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Brief paper

Low-gain integral control of continuous-time linear systems subject to input and output nonlinearities☆

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Abstract

Continuous-time low-gain integral control strategies are presented for tracking of constant reference signals for finite-dimensional, continuous-time, asymptotically stable, single-input single-output, linear systems subject to a globally Lipschitz and non-decreasing input nonlinearity and a locally Lipschitz, non-decreasing and affinely sector-bounded output nonlinearity. Both non-adaptive (but possibly time varying) and adaptive integrator gains are considered. In particular, it is shown that applying error feedback using an integral controller ensures asymptotic tracking of constant reference signals, provided that (a) the steady-state gain of the linear part of the plant is positive, (b) the positive integrator gain is ultimately sufficiently small and (c) the reference value is feasible in a very natural sense. The classes of actuator and sensor nonlinearities under consideration contain standard nonlinearities important in control engineering such as saturation and deadzone.

Keywords: Adaptive control; Continuous-time systems; Input/output nonlinearities; Integral control; Robust control; Tracking

1. Introduction

The synthesis of low-gain integral and proportional-plus-integral controllers for (uncertain) stable plants has received considerable attention in the last 20 years. The following principle is well known (see, for example, Davison 1976; Lunze, 1988): closing the loop around an asymptotically stable, finite-dimensional, continuous-time, single-input, single-output linear plant Σ, with transfer function G, compensated by a pure integral controller with gain k, will result in a stable closed-loop system which achieves asymptotic tracking of arbitrary constant reference signals, provided that $|k|$ is sufficiently small and $kG(0) > 0$. Therefore, if a plant is known to be asymptotically stable and if the sign of $G(0)$ is known (this information can be obtained from plant step response data), then the problem of tracking by low-gain integral control reduces to that of tuning the gain parameter k. The problem of tuning the integrator gain adaptively has been addressed in a number of papers, see Cook (1992) and Miller and Davison 1989, 1993 (with input constraints treated in Miller and Davison (1989, 1993)). Recently, Logemann et al. have developed tuning regulator results for infinite-dimensional systems with input nonlinearities (Logemann & Ryan, 2000; Logemann, Ryan, & Townley, 1998).

In this paper, we present results which show that the above principle remains true if the plant to be controlled is a stable, finite-dimensional single-input, single-output, linear system subject to an input and/or output nonlinearity (see Fig. 1). Precisely, we prove that, if $G(0) > 0$ and if the constant reference signal r is feasible in an entirely natural sense, then there exists a number $k^* > 0$ such that, for all non-decreasing, piecewise continuously differentiable, globally Lipschitz input nonlinearities φ and all non-decreasing, piecewise continuously differentiable, locally Lipschitz and affinely sector-bounded output nonlinearities ψ the following holds: for all positive, bounded and continuous integrator gains $k(\cdot)$ (thus in particular for positive constant gains), the output $y(t)$ of the closed-loop system converges to r as $t \to \infty$, provided that $\limsup_{t\to\infty} k(t) < k^*$ and

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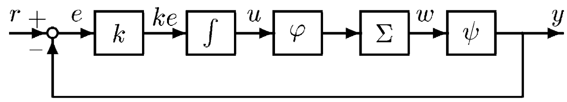

Fig. 1. Low-gain integral control.

k is not of class L^1 (under some additional assumptions on the nonlinearities, results concerning the rate of convergence are derived). When compared with Logemann and Ryan (2000) and Logemann et al. (1998), the novelty in this paper is not only the inclusion of output nonlinearities, but also a different Lyapunov analysis which, for finite-dimensional systems, is more natural and powerful than the (infinite-dimensional) approaches developed in Logemann and Ryan (2000) and Logemann et al. (1998).

Finally, in Section 3.2, we show that one consequence of the above principle is that the following simple adaptation law (introduced in Logemann and Ryan (2000)) $k(t)=1/l(t)$, $\dot{l}(t)=|r-y(t)|$ with $l(0)=l^0>0$, produces an integrator gain k so that the output $y(t)$ of the closed-loop system converges to r as $t\to\infty$.

2. Problem formulation

The problem of tracking constant reference signals $r\in\mathbb{R}$ will be addressed in the context of a class of finite-dimensional (state space $\mathbb{R}^N$) single-input ($u(t)\in\mathbb{R}$), single-output ($y(t)\in\mathbb{R}$), continuous-time (time domain $\mathbb{R}_+ := [0,\infty)$), real linear systems $\Sigma=(A,B,C,D)$ having a nonlinearity in the input and output channel:

$$\dot{x}=Ax+B\varphi(u), \qquad x(0)=x^0\in\mathbb{R}^N,$$

$$w=Cx+D\varphi(u), \qquad y=\psi(w). \tag{1}$$

2.1. The class $\mathscr{S}$ of linear systems

In (1), A is assumed to be Hurwitz, i.e., each eigenvalue of A has negative real part. Furthermore, the transfer function G, given by $G(s)=C(sI-A)^{-1}B+D$, is assumed to satisfy $G(0)>0$. Thus, the underlying class of real linear systems $\Sigma=(A,B,C,D)$ is

$$\mathscr{S} := \{\Sigma=(A,B,C,D)\,|\,A \text{ Hurwitz}, \quad G(0)=D-CA^{-1}B>0\}.$$

If G is the transfer function of a system $\Sigma\in\mathscr{S}$, then it is readily shown that $\mathrm{Re}(G(s)/s)$ is bounded away from $-\infty$ on the open right-half plane and, hence,

$$1+\kappa\,\mathrm{Re}\,\frac{G(s)}{s}\geqslant 0 \qquad \forall s\in\mathbb{C} \quad \text{with } \mathrm{Re}\,s>0 \tag{2}$$

for all sufficiently small $\kappa>0$. We refer to (2) as the positive-real condition. Define

$$\kappa^* := \sup\{\kappa>0\,|\,(2) \text{ holds}\}. \tag{3}$$

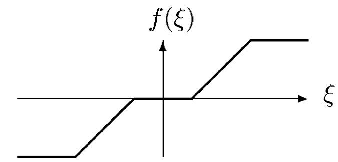

Fig. 2. Nonlinearity with saturation and deadzone.

