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Discrete-time low-gain control of linear systems with input/output nonlinearities

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SUMMARY

Discrete-time low-gain control strategies are presented for tracking of constant reference signals for finite-dimensional, discrete-time, power-stable, single-input, single-output, linear systems subject to a globally Lipschitz, non-decreasing input nonlinearity and a locally Lipschitz, non-decreasing, affinely sector-bounded output nonlinearity (the conditions on the output nonlinearities may be relaxed if the input nonlinearity is bounded). Both non-adaptive and adaptive gain sequences are considered. In particular, it is shown that applying error feedback using a discrete-time 'integral' controller ensures asymptotic tracking of constant reference signals, provided that (a) the steady-state gain of the linear part of the plant is positive, (b) the positive gain sequence is ultimately sufficiently small and (c) the reference value is feasible in a very natural sense. The classes of input and output nonlinearities under consideration contain standard nonlinearities important in control engineering such as saturation and deadzone. The discrete-time results are applied in the development of sampled-data low-gain control strategies for finite-dimensional, continuous-time, exponentially stable, linear systems with input and output nonlinearities. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: adaptive control; discrete-time systems; input/output nonlinearities; integral control; robust control; sampled-data systems; tracking

1. INTRODUCTION

The present paper continues a sequence [1–4] of recent investigations pertaining to the problem of tracking constant reference signals, by low-gain integral control, for linear (uncertain) systems subject to input and/or output nonlinearities. These investigations extend the well-known principle that closing the loop around an exponentially stable, linear, finite-dimensional, continuous-time, single-input, single-output plant Σ_c, with transfer function G_c, compensated by an

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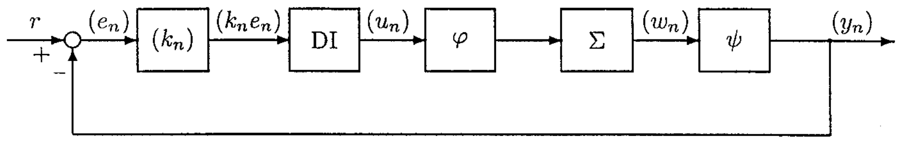

Figure 1. Discrete-time low-gain control with input and output nonlinearities.

integral controller with gain k, will result in a stable closed-loop system which achieves asymptotic tracking of arbitrary constant reference signals, provided that the modulus $|k|$ of the integrator gain k is sufficiently small and $kG_c(0) > 0$ (see References [5–7]). Therefore, if a plant is exponentially stable and if the sign of $G_c(0)$ is known (this information can be obtained from plant step response data), then the problem of tracking by low-gain integral control reduces to that of tuning the gain parameter k. Such a controller design approach ('tuning regulator theory' [5]) has been successfully applied in process control, see, for example, References [8, 9]. Furthermore, the problem of tuning the integrator gain adaptively has been addressed in various papers for finite-dimensional [1, 10–12] and infinite-dimensional systems [2, 3, 13], with input nonlinearities considered in References [2, 3, 11] and both input and output nonlinearities treated in Reference [1]. The purpose of this paper is twofold:

(i) to provide the discrete-time counterparts of the results in Reference [1];
(ii) to apply the discrete-time theory in the derivation of sampled-data control strategies for continuous-time systems.

Whilst the structure of the discrete-time analysis in Section 3 below parallels that of the continuous-time analysis in Reference [1], there are several points where these analyses differ in an essential manner; moreover, the discrete-time low-gain results should be regarded primarily as the main tool in the subsequent derivation, in Section 4, of sampled-data low-gain integral control strategies for a class of continuous-time nonlinear systems which is the main aim of the paper.

With reference to (i) we show that the principle alluded to in the opening paragraph remains true if the plant to be controlled is a discrete-time, power-stable, single-input, single-output, linear system with transfer function G satisfying $G(1) > 0$ and with nonlinearities, φ and ψ, in the input and output channel which belong to a certain class. Figure 1 depicts the control structure schematically, wherein DI denotes the discrete-time 'integral' controller

$$u_{n+1} = u_n + k_n e_n = u_n + k_n(r - \psi(Cx_n + D\varphi(u_n))) = u_n + k_n(r - y_n), \quad u_0 \in \mathbb{R} \tag{1}$$

Precisely, we prove that, if the constant reference signal r is feasible in an entirely natural sense, then there exists a number $k^* > 0$ such that, for all non-decreasing, globally Lipschitz input nonlinearities φ and all non-decreasing, locally Lipschitz and affinely sector-bounded output nonlinearities ψ (the sector-bound assumption on ψ can be removed if φ is bounded) the following holds: for all positive and bounded gain sequences (k_n) (thus in particular for positive constant gains), the output y_n of the closed-loop system converges to r as $n \to \infty$ provided that $\limsup_{n\to\infty} k_n < k^*$ and (k_n) is not of class l^1 (under some additional assumptions on the nonlinearities, results concerning the rate of convergence are derived). Moreover, we show that the following simple adaptation law (a generalization of that introduced in

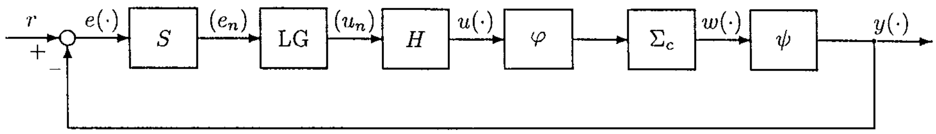

Figure 2. Sampled-data low-gain control with input and output nonlinearities.

Reference [3]):

$$k_n = 1/l_n, \quad l_{n+1} = l_n + \chi(|r - y_n|), \quad l_0 > 0$$

with $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ suitably chosen, produces a gain sequence (k_n) such that the output y_n of the closed-loop system converges to r as $n \to \infty$.

The number k^* is closely related to the supremum κ^* of the set of all numbers $\kappa > 0$ such that the function

$$z \mapsto 1 + \kappa \frac{G(z)}{z-1}$$

is positive-real: for example, if both φ and ψ are globally Lipschitz with Lipschitz constants $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively, then $k^* = \kappa^*/(\lambda_1 \lambda_2)$.

In the sampled-data context (ii), our aim is to establish the efficacy of the control structure in Figure 2, wherein S and H denote standard sampling and hold operations, respectively, and LG represents the (discrete-time) low-gain 'integral' controller given by (1), where the gain sequence $(k_n) \subset \mathbb{R}_+$ is either prescribed or updated adaptively and the nonlinearities in the input and output channel, φ and ψ, have the same properties as above.

We show that tracking of feasible reference values r is achieved by (1) for continuous-time, exponentially stable, single-input, single-output, linear systems Σ_c with transfer function G_c satisfying $G_c(0) > 0$ provided that the positive gain sequence (k_n) is chosen according to the strategies derived in the purely discrete-time setting. The performance of the sampled-data controller is illustrated by means of an example.

2. THE CLASS $\mathcal{N}$ OF INPUT/OUTPUT NONLINEARITIES

For $\lambda > 0$, we introduce the following sets of monotone, non-decreasing nonlinearities:

$$\mathcal{M} := \{ f : \mathbb{R} \to \mathbb{R} \,|\, f \text{ locally Lipschitz and non-decreasing} \}$$

$$\mathcal{M}(\lambda) := \{ f \in \mathcal{M} \,|\, 0 \leqslant (f(\xi) - f(0))\xi \leqslant \lambda \xi^2 \;\; \forall \xi \in \mathbb{R} \}$$

$$\mathcal{M}_L(\lambda) := \{ f \in \mathcal{M} \,|\, f \text{ is globally Lipschitz with Lipschitz constant } \lambda \}$$

Clearly, $\mathcal{M}_L(\lambda) \subset \mathcal{M}(\lambda) \subset \mathcal{M}$.

