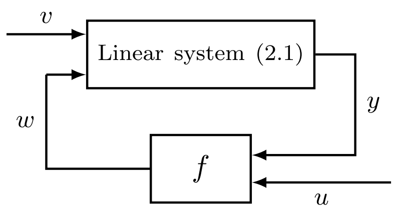

Fig. 2.1. Application of the feedback law $w = f(u, y)$ to system (2.1).

The terms $u(t)$ and $v(t)$ in (1.1) denote exogenous forcing functions (which could be control inputs or disturbances). Throughout, the functions u and v take values in non-empty compact sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n_+$, respectively, where it is assumed that $0 \in V$, and the nonlinearity $f : U \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. The initial-value problem (1.1) has a unique solution $x : \{-1\} \cup \mathbb{N}_0 \to \mathbb{R}^n$. As the system and initial data are nonnegative, it is clear that the solution of (1.1) has values in $\mathbb{R}^n_+$.

System (1.1) can be thought of as the feedback loop obtained by subjecting the linear controlled and observed system

$$x(t+1) = Ax(t) + bw(t) + v(t), \quad y(t) = c^\top x(t) + d^\top x(t-1) \tag{2.1}$$

to the nonlinear feedback $w(t) = f(u(t), y(t))$, see the block diagram in Fig. 2.1. In control theory, such feedback systems are called Lur'e systems. In (2.1), w is a (scalar-valued) input which is available for feedback, whereas y is the measurement, observation or output.

Associated with the quadruple $(A, b, c^\top, d^\top)$ is the rational function $\mathbf{G}$ defined by

$$\mathbf{G}(\zeta) := \left(c^\top + (1/\zeta)d^\top\right)\left(\zeta I - A\right)^{-1} b, \quad \text{where } \zeta \in \mathbb{C}.$$

If $x(0) = x(-1) = 0$ and $v = 0$, then application of the Z-transform to (2.1) shows that

$$(\mathcal{Z}y)(\zeta) = \mathbf{G}(\zeta)(\mathcal{Z}w)(\zeta).$$

The above identity shows that if $x(0) = x(-1) = 0$ (zero initial conditions) and $v = 0$, then the effect of the input w on the output y of system (2.1) is described in the frequency domain by the product of $\mathbf{G}$ and the Z-transform of w. Therefore, $\mathbf{G}$ is called the transfer function of (2.1) with $v = 0$.

We set

$$\|\mathbf{G}\|_{H^\infty} := \sup_{|\zeta|\geq 1} |\mathbf{G}(\zeta)| = \sup_{|\zeta|=1} |\mathbf{G}(\zeta)|,$$

where H^∞ refers to the space of all bounded holomorphic functions defined on the complement of the closed unit disc. If $x(0) = x(-1) = 0$ and $v = 0$ in (2.1), then the associated output $y = y_w$ depends only on w, and

$$\sup\{\|y_w\|_{\ell^2} : \|w\|_{\ell^2} = 1\} = \|\mathbf{G}\|_{H^\infty}, \quad \text{where } \|w\|_{\ell^2} := \sqrt{\sum_{t=0}^{\infty} |w(t)|^2}.$$

The above identity provides an appealing interpretation of $\|\mathbf{G}\|_{H^\infty}$ in time-domain terms. Using the positivity assumptions on A, b, c and d we have that, for all $\zeta \in \mathbb{C}$ such that $|\zeta| = 1$,

$$\begin{aligned}|\mathbf{G}(\zeta)| &\leq \sum_{j=0}^{\infty} \left|(c^\top + (1/\zeta)d^\top)\zeta^{-(j+1)} A^j b\right| \leq \sum_{j=0}^{\infty} \left(|c^\top A^j b| + |d^\top A^j b|\right) \\ &= \sum_{j=0}^{\infty} (c^\top + d^\top) A^j b = \mathbf{G}(1),\end{aligned}$$

where, invoking the asymptotic stability of A, we have used that $(\zeta I - A)^{-1} = \zeta^{-1}(I - \zeta^{-1}A)^{-1} = \sum_{j=0}^{\infty} \zeta^{-(j+1)} A^j$ for all $\zeta \in \mathbb{C}$ such that $|\zeta| = 1$. Consequently,

$$\|\mathbf{G}\|_{H^\infty} = \mathbf{G}(1).$$

We define

$$p := \frac{1}{\mathbf{G}(1)} = \frac{1}{\|\mathbf{G}\|_{H^\infty}}, \qquad \text{where } p := \infty \text{ if } \mathbf{G}(1) = \|\mathbf{G}\|_{H^\infty} = 0.$$

For $\xi \in \mathbb{R}^{2n}$, we write

$$\xi = [\![\xi^0, \xi^{-1}]\!], \quad \xi^0, \xi^{-1} \in \mathbb{R}^n,$$

and define the linear functional $\mathcal{F} : \mathbb{R}^{2n} \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{F}(\xi) &= \mathcal{F}\left([\![\xi^0, \xi^{-1}]\!]\right) = (c^\top + d^\top)(I - A)^{-1}\xi^0 + d^\top \xi^{-1} \\ &= \left((c^\top + d^\top)(I - A)^{-1}, d^\top\right)\xi.\end{aligned}$$

Defining $\phi \in \mathbb{R}^{2n}_+$ by

$$\phi := \left((c^\top + d^\top)(I - A)^{-1}, d^\top\right)^\top, \tag{2.2}$$

it follows that $\mathcal{F}(\xi) = \phi^\top \xi$, and therefore, we will frequently identify $\mathcal{F}$ with the row vector $\phi^\top = \left((c^\top + d^\top)(I - A)^{-1}, d^\top\right)$. Furthermore, for $e \in \mathbb{R}^n$, we shall make use of the matrix

$$\mathcal{O}(e^\top, A) := \begin{pmatrix} e^\top \\ e^\top A \\ \vdots \\ e^\top A^{n-1} \end{pmatrix} \in \mathbb{R}^{n \times n},$$

and introduce the following assumption (cf. [32]):

(O) $\ker \mathcal{O}(c^\top, A) \cap \ker \mathcal{O}(d^\top, A) \cap \mathbb{R}^n_+ = \{0\}$.

As A, b, c and d are non-negative, (O) is equivalent to $\ker \mathcal{O}((c+d)^\top, A) \cap \mathbb{R}^n_+ = \{0\}$.

The matrix $\mathcal{O}(e^\top, A)$ is the so-called *observability matrix* of the observed system

$$x(t+1) = Ax(t), \quad y(t) = e^\top x(t). \tag{2.3}$$

Obviously, the output y of (2.3) is given by $y(t) = e^\top A^t x(0)$. It is well known and not difficult to show that $y(t) = 0$ for all $t \in \mathbb{N}_0$ if, and only if, $x(0) \in \ker \mathcal{O}(e^\top, A)$. In particular, if $x(0) \notin \ker \mathcal{O}(e^\top, A)$, then the corresponding observation is not identically equal to 0. If $\ker \mathcal{O}(e^\top, A)$ is non-trivial, then the non-zero vectors in $\ker \mathcal{O}(e^\top, A)$ are called the *unobservable states* of (2.3). If there exists an unobservable state x^0 of (2.3), then, for every $z \in \mathbb{R}^n$, we have that $e^\top A^t z = e^\top A^t (z + x^0)$ for all $t \in \mathbb{N}_0$, that is, the states z and $z + x^0$ are indistinguishable on the basis of their corresponding output information.

The condition (O) is not very restrictive because the set

$$\Omega := \{(e, A) \in \mathbb{R}^n_+ \times \mathbb{R}^{n\times n}_+ : \ker \mathcal{O}(e^\top, A) \cap \mathbb{R}^n_+ = \{0\}\}$$

is "large" in the sense that Ω is dense in $\mathbb{R}^n_+ \times \mathbb{R}^{n\times n}_+$, Ω is relatively open with respect to $\mathbb{R}^n_+ \times \mathbb{R}^{n\times n}_+$ and the complement of Ω in $\mathbb{R}^n_+ \times \mathbb{R}^{n\times n}_+$ has zero Lebesgue measure. The latter is a trivial consequence of the inclusion

$$\left(\mathbb{R}^n_+ \times \mathbb{R}^{n\times n}_+\right) \setminus \Omega \subset \left(\{0\} \times \mathbb{R}^{n\times n}_+\right) \cup \left(\partial \mathbb{R}^n_+ \times \partial \mathbb{R}^{n\times n}_+\right) \subsetneq \partial\left(\mathbb{R}^n_+ \times \mathbb{R}^{n\times n}_+\right).$$

Assumption (O) shall appear in later persistence and stability results, see Corollaries 3.13 and 4.4, respectively. Roughly speaking, its validity simplifies the verification of persistence properties. The following proposition contains an elementary characterisation of condition (O) and, further, that (O) is sufficient for certain positivity properties of $\mathbf{G}$ and $\mathcal{F}$.

Proposition 2.1. *The following statements hold.*

(1) $\mathbf{G}(1) > 0$ *if, and only if,* $b \notin \ker \mathcal{O}((c+d)^\top, A)$.
(2) *Assumption* (O) *holds if, and only if,* $(c+d)^\top (I - A)^{-1} = (c+d)^\top \sum_{k=0}^{\infty} A^k \gg 0$.
(3) *If* (O) *holds, then* $\mathbf{G}(1) > 0$.
(4) $\mathcal{F}(\xi) \geq 0$ *for all* $\xi \in \mathbb{R}^{2n}_+$.
(5) *If* (O) *holds, then* $\inf_{\xi \in \mathbb{R}^{2n}_+, \|\xi^0\|=1} \mathcal{F}(\xi) > 0$.
(6) *If* $d \gg 0$, *then* $\inf_{\xi \in \mathbb{R}^{2n}_+, \|\xi\|=1} \mathcal{F}(\xi) > 0$.
(7) *Assume that* $b \notin \ker \mathcal{O}((c+d)^\top, A)$. *If* z *is a solution of the linear difference equation*

$$z(t+1) = Az(t) + pb\left(c^\top z(t) + d^\top z(t-1)\right), \tag{2.4}$$

then

$$\mathcal{F}\big(\tilde{z}(t+1)\big) = \mathcal{F}\big(\tilde{z}(t)\big), \quad t \in \mathbb{N}_0, \quad \textit{where} \quad \tilde{z}(t) := [\![z(t), z(t-1)]\!].$$

Statement (7) shows that if $b \notin \ker \mathcal{O}((c+d)^\top, A)$ and z is a solution of (2.4), then $\mathcal{F}$ is constant on $\tilde{z}(\mathbb{N}_0)$.

Proof of Proposition 2.1. (1) As A is asymptotically stable, we have that

$$(I-A)^{-1} = \sum_{j=0}^{\infty} A^j, \tag{2.5}$$

and so,

$$\mathbf{G}(1) = (c+d)^\top \Big(\sum_{j=0}^{\infty} A^j\Big) b \geq 0.$$

Trivially, if $\mathbf{G}(1) = 0$, then $(c+d)^\top A^j b = 0$ for all $j \in \mathbb{N}_0$, and so $b \in \ker \mathcal{O}((c+d)^\top, A)$. Conversely, if $b \in \ker \mathcal{O}((c+d)^\top, A)$, then $(c+d)^\top A^j b = 0$ for all $j \in \{0, 1, \ldots, n-1\}$. An application of the Cayley–Hamilton theorem then shows that $(c+d)^\top A^j b = 0$ for all $j \in \mathbb{N}_0$, implying that $\mathbf{G}(1) = 0$. The claim now follows via contraposition.
(2) Yet again, we will prove the statement by contraposition. If (O) does not hold, then there exists non-zero $z \in \mathbb{R}^n_+$ such that $(c+d)^\top A^j z = 0$ for all $j \in \{0, 1, \ldots, n-1\}$. Hence, by the Cayley–Hamilton theorem, $(c+d)^\top A^j z = 0$ for all $j \in \mathbb{N}_0$, and so, appealing to (2.5), $(c+d)^\top (I-A)^{-1} z = 0$, showing that $(c+d)^\top (I-A)^{-1}$ is not strictly positive.

Conversely, assume that there exists non-zero $z \in \mathbb{R}^n_+$ such that $(c+d)^\top (I-A)^{-1} z = 0$, that is, $(c+d)^\top (I-A)^{-1}$ is not strictly positive. Using (2.5) once more, we see that $(c+d)^\top A^j z = 0$ for all $j \in \mathbb{N}_0$. Consequently, $z \in \ker \mathcal{O}((c+d)^\top, A)$, showing that the intersection $\ker \mathcal{O}(c^\top, A) \cap \ker \mathcal{O}(d^\top, A) \cap \mathbb{R}^n_+$ contains non-zero elements.

(3) This is an immediate consequence of statement (2) as $b > 0$.

(4) Statement (4) is obvious as $(c+d)^\top (I-A)^{-1} \geq 0$ and $d^\top \geq 0$.

(5) This statement follows from statement (2).

(6) Noting that $d \gg 0$ implies that $\phi^\top = \big((c^\top + d^\top)(I-A)^{-1}, d^\top\big) \gg 0$, the claim follows easily.

(7) Assume that $b \notin \ker \mathcal{O}((c+d)^\top, A)$. By statement (1), $\mathbf{G}(1) > 0$, or, equivalently, $p < \infty$. Let z be a solution of $z(t+1) = Az(t) + pb\big(c^\top z(t) + d^\top z(t-1)\big)$. Using that $(I-A)^{-1}A = (I-A)^{-1} - I$, we obtain that, for all $t \in \mathbb{N}_0$,

$$\begin{aligned}
\mathcal{F}\big(\tilde{z}(t+1)\big) &= (c+d)^\top (I-A)^{-1}\big(Az(t) + pb(c^\top z(t) + d^\top z(t-1))\big) + d^\top z(t) \\
&= (c+d)^\top (I-A)^{-1} z(t) + c^\top z(t) + d^\top z(t-1) - (c+d)^\top z(t) \\
&\quad + d^\top z(t) \\
&= (c+d)^\top (I-A)^{-1} z(t) + d^\top z(t-1) \\
&= \mathcal{F}\big(\tilde{z}(t)\big),
\end{aligned}$$

completing the proof. □

3. Boundedness, persistency and excitability

In this section, we explore boundedness, persistency and excitability properties of the Lur'e system (1.1). Frequently, our analysis benefits from the useful and convenient construction of expressing the second-order (or, delayed) Lur'e system (1.1) in first-order form:

$$\tilde{x}(t+1) = \tilde{A}\tilde{x}(t) + \tilde{b} f\big(u(t), \tilde{c}^\top \tilde{x}(t)\big) + \tilde{v}(t), \quad \tilde{x}(0) = \tilde{x}^0 := [\![x^0, x^{-1}]\!], \tag{3.1}$$

where $t \in \mathbb{N}_0$, and

$$\tilde{A} := \begin{pmatrix} A & 0 \\ I & 0 \end{pmatrix}, \quad \tilde{b} := \begin{pmatrix} b \\ 0 \end{pmatrix}, \quad \tilde{c} := \begin{pmatrix} c \\ d \end{pmatrix}, \quad \tilde{x}(t) := \begin{pmatrix} x(t) \\ x(t-1) \end{pmatrix},$$
$$\tilde{v}(t) := \begin{pmatrix} v(t) \\ 0 \end{pmatrix}. \tag{3.2}$$

We note that $\tilde{A}$ is asymptotically stable, $\tilde{A}$, $\tilde{b}$ and $\tilde{c}$ are non-negative and

$$(\zeta I - \tilde{A})^{-1} = \begin{pmatrix} (\zeta I - A)^{-1} & 0 \\ (1/\zeta)(\zeta I - A)^{-1} & (1/\zeta) I \end{pmatrix} \quad \forall\, \zeta \in \mathbb{C},\ \zeta \neq 0,\ \zeta \notin \operatorname{spec}(A).$$

Hence,

$$\begin{aligned}
\tilde{c}^\top (\zeta I - \tilde{A})^{-1} \tilde{b} &= (c^\top + (1/\zeta) d^\top)(\zeta I - A)^{-1} b \\
&= \mathbf{G}(\zeta) \quad \forall\, \zeta \in \mathbb{C},\ \zeta \neq 0,\ \zeta \notin \operatorname{spec}(A).
\end{aligned} \tag{3.3}$$

Moreover,

$$\tilde{c}^\top (I - \tilde{A})^{-1} = \big((c+d)^\top (I-A)^{-1}, d^\top\big) = \phi^\top.$$

Recall that, in (1.1), the functions u and v take values in the compact subsets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n_+$, respectively, where it is assumed that $0 \in V$. We introduce the following assumptions on the nonlinearity f.