Lower bounds and formulae for κ^* may be found in Logemann, Ryan, and Townley (1999). The following lemma will be invoked in a later stability analysis: the lemma is the continuous-time analogue of the discrete-time result in Fliegner, Logemann, and Ryan (2001, Lemma 3.2).

Lemma 1. *Assume that* $\Sigma=(A,B,C,D)\in\mathscr{S}$ *and let* $\Delta>1/\kappa^*$. *Then there exists* $P\in\mathbb{R}^{N\times N}$ *such that* $P=P^{\mathrm{T}}>0$ *and*

$$\begin{bmatrix} PA+A^{\mathrm{T}}P & PA^{-1}B-C^{\mathrm{T}} \\ (A^{-1}B)^{\mathrm{T}}P-C & -2\Delta \end{bmatrix}<0. \tag{4}$$

2.2. Input and output nonlinearities

With the intention of encompassing nonlinearities with sufficiently general regularity properties to capture, for example, saturation and deadzone (see Fig. 2), the following sets of piecewise continuously differentiable, monotone non-decreasing nonlinearities are first introduced:

$$\mathscr{M} := \{f\colon\mathbb{R}\to\mathbb{R}\,|\,f \text{ is piecewise continuously differentiable and non-decreasing}\},$$

$$\mathscr{M}(\lambda) := \{f\in\mathscr{M}\,|\,0\leqslant(f(\xi)-f(0))\xi\leqslant\lambda\xi^2\ \forall\xi\in\mathbb{R}\},$$

$$\mathscr{M}_L(\lambda) := \{f\in\mathscr{M}\,|\,f \text{ is globally Lipschitz with Lipschitz constant } \lambda\}.$$

Clearly, $\mathscr{M}_L(\lambda)\subset\mathscr{M}(\lambda)\subset\mathscr{M}$ and for every $f\in\mathscr{M}$, its left $f'_-(\xi)$ and right $f'_+(\xi)$ derivative exist at every point $\xi\in\mathbb{R}$.

Remark 2. (i) Let $f\in\mathscr{M}(\lambda)$. Then $f(\xi)=f(0)+\gamma(\xi)\xi$ for all ξ, where $\gamma(\xi):=(f(\xi)-f(0))/\xi$ if $\xi\neq 0$ and $\gamma(0):=\lambda$. Clearly, $\gamma(\xi)\in[0,\lambda]$ for all $\xi\in\mathbb{R}$.

(ii) If $f\in\mathscr{M}(\lambda)$, then for each $v\in\mathbb{R}$, there exists $\tilde{\lambda}\geqslant 0$ such that the function $\xi\mapsto f(\xi+v)-f(v)$ is of class $\mathscr{M}(\tilde{\lambda})$.

(iii) If $f\in\mathscr{M}_L(\lambda)$, then for each $v\in\mathbb{R}$, $\xi\mapsto f(\xi+v)-f(v)$ is also of class $\mathscr{M}_L(\lambda)$.

Let $f\in\mathscr{M}$. For $\xi\in\mathbb{R}$, we define

$$f^{\triangledown}(\xi):=\min\{f'_-(\xi),f'_+(\xi)\}.$$

Note that, since f is non-decreasing, $f^{\triangledown}(\xi)\geqslant 0$ for all $\xi\in\mathbb{R}$. Clearly, $f^{\triangledown}(\xi)$ coincides with the derivative $f'(\xi)$ whenever the latter exists. A point $\xi\in\mathbb{R}$ is said to be a *critical point* (and $f(\xi)$ is said to be a *critical value*) of f

if $f^{\nabla}(\xi)=0$. We denote, by $\mathscr{C}(f)$, the set of critical values of f.

The following lemma will be used later. The proof is straightforward and is therefore omitted.

Lemma 3. *Let $f \in \mathscr{M}$ and let $g : \mathbb{R}_+ \to \mathbb{R}$ be absolutely continuous. Then $f \circ g$ is absolutely continuous and*

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ g)(t) = f^{\nabla}(g(t))\dot{g}(t) \quad \textit{for a.a. } t \in \mathbb{R}_+.$$

Finally, we make precise the class $\mathscr{N}$ of input/output nonlinearities:

$$\mathscr{N} := \{(\varphi, \psi) \in \mathscr{M}_L(\lambda_1) \times \mathscr{M}(\lambda_2) \mid \lambda_1 > 0,\ \lambda_2 > 0\}.$$

2.3. The tracking objective and feasibility

Given $\Sigma = (A, B, C, D) \in \mathscr{S}$ and $(\varphi, \psi) \in \mathscr{N}$, the tracking objective is to determine, by feedback, an input u such that, for given $r \in \mathbb{R}$, the output y of (1) has the property $y(t) \to r$ as $t \to \infty$. Clearly, if this objective is achievable, then r is necessarily in the closure of $\operatorname{im} \psi$. We will impose a stronger condition, namely,

$$\Psi^r \cap \bar{\Phi} \neq \emptyset,$$

where $\Psi^r := \{v \in \mathbb{R} \mid \psi(G(0)v) = r\}$, $\Phi := \operatorname{im} \varphi$, $\bar{\Phi} := \operatorname{clos}(\Phi)$, and refer to the set

$$\mathscr{R} := \{r \in \mathbb{R} | \Psi^r \cap \bar{\Phi} \neq \emptyset\}$$

as the set of *feasible reference values*. The next proposition (proof omitted for brevity) shows that $r \in \mathscr{R}$ is close to being a necessary condition for tracking insofar as, if tracking of r is achievable whilst maintaining continuity and boundedness of $\varphi \circ u$, then $r \in \mathscr{R}$.