Remark 2.1

Let $f \in \mathcal{M}$, $v \in \mathbb{R}$ and define $\tilde{f} : \xi \mapsto f(\xi + v) - f(v)$.

(i) If $f \in \mathcal{M}(\lambda)$, then $\tilde{f} \in \mathcal{M}(\tilde{\lambda})$ for some $\tilde{\lambda} > 0$. (ii) If $f \in \mathcal{M}_L(\lambda)$, then $\tilde{f} \in \mathcal{M}_L(\lambda)$.

For locally Lipschitz $f: \mathbb{R} \to \mathbb{R}$, define

$$f^{-}: \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \liminf_{\substack{\theta \to \xi \\ \varepsilon \downarrow 0}} \frac{f(\theta) - f(\theta - \varepsilon)}{\varepsilon}$$

Note that $-(f^{-})(\xi)$ is the Clarke derivative of f at ξ in direction -1 (see Reference [14]). If f is C^1 with derivative f', then $f^{-} \equiv f'$. Let $f \in \mathcal{M}$: a point $\xi \in \mathbb{R}$ is said to be a *critical point* (and $f(\xi)$ is said to be a *critical value*) of f if $f^{-}(\xi) = 0$.‡ We denote, by $\mathscr{C}(f)$, the set of critical values of f. The following two lemmas will be used later. The first result appeared originally in Reference [3], while a proof of the second lemma may be found in Reference [1].

Lemma 2.2

Let $f \in \mathcal{M}_L(\lambda)$ for some $\lambda > 0$. Let $(u_n) \subset \mathbb{R}$ and define $(\delta_n) \subset [0, \lambda]$ by

$$\delta_n := \begin{cases} (f(u_{n+1}) - f(u_n))/(u_{n+1} - u_n), & u_{n+1} \neq u_n \\ \lambda, & u_{n+1} = u_n \end{cases}$$

If (u_n) is convergent and its limit is not a critical point of f, then there exist $\mu > 0$ and $n_0 \in \mathbb{N}$ such that $\delta_n \geqslant \mu$ for all $n \geqslant n_0$.

Lemma 2.3

Let $f: \mathbb{R} \to \mathbb{R}$ be locally Lipschitz. Then

$$f^{-}(\xi) \leqslant \liminf_{h \to 0} \frac{f(\xi + h) - f(\xi)}{h} \quad \forall \xi \in \mathbb{R}$$

Finally, we make precise the class $\mathcal{N}$ (originally introduced in Reference [1]) of input/output nonlinearities considered in this paper: a pair (φ, ψ) is in $\mathcal{N}$ if $\varphi \in \mathcal{M}_L(\lambda_1)$ for some $\lambda_1 > 0$, $\psi \in \mathcal{M}$ and at least one of the following holds: (i) φ is bounded, or (ii) $\psi \in \mathcal{M}(\lambda_2)$ for some $\lambda_2 > 0$. Equivalently,

$$\mathcal{N} := \{(\varphi, \psi) \in \mathcal{M}_L(\lambda_1) \times \mathcal{M} \mid \lambda_1 > 0,\ \varphi \text{ unbounded } \Rightarrow \psi \in \mathcal{M}(\lambda_2) \text{ for some } \lambda_2 > 0\}$$

3. DISCRETE-TIME LOW-GAIN CONTROL

3.1. The class of discrete-time systems and the control objective

We consider a class of finite-dimensional (state space $\mathbb{R}^N$) single-input ($u_n \in \mathbb{R}$), single-output ($y_n \in \mathbb{R}$), discrete-time (time domain $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$), real linear systems $\Sigma = (A, B, C, D)$ having a nonlinearity in the input and output channel:

$$x_{n+1} = Ax_n + B\varphi(u_n), \quad x_0 \in \mathbb{R}^N \tag{2a}$$

$$w_n = Cx_n + D\varphi(u_n) \tag{2b}$$

$$y_n = \psi(w_n) = \psi(Cx_n + D\varphi(u_n)) \tag{2c}$$

‡ If f is merely locally Lipschitz, but not in $\mathcal{M}$, then it would be natural to deem $\xi \in \mathbb{R}$ a critical point of f if 0 belongs to the generalized gradient $\partial f(\xi)$ of f at ξ [14]. We remark that, in the case of $f \in \mathcal{M}$, $0 \leqslant \min\{\partial f(\xi)\} = f^{-}(\xi)$. Therefore, for functions $f \in \mathcal{M}$, the latter concept of a critical point coincides with that given above.

In (2), A is assumed to be power-stable, i.e. each eigenvalue of A has modulus strictly less than one, and (φ, ψ) is assumed to belong to $\mathcal{N}$. Furthermore, the transfer function G of the associated linear system, given by $G(z) = C(zI - A)^{-1}B + D$, is assumed to satisfy $G(1) > 0$. The underlying class of linear discrete-time systems $\Sigma = (A, B, C, D)$ is denoted by

$$\mathcal{S} := \{\Sigma = (A, B, C, D) \mid A \text{ power-stable}, G(1) = C(I - A)^{-1}B + D > 0\}$$

Given $\Sigma = (A, B, C, D) \in \mathcal{S}$ and $(\varphi, \psi) \in \mathcal{N}$, the control objective is to track constant reference values $r \in \mathbb{R}$, i.e. to determine, by feedback, an input sequence $(u_n) \subset \mathbb{R}$ such that, for given $r \in \mathbb{R}$, the output y_n of (2) has the property $y_n \to r$ as $n \to \infty$. If this objective is achievable, then r is necessarily in the closure of $\operatorname{im} \psi$. We will impose a stronger condition, namely,

$$\Psi^r \cap \bar{\Phi} \neq \emptyset, \text{ where } \Psi^r := \{v \in \mathbb{R} \mid \psi(G(1)v) = r\},\ \Phi := \operatorname{im} \varphi, \quad \bar{\Phi} := \operatorname{clos}(\Phi)$$

and refer to the set

$$\mathcal{R} := \{r \in \mathbb{R} \mid \Psi^r \cap \bar{\Phi} \neq \emptyset\}$$

as the set of *feasible reference values*. By a modification of the argument used in establishing the continuous-time result in Proposition 2.5 of Reference [1], it can be shown that $r \in \mathcal{R}$ is close to being a necessary condition for tracking insofar as, if tracking of r is achievable whilst maintaining boundedness of $\varphi(u_n)$, then $r \in \mathcal{R}$. In particular, we have

Proposition 3.1

Let $\Sigma = (A, B, C, D) \in \mathcal{S}$, let $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ and let ψ be continuous and monotone. Let $(u_n) \subset \mathbb{R}$ be such that $\varphi(u_n)$ is bounded. For $x_0 \in \mathbb{R}^N$, let $n \mapsto x_n$ denote the solution of the initial-value problem (2a). Then

$$\lim_{n \to \infty} [\psi(Cx_n + D\varphi(u_n))] = r \ \Rightarrow\ r \in \mathcal{R}$$

To achieve the objective of tracking feasible reference values $r \in \mathcal{R}$, we shall investigate discrete-time 'integral' control action of the form (1) where $(k_n) \subset \mathbb{R}$ is a gain sequence which is either prescribed or determined adaptively.