(N1) $f(w,z) > 0$ for all $w \in U$ and $z > 0$ and
$$\limsup_{z \to \infty} \Big(\max_{w \in U} \frac{f(w,z)}{z}\Big) < p.$$

(N2) (N1) holds, $p < \infty$ and
$$\liminf_{z \downarrow 0} \Big(\min_{w \in U} \frac{f(w,z)}{z}\Big) > p.$$

Conditions (N1) and (N2) were also employed in the continuous-time setting in [32] when studying boundedness and persistence. The interested reader can find a biological interpretation of these assumptions in [14, Remark 4.2]. We note that, by statement (1) of Proposition 2.1, we could replace condition $p < \infty$ in (N2) by $b \notin \ker \mathcal{O}(c^\top + d^\top, A)$.

3.1. Boundedness and persistency

The first result of this subsection will be used as a tool in the analysis of persistence properties of system (1.1). We set

$$y_x(t) := c^\top x(t) + d^\top x(t-1) = \tilde{c}^\top \tilde{x}(t) \quad \forall\, t \in \mathbb{N}_0,$$

where x is the solution of (1.1).

Proposition 3.1. *Consider the system (1.1) and let $\beta > 0$. The following statements hold.*

(1) *If (N1) holds, then there exist $\gamma > 0$ such that, for all $x^{-1}, x^0 \in \mathbb{R}^n_+$ with $\|x^{-1}\|, \|x^0\| \leq \beta$, all $u : \mathbb{N}_0 \to U$, and all $v : \mathbb{N}_0 \to V$, the solution x of (1.1) satisfies*
$$\|x(t-1)\| \leq \gamma, \quad \forall\, t \in \mathbb{N}_0. \tag{3.4}$$

(2) *If (N2) holds, then there exists $\theta > 0$ such that, for all $x^{-1}, x^0 \in \mathbb{R}^n_+$ with $\|x^{-1}\|, \|x^0\| \leq \beta$, all $u : \mathbb{N}_0 \to U$, and all $v : \mathbb{N}_0 \to V$, the solution x of (1.1) satisfies*
$$\min_{w \in U} f(w, y_x(t)) \geq \theta y_x(t), \quad \forall\, t \in \mathbb{N}_0. \tag{3.5}$$

(3) *If (N2) holds, there exists $\eta > 0$ such that, for all $x^{-1}, x^0 \in \mathbb{R}^n_+$ with $\|x^{-1}\|, \|x^0\| \leq \beta$, all $u : \mathbb{N}_0 \to U$, and all $v : \mathbb{N}_0 \to V$, the solution x of (1.1) satisfies*
$$\mathcal{F}(\tilde{x}(t)) \geq \min\big(\mathcal{F}(\tilde{x}(0)), \eta\big), \quad \forall\, t \in \mathbb{N}_0.$$

Proof. (1) Statement (1) follows immediately from [14, Theorem 4.4] applied to the augmented system (3.1). Note that (1.1) and (3.1) have the same nonlinear term f, and $\tilde{c}^\top (I - \tilde{A})^{-1} \tilde{b} = \mathbf{G}(1) = 1/p$ by (3.3).

(2) By statement (1) there exists $\gamma > 0$ such that, for all $x^{-1}, x^0 \in \mathbb{R}^n_+$ with $\|x^{-1}\|, \|x^0\| \leq \beta$ and all functions $u : \mathbb{N}_0 \to U$ and $v : \mathbb{N}_0 \to V$, the solution x of (1.1) satisfies

$$y_x(t) \leq (\|c\| + \|d\|)\gamma \quad \forall\, t \in \mathbb{N}_0. \tag{3.6}$$

On the one hand, by assumption (N2), there exists $0 < y^\sharp < (\|c\| + \|d\|)\gamma$ such that

$$\min_{w \in U} f(w,z) \geq pz \quad \forall\, z \in [0, y^\sharp]. \tag{3.7}$$

On the other hand, using the positivity and continuity of f,

$$\hat{p} := \min\{ f(w,z)/z : w \in U, y^\sharp \leq z \leq (\|c\| + \|d\|)\gamma \} > 0. \tag{3.8}$$

Hence, inequality (3.5) holds with $\theta := \min\{p, \hat{p}\}$.

(3) As in the proof of statement (2), there exists $\gamma > 0$ such that (3.6) is satisfied for all x^0 and x^{-1} such that $\|x^0\|, \|x^{-1}\| \le \beta$. By assumption (N2) there exists $0 < y^\sharp < (\|c\| + \|d\|)\gamma$ such that (3.7) holds. Let $t \in \mathbb{N}_0$. We consider two cases.

Case 1: $y_x(t) < y^\sharp$. Since $f(u(t), y_x(t)) \ge p y_x(t)$, we have

$$\begin{aligned}\mathcal{F}\big(\tilde{x}(t+1)\big) &= (c+d)^\top (I-A)^{-1} x(t+1) + d^\top x(t)\\ &= (c+d)^\top (I-A)^{-1}\big[Ax(t) + bf(u(t), y_x(t)) + v(t)\big] + d^\top x(t)\,.\end{aligned}$$

Hence, using that $(I-A)^{-1}A = (I-A)^{-1} - I$,

$$\begin{aligned}\mathcal{F}\big(\tilde{x}(t+1)\big) &\ge (c+d)^\top (I-A)^{-1}\big(Ax(t) + pby_x(t)\big) + d^\top x(t)\\ &= (c+d)^\top (I-A)^{-1} x(t) - (c+d)^\top x(t) + y_x(t) + d^\top x(t)\\ &= (c+d)^\top (I-A)^{-1} x(t) + d^\top x(t-1).\end{aligned}$$

The above yields that

$$\mathcal{F}\big(\tilde{x}(t+1)\big) \ge \mathcal{F}\big(\tilde{x}(t)\big). \tag{3.9}$$

Case 2: $y^\sharp \le y_x(t) \le (\|c\| + \|d\|)\gamma$. With $\hat{p}$ as in (3.8), we have that

$$\begin{aligned}\mathcal{F}\big(\tilde{x}(t+1)\big) &= (c+d)^\top (I-A)^{-1}\big[Ax(t) + bf(u(t), y_x(t)) + v(t)\big] + d^\top x(t)\\ &\ge \mathbf{G}(1)\hat{p}y^\sharp = \frac{\hat{p}y^\sharp}{p} =: \eta > 0\,.\end{aligned}$$

Together with (3.9) established in Case 1, this shows that

$$\mathcal{F}\big(\tilde{x}(t+1)\big) \ge \min(\mathcal{F}\big(\tilde{x}(t)\big), \eta), \quad t \in \mathbb{N}_0,$$

from which the claim follows. □

Whilst Propositions 2.1 and 3.1 (with $c = 0$) show that there are some strong similarities between the continuous-time and discrete-time cases, see [32, Propositions 3.2, 3.3 and Lemma 4.2], there are also substantial differences. To highlight the differences, we consider the following positive differential-delay system

$$\begin{aligned}&\dot{x}(t) = A_c x(t) + bf(u(t), d^\top x(t-h)) + v(t),\\ &x(t) = \xi(t) \;\; \forall\, t \in [-h, 0], \;\; \xi \in C([-h, 0], \mathbb{R}^n_+),\end{aligned} \tag{3.10}$$

where $A_c \in \mathbb{R}^{n\times n}_+$, $b, d \in \mathbb{R}^n_+$, $b \ne 0$, $d \ne 0$, $h > 0$, $f : \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, and u and v are bounded measurable functions defined on $\mathbb{R}_+$ with values in $\mathbb{R}^m$ and $\mathbb{R}^n_+$, respectively. Obviously, (3.10) is a natural continuous-time time analogue of (1.1) (with $c = 0$).

The following result is an immediate consequence of [32, Proposition 3.2 and Theorem 4.3].

Proposition 3.2. *Assume that in (3.10), A_c is Metzler (all off-diagonal entries are non-negative) and Hurwitz (all eigenvalues have negative real parts). The following statements hold.*

(1) $\ker \mathcal{O}(d^\top, A_c) \cap \mathbb{R}^n_+ = \{0\}$ *if, and only if,* $d^\top e^{A_c t} \gg 0$ *for all* $t > 0$.
(2) *If* $\ker \mathcal{O}(d^\top, A_c) \cap \mathbb{R}^n_+ = \{0\}$, (N2) *holds and* $\xi(0) \ne 0$, *then the solution x of* (3.10) *satisfies* $d^\top x(t) > 0$ *for all* $t \ge 2h$.
(3) *If* $d \gg 0$, (N2) *holds and* $\xi(t) \not\equiv 0$, *then the solution x of* (3.10) *satisfies* $d^\top x(t) > 0$ *for all* $t \ge 2h$.

The following example shows that Proposition 3.2 does not carry over to the discrete-time setting.

Example 3.3. Consider system (1.1) without external forcing (that is, $v = 0$ and $f(w, z) = f(z)$ for all $z \in \mathbb{R}_+$) and the linear and initial data

$$A = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad b = \begin{pmatrix} 1/2\\ 0\\ 1/2\end{pmatrix}, \quad c = 0, \quad d = \begin{pmatrix} 1\\ 1\\ 1\end{pmatrix}, \quad x^0 = \begin{pmatrix} 1\\ 0\\ 0\end{pmatrix}, \quad x^{-1} = 0.$$

It is clear that (O) holds (because $d \gg 0$) and $d^\top A^t = (0, 0, 0)$ for all $t \in \mathbb{N}_0$ with $t \ge 2$. This shows that statement (1) of Proposition 3.2 does not carry over to the discrete-time setting.

Furthermore, for every continuous $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$, it is straightforward to prove by induction that the solution x of (1.1) satisfies

$$\begin{aligned}&x(t) = 0, \quad \text{if } t = 2m+1,\ m \in \mathbb{N}_0\\ \text{and}\quad &x(t) = bf^{t/2}(1), \quad \text{if } t = 2(m+1),\ m \in \mathbb{N}_0.\end{aligned}$$

Consequently, $d^\top x = 0$ if $t = 2m + 1$, $m \in \mathbb{N}_0$, showing that statements (2) and (3) of Proposition 3.2 do not extend to the discrete-time system (1.1). ◊

A key requirement for the stability properties that we will present in Section 4 is that the observation $y_x(t) = c^\top x(t) + d^\top x(t-1)$ associated with system (1.1) is eventually uniformly positive. However, as we have just seen in Example 3.3, conditions (O) and (N2) (or conditions $d \gg 0$ and (N2)) do not guarantee that such a requirement is fulfilled in the discrete-time case for all $(x^0, x^{-1}) \ne (0, 0)$. In order to deal with this issue, we introduce a concept of persistency with respect to a set of initial conditions for the system (1.1).

Definition 3.4. Given a set of initial conditions $\Gamma \subset \mathbb{R}^{2n}_+$, we say that (1.1) is *$\tilde{c}$-persistent with respect to Γ* if there exist $\tau \in \mathbb{N}_0$ and $\delta > 0$ such that the solution x of (1.1) satisfies

$$y_x(t+\tau) = \tilde{c}^\top \tilde{x}(t+\tau) = c^\top x(t+\tau) + d^\top x(t+\tau-1) \ge \delta \quad \forall\, t \in \mathbb{N}_0\,, \tag{3.11}$$

for all $(x^0, x^{-1}) \in \Gamma$ and all functions $u : \mathbb{N}_0 \to U$ and $v : \mathbb{N}_0 \to V$. ◊

As the persistency property $\tilde{c}^\top \tilde{x}(t) \ge \delta$ is required to hold only for all sufficiently large t, it would perhaps be more accurate to call the concept "ultimate $\tilde{c}$-persistency with respect to Γ", but we will refrain from doing so for the sake of simplicity and the avoidance of potentially awkward formulations.

If f is non-decreasing in its second argument (that is, $f(w, z_1) \le f(w, z_2)$ for all $0 \le z_1 \le z_2$ and all $w \in U$), then persistency can be checked without reference to the additive forcing v. This follows because, in this case, it is routine to show that (1.1) is $\tilde{c}$-persistent with respect to Γ if, and only if, there exist $\tau \in \mathbb{N}_0$ and $\delta > 0$ such that the solution x of (1.1) with $v = 0$ satisfies (3.11) for all $(x^0, x^{-1}) \in \Gamma$ and all $u : \mathbb{N}_0 \to U$. However, this equivalence is not in true in general.

Our first approach to $\tilde{c}$-persistency properties of (1.1) is to apply the persistency results of [14, Section 4] for undelayed difference equations to the augmented system (3.1). To this end, we introduce the following condition on $\tilde{A}$, $\tilde{b}$ and $\tilde{c}$ defined in (3.2).

(P1) There exists $\tau \in \mathbb{N}_0$ such that $\tilde{c}^\top\big(\tilde{A} + \tilde{b}\tilde{c}^\top\big)^\tau \gg 0$.

A simple consequence of (P1) is described in the following lemma.

Lemma 3.5. *Assume that there exists $\tau \in \mathbb{N}_0$ such that* (P1) *holds. Then*

$$\tilde{c}^\top\big(\tilde{A} + q\tilde{b}\tilde{c}^\top\big)^{t+\tau} \gg 0 \quad \textit{for all } q > 0 \textit{ and } t \in \mathbb{N}_0.$$

Proof. Let $q > 0$. If $q \ge 1$, then $q - 1 \ge 0$, and thus,

$$\tilde{c}^\top\big(\tilde{A} + q\tilde{b}\tilde{c}^\top\big)^\tau = \tilde{c}^\top\big(\tilde{A} + \tilde{b}\tilde{c}^\top + (q-1)\tilde{b}\tilde{c}^\top\big)^\tau \ge \tilde{c}^\top\big(\tilde{A} + \tilde{b}\tilde{c}^\top\big)^\tau \gg 0.$$

If $q < 1$, then $q^{-1} - 1 > 0$, and hence,

$$\tilde{c}^\top\big(\tilde{A} + q\tilde{b}\tilde{c}^\top\big)^\tau = q^\tau \tilde{c}^\top\big(\tilde{A} + \tilde{b}\tilde{c}^\top + (q^{-1} - 1)\tilde{A}\big)^\tau \ge q^\tau \tilde{c}^\top\big(\tilde{A} + \tilde{b}\tilde{c}^\top\big)^\tau \gg 0.$$

It follows that the matrix $\tilde{A} + q\tilde{b}\tilde{c}^\top$ cannot have any zero columns. Consequently, as $\tilde{c}^\top\big(\tilde{A} + q\tilde{b}\tilde{c}^\top\big)^\tau \gg 0$, we have that $\tilde{c}^\top\big(\tilde{A} + q\tilde{b}\tilde{c}^\top\big)^{\tau+1} \gg 0$. The claim now follows from a straightforward induction argument. □

The next result provides necessary and sufficient conditions for (P1) to hold.