Proposition 4. *Let $\Sigma = (A, B, C, D) \in \mathscr{S}$, let $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ be continuous and let $u : \mathbb{R}_+ \to \mathbb{R}$ be such that $\varphi \circ u$ is continuous and bounded. For $x^0 \in \mathbb{R}^N$, let $x : \mathbb{R}_+ \to \mathbb{R}^N$ denote the bounded solution of the initial-value problem in* (1). *If $\lim_{t\to\infty} [\psi(Cx(t) + D\varphi(u(t)))] = r$, then $r \in \mathscr{R}$.*

3. Integral control

Let $\Sigma = (A, B, C, D) \in \mathscr{S}$ and $(\varphi, \psi) \in \mathscr{N}$. To achieve the objective of tracking feasible reference values $r \in \mathscr{R}$, we will investigate integral control action

$$u(t) = u^0 + \int_0^t k(\tau)[r - \psi(Cx(\tau) + D\varphi(u(\tau)))]\,\mathrm{d}\tau$$

$$= u^0 + \int_0^t k(\tau)[r - y(\tau)]\,\mathrm{d}\tau \tag{5}$$

with control gain k (possibly constant) which is either prescribed or determined adaptively.

3.1. Prescribed gain

Henceforth, we assume that the gain function k: $\mathbb{R}_+ \to (0, \infty)$ satisfies

$$k \in \mathscr{G} := \{g \mid g : \mathbb{R}_+ \to (0, \infty), g \text{ is continuous and bounded}\}.$$

An application of integrator (5) leads to the following system of nonlinear differential equations

$$\dot{x}(t) = Ax(t) + B\varphi(u(t)), \tag{6a}$$

$$\dot{u}(t) = k(t)[r - \psi(Cx(t) + D\varphi(u(t)))], \tag{6b}$$

$$(x(0), u(0)) = (x^0, u^0) \in \mathbb{R}^N \times \mathbb{R}. \tag{6c}$$

The next proposition follows from a routine argument based on standard results from the theory of differential equations.

Proposition 5. *Let $\Sigma = (A, B, C, D) \in \mathscr{S}$ and $(\varphi, \psi) \in \mathscr{N}$, $k \in \mathscr{G}$ and $r \in \mathbb{R}$. For each $(x^0, u^0) \in \mathbb{R}^N \times \mathbb{R}$, the initial-value problem* (6) *has a unique solution $(x, u) : \mathbb{R}_+ \to \mathbb{R}^N \times \mathbb{R}$.*

Before presenting the main results in Theorems 7 and 11 below, we prove a convenient lemma which facilitates the proofs of these theorems. Since the lemma hypothesizes the convergence of $\varphi(u(t))$ as $t \to \infty$, it is not of independent interest.

Lemma 6. *Let $\Sigma = (A, B, C, D) \in \mathscr{S}$, $(\varphi, \psi) \in \mathscr{N}$, $r \in \mathscr{R}$ and $k \in \mathscr{G}$. For $(x^0, u^0) \in \mathbb{R}^N \times \mathbb{R}$, let $(x, u) : \mathbb{R}_+ \to \mathbb{R}^N \times \mathbb{R}$ be the unique solution of the initial-value problem* (6). *Assume that $K(t) := \int_0^t k \to \infty$ as $t \to \infty$ and that $\lim_{t\to\infty} \varphi(u(t))$ exists and is finite. Then*

(i) $\lim_{t\to\infty} \varphi(u(t)) =: \varphi^r \in \Psi^r \cap \bar{\Phi}$,
(ii) $\lim_{t\to\infty} x(t) = -A^{-1}B\varphi^r$,
(iii) $\lim_{t\to\infty} y(t) = r$, *where*

$$y(t) = \psi(Cx(t) + D\varphi(u(t))),$$

(iv) *if* $\Psi^r \cap \bar{\Phi} = \Psi^r \cap \Phi$, *then*

$$\lim_{t\to\infty} \operatorname{dist}(u(t), \varphi^{-1}(\varphi^r)) = 0,$$

(v) *if* $\Psi^r \cap \bar{\Phi} = \Psi^r \cap \operatorname{int}(\Phi)$, *then* $u(\cdot)$ *is bounded.*

Proof. By hypothesis, there exists $\varphi^r \in \mathbb{R}$ such that $\lim_{t\to\infty} \varphi(u(t)) = \varphi^r$ which, together with the Hurwitz property of A, implies $\lim_{t\to\infty} w(t) = G(0)\varphi^r$, where $w = Cx + D\varphi(u)$. Evidently, $\varphi^r \in \bar{\Phi}$ and so, to establish (i), it suffices to show that $\varphi^r \in \Psi^r$. Seeking a contradiction, suppose that $\varphi^r \notin \Psi^r$. This implies that $\theta := (r - \psi(G(0)\varphi^r))/2 \neq 0$. Using continuity of ψ, we obtain for sufficiently large $s > 0$

$$|y(t) - \psi(G(0)\varphi^r)| \leqslant |\theta| \quad \forall t \geqslant s.$$

As a consequence, and noticing that $\dot{u}(t)=k(t)[r-y(t)]=k(t)[2\theta-y(t)+\psi(G(0)\varphi^r)]$, we have

$$-|\theta|k(t) \leqslant \dot{u}(t)-2\theta k(t) \leqslant |\theta|k(t) \quad \forall t \geqslant s.$$

Since $\theta \neq 0$, either $\theta > 0$ or $\theta < 0$. Assume $\theta > 0$. Then $\dot{u}(t) \geqslant \theta k(t)$ for all $t \geqslant s$ which, on integration, yields $u(t)-u(s) \geqslant \theta(K(t)-K(s))$ for all $t \geqslant s$. Since $K(t) \to \infty$ as $t \to \infty$, we conclude that $u(t) \to \infty$ as $t \to \infty$, hence