3.2. *Prescribed gain*

Henceforth, we assume that the gain sequence (k_n) satisfies

$$(k_n) \in \mathcal{G} := \{g \mid g : \mathbb{N}_0 \to (0, \infty) \text{ bounded}\}$$

An application of the control law (1) leads to the following system of nonlinear equations:

$$x_{n+1} = Ax_n + B\varphi(u_n), \quad x_0 \in \mathbb{R}^N \tag{3a}$$

$$u_{n+1} = u_n + k_n(r - \psi(Cx_n + D\varphi(u_n))), \quad u_0 \in \mathbb{R} \tag{3b}$$

If G is the transfer function of a system $\Sigma = (A, B, C, D) \in \mathcal{S}$, then it is readily shown that

$$1 + \kappa \operatorname{Re} \frac{G(z)}{z - 1} \geqslant 0 \quad \forall z \in \mathbb{C} \text{ with } |z| > 1 \tag{4}$$

for all sufficiently small $\kappa > 0$, see [15, Theorem 2.5]. Define

$$\kappa^* := \sup\{\kappa > 0 \mid (4) \text{ holds}\} \tag{5}$$

Lemma 3.2

Assume that $\Sigma = (A, B, C, D) \in \mathscr{S}$ and let $\Delta > 1/\kappa^*$. Then there exists $P \in \mathbb{R}^{N \times N}$ such that $P = P^{\mathrm{T}} > 0$ and

$$\begin{bmatrix} A^{\mathrm{T}}PA - P & A^{\mathrm{T}}P(I-A)^{-1}B + C^{\mathrm{T}} \\ ((I-A)^{-1}B)^{\mathrm{T}}PA + C & ((I-A)^{-1}B)^{\mathrm{T}}P(I-A)^{-1}B - 2\Delta + G(1) \end{bmatrix} < 0$$

Proof. Define

$$H(z) := -C(zI - A)^{-1}(I-A)^{-1}B - G(1)/2 = \begin{cases} \dfrac{G(z) - G(1)}{z-1} - G(1)/2, & z \neq 1 \\ G'(1) - G(1)/2, & z = 1 \end{cases}$$

Let $\tilde{\Delta} \in (1/\kappa^*, \Delta)$. Clearly, H is holomorphic in $\{z \in \mathbb{C} \mid |z| > \alpha\}$ for some $\alpha \in (0,1)$ and the positive-real condition (4) holds for $\kappa = 1/\tilde{\Delta}$. Notice that $\mathrm{Re}(1 - \mathrm{e}^{\mathrm{i}\theta})^{-1} = 1/2$, $\theta \in (0, 2\pi)$. Therefore, $\tilde{\Delta} + \mathrm{Re}\, H(\mathrm{e}^{\mathrm{i}\theta}) \geqslant 0$ for all $\theta \in [0, 2\pi)$ and so $\Delta + \mathrm{Re}\, H(\mathrm{e}^{\mathrm{i}\theta}) > 0$ for all $\theta \in [0, 2\pi)$. Writing

$$M := \begin{bmatrix} 0 & C^{\mathrm{T}} \\ C & -2\Delta + G(1) \end{bmatrix}$$

we may conclude that, for all $\theta \in [0, 2\pi)$,

$$\begin{bmatrix} (\mathrm{e}^{\mathrm{i}\theta}I - A)^{-1}(I-A)^{-1}B \\ 1 \end{bmatrix}^* M \begin{bmatrix} (\mathrm{e}^{\mathrm{i}\theta}I - A)^{-1}(I-A)^{-1}B \\ 1 \end{bmatrix} = -2(\mathrm{Re}\, H(\mathrm{e}^{\mathrm{i}\theta}) + \Delta) < 0$$

and the assertion of the lemma immediately follows from a variant of the Kalman–Yakubovich–Popov Lemma as given by Rantzer [16, Theorem 2]. □

Before presenting the main results, we describe a convenient family of projection operators. Specifically, with each $p \in [0, \infty]$, we associate an operator $\Pi_p : \mathscr{M} \to \mathscr{M}$, with the property $\Pi_p \circ \Pi_p = \Pi_p$ (hence the terminology projection operator), defined as follows:

$$\text{if } p < \infty, \text{ then } \Pi_p f : \xi \mapsto \begin{cases} f(-p), & \xi < -p \\ f(\xi), & |\xi| \leqslant p \\ f(p), & \xi > p \end{cases} ; \quad \text{if } p = \infty, \text{ then } \Pi_p f = \Pi_\infty f := f$$

Furthermore, we denote the l^∞-gain of $\Sigma = (A, B, C, D) \in \mathscr{S}$ by Γ_Σ, and so

$$0 < G(1) \leqslant \Gamma_\Sigma = \sum_{j=0}^{\infty} |CA^jB| + |D| < \infty \tag{6}$$

Theorem 3.3

Let $\Sigma = (A, B, C, D) \in \mathscr{S}$, $(\varphi, \psi) \in \mathscr{N}$, $r \in \mathscr{R}$ and $(k_n) \in \mathscr{G}$. Define

$$p^* := \Gamma_\Sigma \sup_{\xi \in \Phi} |\xi| \in (0, \infty]$$

Let $(x_0, u_0) \in \mathbb{R}^N \times \mathbb{R}$, let $n \mapsto (x_n, u_n)$ be the solution of the initial-value problem (3).

(A) Assume that $K_n := \sum_{j=0}^{n} k_j \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} \varphi(u_n)$ exists and is finite. Then

(a1) $\lim_{n\to\infty} \varphi(u_n) =: \varphi^r \in \Psi^r \cap \bar{\Phi}$,

(a2) $\lim_{n\to\infty} x_n = (I - A)^{-1}B\varphi^r$,

(a3) $\lim_{n\to\infty} y_n = r$, where $y_n = \psi(Cx_n + D\varphi(u_n))$,

(a4) if $\Psi^r \cap \bar{\Phi} = \Psi^r \cap \Phi$, then $\lim_{n\to\infty} \operatorname{dist}(u_n, \varphi^{-1}(\varphi^r)) = 0$,
(a5) if $\Psi^r \cap \bar{\Phi} = \Psi^r \cap \operatorname{int}(\Phi)$, then (u_n) is bounded.

(B) There exists $k^* > 0$, independent of (x_0, u_0) and (k_n), such that the following hold:
(b1) If $\limsup_{n\to\infty} k_n < k^*$, then $\lim_{n\to\infty} \varphi(u_n)$ exists and is finite.
(b2) If $\limsup_{n\to\infty} k_n < k^*$ and $K_n := \sum_{j=0}^{n} k_j \to \infty$ as $n \to \infty$, then statements (a1) to (a5) hold. If, in addition, the nonlinearities φ and ψ are such that $\Psi^r \cap \bar{\Phi} = \Psi^r \cap \Phi$, $\Psi^r \cap \mathscr{C}(\varphi) = \emptyset$ and $r \notin \mathscr{C}(\psi)$, then the convergence in (a1)–(a3) is of order ρ^{-K_n} for some $\rho > 1$.

(C) Let $\lambda_1 > 0$ be a Lipschitz constant for φ. If $\Pi_{(p^*+\delta)}\psi \in \mathscr{M}_L(\lambda_2)$ for some $\delta > 0$ and some $\lambda_2 > 0$, then (b1) and (b2) are valid with $k^* = \kappa^*/(\lambda_1\lambda_2)$, where κ^* is given by (5).

Remark 3.4

(i) Part (A) of Theorem 3.3 has limited practical significance in its own right but is introduced as a convenient artifact that plays a central role in the proof of part (b2) and in the analysis of the adaptive control proposed in Section 3.3 below.