Lemma 3.6. *The following statements hold.*

(1) *If* (P1) *holds, then* $d \gg 0$.

(2) *If $d \gg 0$ and there exists $k \in \mathbb{N}_0$ such that $c^\top(A+bc^\top)^k \gg 0$, then (P1) holds with $\tau = k+1$.*
(3) *If $d \gg 0$ and $Ab > 0$, then (P1) holds with $\tau = 3$.*
(4) *If $c = 0$ and (P1) holds, then $d \gg 0$ and $Ab > 0$.*

Proof. (1) If d is not strictly positive, then

$$\tilde{A} + \tilde{b}\tilde{c}^\top = \begin{pmatrix} A + bc^\top & bd^\top \\ I & 0 \end{pmatrix}$$

has a zero column, and hence $(\tilde{A} + \tilde{b}\tilde{c}^\top)^\tau$ has a zero column for every $\tau \in \mathbb{N}$. Consequently, $\tilde{c}^\top(\tilde{A} + \tilde{b}\tilde{c}^\top)^\tau$ is not strictly positive for every $\tau \in \mathbb{N}_0$, showing that (P1) does not hold. The claim now follows by contraposition.

(2) A standard induction argument shows that, for every $j \in \mathbb{N}_0$, there exist vectors $q_j, r_j \in \mathbb{R}^n_+$ such that

$$\tilde{c}^\top\left(\tilde{A} + \tilde{b}\tilde{c}^\top\right)^{j+1} = \left(c^\top(A+bc^\top)^{j+1} + q_j, (c^\top(A+bc^\top)^j + r_j)bd^\top\right).$$

It follows that if $d \gg 0$ and there exists $k \in \mathbb{N}_0$ such that $c^\top(A+bc^\top)^k \gg 0$, then (P1) holds for $\tau = k+1$.

(3) A routine calculation shows that there exist $q, r \in \mathbb{R}^n_+$ such that

$$\tilde{c}^\top\left(\tilde{A} + \tilde{b}\tilde{c}^\top\right)^3 = \left((d^\top b)d^\top + q^\top, (d^\top Ab)d^\top + r^\top\right).$$

Consequently, if $d \gg 0$ and $Ab > 0$, then the right-hand side is strictly positive, showing that (P1) holds for $\tau = 3$.

(4) Assume that $c = 0$ and (P1) holds, and set

$$\hat{A} := \tilde{A} + \tilde{b}\tilde{c}^\top = \begin{pmatrix} A & bd^\top \\ I & 0 \end{pmatrix}.$$

By statement (1), $d \gg 0$. To show that $Ab > 0$, we argue by contraposition. If $Ab > 0$ does not hold, then $Ab = 0$ as $Ab \geq 0$. A simple induction argument shows that then

$$\hat{A}^{2k+1} = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \quad \forall\, k \in \mathbb{N}_0,$$

with the induction step comprising

$$\hat{A}^{2(k+1)+1} = \hat{A}^{2k+1}\hat{A}^2 = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}\begin{pmatrix} * & Abd^\top \\ * & * \end{pmatrix} = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}\begin{pmatrix} * & 0 \\ * & * \end{pmatrix} = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix},$$

where $*$ stands for certain matrices, the entries and structure of which are irrelevant for the matter under consideration. Therefore, $\tilde{c}^\top\hat{A}^{2k+1} = (*, 0)$ for all $k \in \mathbb{N}_0$, and so, property (P1) cannot hold for odd τ. It follows from Lemma 3.5 that (P1) cannot hold for even τ either, completing the proof. □

We are now in the position to state the first persistency result.

Corollary 3.7. *Consider the system (1.1) and assume that (P1) and (N2) hold. Then (1.1) is $\tilde{c}$-persistent with respect to any compact set $\Gamma \subset \mathbb{R}^{2n}_+$ such that $0 \notin \Gamma$.*

Corollary 3.7 follows from a straightforward application of [14, Theorem 4.4] to the augmented system (3.1).

Observe that in Example 3.3 the conclusion of Corollary 3.7 does not hold. This is explained by the fact that (P1) is not satisfied (as follows from statement (4) of Lemma 3.6 since $c = Ab = 0$).

Whilst the conclusion of Corollary 3.7 is strong in the sense that it guarantees $\tilde{c}$-persistency with respect to every compact set $\Gamma \subset \mathbb{R}^{2n}_+$ not containing 0 (so-called semi-global $c^\top$-persistency), the requirement of strict positivity of d (which, by Lemma 3.6, is necessary for (P1) to hold) is too restrictive for many applications. Therefore, our approach is to identify conditions weaker than (P1) which, together with (N2) (or some variant of it), are sufficient for $\tilde{c}$-persistency with respect sufficiently "large" sets to allow interesting applications. To this end, we introduce the following positivity hypotheses on the linear system:

(P2) There exist $\tau \in \mathbb{N}$ and $\varepsilon > 0$ such that $\tilde{c}^\top(\tilde{A}+\tilde{b}\tilde{c}^\top)^\tau \geq \varepsilon\phi^\top$, where ϕ is defined in (2.2).

The inequality in (P2) can be expressed in the form $\tilde{c}^\top(\tilde{A} + \tilde{b}\tilde{c}^\top)^\tau\xi \geq \varepsilon\mathcal{F}(\xi)$ for all $\xi \in \mathbb{R}^{2n}_+$.

Furthermore, we consider the following variant of hypothesis (N2):

(N3) Hypothesis (N1) holds, $d^\top b > 0$ and

$$\liminf_{z\downarrow 0}\left(\min_{w\in U}\frac{f(w,z)}{z}\right) > \frac{1}{d^\top b}.$$

It follows from an argument similar to that used in the proof of Lemma 3.5 that if (P2) holds, then, for all $q > 0$,

$$\tilde{c}^\top(\tilde{A} + q\tilde{b}\tilde{c}^\top)^\tau \geq \varepsilon_q\phi^\top, \tag{3.12}$$

where $\varepsilon_q := \varepsilon\min\{1, q^\tau\} > 0$. Note that (P2) is implied by (P1), but (P2) is not sufficient for (P1) to hold (see Example 3.8 below). In fact it is easy to show that (P1) is satisfied if, and only if, (P2) holds and $d \gg 0$.

Whilst, (P2) is weaker than (P1), the hypothesis (N3) places a stronger condition on the nonlinearity f than (N2) because $d^\top b \leq \mathbf{G}(1) = 1/p$, and thus, if (N3) holds, then so does (N2). In general, $d^\top b$ and $1/p$ are not equal, but when they are, hypotheses (N2) and (N3) coincide.

Example 3.8. We discuss two examples illustrating condition (P2).
(1) Consider

$$A = \begin{pmatrix} 1/2 & 1 \\ 0 & 1/2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad c = 0, \quad d = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

As

$$(I - A)^{-1} = \begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix},$$

we have that $\phi^\top = \left(d^\top(I-A)^{-1}, d^\top\right) = (2,4,1,0)$. Moreover, $\tilde{c}^\top(\tilde{A} + \tilde{b}\tilde{c}^\top)^2 = (1/2, 1, 1, 0)$, and we conclude that (P2) holds with $\tau = 2$ and $\varepsilon = 1/4$. It follows from Lemma 3.6 that (P1) does not hold because d is not strictly positive.
(2) Let A, b, c and d be given by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad c = 0, \quad d = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \tag{3.13}$$

Then

$$(I - A)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

and $\phi^\top = \left(d^\top(I-A)^{-1}, d^\top\right) = (2,2,1,0,1,1)$. Elementary calculations yield that $\tilde{c}^\top\left(\tilde{A} + \tilde{b}\tilde{c}^\top\right)^7 = (1,1,1,0,2,2)$, showing that (P2) holds with $\tau = 7$ and $\varepsilon = 1/2$. Moreover, 7 is the smallest value of τ such that (P2) is satisfied. As d is not strictly positive, Lemma 3.6 implies that (P1) does not hold. ◊

The next result provides a sufficient condition for (P2).

Lemma 3.9. *If there exist $k, l \in \mathbb{N}_0$ and $\delta > 0$ such that*

$$\left(d^\top, d^\top\right)\left(\tilde{A}+\tilde{b}\tilde{c}^\top\right)^l \geq \delta\phi^\top \quad \text{and} \quad \mu := \min\{d^\top A^k b, d^\top A^{k+1} b\} > 0, \tag{3.14}$$

then (P2) holds with $\tau = k + l + 3$ and $\varepsilon = \delta\mu$.

Assume that there exists $\rho > 0$ such that $d^\top \geq \rho c^\top$ (for example, when $c = 0$). Then it is sometimes easier to verify (3.14) than (P2), see part (1) of Example 3.10 below. Furthermore,

$$\left(d^\top, d^\top\right)(\tilde{A} + \tilde{b}\tilde{c}^\top) \geq \left(\rho c^\top, d^\top\right)(\tilde{A} + \tilde{b}\tilde{c}^\top) \geq \min\{1,\rho\}\tilde{c}^\top(\tilde{A} + \tilde{b}\tilde{c}^\top),$$

and thus (P2) implies that the first inequality in (3.14) holds. However, (P2) does not imply the existence of an integer k such that the second inequality in (3.14) is satisfied as part (2) of Example 3.10 below shows.

Example 3.10. (1) Consider A, b, c and d as given in (3.13) with $\phi^\top = (2,2,1,0,1,1)$. It is straightforward to show that $(d^\top, d^\top)(\tilde{A} + \tilde{b}\tilde{c}^\top)^3 = (1,1,1,0,2,2) \geq (1/2)\phi^\top$ and $d^\top A b = d^\top A^2 b = 1 > 0$, showing that (3.14) holds with $l = 3$, $\delta = 1/2$ and $k = 1$. Recall from part (2) of Example 3.8 that 7 is the smallest value of τ such that (P2) is satisfied.
(2) Consider

$$A = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c = 0, \quad d = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where $0 < a < 1$. Elementary calculations show that (P2) holds with $\tau = 4$ and $\varepsilon = (1 - a^2)a^4$. But, for all $k \in \mathbb{N}_0$,

$$d^\top A^{2k} b = 0 \quad \text{and} \quad d^\top A^{2k+1} b = a^{2k+1},$$

showing that $\min\{d^\top A^k b, d^\top A^{k+1} b\} = 0$ for all $k \in \mathbb{N}_0$. It follows that condition (3.14) is not necessary for (P2) to hold. $\diamondsuit$

Proof of Lemma 3.9. Since

$$\tilde{A} + \tilde{b}\tilde{c}^\top = \begin{pmatrix} A + bc^\top & bd^\top \\ I & 0 \end{pmatrix} \geq \begin{pmatrix} A & bd^\top \\ I & 0 \end{pmatrix},$$

a routine induction argument shows that

$$(\tilde{A} + \tilde{b}\tilde{c}^\top)^{j+3} \geq \begin{pmatrix} A^{j+1}bd^\top & A^{j+2}bd^\top \\ A^j bd^\top & A^{j+1}bd^\top \end{pmatrix} \quad \forall\, j \in \mathbb{N}_0.$$

Consequently, for all $j \in \mathbb{N}_0$,

$$\tilde{c}^\top(\tilde{A} + \tilde{b}\tilde{c}^\top)^{j+3} \geq (0, d^\top)(\tilde{A} + \tilde{b}\tilde{c}^\top)^{j+3} \geq \big((d^\top A^j b)d^\top, (d^\top A^{j+1} b)d^\top\big),$$

and thus

$$\tilde{c}^\top(\tilde{A} + \tilde{b}\tilde{c}^\top)^{k+3} \geq \mu(d^\top, d^\top).$$

Multiplying the above inequality from the right by $(\tilde{A} + \tilde{b}\tilde{c}^\top)^l$ leads to

$$\tilde{c}^\top(\tilde{A} + \tilde{b}\tilde{c}^\top)^{k+l+3} \geq \mu(d^\top, d^\top)(\tilde{A} + \tilde{b}\tilde{c}^\top)^l \geq \delta\mu\phi^\top,$$

showing that (P2) is satisfied with $\tau = k + l + 3$ and $\varepsilon = \delta\mu$. $\square$

Whilst Lemma 3.9 constitutes a sufficient condition for (P2) to hold, the next result provides, under the assumption that $(c + d)^\top b \neq 0$, a necessary condition for (P2).

Lemma 3.11. *Assume that $(c + d)^\top b \neq 0$. If (P2) is satisfied, then $(A + bc^\top)b \neq 0$.*

Proof. We prove the claim by contraposition. To this end, we assume that $(A + bc^\top)b = 0$. We have to show that (P2) is not satisfied. Since $(A + bc^\top)b = 0$, it follows that $Ab = 0$ and $(c^\top b)b = 0$. As $b \neq 0$, the latter identity implies that $c^\top b = 0$. Next we note that

$$\begin{aligned} \phi^\top \begin{pmatrix} b \\ 0 \end{pmatrix} &= \big((c + d)^\top (I - A)^{-1}, d^\top\big) \begin{pmatrix} b \\ 0 \end{pmatrix} \\ &= (c + d)^\top \sum_{j=0}^{\infty} A^j b = (c + d)^\top b > 0. \end{aligned} \tag{3.15}$$

Furthermore, as $d \neq 0$,

$$\phi^\top \begin{pmatrix} 0 \\ d \end{pmatrix} = \big((c + d)^\top (I - A)^{-1}, d^\top\big) \begin{pmatrix} 0 \\ d \end{pmatrix} = \|d\|^2 > 0. \tag{3.16}$$

On the other hand, a straightforward induction argument shows that there exist $h_k \in \mathbb{R}^n_+$ and $\lambda_k \in \mathbb{R}_+$ such that

$$\tilde{c}^\top\big(\tilde{A} + \tilde{b}\tilde{c}^\top\big)^{2k} = \big(h_k^\top A + \lambda_k c^\top, *\big) \quad \forall\, k \in \mathbb{N},$$

and, furthermore,

$$\tilde{c}^\top\big(\tilde{A} + \tilde{b}\tilde{c}^\top\big)^{2k+1} = (*, 0) \quad \forall\, k \in \mathbb{N}_0.$$

Consequently, for all $k \in \mathbb{N}_0$,

$$\tilde{c}^\top\big(\tilde{A} + \tilde{b}\tilde{c}^\top\big)^{2k} \begin{pmatrix} b \\ 0 \end{pmatrix} = 0 \quad \text{and} \quad \tilde{c}^\top\big(\tilde{A} + \tilde{b}\tilde{c}^\top\big)^{2k+1} \begin{pmatrix} 0 \\ d \end{pmatrix} = 0.$$

Comparing this to (3.15) and (3.16) shows that (P2) is not satisfied. $\square$

The following theorem, the proof of which can be found in the Appendix, contains a number of $\tilde{c}$-persistency results invoking the above hypotheses (P2), (N2) and (N3). Recall the notation $y_x(t) = c^\top x(t) + d^\top x(t-1) = \tilde{c}^\top \tilde{x}(t)$, where x is the solution of (1.1).