$$\sup \bar{\Phi} = \lim_{t\to\infty} \varphi(u(t)) = \varphi^r. \tag{7}$$

Let $\varphi^* \in \Psi^r \cap \bar{\Phi}$. Since $\theta > 0$, it follows $\varphi^* > \varphi^r$ and, consequently, by (7), $\sup \bar{\Phi} < \varphi^*$, a contradiction. A similar argument shows that the assumption $\theta < 0$ also leads to a contradiction. Therefore, we may conclude $\varphi^r \in \Psi^r \cap \bar{\Phi}$ which is statement (i). Statement (ii) follows from (i) and the Hurwitz property of A. Statement (iii) is a consequence of (i), (ii) and continuity of ψ. Next, we establish statement (iv). Assume $\Psi^r \cap \bar{\Phi} = \Psi^r \cap \Phi$ which, together with (i), implies the existence of $\xi^* \in \mathbb{R}$ such that $\varphi^r = \varphi(\xi^*)$. Seeking a contradiction, suppose that $\operatorname{dist}(u(t), \varphi^{-1}(\varphi^r)) \nrightarrow 0$ as $t \to \infty$. Then there exist $\varepsilon > 0$ and a sequence $(t_n) \in \mathbb{R}_+$ with $t_n \to \infty$ as $n \to \infty$, such that

$$\operatorname{dist}(u(t_n), \varphi^{-1}(\varphi^r)) \geqslant \varepsilon. \tag{8}$$

If the sequence $(u(t_n))$ is bounded, we may assume without loss of generality that it converges to a finite limit u_∞. By continuity, $\varphi(u_\infty)=\varphi^r$ and so $u_\infty \in \varphi^{-1}(\varphi^r)$. This contradicts (8). Therefore, we may assume that $(u(t_n))$ is unbounded. Extracting a subsequence if necessary, we may then assume that either $u(t_n) \to \infty$ or $u(t_n) \to -\infty$ as $n \to \infty$: if the former holds, then $u(t_n) > \xi^*$ for all n sufficiently large; if the latter holds, then $u(t_n) < \xi^*$ for all n sufficiently large. In either case, by monotonicity of φ it follows that $\varphi(u(t_n)) = \varphi(\xi^*) = \varphi^r$ for all n sufficiently large. Clearly, this contradicts (8) and so statement (iv) must hold.

To prove (v), assume that $\Psi^r \cap \bar{\Phi} = \Psi^r \cap \operatorname{int}(\Phi)$ and, for contradiction, suppose that u is unbounded. Then there exists a sequence $(t_n) \subset (0,\infty)$ with $t_n \to \infty$ and $|u(t_n)| \to \infty$ as $n \to \infty$. By monotonicity of φ and (i), it then follows that either $\varphi^r = \sup \Phi$ or $\varphi^r = \inf \Phi$, contradicting the fact that $\varphi^r \in \Psi^r \cap \operatorname{int}(\Phi) \subset \operatorname{int}(\Phi)$. Therefore, u is bounded. This completes the proof of the lemma. □

The next theorem forms the core of the paper: it contains the main non-adaptive low-gain tracking result, and, in combination with Lemma 6, is crucial in the proof of the adaptive counterpart in Theorem 11. First we recall that, if functions f_1, f_2 are such that $f_1(t), f_2(t) \to 0$ and $f_1(t) = O(f_2(t))$ as $t \to \infty$, then the convergence of f_1 is said to be of order f_2.

Theorem 7. *Let $\Sigma = (A,B,C,D) \in \mathscr{S}$, $(\varphi,\psi) \in \mathscr{N}$ and $r \in \mathscr{R}$.*

There exists $k^* > 0$ such that, for all $k \in \mathscr{G}$ and $(x^0,u^0) \in \mathbb{R}^N \times \mathbb{R}$, the unique solution $(x,u): \mathbb{R}_+ \to \mathbb{R}^N \times \mathbb{R}$ of the initial-value problem (6) is such that the following hold.

(i) *If $\limsup_{t\to\infty} k(t) < k^*$, then $\lim_{t\to\infty} \varphi(u(t))$ exists and is finite.*
(ii) *If $\limsup_{t\to\infty} k(t) < k^*$ and $K(t) := \int_0^t k \to \infty$ as $t \to \infty$, then statements* (i) *to* (v) *of Lemma* 6 *hold. If, in addition, $\Psi^r \cap \bar{\Phi} = \Psi^r \cap \Phi$, $\Psi^r \cap \mathscr{C}(\varphi) = \emptyset$ and $r \notin \mathscr{C}(\psi)$, then the convergence in statements* (i) *to* (iii) *of Lemma* 6 *is of order $\exp(-\rho K(\cdot))$ for some $\rho > 0$.*

Moreover, if $\psi \in \mathscr{M}_L(\lambda_2)$, then (i) and (ii) *are valid with $k^* = \kappa^*/(\lambda_1\lambda_2)$, where κ^* is given by* (3) *and $\lambda_1 > 0$ is such that $\varphi \in \mathscr{M}_L(\lambda_1)$.*

Remark 8. Theorem 7 identifies conditions on the integrator gain k under which convergence of $\varphi \circ u$, the main prerequisite for an application of Lemma 6, is implied. An immediate consequence of Theorem 7 is the following: if $k \in \mathscr{G}$ is chosen such that, as $t \to \infty$, $k(t)$ tends to zero *sufficiently slowly* (in the sense that $k \notin L^1(\mathbb{R}_+, \mathbb{R}_+)$), then the tracking objective is achieved (this strategy is independent of $k^* > 0$). If $\psi \in \mathscr{M}_L(\lambda_2)$ for some $\lambda_2 > 0$, then we may infer that the tracking objective is achievable by *constant* gain $k \in (0, \kappa^*/(\lambda_1\lambda_2))$ and, moreover, under additional conditions on the nonlinearities, the convergence is exponential.