(ii) Part (B) of Theorem 3.3 contains the main tracking result in the non-adaptive situation. It asserts the existence of a positive constant $k^* > 0$ (independent of $(k_n) \in \mathscr{G}$ and the initial data (x_0, u_0)) such that, if $(k_n) \in \mathscr{G}$ is chosen to be ultimately strictly bounded above by k^* and, in addition, (k_n) is *not* of class l^1, then statements (a1)–(a5) hold, in particular, tracking of arbitrary feasible reference values is achieved; furthermore, if the sets $\Psi^r \cap \bar{\Phi}$ and $\Psi^r \cap \Phi$ coincide, if Ψ^r contains no critical values of φ and if r is not a critical value of ψ, then (b2) provides an estimate of the rate of convergence in (a1)–(a3). An immediate consequence of part (B) of Theorem 3.3 is the following: if $(k_n) \in \mathscr{G}$ is chosen such that, as $n \to \infty$, k_n tends to zero *sufficiently slowly* (in the sense that $(k_n) \notin l^1(\mathbb{R}_+)$), then the tracking objective is achieved.

(iii) Under the assumption that $\Pi_{(p^*+\delta)}\psi \in \mathscr{M}_L(\lambda_2)$ for some $\delta, \lambda_2 > 0$ (which, since $(\varphi, \psi) \in \mathscr{N}$, is equivalent to requiring that $\psi \in \mathscr{M}_L(\lambda_2)$ for some $\lambda_2 > 0$ if φ is unbounded), part (C) of Theorem 3.3 provides a formula for k^*, namely $k^* = \kappa^*/(\lambda_1\lambda_2)$. If $p^* < \infty$ (or, equivalently, if φ is bounded) and if ψ is continuously differentiable in $(p^* - \varepsilon, p^* + \varepsilon)$ and in $(-p^* - \varepsilon, -p^* + \varepsilon)$ for some $\varepsilon > 0$, then part (C) of Theorem 3.3 remains true with $\delta = 0$. If sufficient system information is available *a priori* in order to compute the quantities κ^*, λ_1 and λ_2, then, by part C of Theorem 3.3, we may infer that the tracking objective is achievable by *constant* gain $k \in (0, \kappa^*/(\lambda_1\lambda_2))$ and, moreover, under additional conditions on the nonlinearities (namely, $\Psi^r \cap \bar{\Phi} = \Psi^r \cap \Phi$, $\Psi^r \cap \mathscr{C}(\varphi) = \emptyset$ and $r \notin \mathscr{C}(\psi)$), the convergence is of exponential order ρ^{-nk}.

Proof of Theorem 3.3. (A) By hypothesis, there exists $\varphi^r \in \mathbb{R}$ such that $\lim_{n\to\infty} \varphi(u_n) = \varphi^r$ which, together with the power-stability of A, implies $\lim_{n\to\infty} w_n = G(1)\varphi^r$. Evidently, $\varphi^r \in \bar{\Phi}$ and so, to establish (a1), it suffices to show that $\varphi^r \in \Psi^r$. Seeking a contradiction, suppose that $\varphi^r \notin \Psi^r$. This implies that $\theta := (r - \psi(G(1)\varphi^r))/2 \neq 0$. Using continuity of ψ, we obtain for sufficiently large $n_0 \in \mathbb{N}_0$

$$|y_n - \psi(G(1)\varphi^r)| \leqslant |\theta| \quad \forall n \geqslant n_0$$

As a consequence, and noticing that $u_{n+1} - u_n = k_n(r - y_n) = k_n(2\theta - y_n + \psi(G(1)\varphi^r))$, we have

$$\theta(u_{n+1} - u_n) \geqslant \theta^2 k_n \quad \forall n \geqslant n_0$$

which, on summation, yields

$$\theta[u_{n+1+n_0} - u_{n_0}] \geqslant \theta^2[K_{n+n_0} - K_{n_0-1}] \quad \forall n \in \mathbb{N}_0$$

Since $K_n \to \infty$ as $n \to \infty$, we conclude that $\theta u_n \to \infty$ as $n \to \infty$, hence

$$\lim_{n\to\infty} \varphi(u_n) = \varphi^r = \begin{cases} \sup \bar{\Phi} & \text{if } \theta > 0 \\ \inf \bar{\Phi} & \text{if } \theta < 0 \end{cases} \tag{7}$$

Let $\varphi^* \in \Psi^r \cap \bar{\Phi}$. Then, by (7), $\theta\varphi^* \leqslant \theta\varphi^r$, and, by definition of Ψ^r, $\psi(G(1)\varphi^*) = r$. This, together with the monotonicity of ψ, yields the contradiction

$$\theta\psi(G(1)\varphi^r) \geqslant \theta\psi(G(1)\varphi^*) = \theta r = 2\theta^2 + \theta\psi(G(1)\varphi^r)$$

Therefore, we may conclude $\varphi^r \in \Psi^r \cap \bar{\Phi}$ which is statement (a1). Statement (a2) follows from (a1) and the power-stability of A. Statement (a3) is a consequence of (a1), (a2) and continuity of ψ. Next, we establish statement (a4). Assume $\Psi^r \cap \bar{\Phi} = \Psi^r \cap \Phi$ which, together with (a1), implies the existence of $\xi^* \in \mathbb{R}$ such that $\varphi^r = \varphi(\xi^*)$. Seeking a contradiction, suppose that $\operatorname{dist}(u_n, \varphi^{-1}(\varphi^r)) \not\to 0$ as $n \to \infty$. Then there exist $\varepsilon > 0$ and a subsequence (u_{n_k}) of (u_n) such that

$$\operatorname{dist}(u_{n_k}, \varphi^{-1}(\varphi^r)) \geqslant \varepsilon \quad \forall k \tag{8}$$

If the sequence (u_{n_k}) is bounded, we may assume without loss of generality that it converges to a finite limit u_∞. By continuity, $\varphi(u_\infty) = \varphi^r$ and so $u_\infty \in \varphi^{-1}(\varphi^r)$. This contradicts (8). Therefore, we may assume that (u_{n_k}) is unbounded. Extracting a subsequence if necessary, we may then assume that either $u_{n_k} \to \infty$ or $u_{n_k} \to -\infty$ as $k \to \infty$: if the former holds, then $u_{n_k} > \xi^*$ for all k sufficiently large; if the latter holds, then $u_{n_k} < \xi^*$ for all k sufficiently large. In either case, by monotonicity of φ it follows that $\varphi(u_{n_k}) = \varphi(\xi^*) = \varphi^r$ for all k sufficiently large. Clearly, this contradicts (8) and so statement (a4) must hold.

Now, assume that $\Psi^r \cap \bar{\Phi} = \Psi^r \cap \operatorname{int}(\Phi)$ and, for contradiction, suppose that (u_n) is unbounded. Then there exists a sequence $(n_k) \subset \mathbb{N}$ with $n_k \to \infty$ and $|u_{n_k}| \to \infty$ as $k \to \infty$. By monotonicity of φ and (a1), it then follows that either $\varphi^r = \sup \Phi$ or $\varphi^r = \inf \Phi$, contradicting the fact that $\varphi^r \in \Psi^r \cap \operatorname{int}(\Phi) \subset \operatorname{int}(\Phi)$. Therefore, (u_n) is bounded. This completes the proof of part (A).