Theorem 3.12. *The following statements hold for the system (1.1).*

(1) *If $d^\top b > 0$, $f(w, z) > 0$ for all $z > 0$ and $w \in U$ and $\min\{c^\top x^0 + d^\top x^{-1}, d^\top x^0\} > 0$, then the solution x of (1.1) satisfies $y_x(t) > 0$ for all $t \in \mathbb{N}_0$.*
(2) *If $\min\{c^\top b, d^\top b\} > 0$, $f(w, z) > 0$ for all $z > 0$ and $w \in U$ and $c^\top x^0 + d^\top x^{-1} > 0$, then the solution x of (1.1) satisfies $y_x(t) > 0$ for all $t \in \mathbb{N}_0$.*
(3) *If hypothesis (N3) holds, then (1.1) is $\tilde{c}$-persistent with respect to any non-empty compact subset Γ of*

$$\begin{aligned} \Gamma' := \{\, [\![\xi^0, \xi^{-1}]\!] \in \mathbb{R}^{2n}_+ \ : \ &\min\{c^\top \xi^0 + d^\top \xi^{-1}, e^\top \xi^0\} \\ &+ \min\{e^\top \xi^0, e^\top A\xi^0\} > 0 \,\}, \end{aligned} \tag{3.17}$$

where $e := A^\top c + d \in \mathbb{R}^n_+$.
(4) *If hypotheses (N2) and (P2) hold, then (1.1) is $\tilde{c}$-persistent with respect to any non-empty compact subset Γ of*

$$\Gamma'' := \big\{\xi = [\![\xi^0, \xi^{-1}]\!] \in \mathbb{R}^{2n}_+ \ : \ \mathcal{F}(\xi) > 0\big\}. \tag{3.18}$$

Note that the set Γ' can be expressed in form

$$\Gamma' = \big\{[\![\xi^0, \xi^{-1}]\!] \in \mathbb{R}^{2n}_+ \ : \ \min\{e^\top \xi^0, (c^\top + e^\top A)\xi^0 + d^\top \xi^{-1}\} > 0\big\}.$$

We remark that the subsets Γ' and Γ'' of $\mathbb{R}^{2n}_+$ are "large" in the following two senses:
(a) Γ' and Γ'' are dense in $\mathbb{R}^{2n}_+$ and relatively open with respect to $\mathbb{R}^{2n}_+$;
(b) the complements $\mathbb{R}^{2n}_+ \backslash \Gamma'$ and $\mathbb{R}^{2n}_+ \backslash \Gamma''$ are null sets with respect to Lebesgue measure as

$$\mathbb{R}^{2n}_+ \backslash \Gamma' \subset \partial\mathbb{R}^{2n}_+ \quad \text{and} \quad \mathbb{R}^{2n}_+ \backslash \Gamma'' \subset \partial\mathbb{R}^n_+ \times \partial\mathbb{R}^n_+ \subsetneq \partial\mathbb{R}^{2n}_+.$$

We proceed to state and prove the following important corollary of Theorem 3.12. For which purpose, we set

$$P := (I, 0) \in \mathbb{R}^{n \times 2n}$$

and note that $P[\![\xi^0, \xi^{-1}]\!] = \xi^0$ for all $\xi^0, \xi^{-1} \in \mathbb{R}^n$.

Corollary 3.13. *Assume that hypotheses (O) and (N2) are satisfied. If (P2) holds, then (1.1) is $\tilde{c}$-persistent with respect to any non-empty compact subset $\Gamma \subset \mathbb{R}^{2n}_+$ such that $0 \notin P\Gamma$.*

Proof. Let $\Gamma \subset \mathbb{R}^{2n}_+$ be non-empty, compact and such that $0 \notin P\Gamma$. Then, obviously, for all $\xi^0, \xi^{-1} \in \mathbb{R}^n_+$,

$$[\![\xi^0, \xi^{-1}]\!] \in \Gamma \quad \Rightarrow \quad \xi^0 \neq 0.$$

Invoking statement (5) of Proposition 2.1, we conclude that $\mathcal{F}(\xi) > 0$ for all $\xi \in \Gamma$, showing that $\Gamma \subset \Gamma''$. The claim now follows from statement (4) of Theorem 3.12. $\square$

If the condition $0 \notin P\Gamma$ is not satisfied, then $\tilde{c}$-persistency with respect to Γ may fail to hold. Indeed, take any system of the form (1.1) such that (O), (P2) and (N2) are satisfied, d is not strictly positive, $f(u^e, 0) = 0$ for some $u^e \in U$, $u(t) \equiv u^e$ and $v(t) \equiv 0$: if $x^0 = 0$ and $x^{-1} \neq 0$ is such that $d^\top x^{-1} = 0$, then $x(t) \equiv 0$ is the solution of (1.1), and hence (1.1) is not $\tilde{c}$-persistent with respect to compact sets $\Gamma \subset \mathbb{R}^{2n}_+$ which contain $[\![x^0, x^{-1}]\!]$.

As the following example shows, (P2) does not imply (O), and therefore, the explicit requirement in Corollary 3.13 that (O) holds is not redundant.

Example 3.14. Consider

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad c = 0, \quad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then,

$$\phi = (d^\top(I-A)^{-1}, d^\top) = (0,2,0,1) \quad \text{and} \quad \tilde{c}^\top(\tilde{A}+\tilde{b}\tilde{c}^\top)^2 = (0,1/2,0,1),$$

showing that (P2) holds with $\tau = 2$ and $\varepsilon = 1/4$. Furthermore,

$$\ker \mathcal{O}(c^\top, A) = \mathbb{R}^n \quad \text{and} \quad \ker \mathcal{O}(d^\top, A) = \ker \begin{pmatrix} 0 & 1 \\ 0 & 1/2 \end{pmatrix} = \{(z,0)^\top : z \in \mathbb{R}\},$$

and thus, (O) is not satisfied. We therefore cannot expect that the conclusions of Corollary 3.13 hold, and indeed they do not, as we now show. To this end, we note that in this example $p = 1/\mathbf{G}(1) = 1/2$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and such that (N2) holds with $p = 1/2$ and $f(0) = 0$. Trivially, the singleton $\Gamma := \{(1,0,1,0)^\top\} \subset \mathbb{R}^4_+$ is compact and such that $0 \notin P\Gamma$. It is straightforward to see that the solution x of (1.1) with $v = 0$ and $x^0 = x^{-1} = (1,0)^\top$ satisfies $x(t) = 0$ for all $t \in \mathbb{N}$, showing that (1.1) is not $\tilde{c}$-persistent with respect to Γ. ◊

Next, we provide a simple example to illustrate Theorem 3.12.

Example 3.15. Consider the special case of (1.1) wherein the quadruple $(A, b, c^\top, d^\top)$ satisfies the conditions

$$Ab = 0, \quad c^\top b = 0 \quad \text{and} \quad d^\top b > 0, \tag{3.19}$$

and thus, $p = (d^\top b)^{-1} < \infty$. We assume that f satisfies hypothesis (N3) (which coincides with (N2)).
(a) Statement (3) of Theorem 3.12 guarantees that (1.1) is $\tilde{c}$-persistent with respect to any non-empty compact subset of the set Γ'. In the specific example given by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = d = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

the set Γ' can be expressed as

$$\Gamma' = \left\{(z_1, z_2, z_3, z_4)^\top \in \mathbb{R}^4_+ \;:\; z_1 > 0 \text{ and } z_2 + z_3 > 0\right\}.$$

(b) Returning to the class of quadruples $(A, b, c^\top, d^\top)$ given by (3.19), we note that it follows from Lemma 3.11 that hypothesis (P2) is not satisfied. In general, as statement (4) of Theorem 3.12 is only a sufficient condition, this does not imply that its persistency conclusion fails to hold. We will now show that, in the special scenario determined by (3.19), the conclusion of statement (4) of Theorem 3.12 is not valid, that is, there exist a compact subset Γ of Γ'' and a nonlinearity f satisfying (N2) such that (1.1) is not $\tilde{c}$-persistent with respect to Γ, where Γ'' is defined as in statement (4) of Theorem 3.12. To this end, choose $\xi^{-1} \in \mathbb{R}^n_+$ such that $d^\top \xi^{-1} = 0$. It is clear that $\xi := [\![b, \xi^{-1}]\!] \notin \Gamma'$, so part (a) does not apply. However, as

$$\mathcal{F}(\xi) = (c+d)^\top (I-A)^{-1} b + d^\top \xi^{-1} = d^\top b > 0,$$

it follows that $\xi \in \Gamma''$. We show that, given any nonlinearity f such that (N2) holds and $f(0, u^e) = 0$ for some $u^e \in U$, the system (1.1) is not $\tilde{c}$-persistent with respect to $\{\xi\}$. Let x be the solution of (1.1) with $u(t) \equiv u^e$ and $v(t) \equiv 0$ and such that $x(0) = b$ and $x(-1) = \xi^{-1}$. A routine induction argument shows that there exists $\lambda_t \geq 0$ such that

$$\tilde{x}(2t) = \lambda_t \begin{pmatrix} b \\ 0 \end{pmatrix} \quad \forall\, t \in \mathbb{N}.$$

Hence, $\tilde{c}^\top \tilde{x}(2t) = (c^\top, d^\top)\tilde{x}(2t) = 0$ for all $t \in \mathbb{N}$, establishing that the system is not $\tilde{c}$-persistent with respect to $\{\xi\}$. ◊

3.2. Excitability

In this section we show that even for sets Γ of initial conditions for which $0 \in P\Gamma$, the system (1.1) may be "kicked" into a persistency regime by discrete-time "delta functions" of the form

$$v = \theta_{t_0}\eta, \quad \text{where} \quad t_0 \in \mathbb{N}_0, \quad \eta \in \mathbb{R}^n_+ \quad \text{and} \quad \theta_{t_0}(t) := \begin{cases} 1, & t = t_0 \\ 0, & t \neq t_0\,. \end{cases}$$

This is addressed in the following definition.

Definition 3.16. Assume that there exists $\rho > 0$ such that $\{z \in \mathbb{R}^n_+ : \|z\| \leq \rho\} \subset V$. Given $\Gamma \subset \mathbb{R}^{2n}_+$ and $t_0 \in \mathbb{N}_0$, we say that (1.1) is
(a) *$\tilde{c}$-excitable by small inputs with respect to Γ and time t_0* if, for every $\varepsilon \in (0, \rho]$, there exist $\tau \in \mathbb{N}_0$, $\delta > 0$ and $\eta \in \mathbb{R}^n_+$ satisfying $\|\eta\| \leq \varepsilon$ and such that the solution x of (1.1) with $v = \theta_{t_0}\eta$ satisfies

$$\begin{aligned} y_x(t+t_0+\tau) &= \tilde{c}^\top \tilde{x}(t+t_0+\tau) \\ &= c^\top x(t+t_0+\tau) + d^\top x(t+t_0+\tau-1) \geq \delta \quad \forall\, t \in \mathbb{N}_0 \end{aligned} \tag{3.20}$$

for all $(x^0, x^{-1}) \in \Gamma$ and all functions $u : \mathbb{N}_0 \to U$.
(b) *strictly $\tilde{c}$-excitable by small inputs with respect to Γ and time t_0* if, for every $\varepsilon \in (0, \rho]$, there exist $\tau \in \mathbb{N}_0$ and $\delta > 0$ such that the solution x of (1.1) satisfies (3.20) for all $(x^0, x^{-1}) \in \Gamma$, all functions $u : \mathbb{N}_0 \to U$ and all $v = \theta_{t_0}\eta$ with $\varepsilon/2 \leq \|\eta\| \leq \varepsilon$. ◊

Trivially, strict $\tilde{c}$-excitability implies $\tilde{c}$-excitability, but not necessarily conversely. The former concept is independent of the "direction" of the vector η: for every forcing function $v = \theta_{t_0}\eta$ such that $\varepsilon/2 \leq \|\eta\| \leq \varepsilon$, the inequality (3.20) holds (provided that the initial condition is in Γ). We recall that the excitability concept introduced in [10] refers to an "excitation of all states" property (by application of a suitable non-negative input under zero initial conditions). The notion of $\tilde{c}$-excitability by small inputs, although quite different, is inspired by this concept.

The following corollary provides sufficient conditions for $\tilde{c}$-excitability. This result shows the strength of the $\tilde{c}$-excitability concepts, namely, that under suitable conditions, (strict) $\tilde{c}$-excitability holds with respect to arbitrary compact sets of initial conditions, as was already hinted at the beginning of this section.

Corollary 3.17. *Assume that hypotheses (N2) and (P2) are satisfied, and there exists $\rho > 0$ such that $\{z \in \mathbb{R}^n_+ : \|z\| \leq \rho\} \subset V$. Let $\Gamma \subset \mathbb{R}^{2n}_+$ be compact and $t_0 \in \mathbb{N}_0$. Then (1.1) is $\tilde{c}$-excitable by small inputs with respect to Γ and time t_0. Under the additional assumption that hypothesis (O) holds, (1.1) is strictly $\tilde{c}$-excitable by small inputs with respect to Γ and time t_0.*

Proof. Let $\Gamma \subset \mathbb{R}^{2n}_+$ be compact, $t_0 \in \mathbb{N}_0$ and $\varepsilon \in (0, \rho]$. By statement (1) of Proposition 3.1 there exists $\gamma > 0$ such that the solution x of (1.1) satisfies

$$x(t) \in \{z \in \mathbb{R}^n_+ : \|z\| \leq \gamma\} =: C \quad \forall\, t \in \mathbb{N}_0$$

for all $[\![x^0, x^{-1}]\!] \in \Gamma$ and all functions $u : \mathbb{N}_0 \to U$ and $v : \mathbb{N}_0 \to V$. As $d \neq 0$, there exists $\eta \in \mathbb{R}^n_+$ such that $\|\eta\| \leq \varepsilon$ and $d^\top \eta > 0$. Noting that

$$\mathcal{F}([\![\eta, 0]\!]) = (c+d)^\top (I-A)^{-1}\eta \geq d^\top \sum_{j=0}^{\infty} A^j \eta \geq d^\top \eta > 0,$$

it follows that the compact set

$$\hat{\Gamma} := \left\{[\![\xi^0, \xi^{-1}]\!] : \xi^0, \xi^{-1} \in C \text{ and } \xi^0 \geq \eta\right\} \subset \mathbb{R}^{2n}_+$$

is contained in Γ''. Consequently, by statement (4) of Theorem 3.12, system (1.1) is $\tilde{c}$-persistent with respect to $\hat{\Gamma}$. This means that there exist $\delta > 0$ and $\tau \in \mathbb{N}_0$ such that the solution x of (1.1) satisfies

$$y_x(t+\tau) = c^\top x(t+\tau) + d^\top x(t+\tau-1) \geq \delta \quad \forall\, t \in \mathbb{N}_0 \tag{3.21}$$

for all $[\![x^0, x^{-1}]\!] \in \hat{\Gamma}$, all functions $u : \mathbb{N}_0 \to U$ and $v : \mathbb{N}_0 \to V$.