Proof of Theorem 7. Since $r \in \mathscr{R}$, it follows that $\Psi^r \neq \emptyset$. Let $\varphi^* \in \Psi^r$; since $\psi \in \mathscr{M}(\lambda_2)$ for some $\lambda_2 > 0$, the function

$$\tilde{\psi}: \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \psi(\xi + G(0)\varphi^*) - r \tag{9}$$

is in $\mathscr{M}(\tilde{\lambda}_2)$ for some $\tilde{\lambda}_2 > 0$ (cf. Remark 2(ii)). Therefore, by Remark 2(i),

$$\tilde{\psi}(\xi) = \gamma(\xi)\xi \quad \forall \xi \in \mathbb{R}, \tag{10}$$

where

$$\gamma(0) := \tilde{\lambda}_2 \quad \text{and} \quad \gamma(\xi) := \tilde{\psi}(\xi)/\xi \quad \text{for all } \xi \neq 0. \tag{11}$$

Note that $0 \leqslant \gamma(\xi) \leqslant \tilde{\lambda}_2$ for all $\xi \in \mathbb{R}$. Now define $\lambda := \lambda_1\tilde{\lambda}_2$ and $k^* := \kappa^*/\lambda$, where κ^* is given by (3). Let $(x^0,u^0,k) \in \mathbb{R}^N \times \mathbb{R} \times \mathscr{G}$ and let $(x,u): \mathbb{R}_+ \to \mathbb{R}^N \times \mathbb{R}$ be the unique solution of the initial-value problem (6).

(i) Assume that $\limsup_{t\to\infty} k(t) < k^*$. Choose $k_* > 0$ such that

$$\limsup_{t\to\infty} k(t) < k_* < k^* \tag{12}$$

and fix $\Delta := 1/(k_*\lambda)$. By Lemma 1 there exists a $P \in \mathbb{R}^{N\times N}$, with $P = P^{\mathrm{T}} > 0$, such that

$$\Lambda := \begin{bmatrix} PA + A^{\mathrm{T}}P & PA^{-1}B - C^{\mathrm{T}} \\ (A^{-1}B)^{\mathrm{T}}P - C & -2/\Delta \end{bmatrix} < 0.$$

Introduce new variables

$$z(t) := x(t) + A^{-1}B\varphi(u(t)), \quad v(t) := \varphi(u(t)) - \varphi^* \tag{13}$$

and notice that, in terms of the new variables, w is given by

$$w = Cx + D\varphi(u) = Cz + G(0)(v + \varphi^*) = \tilde{w} + G(0)\varphi^*,$$

where $\tilde{w} := Cz + G(0)v$. By Lemma 3, $\dot{v}(t) = \varphi^{\triangledown}(u(t))\dot{u}(t)$ for almost all $t \in [0, \infty)$. Using (9), we obtain

$$\dot{z}(t) = Az(t) - k(t)\varphi^{\triangledown}(u(t))A^{-1}B\tilde{\psi}(\tilde{w}(t)),$$

$$\dot{v}(t) = -k(t)\varphi^{\triangledown}(u(t))\tilde{\psi}(\tilde{w}(t)) \quad \text{for a.a. } t \geqslant 0.$$

Invoking (10),

$$\dot{z}(t) = Az(t) + A^{-1}B\zeta(t), \quad \dot{v}(t) = \zeta(t) \quad \text{for a.a. } t \geqslant 0, \tag{14}$$

where

$$\zeta(t) := -k(t)\eta(t)\tilde{w}(t),$$

$$\eta(t) := \varphi^{\triangledown}(u(t))\gamma(\tilde{w}(t)). \tag{15}$$

Note that

$$0 \leqslant \eta(t) \leqslant \lambda_1\tilde{\lambda}_2 = \lambda \quad \forall t \geqslant 0. \tag{16}$$

We will investigate asymptotic properties of (z, v) using a Lyapunov approach. Define the absolutely continuous function $V : \mathbb{R}_+ \to \mathbb{R}_+$, $t \mapsto \langle z(t), Pz(t)\rangle + G(0)v^2(t)$. Then, using (14) and (15)

$$\begin{aligned}\dot{V}(t) &= \langle z(t), (PA + A^{\mathrm{T}}P)z(t)\rangle \\ &\quad + 2(A^{-1}B)^{\mathrm{T}}Pz(t)\zeta(t) + 2G(0)v(t)\zeta(t) \\ &= \langle z(t), (PA + A^{\mathrm{T}}P)z(t)\rangle \\ &\quad + 2[(A^{-1}B)^{\mathrm{T}}P - C]z(t)\zeta(t) + 2\tilde{w}(t)\zeta(t) \\ &= \langle [z^{\mathrm{T}}(t), \zeta(t)]^{\mathrm{T}}, \Lambda[z^{\mathrm{T}}(t), \zeta(t)]^{\mathrm{T}}\rangle \\ &\quad + 2\Delta\zeta^2(t) - 2k(t)\eta(t)\tilde{w}^2(t) \\ &\leqslant -\alpha[\|z(t)\|^2 + \zeta^2(t)] - 2k(t)\eta(t) \\ &\quad [1 - k(t)\eta(t)\Delta]\tilde{w}^2(t) \quad \text{for a.a. } t \geqslant 0,\end{aligned}$$

where $\alpha = 1/\|\Lambda^{-1}\|$. Invoking (12) and (16), we see that there exists $T_1 \geqslant 0$ such that

$$\sup_{t \geqslant T_1} k(t)\eta(t)\Delta < 1. \tag{17}$$

Consequently, there exists $\beta_1 > 0$ such that

$$\dot{V}(t) \leqslant -\alpha\|z(t)\|^2 - \alpha\zeta^2(t) - \beta_1 k(t)\eta(t)\tilde{w}^2(t) \quad \text{for a.a. } t \geqslant T_1. \tag{18}$$