(B) Let $\varepsilon > 0$. Then power-stability of A implies that

$$|w_n| = |Cx_n + D\varphi(u_n)| \leqslant |CA^n x_0| + \Gamma_\Sigma \sup_{\xi \in \Phi} |\xi| \leqslant p^* + \varepsilon =: q \in (0, \infty]$$

for all n sufficiently large. Hence, there exists $n_0 \in \mathbb{N}_0$ such that

$$\psi(w_n) = (\Pi_q \psi)(w_n) \quad \forall n \geqslant n_0 \tag{9}$$

Define $\hat{\Psi}^r := \{\xi \in \mathbb{R} \mid (\Pi_q\psi)(G(1)\xi) = r\}$. We claim that

$$\hat{\Psi}^r \cap \bar{\Phi} = \Psi^r \cap \bar{\Phi} \tag{10}$$

To see this, note that, using (6), we have $G(1)|\xi| \leqslant p^*$ for all $\xi \in \bar{\Phi}$: therefore, $(\Pi_q\psi)(G(1)\xi) = \psi(G(1)\xi)$ for all $\xi \in \bar{\Phi}$ and so $\Psi^r \cap \bar{\Phi} = \hat{\Psi}^r \cap \bar{\Phi}$. This establishes (10).

Since $r \in \mathscr{R}$, we may conclude from (10) that $\hat{\Psi}^r \neq \emptyset$. Let $\varphi^* \in \hat{\Psi}^r$; since $\Pi_q\psi \in \mathscr{M}(\lambda_2)$ for some $\lambda_2 > 0$, the function

$$\tilde{\psi}: \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto (\Pi_q\psi)(\xi + G(1)\varphi^*) - r \tag{11}$$

is in $\mathscr{M}(\tilde{\lambda}_2)$ for some $\tilde{\lambda}_2 > 0$ (cf. Remark 2.1(i)). Define $\gamma : \mathbb{R} \to \mathbb{R}_+$ by

$$\gamma(\xi) = \begin{cases} \tilde{\psi}(\xi)/\xi, & \xi \neq 0 \\ \liminf_{h \to 0} \psi(h)/h, & \xi = 0 \end{cases} \tag{12}$$

Observe that γ is lower semicontinuous with $0 \leqslant \gamma(\xi) \leqslant \tilde{\lambda}_2$ and $\tilde{\psi}(\xi) = \gamma(\xi)\xi$ for all $\xi \in \mathbb{R}$.

Let κ^* be given by (5), let $\lambda_1 > 0$ be a Lipschitz constant for φ and define

$$\lambda := \lambda_1 \tilde{\lambda}_2, \qquad k^* := \kappa^*/\lambda$$

Let $(k_n) \in \mathscr{G}$ be such that $\limsup_{n\to\infty} k_n < k^*$. Choose $k_* > 0$ such that

$$\limsup_{n \to \infty} k_n < k_* < k^* \tag{13}$$

By Lemma 3.2 there exists a $P \in \mathbb{R}^{N \times N}$, with $P = P^{\mathrm{T}} > 0$, such that

$$\Lambda := \begin{bmatrix} A^{\mathrm{T}}PA - P & A^{\mathrm{T}}P(I-A)^{-1}B + C^{\mathrm{T}} \\ ((I-A)^{-1}B)^{\mathrm{T}}PA + C & ((I-A)^{-1}B)^{\mathrm{T}}P(I-A)^{-1}B - 2/(k_*\lambda) + G(1) \end{bmatrix} < 0$$

Introduce new variables

$$z_n := x_n - (I-A)^{-1}B\varphi(u_n), \quad v_n := \varphi(u_n) - \varphi^* \quad \forall n \geqslant \mathbb{N}_0$$

and notice that, in terms of the new variables, w_n is given by

$$w_n = Cx_n + D\varphi(u_n) = Cz_n + G(1)(v_n + \varphi^*) = \tilde{w}_n + G(1)\varphi^*, \quad \text{where } \tilde{w}_n := Cz_n + G(1)v_n$$

Define the sequence $(\delta_n) \subset [0, \lambda_1]$ (as in Lemma 2.2) by

$$\delta_n := \begin{cases} (\varphi(u_{n+1}) - \varphi(u_n))/(u_{n+1} - u_n), & u_{n+1} \neq u_n \\ \lambda_1, & u_{n+1} = u_n \end{cases} \tag{14}$$

Using (9) and (11), an easy calculation yields

$$z_{n+1} = Az_n + \delta_n k_n (I-A)^{-1}B\tilde{\psi}(\tilde{w}_n) \quad \forall n \geqslant n_0$$

$$v_{n+1} = v_n - \delta_n k_n \tilde{\psi}(\tilde{w}_n) \quad \forall n \geqslant n_0$$

Equivalently,

$$z_{n+1} = Az_n + (I-A)^{-1}B\zeta_n \quad \forall n \geqslant n_0 \tag{15a}$$

$$v_{n+1} = v_n - \zeta_n \quad \forall n \geqslant n_0 \tag{15b}$$

where,

$$\zeta_n := k_n \eta_n \tilde{w}_n \quad \text{with} \quad 0 \leqslant \eta_n := \delta_n \gamma(\tilde{w}_n) \leqslant \lambda_1 \tilde{\lambda}_2 = \lambda \quad \forall n \geqslant n_0 \tag{16}$$

We shall investigate asymptotic properties of (z_n, v_n) using a Lyapunov approach. Define the sequence (V_n) by

$$V_n = \langle z_n, Pz_n \rangle + G(1)v_n^2$$

Then, using (15) and (16)

$$\begin{aligned} V_{n+1} - V_n &= \langle z_n, (A^{\mathrm{T}}PA - P)z_n \rangle + 2((I-A)^{-1}B)^{\mathrm{T}}PAz_n\zeta_n \\ &\quad + [((I-A)^{-1}B)^{\mathrm{T}}P(I-A)^{-1}B + G(1)]\zeta_n^2 - 2G(1)v_n\zeta_n \\ &= \langle [z_n^{\mathrm{T}}, \zeta_n]^{\mathrm{T}}, \Lambda[z_n^{\mathrm{T}}, \zeta_n]^{\mathrm{T}} \rangle + (2/(k_*\lambda))\zeta_n^2 - 2[Cz_n + G(1)v_n]\zeta_n \\ &\leqslant -\alpha_0[\|z_n\|^2 + \zeta_n^2] - 2k_n\eta_n[1 - k_n\eta_n/(k_*\lambda)]\tilde{w}_n^2 \quad \forall n \geqslant n_0 \end{aligned}$$

where $\alpha_0 = 1/\|\Lambda^{-1}\|$. Invoking (13) and (16), we see that there exists $n_1 \geqslant n_0$ such that

$$\sup_{n \geqslant n_1} k_n\eta_n < k_*\lambda \tag{17}$$

Consequently, there exists $\alpha_1 > 0$ such that

$$V_{n+1} - V_n \leqslant -\alpha_0[\|z_n\|^2 + \zeta_n^2] - \alpha_1 k_n\eta_n\tilde{w}_n^2 \quad \forall n \geqslant n_1 \tag{18}$$

Introduce non-negative sequences (g_n) and (W_n^β) (parametrized by $\beta > 1$) defined by

$$g_n := \sum_{j=0}^{n-1} k_j\eta_j \quad \text{and} \quad W_n^\beta := \beta^{2g_n}V_n \quad \forall n \geqslant n_2 := n_1 + 1$$

Recalling that $\tilde{w}_n = Cz_n + G(1)v_n$, it follows that, for some constant $\alpha_2 > 0$,

$$V_n \leqslant \alpha_2[\|z_n\|^2 + \tilde{w}_n^2] \quad \forall n \in \mathbb{N}_0$$