Now let $[\![x^0, x^{-1}]\!] \in \Gamma$, $u : \mathbb{N}_0 \to U$ be arbitrary, $v = \theta_{t_0}\eta$ and x be the corresponding solution of system (1.1). Then $x(t_0+1), x(t_0) \in C$ and $x(t_0+1) \geq v(t_0) = \eta$, implying that $[\![x(t_0+1), x(t_0)]\!] \in \hat{\Gamma}$. Setting, for all $t \in \mathbb{N}_0$,

$$\hat{x}(t) := x(t+t_0+1), \quad \hat{u}(t) := u(t+t_0+1), \quad \hat{v}(t) := v(t+t_0+1) = \theta_{t_0}(t+t_0+1)\eta,$$

we have that $\hat{x}$ is the solution of (1.1) corresponding to the forcing functions $\hat{u}$ and $\hat{v} = 0$ and such that $[\![\hat{x}(0), \hat{x}(-1)]\!] = [\![x(t_0+1), x(t_0)]\!] \in \hat{\Gamma}$. Consequently, invoking (3.21),

$$y_x(t+t_0+\tau+1) = y_{\hat{x}}(t+\tau) \geq \delta \quad \forall\, t \in \mathbb{N}_0,$$

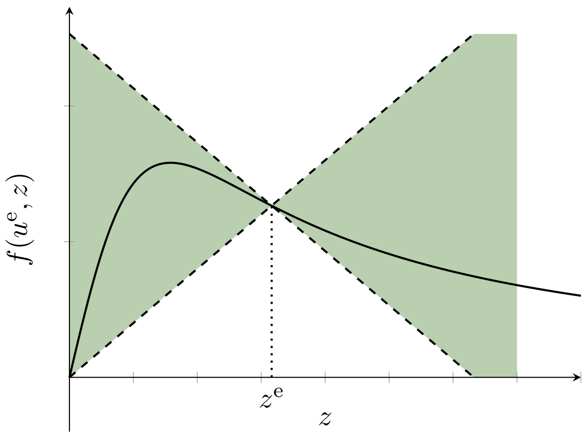

Fig. 4.1. Illustration of the sector condition (4.1). The dashed lines have slope $\pm p$.

showing that (1.1) is $\tilde{c}$-excitable by small inputs with respect to Γ and time t_0.

Finally, assume that hypothesis (O) holds. Changing the definition of $\hat{\Gamma}$ to

$$\hat{\Gamma} := \{ [\![\xi^0, \xi^{-1}]\!] : \xi^0, \xi^{-1} \in C \text{ and } \|\xi^0\| \geq \varepsilon/2 \} \subset \mathbb{R}^{2n}_+$$

we have that $\hat{\Gamma}$ is compact and $0 \notin P\hat{\Gamma}$. Corollary 3.13 guarantees that (1.1) is $\tilde{c}$-persistent with respect to $\hat{\Gamma}$. The above argument can now be used to establish that (1.1) is strictly $\tilde{c}$-excitable by small inputs with respect to Γ and time t_0. $\square$

4. Stability

In this section, we show that, under suitable assumptions on the nonlinearity f, the boundedness and persistence results of Section 3 ensure certain stability properties of (1.1). Recall that model (1.1) contains external forcing terms u and v and, consequently, all stability notions must incorporate these terms. As such, we appeal to the so-called Input-to-State Stability (ISS) paradigm of nonlinear control theory as a suitable analytical framework, see the survey articles [36, 37] and the recent monograph [38].

Given that we may express (1.1) as an augmented (and undelayed) Lur'e system (3.1), our approach to stability is to leverage ideas and methods from the semi-global stability theory developed in [14,15]. To this end, we introduce the following hypothesis (cf. [14, hypothesis (N3)]).

(N4) Hypothesis (N2) holds and there exists $u^e \in U$ such that

$$\left| f(u^e, z) - f(u^e, z^e) \right| < p \left| z - z^e \right| \quad \forall\, z > 0,\ z \neq z^e\,, \tag{4.1a}$$

where $z^e > 0$ is the unique positive solution of $f(u^e, z) = pz$, and

$$\limsup_{z \to z^e} \frac{\left| f(u^e, z) - f(u^e, z^e) \right|}{\left| z - z^e \right|} < p. \tag{4.1b}$$

The existence of positive z^e such that $f(u^e, z^e) = pz^e$ follows from the continuity of f and hypothesis (N2). The uniqueness of z^e is a consequence of the inequality (4.1a). The inequality (4.1a) is a so-called sector condition and means that the graph of $z \mapsto f(z, u^e) - f(z^e, u^e)$ is strictly "sandwiched" between the straight lines $z \mapsto \pm p(z - z^e)$. The condition (4.1b) means that $z \mapsto f(u^e, z)$ is non-tangential to these lines at $z = z^e$. For reasons which will be explained below, u^e is referred to as an equilibrium inducing vector. An illustration of condition (N4) is given in Fig. 4.1.

A number of sufficient conditions on the nonlinearity f for the sector condition (4.1a) in (N4) to hold can be found in [15, Lemmas 5.4, 6.3]. In the typical case that $z \mapsto f(w, z)$ is differentiable at $z = z^e$, verifying condition (4.1b) essentially involves studying the absolute value of the partial derivative $|\partial f / \partial z|$ at $z = z^e$, and may be investigated directly. Calculations of this type appear across the examples in [14,15,32], such as [14, Example 6.1] and [32, Example 5.5].

Under hypothesis (N4), it is routine to establish that

$$x^e := (I - A)^{-1} bpz^e > 0 \tag{4.2}$$

satisfies $(c^\top + d^\top) x^e = z^e$, and is an equilibrium of (1.1) with $u(t) \equiv u^e$ and $v(t) \equiv 0$. Equivalently,

$$\tilde{x}^e := [\![x^e, x^e]\!] = (I - \tilde{A})^{-1} \tilde{b} pz^e > 0$$

satisfies $\tilde{c}^\top \tilde{x}^e = z^e$, and is an equilibrium of the augmented Lur'e system (3.1) with $u(t) \equiv u^e$ and $\tilde{v}(t) \equiv 0$, cf. [14, Lemma 5.1] or [15, Lemmas 5.1-5.3].

The following theorem is the main stability result of this paper.

Theorem 4.1. *Let $\Gamma \subset \mathbb{R}^{2n}_+$ be non-empty and compact and assume that (N4) holds. If (1.1) is $\tilde{c}$-persistent with respect to Γ, then there exist constants $M \geq 1$, $N > 0$, $r > 0$ and $\kappa \in (0, 1)$ such that, for all $[\![x^0, x^{-1}]\!] \in \Gamma$, all $u : \mathbb{N}_0 \to U$ and all $v : \mathbb{N}_0 \to V$, the solution x of (1.1) satisfies*

$$\|x(t) - x^e\| \leq M\kappa^t \|\tilde{x}^0 - \tilde{x}^e\| + N \left(\|v\|_{\ell^\infty(0,t)} + \|\beta_r \circ u\|_{\ell^\infty(0,t)} \right) \quad \forall t \in \mathbb{N}_0, \tag{4.3}$$

where $\tilde{x}^0 := [\![x^0, x^{-1}]\!]$, $\|v\|_{\ell^\infty(0,t)} := \max\{\|v(s)\| : s = 0, 1, \ldots, t\}$ and

$$\beta_r(w) := \max_{0 \leq z \leq r} |f(u^e, z) - f(w, z)| \quad \forall\, w \in U.$$

Theorem 4.1 can be proved by arguments very similar to those used in [14, proof of statement (a) of Theorem 5.2], applied to the augmented system (3.1) with $r = 2\gamma\|\tilde{c}\|$, where γ is as in (3.4). The details are left to the interested reader. We remark that, for simplicity, we have appealed to the stability approach taken in [14] which yields the exponential input-to-state stability estimate (4.3), namely the exponential decay in the contribution of the initial state and the linearly bounded contribution of the forcing functions u and v. If, instead, the approach of [15] had been invoked, then condition (4.1b) in (N4) could have been omitted, at the expense of a slower, non-exponential, decay of the contribution of the initial state in the estimate of $\|x(t) - x^e\|$ (cf. [15, inequality (5.6)]).

If, in Theorem 4.1, the nonlinearity f is globally Lipschitz in its first variable, that is, there exists $\lambda > 0$ such that

$$|f(w_1, z) - f(w_2, z)| \leq \lambda \|w_1 - w_2\| \quad \forall\, z \in \mathbb{R}_+,\ \forall\, w_1, w_2 \in U,$$

then the constant r becomes redundant and (4.3) simplifies to

$$\|x(t) - x^e\| \leq M\kappa^t \|\tilde{x}^0 - \tilde{x}^e\| + N \left(\|v\|_{\ell^\infty(0,t)} + \lambda \|u^e - u\|_{\ell^\infty(0,t)} \right) \quad \forall\, t \in \mathbb{N}_0,$$

An application of Corollary 3.7, Theorem 4.1 and [14, Statement (b) of Theorem 5.2] leads to the following stability result.

Corollary 4.2. *Assume that (P1) and (N4) are satisfied. Then the following statements hold.*

1. *For every non-empty compact set $\Gamma \subset \mathbb{R}^{2n}_+$ such that $0 \notin \Gamma$, there exist $M \geq 1$, $N > 0$, $r > 0$ and $\kappa \in (0, 1)$ such that, for all $[\![x^0, x^{-1}]\!] \in \Gamma$, all $u : \mathbb{N}_0 \to U$ and all $v : \mathbb{N}_0 \to V$, the solution x of (1.1) satisfies (4.3).*
2. *For all $x^0, x^{-1} \in \mathbb{R}^n_+$, $[\![x^0, x^{-1}]\!] \neq 0$, all $u : \mathbb{N}_0 \to U$, and all $v : \mathbb{N}_0 \to V$, if $u(t) \to u^e$ and $v(t) \to 0$ as $t \to \infty$, then the solution x of (1.1) has the convergence property $x(t) \to x^e$ as $t \to \infty$.*

The next result is an immediate consequence of Theorems 3.12 and 4.1.

Corollary 4.3. *Let $\Gamma \subset \mathbb{R}_+^{2n}$ be non-empty and compact, let the subsets Γ' and Γ'' of $\mathbb{R}_+^{2n}$ be given by (3.17) and (3.18), respectively, and assume that (N4) is satisfied. If either*
(a) *(N3) holds and $\Gamma \subset \Gamma'$, or*
(b) *(P2) holds and $\Gamma \subset \Gamma''$,*

then there exist $M \geq 1$, $N > 0$, $r > 0$ and $\kappa \in (0,1)$ such that, for all $[\![x^0, x^{-1}]\!] \in \Gamma$, all $u : \mathbb{N}_0 \to U$ and all $v : \mathbb{N}_0 \to V$, the solution x of (1.1) satisfies (4.3).

Finally, under the assumption that (O), (P2) and (N4) hold, we derive stability and convergence properties which are similar (but not identical) to those in Corollary 4.2.

Corollary 4.4. *Assume that (O), (P2) and (N4) are satisfied. Then the following statements hold.*

(1) *For every non-empty compact set $\Gamma \subset \mathbb{R}_+^{2n}$ such that $0 \notin P\Gamma$, there exist $M \geq 1$, $N > 0$, $r > 0$ and $\kappa \in (0,1)$ such that, for all $[\![x^0, x^{-1}]\!] \in \Gamma$, all $u : \mathbb{N}_0 \to U$ and all $v : \mathbb{N}_0 \to V$, the solution x of (1.1) satisfies (4.3).*
(2) *For all $x^0, x^{-1} \in \mathbb{R}_+^n$, $x^0 \neq 0$, all $u : \mathbb{N}_0 \to U$, and all $v : \mathbb{N}_0 \to V$, if $u(t) \to u^{\mathrm{e}}$ and $v(t) \to 0$ as $t \to \infty$, then the solution x of (1.1) has the convergence property $x(t) \to x^{\mathrm{e}}$ as $t \to \infty$.*

Proof. (1) This statement follows from Corollary 3.13 and Theorem 4.1.
(2) Let $x^0, x^{-1} \in \mathbb{R}_+^n$, $x^0 \neq 0$, set $\tilde{x}^0 := [\![x^0, x^{-1}]\!]$, and let $u : \mathbb{N}_0 \to U$ and $v : \mathbb{N}_0 \to V$ be such that $u(t) \to u^{\mathrm{e}}$ and $v(t) \to 0$ as $t \to \infty$. By statement (1) of Proposition 3.1, there exists $\gamma > 0$ such that the solution x of (1.1) satisfies $\|\tilde{x}(t)\| \leq \gamma$ for all $t \in \mathbb{N}_0$. By Corollary 3.13, (1.1) is $\tilde{c}$-persistent with respect to the set $\{\tilde{x}^0\}$ and consequently, there exists $\delta \in (0, \gamma/2)$ and $\tau \in \mathbb{N}_0$ such that $\|\tilde{x}(t+\tau)\| \geq 2\delta$ for all $t \in \mathbb{N}_0$, where $\tilde{x}(t) = [\![x(t), x(t-1)]\!]$. Hence, for every $t \in \mathbb{N}_0$, we have that $\|x(t+\tau)\| \geq \delta$ or $\|x(t+1+\tau)\| \geq \delta$. Setting

$$\Gamma := \{[\![\xi^0, \xi^{-1}]\!] \in \mathbb{R}_+^{2n} : \|\xi^0\| \geq \delta \text{ and } \left\|[\![\xi^0, \xi^{-1}]\!]\right\| \leq \gamma\} \quad \text{and}$$
$$\Theta := \{t \in \mathbb{N}_0 : \tilde{x}(t) \in \Gamma\}, \tag{4.4}$$

it is clear that Γ is compact, $0 \notin P\Gamma$, and

$$\{t, t+1\} \cap \Theta \neq \emptyset \quad \text{for all } t \in \mathbb{N}_0 \text{ such that } t \geq \tau. \tag{4.5}$$

For $s \in \mathbb{N}_0$, let z_s denote the solution of the initial-value problem

$$z(t+1) = Az(t) + bf((T_s u)(t), c^\top z(t) + d^\top z(t-1)) + (T_s v)(t),$$
$$z(0) = x(s),\ z(-1) = x(s-1),\ t \in \mathbb{N}_0, \tag{4.6}$$

where T_s denotes the translation operator given by $(T_s u)(t) = u(t+s)$. We note that

$$x(t+s) = z_s(t) \quad \forall\, t, s \in \mathbb{N}_0.$$

An application of statement (1), in the context of system (4.6) and the set Γ given in (4.4), shows that there exist constants $M \geq 1$, $N > 0$, $r > 0$ and $\kappa \in (0,1)$ such that

$$\|x(t+s) - x^{\mathrm{e}}\| = \|z_s(t) - x^{\mathrm{e}}\| \leq M\kappa^t \|\tilde{x}(s) - \tilde{x}^{\mathrm{e}}\| + N\|\beta_r \circ T_s u\|_{\ell^\infty}$$
$$+ N\|T_s v\|_{\ell^\infty} \quad \forall\, t \in \mathbb{N}_0,\ \forall\, s \in \Theta, \tag{4.7}$$

where $\|\cdot\|_{\ell^\infty}$ denotes the supremum norm on the space of bounded functions defined on $\mathbb{N}_0$. Given $\varepsilon > 0$, there exists $\theta \in \Theta$ such that $N\left(\|T_\theta v\|_{\ell^\infty} + \|\beta_r \circ T_\theta u\|_{\ell^\infty}\right) \leq \varepsilon/2$ for all $t \in \mathbb{N}_0$ as follows from (4.5) and the hypothesis that $u(t) \to u^{\mathrm{e}}$ and $v(t) \to 0$ as $t \to \infty$. Finally, choosing $\sigma \in \mathbb{N}_0$ such that $M\kappa^\sigma \|\tilde{x}(\theta) - \tilde{x}^{\mathrm{e}}\| \leq \varepsilon/2$, it follows from (4.7) with $s = \theta$ that $\|x(t) - x^{\mathrm{e}}\| \leq \varepsilon$ for all $t \in \mathbb{N}_0$ such that $t \geq \sigma + \theta$, completing the proof. □

5. Examples

We illustrate our results through three examples: a stage-structured population model with delay, and two models relating to spatially structured population models.