It follows that $z \in L^2(\mathbb{R}_+, \mathbb{R}^N)$ and $\zeta \in L^2(\mathbb{R}_+, \mathbb{R})$. Since $k\eta v = -(G(0))^{-1}[k\eta Cz + \zeta]$, we may conclude that $k\eta v \in L^2(\mathbb{R}_+, \mathbb{R})$ and thus

$$k\eta v Cz \in L^1(\mathbb{R}_+, \mathbb{R}). \tag{19}$$

Moreover, using (18) and the definition of $\tilde{w}$, we obtain

$$\dot{V}(t) \leqslant -2\beta_1 G(0)k(t)\eta(t)v(t)Cz(t) - \beta_1 G^2(0)k(t)\eta(t)v^2(t) \quad \text{for a.a. } t \geqslant T_1$$

and so we have

$$0 \leqslant V(t) \leqslant V(T_1) - 2\beta_1 G(0)\int_{T_1}^t k(s)\eta(s)v(s)Cz(s)\,\mathrm{d}s - \beta_1 G^2(0)\int_{T_1}^t k(s)\eta(s)v^2(s)\,\mathrm{d}s \quad \forall t \geqslant T_1,$$

which, together with (19), implies that

$$k\eta v^2 \in L^1(\mathbb{R}_+, \mathbb{R}). \tag{20}$$

From (19) and (20), we may infer that $v\zeta = -k\eta v[Cz + G(0)v] \in L^1(\mathbb{R}_+, \mathbb{R})$. Multiplying the second of equations (14) by $v(t)$ and integrating yields

$$\lim_{t\to\infty} v^2(t) = v^2(T_1) + 2\lim_{t\to\infty}\int_{T_1}^t v\zeta = c$$

for some constant $c \in [0, \infty)$. By continuity of v it follows that either $\lim_{t\to\infty} v(t) = \sqrt{c}$ or $\lim_{t\to\infty} v(t) = -\sqrt{c}$. This establishes statement (i).

(ii) Assume that $\limsup_{t\to\infty} k(t) < k^*$ and $K(t) \to \infty$ as $t \to \infty$. Then it immediately follows from Lemma 6 and (i) that statements (i)–(v) of Lemma 6 hold; in particular, there exists $\varphi^r \in \Psi^r \cap \bar{\Phi}$ such that $\lim_{t\to\infty} \varphi(u(t)) = \varphi^r$.

To show that the convergence is of order $\exp(-\rho K(\cdot))$ for some $\rho > 0$, we make use of inequality (18) with φ^* in (9) and (13) replaced by φ^r. Introduce functions $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ and $W_\nu : \mathbb{R}_+ \to \mathbb{R}_+$ (parameterized by $\nu > 0$) defined by

$$\theta(t) := \int_0^t k\eta \quad \text{and} \quad W_\nu(t) := \exp(2\nu\theta(t))V(t).$$

Noting that, for some constant $\beta_2 > 0$,

$$V(t) = \langle z(t), Pz(t)\rangle + [\tilde{w}(t) - Cz(t)]^2/G(0) \leqslant \beta_2[\|z(t)\|^2 + \tilde{w}^2(t)] \quad \forall t \in \mathbb{R}_+$$

and invoking (17) and (18), we have

$$\begin{aligned}\dot{W}_\nu(t) &= \exp(2\nu\theta(t))[\dot{V}(t) + 2\nu k(t)\eta(t)V(t)] \\ &\leqslant \exp(2\nu\theta(t))[-(\alpha - (2\beta_2\nu)/\Delta)\|z(t)\|^2 \\ &\quad - k(t)\eta(t)(\beta_1 - 2\beta_2\nu)\tilde{w}^2(t)] \quad \text{for a.a. } t \geqslant T_1.\end{aligned}$$

Choose $\nu > 0$ sufficiently small so that $0 < 2\beta_2\nu \leqslant \min\{\alpha\Delta, \beta_1/2\}$, in which case

$$\dot{W}_\nu(t) \leqslant -\tfrac{1}{2}\beta_1\exp(2\nu\theta(t))k(t)\eta(t)\tilde{w}^2(t) \quad \text{for a.a. } t \geqslant T_1,$$

whence boundedness of W_ν which, in turn, implies the existence of a constant $L > 0$ such that

$$\|\exp(\nu\theta(t))z(t)\| \leqslant L,$$

$$|\exp(\nu\theta(t))v(t)| \leqslant L; \quad \forall t \in \mathbb{R}_+. \tag{21}$$

Notice that, by hypothesis, $\varphi^r \in \Phi$ and $\varphi^r \notin \mathscr{C}(\varphi)$. Thus, by monotonicity, the preimage $\varphi^{-1}(\varphi^r)$ is a singleton $\{u^r\}$ and $\varphi^{\triangledown}(u^r) > 0$. By Lemma 6(iv), $u(t) \to u^r$ as $t \to \infty$. Since φ is piecewise continuously differentiable, there exists $T_2 \geqslant T_1$ such that

$$\varphi^{\triangledown}(u(t)) \geqslant \min\{\varphi'_-(u^r), \varphi'_+(u^r)\}/2 > 0 \quad \forall t \geqslant T_2. \tag{22}$$

Since by assumption $r \notin \mathscr{C}(\psi)$, we have that

$\tilde{\psi}^{\nabla}(0) = \psi^{\nabla}(G(0)\varphi^r) > 0.$

We claim that there exist $T_3 \geqslant T_2$ and $0 < \mu < \lambda$ such that $\mu \leqslant \eta(t)$ for all $t \geqslant T_3$. Seeking a contradiction, suppose the claim is not true. Then, by (22), there is a sequence $t_n \to \infty$ such that $0 \leqslant \gamma(\tilde{w}(t_n)) \leqslant 1/n$. Define $h_n := \tilde{w}(t_n) = Cz(t_n) + G(0)v(t_n)$. By statements (i) and (ii) of Lemma 6 we know that $h_n \to 0$ for $n \to \infty$. Now, using (11),

$$0 = \lim_{n\to\infty} \gamma(h_n) = \lim_{n\to\infty} \frac{\tilde{\psi}(h_n)}{h_n} \geqslant \liminf_{h\to 0} \frac{\tilde{\psi}(h)}{h}$$
$$= \min\{\tilde{\psi}'_-(0), \tilde{\psi}'_+(0)\} = \tilde{\psi}^{\nabla}(0) > 0,$$

a contradiction. Thus, there exist $\mu \in (0, \lambda)$ and $T_3 \geqslant T_2$ such that $\mu \leqslant \eta(t)$ for all $t \geqslant T_3$. As a consequence, setting $\rho := \nu\mu$, there exists $L_1 > 0$ such that $\exp(\rho K(t)) \leqslant L_1 \exp(\nu\theta(t))$ for all $t \geqslant 0$, and so, by (21), there exists $L_2 > 0$ such that

$$\|\exp(\rho K(t))z(t)\| \leqslant L_2,$$
$$|\exp(\rho K(t))v(t)| \leqslant L_2 \quad \forall t \in \mathbb{R}_+. \tag{23}$$

This shows that the convergence in (i) and (ii) of Lemma 6 is of order $\exp(-\rho K(\cdot))$. Convergence of order $\exp(-\rho K(\cdot))$ in (iii) of Lemma 6 follows by a routine argument.