Invoking (18), we have for all $n \geqslant n_2$

$$\begin{aligned} W_{n+1}^\beta - W_n^\beta &= \beta^{2g_{n+1}}(V_{n+1} - V_n) + (\beta^{2g_{n+1}} - \beta^{2g_n})V_n = \beta^{2g_{n+1}}[V_{n+1} - V_n + (1 - 1/\beta^{2k_n\eta_n})V_n] \\ &\leqslant -\beta^{2g_{n+1}}[(\alpha_0 - (1 - 1/\beta^{2k_n\eta_n})\alpha_2)\|z_n\|^2 + (\alpha_1 k_n\eta_n - (1 - 1/\beta^{2k_n\eta_n})\alpha_2)\tilde{w}_n^2] \end{aligned}$$

By (13) and (17),

$$(1 - 1/\beta^{2k_n\eta_n}) \leqslant 2k_n\eta_n \ln\beta \leqslant 2k^*\lambda \ln\beta \quad \forall n \geqslant n_2$$

and so

$$W_{n+1}^\beta - W_n^\beta \leqslant -\beta^{2g_{n+1}}[(\alpha_0 - 2\alpha_2 k^*\lambda\ln\beta)\|z_n\|^2 + k_n\eta_n(\alpha_1 - 2\alpha_2\ln\beta)\tilde{w}_n^2]$$

Writing $\alpha = \min\{\alpha_0, \alpha_1\}/2$ and fixing $\beta > 1$ sufficiently close to 1 we have

$$W_{n+1}^\beta - W_n^\beta \leqslant -\alpha\beta^{2g_{n+1}}[\|z_n\|^2 + k_n\eta_n\tilde{w}_n^2] \quad \forall n \geqslant n_2$$

implying boundedness of (W_n^β), and hence boundedness of $(\beta^{g_{n+1}}z_n)$ and $(\beta^{g_{n+1}}v_n)$; moreover,

$$\sum_{n=n_2}^{\infty} \beta^{2g_{n+1}}k_n\eta_n\tilde{w}_n^2 < \infty \tag{19}$$

Note that

$$\sum_{n=1}^{\infty} \beta^{-2g_{n+1}}k_n\eta_n = \sum_{n=1}^{\infty} \beta^{-2g_{n+1}}(g_{n+1} - g_n) \leqslant \sum_{n=1}^{\infty} \int_{g_n}^{g_{n+1}} \beta^{-2t}\,\mathrm{d}t < \infty$$

which, when combined with (19), yields

$$\sum_{n=n_2}^{\infty} k_n\eta_n|\tilde{w}_n| = \sum_{n=n_2}^{\infty} \beta^{-g_{n+1}}\sqrt{k_n\eta_n}\sqrt{k_n\eta_n}\,\beta^{g_{n+1}}|\tilde{w}_n|$$
$$\leqslant \frac{1}{2}\sum_{n=n_2}^{\infty} \beta^{-2g_{n+1}}k_n\eta_n + \frac{1}{2}\sum_{n=n_2}^{\infty} \beta^{2g_{n+1}}k_n\eta_n\tilde{w}_n^2 < \infty$$

and so $(\zeta_n) = (k_n\eta_n\tilde{w}_n) \in l^1(\mathbb{R})$. Invoking (15b), we may infer that (v_n) is convergent: therefore, $\lim_{n\to\infty}\varphi(u_n)$ exists and is finite. This establishes (b1). Furthermore, we have established the following fact which we record for later reference:

$$\lim_{n\to\infty} k_n = 0 \Rightarrow (\zeta_n) = (k_n\eta_n\tilde{w}_n) \in l^1(\mathbb{R}) \tag{20}$$

To prove (b2), assume that $\limsup_{n\to\infty} k_n < k^*$ and $K_n \to \infty$ as $n \to \infty$. Then it immediately follows from (A) and (b1) that (a1)–(a5) hold; in particular, there exists $\varphi^r \in \Psi^r \cap \bar{\Phi}$ such that $\lim_{n\to\infty}\varphi(u_n) = \varphi^r$. In the above argument establishing statement (b1), φ^* is an arbitrary element of $\Psi^r \cap \bar{\Phi} = \hat{\Psi}^r \cap \bar{\Phi}$: henceforth, we posit $\varphi^* = \varphi^r$.

Now assume that $\Psi^r \cap \bar{\Phi} = \Psi^r \cap \Phi$, $\Psi^r \cap \mathscr{C}(\varphi) = \emptyset$ and $r \notin \mathscr{C}(\psi)$: we will show that the convergence in (a1)–(a3) is of order ρ^{-K_n} for some $\rho > 1$. By hypothesis, $\varphi^r \in \Phi$ and $\varphi^r \notin \mathscr{C}(\varphi)$. Thus, by monotonicity of φ, the preimage $\varphi^{-1}(\varphi^r)$ is a singleton $\{u^r\}$. Moreover, by (a4), $u_n \to u^r$ as $n \to \infty$. By Lemma 2.2, there exist $\mu_0 > 0$ and $n_3 \geqslant n_2$ such that

$$0 < \mu_0 \leqslant \delta_n \quad \forall n \geqslant n_3$$

Since by assumption $r \notin \mathscr{C}(\psi)$, we have that

$$\tilde{\psi}^-(0) = (\Pi_q\psi)^-(G(1)\varphi^r) = \psi^-(G(1)\varphi^r) > 0$$

and so, invoking lower semicontinuity of γ, Lemma 2.3 and the fact that $\lim_{n\to\infty}\tilde{w}_n = 0$, there exists $n_4 \geqslant n_3$ such that

$$\gamma(\tilde{w}_n) \geqslant \gamma(0)/2 \geqslant \tilde{\psi}^-(0)/2 =: \mu_1 > 0 \quad \forall n \geqslant n_4$$

Therefore, writing $\mu := \mu_0\mu_1 > 0$, we have $0 < \mu \leqslant \eta_n = \delta_n\gamma(\tilde{w}_n)$ for all $n \geqslant n_4$, and so

$$g_{n+1} = \sum_{j=0}^{n} k_j\eta_j \geqslant \sum_{j=0}^{n_4-1} k_j\eta_j + \mu\sum_{j=n_4}^{n} k_j = \text{const.} + \mu K_n \quad \forall n \geqslant n_4$$

Define $\rho := \beta^\mu > 1$. Since the sequences $(\beta^{g_{n+1}}z_n)$ and $(\beta^{g_{n+1}}v_n)$ are bounded, we may now conclude boundedness of $(\rho^{K_n}z_n)$ and $(\rho^{K_n}v_n)$ (whence boundedness of $(\rho^{K_n}\tilde{w}_n)$). Convergence of order ρ^{-K_n} in (a1) and (a2) immediately follows. Recall that $\tilde{\psi} \in \mathscr{M}(\tilde{\lambda}_2)$ and $\tilde{\psi}(0) = 0$. Thus

$$|e_n| = |r - y_n| = |r - \psi(w_n)| = |\tilde{\psi}(\tilde{w}_n)| \leqslant \tilde{\lambda}_2|\tilde{w}_n| \quad \forall n \in \mathbb{N}_0$$

and so the sequence $(\rho^{K_n}e_n)$ is bounded, implying convergence of order ρ^{-K_n} in (a3).