Example 5.1 (*A Stage-structured Population Model with Delay*). Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^\top$ denote the population at time-step $t \in \mathbb{N}_0$ of a single, local, stage-structured population with n stages, where $n \geq 2$. Inspired by the model appearing in [15, Example 6.1], assume that x is governed by: for $t \in \mathbb{N}_0$

$$\left.\begin{aligned} x_1(t+1) &= s_1 x_1(t) + g\big(u(t), d^\top x(t-1)\big) d^\top x(t-1) + v_1(t) \\ x_k(t+1) &= s_k x_k(t) + h_{k-1} x_{k-1}(t) + v_k(t) \quad k \in \{2, 3, \ldots, n\}\,. \end{aligned}\right\} \tag{5.1}$$

Here s_k and h_k are probabilities (or proportions) denoting survival (or stasis) within stage-classes and movement into subsequent stage-classes, respectively, which we, therefore, assume satisfy $h_i \in (0,1)$ for $i \in \{1, 2, \ldots, n-1\}$, $s_j \in [0,1)$ for $j \in \{1, 2, \ldots, n\}$ and $s_i + h_i \leq 1$ for all $i \in \{1, 2, \ldots, n-1\}$. We further assume that $s_n > 0$ and $g : \mathbb{R}_+^2 \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. Writing

$$d = \big(d_1, \ldots, d_n\big)^\top \in \mathbb{R}_+^n, \tag{5.2}$$

the nonnegative constants d_k model the fecundity of the kth stage-class. For fixed $w \in \mathbb{R}_+^2$, the function $g(w, \cdot)$ represents the density-dependent per-capita survival probability of new individuals. Consequently, the product term

$$g\big(u(t), d^\top x(t-1)\big) d^\top x(t-1) = g\Big(u(t), \sum_{k=1}^n d_k x_k(t-1)\Big) \sum_{k=1}^n d_k x_k(t-1)$$

in (5.1) captures the recruitment into the population at time-step $t+1$. Here, the forcing function u represents the effects of temporal environmental or demographic fluctuations which are assumed to affect recruitment only (in this example it takes values in $\mathbb{R}_+^2$). The exogenous additive forcing variable v represents structured migration into the population. We assume, as is typical for such structured models, that reproduction adds individuals into the first stage-class, perhaps representing the number of eggs, juveniles or seeds, in an insect, animal or plant model, respectively.

The model (5.1) differs from that in [15, Example 6.1] via the inclusion of the delayed recruitment, which is biologically plausible when gestation/reproduction is longer than a single time-step. System (5.1) may be written in the form (1.1) with

$$A := \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ h_1 & s_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & h_{n-1} & s_n \end{pmatrix}, \quad b := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad c := 0, \quad f(w, z) := g(w, z) z,$$

d given by (5.2) and $v(t) := \big(v_1(t), v_2(t), \ldots, v_n(t)\big)^\top$.

As $s_k < 1$ for all $k = 1, \ldots, n$, the matrix A is asymptotically stable. To simplify the presentation, we assume that $d_1 = \cdots = d_{n-1} = 0$ and $d_n > 0$, so that only the final stage class is reproductively active. Elementary calculations give that

$$\mathbf{G}(1) = \frac{d_n \prod_{k=1}^{n-1} h_k}{\prod_{k=1}^{n} (1 - s_k)} > 0,$$

so that $p = 1/\mathbf{G}(1)$ is finite.

We examine the extent to which the various hypotheses hold. A calculation shows that the $n \times n$-matrix $\mathcal{O}(d^\top, A)$ is lower triangular with the kth anti-diagonal entry given by

$$\zeta_k = d_n \prod_{j=0}^{n-k} h_j > 0, \quad \text{where } h_0 := 1,$$

and the anti-diagonal is indexed from bottom left to top right. Consequently, $\mathcal{O}(d^\top, A)$ has full rank and, therefore, property (O) holds. Since $c = 0$ and d is not strictly positive, it follows from Lemma 3.6 that property (P1) does not hold. We claim that property (P2) is satisfied. Whilst this may be verified by direct calculations, we provide an alternative argument based on Lemma 3.9. For which purpose, observe first that $d^\top A^k b = d_n[A^k]_{(n,1)} > 0$ if $k \geq n-1$, where $[A^k]_{(i,j)}$ denotes the entry of A^k in position (i,j). In particular, $\mu := \min(d^\top A^{n-1}b, d^\top A^n b) > 0$ (see (3.14)). Moreover, let Δ_{ij} denote the $n\times n$-matrix with single non-zero element equal to 1 in position (i,j). A routine induction argument shows that

$$(\tilde{A} + \tilde{b}\tilde{c}^\top)^q \geq \begin{pmatrix} A^q & A^{q-1}d_n\Delta_{1n} \\ * & * \end{pmatrix} \quad \forall\, q \in \mathbb{N},$$

so that

$$\big(d^\top, d^\top\big)(\tilde{A} + \tilde{b}\tilde{c}^\top)^q \geq \big(d^\top A^q, d^\top A^{q-1}d_n\Delta_{1n}\big) \quad \forall\, q \in \mathbb{N}.$$

The structure of A and d gives $d^\top A^{n-1} \gg 0$ and $d^\top A^{n-1}d_n\Delta_{1n} \geq \delta d^\top$ for a suitable constant $\delta > 0$. As ϕ is of the form $\phi = \big(*, d^\top\big)^\top$, it follows that there exists a constant $\varepsilon \in (0,\delta]$ such that $\big(d^\top, d^\top\big)(\tilde{A} + \tilde{b}\tilde{d}^\top)^{n-1} \geq \varepsilon\phi^\top$. This, together with the positivity of μ, and subsequent application of Lemma 3.9 shows that (P2) holds.

As usual, whether or not properties (N1), (N2) and (N4) are satisfied depends on the interplay between the nonlinearity f and the constant p. Note that, since $d^\top b = 0$, property (N3) does not hold.

As a numerical example, we consider a population stratified into $n = 4$ stages, with model data

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0.62 & 0.16 & 0 & 0 \\ 0 & 0.36 & 0.23 & 0 \\ 0 & 0 & 0.53 & 0.35 \end{pmatrix}, \quad d_4 = 30;$$

which gives rise to

$$\mathbf{G}(1) = d^\top(I-A)^{-1}b = 8.4413 = 1/p, \quad \text{equivalently}, \quad p = 0.1185.$$

We consider the Beverton–Holt type nonlinearity

$$f(w,z) := g(w,z)z, \quad g(w,z) := \frac{\lambda w_1}{1+\kappa w_2 z} \quad \forall\, z \geq 0. \tag{5.3}$$

Here $\lambda \in (0,1)$ denotes the per capita survival probability at low population abundance when $w_1 = 1$; the constant $\kappa > 0$ plays the role of moderating survival probabilities at higher population abundance. For fixed non-empty, compact $U \subseteq (0,1/\lambda)\times(0,\infty)$, arguments as in the proofs of [14, Proposition 4.3] and [15, Lemma 5.4] show that f satisfies (N1), (N2) and (N4) if

$$\min_{w\in U} g(w,0) = \lambda \min_{(w_1,w_2)\in U} w_1 > p\,. \tag{5.4}$$

In this case, routine calculations show that, for fixed equilibrium inducing vector $u^e = (u_1^e, u_2^e)^\top \in U$,

$$z^e := \frac{\lambda u_1^e - p}{p\kappa u_2^e},$$

is the unique positive solution of $pz = f(u^e, z)$ and, consequently,

$$x^e := (I-A)^{-1}bpz^e = \begin{pmatrix} 1 \\ 0.7381 \\ 0.3451 \\ 0.2814 \end{pmatrix} \frac{\lambda u_1^e - p}{\kappa u_2^e},$$

is the corresponding unique non-zero equilibrium population of (5.1) with $u(t) \equiv u^e$ and $v(t) \equiv 0$. We see that variations of u_1^e and u_2^e from 1 have the effect of replacing λ and κ by λu_1^e and κu_2^e, respectively. In particular, z^e, and hence also x^e, is a decreasing function of the parameter κ and of u_2^e.

We take

$$u^e := (1,1)^\top, \quad \lambda = 0.95 \quad \text{and} \quad \kappa = 0.5,$$

with the graph of f plotted in Fig. 5.1(a), showing that the sector condition (N4) holds. To illustrate the persistence result Corollary 3.13, Fig. 5.1(b) plots in grayscale the x_4-component of 30 solutions of (5.1) with $u := u^e$, $v = 0$ and (pseudo)random initial conditions such that

$$[\![x(0), x(-1)]\!] \in \Gamma := [0.1,3]^4 \times [0,2]^4.$$

Evidently, Γ is a non-empty, compact subset of $\mathbb{R}^8_+$ with $0 \notin P\Gamma$. For clarity, the vertical axis in Fig. 5.1(b) has a logarithmic scale, and we comment that the purpose is to visualise a system-level property. Indeed, $\tilde{c}$-persistency is observed. Moreover, the hypotheses of Corollary 4.4 are also satisfied, and convergence over time of $x_4(t)$ to x_4^e is observed.

Next, we take $U := [0.7, 1.3]^2$, and note that (5.4) is satisfied for the given numerical scenario ($\lambda = 0.95$ and $p = 0.1185$). We consider convergent u of the form

$$u(t) = \begin{cases} u^e & 0 \leq t \leq 10 \\ \left(I + \theta(0.95)^{t-10}\begin{pmatrix} \cos(0.3t) & 0 \\ 0 & \sin(0.3t)\end{pmatrix}\right)u^e & t > 10, \end{cases}$$

where $\theta \in [0, 0.3]$ plays the role of an amplitude parameter. We assume that $v = 0$ and that $x(-1) = x(0) = x^e$. The interpretation is that a disturbance from $(u,v,x) = (u^e, 0, x^e)$ occurs after $t = 10$. The first component $x_1(t)$ of $x(t)$ is plotted against t in Fig. 5.1(c) for varying θ. Two phenomena are observed. First, the transient deviation $|x_1(t) - x_1^e|$ increases as $\|u(t) - u^e\|$ increases, facilitated by increasing θ, as expected from the inequality (4.3). Furthermore, for each value of $\theta > 0$ considered, convergence $|x_1(t) - x_1^e| \to 0$ as $t \to \infty$ is observed, in accordance with statement (2) of Corollary 4.4.

Finally, we perform numerical simulations with $x(-1) = x(0) = x^e$, $u(t) \equiv u^e = (1,1)^\top$ and v given by

$$v(t) = \theta(0,1,0,0)^\top(1+\sin(0.4t)),$$

where θ again plays the role of an amplitude parameter. Graphs of $\|x(t) - x^e\|_1$ against t for increasing θ are plotted in Fig. 5.1(d). We see that $\|x(t) - x^e\|_1$ increases as θ (and hence $v(t)$) increases, again in accordance with the estimate (4.3). Note that the function $z \mapsto f(u^e, z)$ given by (5.3) is non-decreasing, and thus, if $u(t) \equiv u^e$, the difference equation (5.1) is a monotone control system in the sense of [11] (with respect to the nonnegative orthant $\mathbb{R}^4_+$ and input variable v). The monotonicity properties observed in Fig. 5.1(d) is in accordance with the predictions provided by the theory monotone control systems [11]. However, we emphasise that the theory developed in Sections 3 and 4 is not contingent on a monotone control systems structure. ♢

Example 5.2 (*A Spatially and Stage-structured Population Model*)**.** Some species have breeding areas where individuals move for reproduction [39]. The current example considers a population with two age classes (juveniles and adults) spatially structured across three patches: one is the breeding site and the other two (labelled 1 and 2) are feeding sites. We model the following basic processes: adults produce juveniles, denoted J, in the breeding site; juveniles become adults and move to site 1 or 2, denoted A_1 and A_2, respectively; moreover, adults in patch 1 or 2 remain in that site or move to the other. Formalising the verbal model description in [39], we arrive at the following system of nonlinear difference equations:

$$\left.\begin{aligned} J(t+1) &= f\big(A_1(t) + A_2(t)\big), \\ A_1(t+1) &= (1-s)\eta J(t) + (1-r_1)\lambda A_1(t) + r_2\mu A_2(t), \\ A_2(t+1) &= s\eta J(t) + r_1\lambda A_1(t) + (1-r_2)\mu A_2(t), \end{aligned}\right\} \tag{5.5}$$

where $r_1, r_2, s, \eta, \lambda, \mu \in [0,1]$ are constants and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonlinearity, the biological interpretation of which is given in Table 5.1.