Finally, assume that $\psi \in \mathscr{M}_L(\lambda_2)$ for some $\lambda_2 > 0$. By Remark 2(iii) it follows that, for any fixed $r \in \mathscr{R}$, the function $\tilde{\psi} : \mathbb{R} \to \mathbb{R}, \xi \mapsto \psi(\xi + G(0)\varphi^r) - r$ is also in $\mathscr{M}_L(\lambda_2)$. The argument used above in the proof of (i) and (ii) applies *mutatis mutandis* to conclude that (i) and (ii) hold with $k^* = \kappa^*/(\lambda_1\lambda_2)$. □

Remark 9. Note that (14) can be written in the form

$$\frac{\mathrm{d}\tilde{z}}{\mathrm{d}t}(t) = \tilde{A}\tilde{z}(t) - \tilde{B}\tilde{\varphi}(t, \tilde{C}\tilde{z}(t)), \tag{24}$$

where

$$\tilde{z} := \begin{pmatrix} z \\ v \end{pmatrix}, \quad \tilde{A} := \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{B} := \begin{pmatrix} A^{-1}B \\ 1 \end{pmatrix},$$

$\tilde{C} := (C, G(0))$ and $\tilde{\varphi}(t, \xi) := k(t)\eta(t)\xi$. Clearly,

$$0 \leqslant \tilde{\varphi}(t, \xi)\xi \leqslant k_*\lambda\xi^2 < \kappa^*\xi^2, \quad \forall t \geqslant T_1, \ \forall \xi \in \mathbb{R}. \tag{25}$$

Moreover, $\tilde{G}(s) := \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} = G(s)/s$, and so

$$1 + \kappa^* \operatorname{Re} \tilde{G}(s) \geqslant 0, \quad \forall s \in \mathbb{C} \text{ with } \operatorname{Re} s > 0. \tag{26}$$

In view of (24) to (26), it might appear that the Lyapunov analysis of (14) in the proof of statement (i) of Theorem 7 can be replaced by an application of the circle criterion. Since $k(t)\eta(t)$ might be equal to 0 for some values of t (indeed $k\eta$ might vanish on subintervals of $\mathbb{R}_+$) and $\tilde{G}$ has a pole at 0, it is clear that the circle criterion, as presented in standard texts such as (Brockett, 1970; Khalil, 1996; Vidyasagar, 1993; Willems, 1970), does not apply to (24). Moreover, the relaxed circle criterion in Aeyels, Sepulchre, and Peuteman (1998) does not encompass the case wherein $k(t)\eta(t) \to 0$ as $t \to \infty$ (see (69) in Aeyels et al., 1998) and so again does not apply to (24). Less importantly, we mention that the above references assume minimality of the underlying linear state–space system, an assumption which we have not imposed. Nevertheless, a suitable extension of the circle criterion (which does not seem to be available in the literature) can indeed be used to derive statement (i). However, for gain functions k converging to 0, this approach does not yield the claim in statement (ii) of Theorem 7 relating to convergence of order $\exp(-\rho K(\cdot))$, a result which is crucial in the proof of the adaptive result given in Theorem 11.

3.2. Adaptive gain

Whilst Theorem 7 identifies conditions under which the tracking objective is achieved through the use of a prescribed gain function, the resulting control strategy is somewhat unsatisfactory insofar as the gain function is selected a priori: no use is made of the output information from the plant to update the gain. We now consider the possibility of exploiting this output information to generate, by feedback, an appropriate gain function. Let $\mathscr{L}$ denote the class of locally Lipschitz functions $\mathbb{R}_+ \to \mathbb{R}_+$ with value zero only at zero and with linear growth near zero, specifically:

$$\mathscr{L} := \Big\{ f \mid f : \mathbb{R}_+ \to \mathbb{R}_+, f \text{ locally Lipschitz},$$
$$f^{-1}(0) = \{0\}, \ \liminf_{\xi \downarrow 0} \xi^{-1} f(\xi) > 0 \Big\}.$$

Let $\chi \in \mathscr{L}$ and let the gain $k(\cdot)$ be generated by the following adaptation law:

$$k(t) = 1/l(t), \quad \dot{l}(t) = \chi(|r - y(t)|), \quad l(0) = l^0 > 0. \tag{27}$$

This leads to the feedback system

$$\dot{x}(t) = Ax(t) + B\varphi(u(t)), \tag{28a}$$

$$\dot{u}(t) = k(t)[r - \psi(Cx(t) + D\varphi(u(t)))], \tag{28b}$$

$$\dot{k}(t) = -k^2(t)\chi(|r - \psi(Cx(t) + D\varphi(u(t)))|), \tag{28c}$$

$$(x(0), u(0), k(0)) = (x^0, u^0, k^0) \in \mathbb{R}^N \times \mathbb{R} \times (0, \infty). \tag{28d}$$

The following proposition can be obtained by a routine argument.

Proposition 10. *Let* $\Sigma = (A, B, C, D) \in \mathscr{S}$, $(\varphi, \psi) \in \mathscr{N}$, $\chi \in \mathscr{L}$ *and* $r \in \mathbb{R}$.

For each $(x^0, u^0, k^0) \in \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$, *the initial-value problem* (28) *has a unique solution* $(x, u, k) : \mathbb{R}_+ \to \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$.

We now arrive at the main adaptive tracking result.