(C) Let $\lambda_1 > 0$ be a Lipschitz constant for φ. By hypothesis, $\Pi_{(p^*+\delta)}\psi \in \mathscr{M}_L(\lambda_2)$ for some $\delta > 0$ and some $\lambda_2 > 0$. By Remark 2.1(ii) it follows that, for any fixed $r \in \mathscr{R}$, the function

$$\tilde{\psi}: \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \Pi_{(p^*+\delta)}\psi(\xi + G(1)\varphi^r) - r$$

is also in $\mathscr{M}_L(\lambda_2)$. The argument used in the proof of part (B) above applies *mutatis mutandis* (specifically, on replacing ε by δ, φ^* by φ^r and $\tilde{\lambda}_2$ by λ_2) to conclude that (b1) and (b2) hold with $k^* = \kappa^*/(\lambda_1\lambda_2)$. □

3.3. Adaptive gain

Whilst Theorem 3.3(B) (see also Remark 3.4(ii)) identifies conditions under which the tracking objective is achieved through the use of a prescribed gain sequence, the resulting control strategy is somewhat unsatisfactory insofar as the gain sequence is selected *a priori*: no use is made of the output information from the plant to update the gain. We now consider the possibility of exploiting this output information to generate, by feedback, an appropriate gain sequence. Let $\mathscr{L}$ denote the class of locally Lipschitz functions mapping from $\mathbb{R}_+$ into $\mathbb{R}_+$ with value zero only at zero and which satisfy a particular growth condition near zero, specifically:

$$\mathscr{L} := \{ f: \mathbb{R}_+ \to \mathbb{R}_+ \mid f \text{ locally Lipschitz}, f^{-1}(0) = \{0\}, \liminf_{\xi \downarrow 0} \xi^{-1} f(\xi) > 0\}$$

Let $\chi \in \mathscr{L}$ and let the gain sequence (k_n) be generated by the following adaptation law:

$$k_n = (l_n)^{-1}, \quad l_{n+1} = l_n + \chi(|r - y_n|), \quad l_0 > 0 \tag{21}$$

This leads to the feedback system

$$x_{n+1} = Ax_n + B\varphi(u_n), \quad x_0 \in \mathbb{R}^N \tag{22a}$$

$$u_{n+1} = u_n + l_n^{-l}(r - \psi(Cx_n + D\varphi(u_n))), \quad u_0 \in \mathbb{R} \tag{22b}$$

$$l_{n+1} = l_n + \chi(|r - \psi(Cx_n + D\varphi(u_n))|), \quad l_0 \in (0, \infty) \tag{22c}$$

Corollary 3.5

Let $\Sigma = (A, B, C, D) \in \mathscr{S}$, $(\varphi, \psi) \in \mathscr{N}$, $\chi \in \mathscr{L}$ and $r \in \mathscr{R}$. Assume furthermore that, if φ is unbounded, then there exists $\delta > 0$ such that $\chi(\xi) \geqslant \delta\xi$ for all $\xi \in \mathbb{R}_+$.

For each $(x_0, u_0, l_0) \in \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$, the solution $n \mapsto (x_n, u_n, l_n)$ of the initial-value problem (22) is such that statements (a1)–(a5) of Theorem 3.3 hold. Moreover, if $\Psi^r \cap \bar{\Phi} = \Psi^r \cap \Phi$ and $\Psi^r \cap \mathscr{C}(\varphi) = \emptyset$, then the non-increasing gain sequence $(k_n) = (l_n^{-1})$ converges to a positive value.

Proof. Let $(x_0, u_0, l_0) \in \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$. Since (l_n) is non-decreasing, either $l_n \to \infty$ as $n \to \infty$ (*Case* 1), or $l_n \to \hat{l} \in (0, \infty)$ as $n \to \infty$ (*Case* 2). We consider these two cases separately.

Case 1: In this case, $k_n \downarrow 0$ as $n \to \infty$ and by Theorem 3.3(b1), $(\varphi(u_n))$ and consequently (x_n) converge as $n \to \infty$, and so are bounded sequences. Therefore, there exists $\alpha > 0$ such that

$$k_n = l_n^{-1} \geqslant (l_0 + \alpha n)^{-1} \quad \forall n \in \mathbb{N}_0 \tag{23}$$

and so $K_n \to \infty$ as $n \to \infty$. Therefore, by Theorem 3.3(b2), statements (a1)–(a5) of Theorem 3.3 hold.

Case 2: In this case, $k_n \to \hat{k} := 1/\hat{l} > 0$ as $n \to \infty$ and so $K_n \to \infty$ as $n \to \infty$. In order to conclude that statements (a1)–(a5) of Theorem 3.3 hold, it suffices to show (by part (A) of Theorem 3.3) that (u_n) converges to a finite limit as $n \to \infty$: in view of (22b), we will establish the latter convergence by showing that $(e_n) := (r - y_n)$ is of class $l^1(\mathbb{R})$. Note that, using (22c) and boundedness of (l_n), we may conclude that

$$\sum_{n=0}^{\infty} \chi(|e_n|) < \infty \tag{24}$$

First assume that φ is unbounded. Then, by hypothesis, $\chi(|e_n|) \geqslant \delta |e_n|$ for all $n \in \mathbb{N}_0$ which, together with (24), implies that $(e_n) \in l^1(\mathbb{R})$. Next, assume φ is bounded. Then, by the power-stability of A, it follows that (e_n) is bounded. By (24) we have $\chi(|e_n|) \to 0$ as $n \to \infty$ which, recalling that $\chi^{-1}(0) = \{0\}$ and invoking boundedness of (e_n), implies that $|e_n| \to 0$ as $n \to \infty$. Since $\liminf_{\xi \downarrow 0} \xi^{-1}\chi(\xi) > 0$, we may infer the existence of $n_0 \in \mathbb{N}_0$ and $\delta > 0$ such that $\chi(|e_n|) \geqslant \delta |e_n|$ for all $n \geqslant n_0$ which, together with (24), implies that $(e_n) \in l^1(\mathbb{R})$. Therefore, (u_n) converges to a finite limit as $n \to \infty$. Hence, the hypotheses of part (A) of Theorem 3.3 are satisfied and so statements (a1) to (a5) of Theorem 3.3 hold.

Finally, assume that $\Psi^r \cap \bar{\Phi} = \Psi^r \cap \Phi$, $\Psi^r \cap \mathscr{C}(\varphi) = \emptyset$. We will show that the monotone gain sequence (k_n) converges to a positive value. By boundedness of (e_n) and the local Lipschitz property of $\chi \in \mathscr{L}$, there exists $\hat{\lambda} > 0$ such that $\chi(|e_n|) \leqslant \hat{\lambda}|e_n|$ for all $n \in \mathbb{N}_0$. Seeking a contradiction, suppose that $k_n \downarrow 0$ as $n \to \infty$. Noting that

$$k_{n+1} = k_n(1 + k_n\chi(|e_n|))^{-1} = k_0\left(\prod_{j=0}^{n} (1 + k_j\chi(|e_j|))\right)^{-1} \quad \forall n \in \mathbb{N}_0$$

it then follows that

$$\hat{\lambda}\sum_{j=0}^{n} k_j|e_j| \geqslant \sum_{j=0}^{n} k_j\chi(|e_j|) \geqslant \sum_{j=0}^{n} \log(1 + k_j\chi(|e_j|)) = \log\left(\prod_{j=0}^{n} (1 + k_j\chi(|e_j|))\right) \to \infty$$

as $n \to \infty$. Therefore, $(k_n e_n) \notin l^1(\mathbb{R})$. Recalling that $\varphi(u_n) \to \varphi^r \in \Psi^r \cap \bar{\Phi}$ and setting $\varphi^* = \varphi^r$, define $\tilde{\psi}$, γ and $(\tilde{w}_n)$ as in the proof of Theorem 3.3(b1). Observe that $-e_n = \tilde{\psi}(\tilde{w}_n) = \gamma(\tilde{w}_n)\tilde{w}_n$ for all n sufficiently large. Since $(k_n e_n) \notin l^1(\mathbb{R})$, it follows that $(k_n\gamma(\tilde{w}_n)\tilde{w}_n) \notin l^1(\mathbb{R})$. By hypothesis, $\Psi^r \cap \mathscr{C}(\varphi) = \emptyset$ and so $\varphi^r \notin \mathscr{C}(\varphi)$. Therefore, by monotonicity of φ, $\varphi^{-1}(\varphi^r)$ is a singleton: it follows that (u_n) is convergent. By Lemma 2.2, there exist $\mu_0 > 0$ and $n_0 \in \mathbb{N}_0$ such that $0 < \mu_0 \leqslant \delta_n$ for all $n \geqslant n_0$, where (δ_n) is defined as in (14). It now follows that $(k_n\delta_n\gamma(\tilde{w}_n)\tilde{w}_n) = (k_n\eta_n\tilde{w}_n) \notin l^1(\mathbb{R})$, which contradicts (20). Therefore, (k_n) converges to a positive limit. □