System (5.5) extends the Allen–Clark model, a scalar difference equation of the form

$$z(t+1) = \alpha z(t) + \beta f(z(t-1)),$$

(named after Allen [40] and Clark [41], see also the bibliographical notes in [42] for other early contributors). Indeed, after elimination

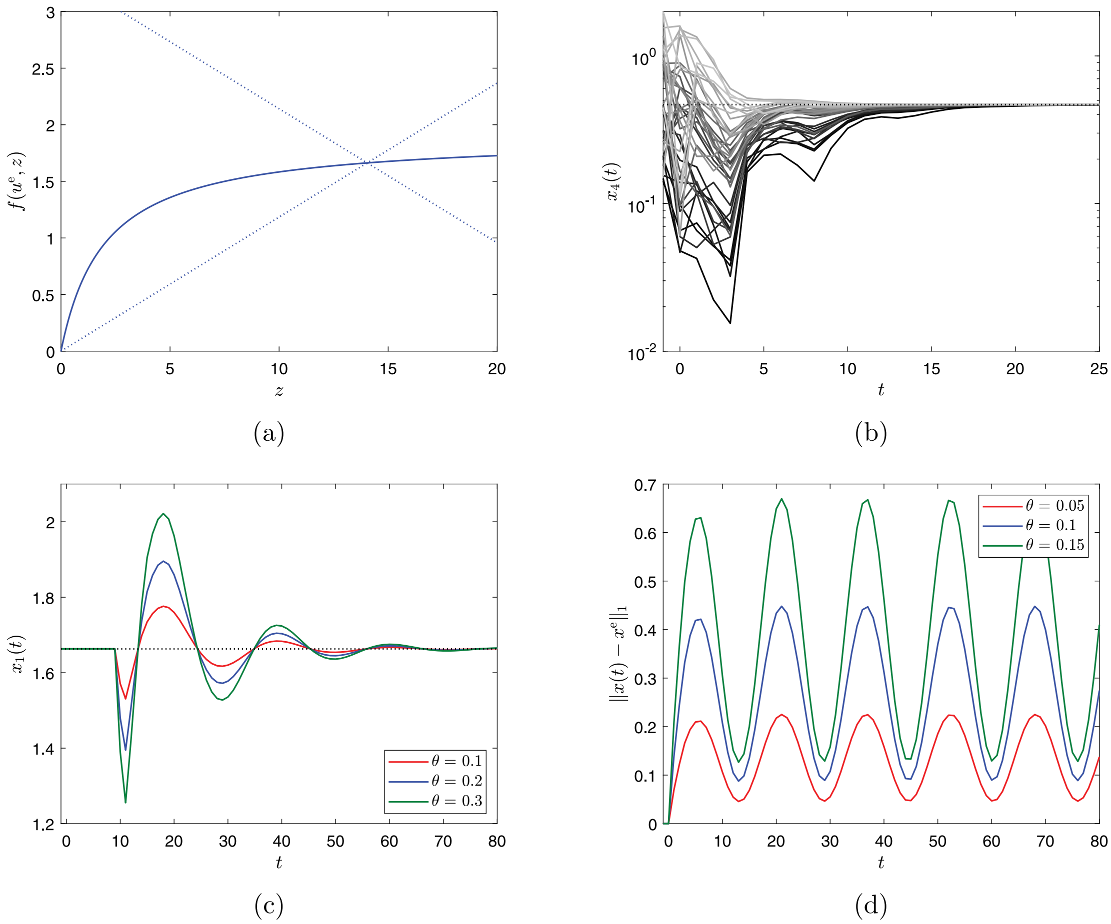

Fig. 5.1. Numerical simulations illustrating Example 5.1. (a) Graph of the nonlinearity $z \mapsto f(u^e, z)$ in (5.3). The straight lines have slope $\pm p$ and intersect the graph of f at $z = z^e$. (b) Plots of $x_4(t)$ against t for varying $[\![x(0), x(-1)]\!] \in \Gamma$, with logarithmic scale on the vertical axis. The dotted line is x_4^e. (c) Plots of $x_1(t)$ against t for varying θ. (d) Plots of $\|x(t) - x^e\|_1$ against t for varying θ.

Table 5.1
Interpretation of symbols in (5.5).

Symbol	Interpretation (each per time-step)
f	Recruitment function of new juveniles
r_i	Proportion of adults who move from site i to $3-i$
s	Proportion of juveniles who move to patch 2
η	Survival probability of juvenile transition to feeding sites
λ	Survival probability of adults in site 1
μ	Survival probability of adults in site 2

of $J(t)$ from (5.5), we recover the above equation with $z(t) = A_1(t)$ or $z(t) = A_1(t) + A_2(t)$ by taking $s = \mu = r_1 = r_2 = 0$ or $\mu = \lambda$, respectively.

The dispersal is symmetric between the feeding sites if $r_1 = r_2 =: r$, which we shall assume. To further simplify the presentation, we assume that $\eta = 1$. Eliminating $J(t)$ from (5.5) yields a delayed Lur'e system of the form (1.1) with $n = 2$, and

$$x(t) := \begin{pmatrix} A_1(t) \\ A_2(t) \end{pmatrix}, \quad A := \begin{pmatrix} (1-r)\lambda & r\mu \\ r\lambda & (1-r)\mu \end{pmatrix}, \quad b := \begin{pmatrix} 1-s \\ s \end{pmatrix},$$
$$c := 0, \quad d := \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We assume that

$$0 < \lambda < 1 \quad \text{and} \quad 0 < \mu < 1,$$

so that the column sums of the matrix A are smaller than 1 and, therefore, the matrix A is asymptotically stable, see [43, Theorem 8.1.22]. An application of statement (3) of Lemma 3.6 yields that property (P1) holds for all $r \in [0,1]$ and all $0 < \lambda, \mu < 1$. Consequently, the conclusions of Corollaries 3.7 and 4.2 hold, provided that the nonlinearity f satisfies (N2) and (N4), respectively.

We note the case $r = 0$ corresponds to the absence of redistribution of adults between the feeding patches. Ecological corridors or stepping stones are used in ecosystems management to promote connectivity between patches [44,45], that is, increasing the value of r. In the following, we use the results of Sections 3 and 4 to investigate how persistency of the total adult population and the asymptotic total adult population are affected by changes in the parameter r.

We start with persistency. Routine calculations yield

$$\mathbf{G}(1) = \frac{C_1 r + C_2}{C_3 r + C_4} =: F(r),$$

with

$$C_1 := \lambda + \mu, \quad C_2 := s\mu - \mu + 1 - \lambda s, \quad C_3 := \lambda + \mu - 2\lambda\mu \quad \text{and}$$
$$C_4 := \lambda\mu - \mu - \lambda + 1.$$

We compute that

$$F'(r) = \frac{C_1 C_4 - C_2 C_3}{(C_3 r + C_4)^2},$$

and

$$C_1 C_4 - C_2 C_3 = (\lambda - \mu)(\lambda\mu - \lambda + \lambda s + \mu s - 2\lambda\mu s). \tag{5.6}$$

The sign of $C_1 C_4 - C_2 C_3$ determines how F changes with increasing r, and can take both signs depending on the interplay between λ, μ and s. We note that $C_1 C_4 - C_2 C_3$ is symmetric in λ and μ under the transformation $s \mapsto 1 - s$. This is intuitively obvious as swapping λ and

μ, and replacing s by $1-s$ amounts to swapping the labels of feeding sites 1 and 2. Furthermore, as $p = p(r) = 1/\mathbf{G}(1) = 1/F(r)$, an increase (decrease) in F leads to a decrease (increase) in p. Assuming that

$$\limsup_{z\to\infty} \frac{f(z)}{z} = 0,$$

the hypothesis (N2) becomes less (more) restrictive if $F(r)$ is increasing (decreasing).

An inspection of (5.6) reveals several special cases, which we proceed to highlight. First, if $\lambda = \mu$, then $C_1C_4 - C_2C_3 = 0$ and hence, F is independent of r. In this case, the dynamics for $A_1 + A_2$ are independent of the feeding site structure (essentially because both sites are the same). Second, if $s \approx 0$, meaning juveniles strongly prefer site 1, then

$$C_1C_4 - C_2C_3 \approx \lambda(\lambda-\mu)(\mu-1)$$

which is positive when $\mu > \lambda$ and negative when $\mu < \lambda$. Third, and conversely to the previous case, if $s \approx 1$, meaning juveniles strongly prefer site 2, then

$$C_1C_4 - C_2C_3 \approx \mu(\lambda-\mu)(1-\lambda)$$

which is positive when $\lambda > \mu$ and negative when $\lambda < \mu$. These last two observations indicate that, in case juveniles prefer feeding site i, where $i = 1, 2$, it is the survivability of adults in site $3-i$ in relation to that in site i which determines the effect of an increase of dispersal on the persistence of the adult population $A_1 + A_2$: if juveniles strongly prefer site i and the survival probability of adults in site $3-i$ is greater than in site i, then promoting dispersal between the feeding sites facilitates the persistence of the adult population. This could be considered an indirect rescue effect: emigrants from surrounding population sites may reduce the probability of local extinction [46]; here, the rescue effect is indirect in the sense that it is the greater survival probability of adults in site $3-i$ which keeps the number of juveniles in the breeding site and, as a consequence, the number of adults arriving at site i above certain levels.

Finally, if $s = 1/2$, meaning that juveniles do not have preference for a feeding area, then

$$C_1C_4 - C_2C_3 = \frac{-(\lambda-\mu)^2}{2} \le 0.$$

Hence, in this case, increasing the connectivity between the feeding sites has a potentially negative effect on the persistence of the adult population.

Figs. 5.2(a) and 5.2(b) contain surface plots of $C_1C_4 - C_2C_3$, viewed as a function of λ and μ, for $s = 0.9$ and $s = 0.55$, respectively. The plots were routinely computed using the `surf` command in Matlab. The contour $C_1C_4 - C_2C_3 = 0$ is shown in black. Interestingly, whilst there are regions where $C_1C_4 - C_2C_3$ is positive, the value of $C_1C_4 - C_2C_3$ is "small" here, and these regions are much smaller comparatively to where $C_1C_4 - C_2C_3$ is negative, particularly when $s = 0.55$, which is close to $1/2$, where $F'(r)$ is always non-positive. Roughly speaking, $C_1C_4 - C_2C_3 = 0$ is "more likely" to be negative it appears. Fig. 5.2(c) plots graphs of $F(r)$ against r for two (λ, μ) pairs, where $F'(r)$ takes different signs, both for $s = 0.9$.

Next, we study the response of the (nonzero) asymptotic total population of adults to an increase of the dispersal rate r. There has been a growing interest on understanding the effect of dispersal on the asymptotic population size because of its practical implications; see, for example [33,34,47–51]. In particular, in [34] it is shown that for a two-patch population model with local Beverton–Holt dynamics and symmetric dispersal there exist four possible response scenarios of the total population size to an increases of the dispersal rate: monotonically beneficial, monotonically detrimental, unimodally beneficial, and beneficial turning detrimental. In the first two, the response of the total population is monotonic, whereas in the last two, the response is unimodal, that is, total population increases until it reaches a global maximum and then it decreases. Moreover, exactly the same four response scenarios are possible for a continuous-time model with logistic growth [33,34] and when dispersal is asymmetric [35].

Assume that f satisfies hypothesis (N4) for all r in some non-empty interval $J \subseteq [0,1]$. In particular, for $r \in J$, it follows that $x^{\mathrm{e}} = x^{\mathrm{e}}(r)$ given by (4.2) is the unique nonzero equilibrium of (1.1), that is, the nonzero asymptotic adult population. Recall that in this example the forcing function u is not present (as $f(w,z) = f(z)$) and $v(t) \equiv 0$. The asymptotic total population of adults is given by

$$x^{\mathrm{e}}_1 + x^{\mathrm{e}}_2 = (1,1)x^{\mathrm{e}} = d^{\top}(I-A)^{-1}bpz^{\mathrm{e}} = \mathbf{G}(1)pz^{\mathrm{e}} = z^{\mathrm{e}}, \tag{5.7}$$

where $z^{\mathrm{e}} = z^{\mathrm{e}}(r)$ is the unique positive solution of

$$F(r)f(z) = z, \quad \text{equivalently,} \quad F(r)\frac{f(z)}{z} = 1 \quad z > 0. \tag{5.8}$$

Assume that

$$z \mapsto f(z)/z \quad \text{is non-increasing}. \tag{5.9}$$

The above condition includes unimodal and non-decreasing functions, such as Ricker and Beverton–Holt nonlinearities, respectively. Thus, we conclude from (5.8) that if $r \mapsto F(r)$ is increasing (decreasing), then $z^{\mathrm{e}}(r)$ is increasing (decreasing) and, hence, so is the asymptotic population size by (5.7). Since the sign of $F'(r)$ is independent of r, we obtain that the only possible response scenarios for model (5.5) are monotonically beneficial, in which increasing dispersal monotonically increases the asymptotic adult population size, and monotonically detrimental, in which increasing dispersal monotonically decreases the asymptotic adult population size. Interestingly, to summarise, there are only two response scenarios for model (5.5) with symmetric dispersal and nonlinear term satisfying (5.9), rather than the four scenarios for the model considered in [34].

The Beverton–Holt nonlinearity (cf. (5.3))

$$f(z) = \frac{az}{1+bz} \quad z \ge 0,$$

for positive constants $a, b > 0$, satisfies condition (5.9). Moreover, property (N4) holds if, and only if, $F(r)a > 1$, in which case

$$z^{\mathrm{e}} = \frac{F(r)a - 1}{b}.$$

A surface plot of $(F(r)a-1)/b$ against varying r and s is shown in Fig. 5.2(d) for

$$\mu = 0.7, \quad \lambda = 0.9, \quad a = 1.1, \quad b = 0.01.$$

Observe how dispersal can be monotonically detrimental for s close to 1, and monotonically beneficial for s close to 0; but there are no unimodal responses of the total adult population to an increase of dispersal as in [34]. $\diamondsuit$

Our third and final example is closely related to Example 5.2, but here assumption (P1) is not satisfied.

Example 5.3 (*A Spatially Structured Population Model with Dormants*). Consider a population structured in three groups or classes: juveniles, adults and dormants, denoted J, A, and D, respectively. We consider the following basic processes: adults produce juveniles; juveniles become adults or dormants; adults remain as adults or become dormants; and dormants remain as dormants or become adults. This model is mathematically described by

$$\left.\begin{aligned} &J(t+1) = f\big(A(t)\big), \\ &A(t+1) = (1-s)\eta J(t) + (1-r_1)\lambda A(t) + r_2\mu D(t), \\ &D(t+1) = s\eta J(t) + r_1\lambda A(t) + (1-r_2)\mu D(t), \end{aligned}\right\} \tag{5.10}$$

where $r_1, r_2, s, \eta, \lambda, \mu \in [0,1]$ measure mortality and dispersal/redistribution. When $r_1 = r_2$, the redistribution between the adult and dormant groups is symmetric.

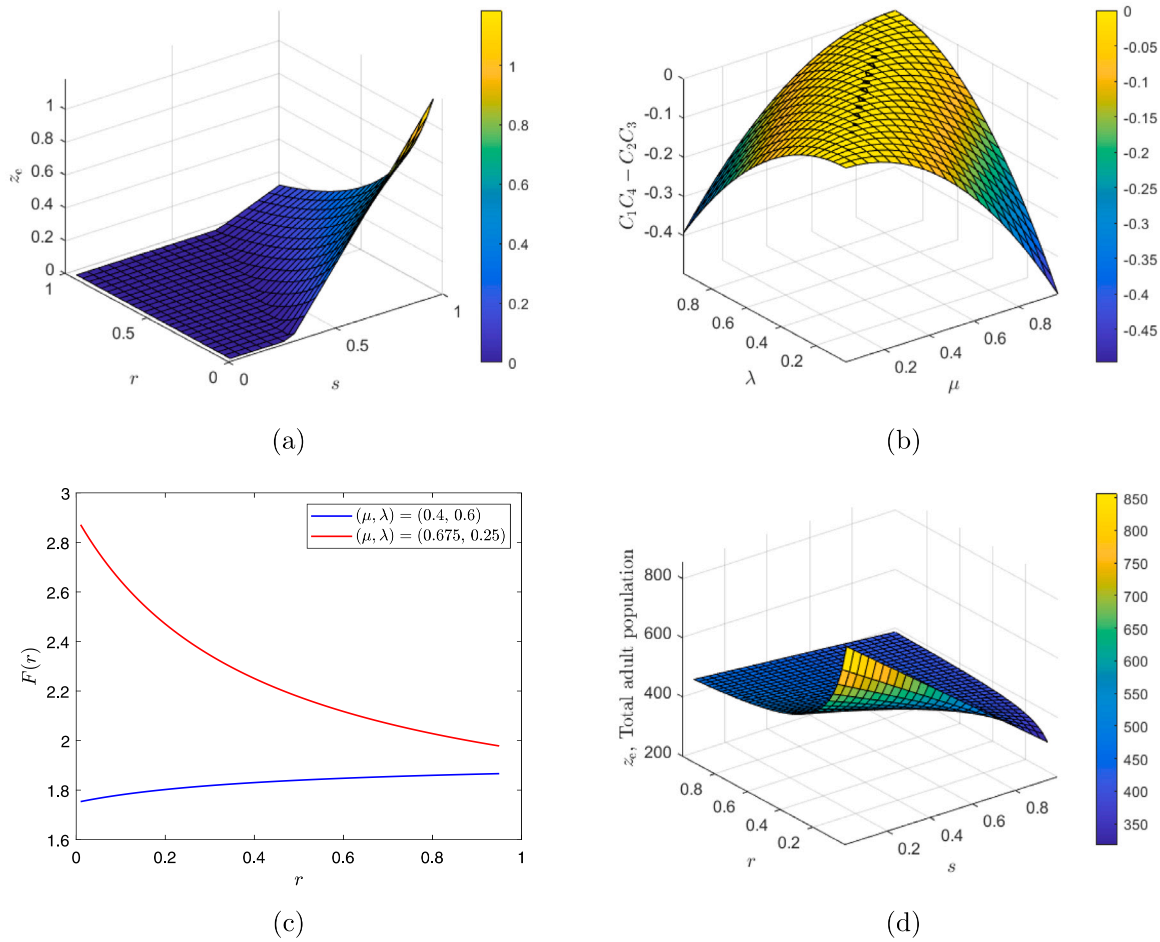

Fig. 5.2. Numerical simulations illustrating Example 5.2. (a) and (b) Surface plots of $C_1C_4 - C_2C_3$ in (5.6) against varying λ and μ for $s = 0.9$ and $s = 0.55$. (c) Graphs of $F(r)$ against r for different (μ, λ) pairs, with fixed $s = 0.9$. (d) Surface plot of asymptotic adult population size z^e against varying r and s, here $\lambda = 0.9$, $\mu = 0.7$, and $f(z) = \frac{1.1z}{1+0.01z}$.