Theorem 11. *Let* $\Sigma = (A, B, C, D) \in \mathscr{S}$, $(\varphi, \psi) \in \mathscr{N}$, $\chi \in \mathscr{L}$ *and* $r \in \mathscr{R}$. *Assume further that, if* φ *is unbounded, then there exists* $\delta > 0$ *such that* $\chi(\xi) \geqslant \delta\xi$ *for all* $\xi \in \mathbb{R}_+$. *For* *each* $(x^0, u^0, k^0) \in \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$, *the unique solution* $(x, u, k) : \mathbb{R}_+ \to \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$ *of the initial-value problem* (28) *is such that statements* (i) *to* (v) *of Lemma* 6 *hold. Moreover, if* $\Psi^r \cap \bar{\Phi} = \Psi^r \cap \Phi$, $\Psi^r \cap \mathscr{C}(\varphi) = \emptyset$ *and* $r \notin \mathscr{C}(\psi)$, *then the monotone gain k converges to a positive value.*

Proof. Let $(x^0, u^0, k^0) \in \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$ and let $(x, u, k) : \mathbb{R}_+ \to \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$ be the unique solution of the initial-value problem (28). For all $t \in \mathbb{R}_+$, let $K(t) = \int_0^t k$, $l(t) = 1/k(t)$ and $e(t) = r - \psi(Cx(t) + D\varphi(u(t)))$. Since l is a continuous non-decreasing function, either $l(t) \to \infty$ as $t \to \infty$ (Case 1), or $l(t) \to \hat{l} \in (0, \infty)$ as $t \to \infty$ (Case 2).

Case 1: In this case, $k(t) \downarrow 0$ as $t \to \infty$ and by Theorem 7(i), $\varphi(u(t))$ and consequently $x(t)$ converge as $t \to \infty$, and so, in particular, $\varphi \circ u$ and x are bounded functions. Therefore, there exists $\beta > 0$ such that $\dot{l}(t) = \chi(|e(t)|) \leqslant \beta$ for all $t \geqslant 0$, whence

$$k(t) = 1/l(t) \geqslant 1/(l^0 + \beta t) \quad \forall t \geqslant 0 \tag{29}$$

and so $K(t) \to \infty$ as $t \to \infty$. Therefore, by Theorem 7(ii), statements (i)–(v) of Lemma 6 hold.

Case 2: In this case, $k(t) \to \hat{k} := 1/\hat{l} > 0$ as $t \to \infty$ and so $K(t) \to \infty$ as $t \to \infty$. Therefore, in order to conclude that statements (i)–(v) of Lemma 6 hold, it remains to show that $\varphi(u(t))$ converges to a finite limit as $t \to \infty$. It suffices to establish that $e \in L^1(\mathbb{R}_+, \mathbb{R})$, in which case, by (28b), $u(t)$, and hence $\varphi(u(t))$, converges to a finite limit as $t \to \infty$. By boundedness of l and (27),

$$\int_0^\infty \chi(|e(t)|)\,\mathrm{d}t < \infty. \tag{30}$$

First assume φ is unbounded. Then, by hypothesis, $\chi(|e(t)|) \geqslant \delta|e(t)|$ for all $t \geqslant 0$ which, together with (30), implies that $e \in L^1(\mathbb{R}_+, \mathbb{R})$. Next, assume φ is bounded. Then, by the Hurwitz property of A, (28a) and (28b), it follows that e is uniformly continuous. Moreover, (30) holds and so, by Barbălat's Lemma (Khalil, 1996), $\chi(|e(t)|) \to 0$ as $t \to \infty$ which, recalling that $\chi^{-1}(0) = \{0\}$, implies that $e(t) \to 0$ as $t \to \infty$. Since $\liminf_{\xi \downarrow 0} \xi^{-1}\chi(\xi) > 0$, we may infer the existence of $T \in \mathbb{R}_+$ and $\delta > 0$ such that $\chi(|e(t)|) \geqslant \delta|e(t)|$ for all $t \geqslant T$ which, together with (30), implies that $e \in L^1(\mathbb{R}_+, \mathbb{R})$. Therefore, $u(t)$ converges to a finite limit as $t \to \infty$. Hence, the hypotheses of Lemma 6 are satisfied and so statements (i)–(v) of Lemma 6 hold.

Finally, assume that $\Psi^r \cap \bar{\Phi} = \Psi^r \cap \Phi$, $\Psi^r \cap \mathscr{C}(\varphi) = \emptyset$ and $r \notin \mathscr{C}(\psi)$. We will show that the monotone gain k converges to a positive value. Seeking a contradiction, suppose that the monotone function l is unbounded (equivalently, $k(t) \downarrow 0$ as $t \to \infty$). Then the hypothesis of Theorem 7(i) is satisfied: therefore, $\varphi \circ u$ and x are bounded and so (29) holds for some $\beta > 0$. Therefore, $K(t) \to \infty$ as $t \to \infty$ and so, by Theorem 7(ii), $e(t) \to 0$ as $t \to \infty$, and the convergence is of order $\exp(-\rho K(\cdot))$ for some $\rho > 0$. This, together with the fact that χ is locally Lipschitz with $\chi(0) = 0$, implies the existence of a constant $L > 0$ such that

$$\chi(|e(t)|) \leqslant L\exp(-\rho K(t)) \quad \forall t \in \mathbb{R}_+. \tag{31}$$

Combining (28c) and (31), we obtain

$$-\dot{k}(t)/k(t) \leqslant Lk(t)\exp(-\rho K(t)) = L\dot{K}(t)\exp(-\rho K(t)) \quad \forall t \in \mathbb{R}_+,$$

which, on integration, yields

$$\ln(k^0/k(t)) \leqslant (L/\rho)(1 - \exp(-\rho K(t))) \leqslant L/\rho \;\; \forall t \in \mathbb{R}_+$$

contradicting the supposition that $k(t) \downarrow 0$ as $t \to \infty$. □

Remark 12. Finally, we remark that, if the input nonlinearity $\varphi \in \mathscr{M}_L(\lambda_1)$ is bounded, then the assertions of Theorems 7 and 11 remain valid for all output nonlinearities ψ in $\mathscr{M}$ (i.e., the sector condition on ψ can be removed in the case of bounded φ). For the sake of brevity, we omit details.

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