Eliminating $J(t)$ from (5.10) yields a delayed Lur'e system of the form (1.1) with $n = 2$, and

$$x(t) := \begin{pmatrix} A(t) \\ D(t) \end{pmatrix}, \quad A := \begin{pmatrix} (1-r_1)\lambda & r_2\mu \\ r_1\lambda & (1-r_2)\mu \end{pmatrix}, \quad b := \begin{pmatrix} (1-s)\eta \\ s\eta \end{pmatrix},$$

$$c = 0, \quad d := \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The models (5.5) and (5.10) are structurally very similar and differ only by the form of the nonlinear recruitment term. However, by Lemma 3.6, hypothesis (P1) does not hold here because d is not strictly positive. We assume that

$$r_1, r_2, \lambda, \mu \in (0,1), \quad s \in [0,1) \quad \text{and} \quad \eta \in (0,1].$$

As in Example 5.2, it follows that A is asymptotically stable. A straightforward calculation gives that

$$\phi^\top = \left((c+d)^\top (I-A)^{-1}, d^\top\right) = (*, *, 1, 0),$$

where $*$ denotes a nonnegative entry. Thus, there exists $\varepsilon > 0$ such that

$$\tilde{c}^\top (\tilde{A} + \tilde{b}\tilde{c}^\top)^2 = \left(\lambda(1-r_1), \mu r_2, \eta(1-s), 0\right) \geq \varepsilon \phi^\top,$$

showing that hypothesis (P2) is satisfied. Furthermore, it is straightforward to check that condition (O) also holds. Consequently, the conclusions of Corollaries 3.13 and 4.4 hold, provided that the nonlinearity f satisfies (N2) and (N4), respectively. Finally, if we take $r := r_1 = r_2$ and $\eta = 1$ to address the effect of an increase of dispersal on the persistency and the asymptotic size of the adult population as in the previous example, we obtain similar results to the ones presented there. Indeed, it is not hard to check that the response of $\mathbf{G}(1)$ to an increase of r is monotone (decreasing or increasing depending on the values of λ, μ and s). Consequently, if (5.9) holds, the response of the asymptotic adult population size is monotone (decreasing or increasing depending on the values of λ, μ and s), but never unimodal in contrast with the behaviour of the model studied in the paper [34]. ◊

6. Summary

Boundedness, persistence, excitability, and stability and convergence properties of a class of forced, delayed, positive discrete-time Lur'e systems have been considered. The inclusion of a delay distinguishes the present work from earlier papers [14,15,22] by the authors. Similarly, the excitability property, presented in Section 3.2 and which was not considered in [14,15,22], is a novelty of the present work. We have demonstrated that whilst there is some overlap between boundedness and stability properties, there are significant differences in terms of persistency properties, where we have consequently focussed our attention. To summarise, delays can be detrimental to ensuring $\tilde{c}$-persistency for the models under consideration. Whilst condition (P1) is known from [14] to be sufficient for $\tilde{c}^\top$-persistence of the first-order version of (1.1), namely,

$$\tilde{x}(t+1) = \tilde{A}x(t) + \tilde{b}f(u(t), \tilde{c}^\top \tilde{x}(t)) + \tilde{v}(t),$$

with the augmented quantities $\tilde{A}$, $\tilde{b}$, $\tilde{c}$, $\tilde{x}$ and $\tilde{v}$ given by (3.2), we have seen that (P1) is quite restrictive due to the special structure of $\tilde{A}$, $\tilde{b}$ and $\tilde{c}$. Therefore, new approaches to persistency have been developed, namely in terms of the hypotheses (P2) or (N3). We have presented a number of necessary and sufficient conditions for (P2), and our main result on persistence is Theorem 3.12. As hypothesis (P2) is strictly weaker than (P1), Theorem 3.12 is applicable when the results of [14] are not — this is a strength of the current work. Further strengths of the work are the inclusion of control/disturbance terms in (1.1), distinguishing our work from much of the current literature and facilitating the modelling of (possibly anthropogenic) interventions, and our treatment of excitability.

We summarise some of the merits and limitations of work. In terms of merits, a detailed study of the forced, nonlinear difference equation (1.1) has been conducted, with readily checkable sufficient conditions for a range of relevant and significant qualitative and quantitative properties provided. In terms of limitations, here we have focussed on a unit delay in (1.1). A natural question is the effect of potentially larger delays, that is, the qualitative properties of

$$x(t+1) = Ax(t) + bf\Big(u(t), \sum_{j=0}^{h} c_j^\top x(t-j)\Big) + v(t) \quad t \in \mathbb{N}_0\,, \tag{6.1}$$

where $h \in \mathbb{N}$ is arbitrary. Given the differences in dynamical behaviour between the undelayed case (see [14,15,22]) and unit delay considered in this paper, a thorough study of (6.1) requires more research and is, therefore, beyond the scope of the present work.

In closing we comment that our examples have not focussed in depth on the Allen–Clark model (although it is related to the models appearing in Examples 5.2 and 5.3, as commented there), which is a common and important example of delayed nonlinear difference equations arising frequently in mathematical biology or ecology. Whilst the results of the present work are applicable to the Allen–Clark model, the special structure of its first-order formulation is best exploited by a bespoke persistency and stability theory (different to that pursued in this paper). This theory will appear in the forthcoming paper [52] which considers a large class of scalar higher-order system containing the Allen–Clark model and the scalar version of (6.1) as special cases.

CRediT authorship contribution statement

Daniel Franco: Writing – review & editing, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Chris Guiver:** Writing – review & editing, Writing – original draft, Software, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Hartmut Logemann:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Juan Perán:** Writing – review & editing, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The authors have no relevant financial or non-financial interests to disclose.

Data availability

No data was used for the research described in the article.

Declaration of Generative AI and AI-assisted technologies in the writing process

No generative AI was used in the production of this work.

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Appendix. Proof of Theorem 3.12

Proof of Theorem 3.12. Let u and v be arbitrary functions from $\mathbb{N}_0$ to U and V, respectively. Every solution of (1.1) satisfies

$$y_x(t+2) \geq d^\top x(t+1) \geq d^\top bf(u(t), y_x(t)), \quad \forall\, t \in \mathbb{N}_0. \tag{A.1}$$

(1) Let $x^0, x^{-1} \in \mathbb{R}^n_+$ be such that $\min\{c^\top x^0 + d^\top x^{-1}, d^\top x^0\} > 0$, and let x be the solution of (1.1) satisfying $x(0) = x^0$ and $x(-1) = x^{-1}$. The claim follows from an induction argument applied to inequality (A.1).
(2) Let $x^0, x^{-1} \in \mathbb{R}^n_+$ be such that $c^\top x^0 + d^\top x^{-1} > 0$, and let x be the solution of (1.1) satisfying $x(0) = x^0$ and $x(-1) = x^{-1}$. As $x(1) \geq bf(u(0), c^\top x^0 + d^\top x^{-1})$, we have that $c^\top x(1) \geq c^\top bf(u(0), c^\top x^0 + d^\top x^{-1}) > 0$, implying that $y_x(1) > 0$. Invoking (A.1), we conclude that $y_x(t) > 0$ for all $t \in \mathbb{N}_0$.
(3) Let $\Gamma \subset \Gamma'$ be non-empty and compact and let $[\![x^0, x^{-1}]\!] \in \Gamma$. By hypothesis (N3) and the continuity of f, there exists $z_0 > 0$ such that

$$(d^\top b) \min_{w \in U} f(w, z) \geq z \quad \forall\, z \in [0, z_0]. \tag{A.2}$$

Let x be the solution of (1.1) such that $x(0) = x^0$ and $x(-1) = x^{-1}$. By statement (2) of Proposition 3.1 there exists a constant $\theta > 0$ (depending on Γ, U and V, but not on x^0, x^{-1}, u or v) such that

$$f(u(t), y_x(t)) \geq \theta y_x(t) \quad \forall\, t \in \mathbb{N}_0.$$

Hence, if $y_x(t) \geq z_0$, then, by (A.1),

$$y_x(t+2) \geq d^\top bf(u(t), y_x(t)) \geq d^\top b\theta y_x(t) \geq (d^\top b)\theta z_0 > 0.$$

On the other hand, if $y_x(t) \leq z_0$, then it follows from (A.1) and (A.2) that

$$y_x(t+2) \geq d^\top bf(u(t), y_x(t)) \geq y_x(t).$$

Combining the above two inequalities, we obtain

$$y_x(t+2) \geq \min\{y_x(t), \eta\} \quad \forall\, t \in \mathbb{N}_0, \quad \text{where } \eta := (d^\top b)\theta z_0 > 0. \tag{A.3}$$

Consequently,

$$y_x(t) \geq \min\{c^\top x^0 + d^\top x^{-1}, e^\top x^0, \eta\} \quad \forall\, t \in \mathbb{N}_0. \tag{A.4}$$

Furthermore, a simple induction argument based on (A.3) gives

$$y_x(t) \geq \min\{e^\top x^0, c^\top x(2) + d^\top x(1), \eta\} \quad \forall\, t \in \mathbb{N}.$$

Since

$$c^\top x(2) + d^\top x(1) \geq c^\top Ax(1) + d^\top Ax^0 \geq c^\top A^2 x^0 + d^\top Ax^0 = e^\top Ax^0,$$

we conclude that

$$y_x(t) \geq \min\{e^\top x^0, e^\top Ax^0, \eta\} \quad \forall\, t \in \mathbb{N}. \tag{A.5}$$

Defining $\psi_1, \psi_2 : \mathbb{R}^{2n} \to \mathbb{R}$ by

$$\psi_1\big([\![\xi^0, \xi^{-1}]\!]\big) := \min\{c^\top \xi^0 + d^\top \xi^{-1}, e^\top \xi^0\},$$
$$\psi_2\big([\![\xi^0, \xi^{-1}]\!]\big) := \min\{e^\top \xi^0, e^\top A\xi^0\}; \quad \forall\, \xi^0, \xi^{-1} \in \mathbb{R}^n,$$

we have that

$$\psi_1\big([\![\xi^0, \xi^{-1}]\!]\big) + \psi_2\big([\![\xi^0, \xi^{-1}]\!]\big) > 0 \quad \forall\, [\![\xi^0, \xi^{-1}]\!] \in \Gamma'. \tag{A.6}$$

It follows from (A.4) and (A.5) that

$$2y_x(t) \geq \min\big\{\psi_1\big([\![x^0, x^{-1}]\!]\big), \eta\big\} + \min\big\{\psi_2\big([\![x^0, x^{-1}]\!]\big), \eta\big\} \quad \forall\, t \in \mathbb{N}. \tag{A.7}$$

As $\Gamma \subset \Gamma'$, Γ is compact and ψ_1 and ψ_2 are continuous, it follows from (A.6) that there exists $\varepsilon > 0$ such that

$$\psi_1\big([\![\xi^0, \xi^{-1}]\!]\big) + \psi_2\big([\![\xi^0, \xi^{-1}]\!]\big) \geq \varepsilon \quad \forall\, [\![\xi^0, \xi^{-1}]\!] \in \Gamma.$$

Together with (A.7) this leads to

$$2y_x(t) \geq \min\{\varepsilon, \eta\} > 0 \quad \forall\, t \in \mathbb{N}.$$

(4) Let $\Gamma \subset \Gamma''$ be non-empty and compact, $[\![x^0, x^{-1}]\!] \in \Gamma$, and let x be the solution of (1.1) satisfying $x(0) = x^0$ and $x(-1) = x^{-1}$. By statement (2) of Proposition 3.1 there exists $\theta > 0$ (depending on Γ, U and V, but not on x^0, x^{-1}, u or v) such that

$$f(u(t), \tilde{c}^\top \tilde{x}(t)) = f(u(t), y_x(t)) \geq \theta y_x(t) = \theta \tilde{c}^\top \tilde{x}(t) \quad \forall\, t \in \mathbb{N}_0,$$

where we recall that $\tilde{x}(t) = [\![x(t), x(t-1)]\!]$. Using the above inequality in combination with (3.1) yields

$$\tilde{x}(t+\tau) \geq \left(\tilde{A} + \theta \tilde{b}\tilde{c}^\top\right)\tilde{x}(t+\tau-1) \geq \cdots \geq \left(\tilde{A} + \theta \tilde{b}\tilde{c}^\top\right)^\tau \tilde{x}(t) \quad \forall\, t \in \mathbb{N}_0.$$

It follows from (3.12), a consequence of (P2), that

$$y_x(t+\tau) = \tilde{c}^\top \tilde{x}(t+\tau) \geq \tilde{c}^\top (\tilde{A} + \theta \tilde{b}\tilde{c}^\top)^\tau \tilde{x}(t) \geq \varepsilon_\theta \phi^\top \tilde{x}(t) = \varepsilon_\theta \mathcal{F}(\tilde{x}(t)) \quad \forall t \in \mathbb{N}_0,$$

where $\varepsilon_\theta := \varepsilon \min\{1, \theta^\tau\} > 0$. Appealing to statement (3) of Proposition 3.1, we conclude that there exists $\eta > 0$ such that

$$y_x(t+\tau) \geq \varepsilon_\theta \min\{\mathcal{F}(\tilde{x}(0)), \eta\} \quad \forall\, t \in \mathbb{N}_0.$$

Finally, $\mu := \min_{\xi \in \Gamma} \mathcal{F}(\xi) > 0$, and thus,

$$y_x(t+\tau) \geq \varepsilon_\theta \min\{\mu, \eta\} > 0 \quad \forall\, t \in \mathbb{N}_0,$$

completing the proof. □

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