4. SAMPLED-DATA LOW-GAIN CONTROL OF CONTINUOUS-TIME SYSTEMS

In this section we apply the results of Section 3 to solve the continuous-time low-gain tracking problem, by sampled-data 'integral' control, for the class of systems introduced below.

4.1. The class of continuous-time systems and the tracking objective

Tracking results will be derived for a class of finite-dimensional (state space $\mathbb{R}^N$) single-input ($u(t) \in \mathbb{R}$), single-output ($y(t) \in \mathbb{R}$), continuous-time (time domain $\mathbb{R}_+ := [0, \infty)$), real linear systems $\Sigma_c = (A_c, B_c, C_c, D_c)$ having a nonlinearity in the input and output channel:

$$\dot{x} = A_c x + B_c\varphi(u), \qquad x_0 := x(0) \in \mathbb{R}^N \tag{25a}$$

$$y = \psi(C_c x + D_c\varphi(u)) \tag{25b}$$

In (25), it is assumed that A_c is Hurwitz, i.e. each eigenvalue of A_c has negative real part and $(\varphi, \psi) \in \mathscr{N}$. Furthermore, the transfer function G_c, given by $G_c(s) = C_c(sI - A_c)^{-1}B_c + D_c$, is assumed to satisfy $G_c(0) > 0$. The underlying class of real continuous-time linear systems $\Sigma_c = (A_c, B_c, C_c, D_c)$ is denoted

$$\mathscr{S}_c := \{\Sigma_c = (A_c, B_c, C_c, D_c) \mid A_c \text{ Hurwitz, } G_c(0) = D_c - C_c A_c^{-1} B_c > 0\}$$

Given $\Sigma_c = (A_c, B_c, C_c, D_c) \in \mathscr{S}_c$, $(\varphi, \psi) \in \mathscr{N}$ and $\tau > 0$, the objective is to determine a sequence $(u_n) \subset \mathbb{R}$ such that, for given $r \in \mathbb{R}$, the output y of (25), resulting from the input u given by

$$u(t) = u_n \quad \text{for } t \in [n\tau, (n+1)\tau),\ n \in \mathbb{N}_0 \tag{26}$$

has the property $y(t) \to r$ as $t \to \infty$. In the sequel, we require $r \in \mathscr{R}_c$, where

$$\mathscr{R}_c := \{r \in \mathbb{R} \,|\, \Psi_c^r \cap \bar{\Phi} \neq \emptyset\} \text{ with } \Psi_c^r := \{v \in \mathbb{R} \,|\, \psi(G_c(0)v) = r\}, \quad \Phi := \operatorname{im} \varphi, \quad \bar{\Phi} := \operatorname{clos}(\Phi)$$

Again, it may be shown that $r \in \mathscr{R}_c$ is close to being necessary for tracking insofar as, if tracking of r is achievable whilst maintaining boundedness of $\varphi(u)$, then $r \in \mathscr{R}_c$ (see Reference [1]).

Given a sequence $(u_n) \subset \mathbb{R}$, we define a continuous-time signal $u(\cdot)$ by the standard hold operation (26), where $\tau > 0$ denotes the sampling period. If this signal is applied to the continous-time system (25), then, defining

$$x_n := x(n\tau) \quad \text{and} \quad y_n := y(n\tau) \quad \forall n \in \mathbb{N}_0 \tag{27}$$

it follows that

$$x_{n+1} = Ax_n + B\varphi(u_n), \quad x_0 := x(0) \in \mathbb{R}^N, \quad y_n = \psi(Cx_n + D\varphi(u_n)) \tag{28}$$

where

$$A = e^{A_c\tau},\ B = (e^{A_c\tau} - I)A_c^{-1}B_c,\ C = C_c \text{ and } D = D_c \tag{29}$$

Since A_c is Hurwitz, the matrix A is power stable for arbitrary $\tau > 0$. Furthermore, the transfer function G of $\Sigma = (A, B, C, D)$ satisfies $G(1) = G_c(0)$ implying that $\Psi^r = \Psi_c^r$ and $\mathscr{R} = \mathscr{R}_c$, where $\Psi^r = \{v \in \mathbb{R} \,|\, \psi(G(1)v) = r\}$ and $\mathscr{R}$ denotes the set of feasible reference values for the discretization (28) of (25).

4.2. Prescribed gain

We first treat the case of a prescribed gain sequence $(k_n) \in \mathscr{G}$. For this purpose, let $\tau > 0$ be arbitrary and consider the following sampled-data low-gain controller for (25):

$$u(t) = u_n, \quad \text{for } t \in [n\tau, (n+1)\tau),\ n \in \mathbb{N}_0 \tag{30a}$$

$$y_n = y(n\tau), \quad n \in \mathbb{N}_0 \tag{30b}$$

$$u_{n+1} = u_n + k_n(r - y_n), \quad u_0 \in \mathbb{R} \tag{30c}$$

Theorem 4.1

Let $\Sigma_c = (A_c, B_c, C_c, D_c) \in \mathscr{S}_c$, $(\varphi, \psi) \in \mathscr{N}$, $\tau > 0$ and $r \in \mathscr{R}_c$. Define

$$p^* := \Gamma_\Sigma \sup_{\xi \in \Phi} |\xi| \in (0, \infty]$$

where $\Sigma = (A, B, C, D)$ denotes the discretization (29) of $\Sigma_c = (A_c, B_c, C_c, D_c) \in \mathscr{S}_c$ and Γ_Σ is given by (6).

(A) There exists $k^* > 0$, independent of (x_0, u_0) and $(k_n) \in \mathscr{G}$, such that for all $(k_n) \in \mathscr{G}$ with the properties $\limsup_{n\to\infty} k_n < k^*$ and $K_n := \sum_{j=0}^n k_j \to \infty$ as $n \to \infty$, and for every $(x_0, u_0) \in \mathbb{R}^N \times \mathbb{R}$ the following hold for the solution $(x, u): \mathbb{R}_+ \to \mathbb{R}^N \times \mathbb{R}$ of the closed-loop system given by (25) and (30):

(i) $\lim_{t\to\infty}\varphi(u(t)) =: \varphi^r \in \Psi^r_c \cap \bar{\Phi}$,
(ii) $\lim_{t\to\infty} x(t) = -A_c^{-1}B_c\varphi^r$,
(iii) $\lim_{t\to\infty} y(t) = r$, where $y(t) = \psi(C_c x(t) + D_c\varphi(u(t)))$,
(iv) if $\Psi^r_c \cap \bar{\Phi} = \Psi^r_c \cap \Phi$, then $\lim_{t\to\infty}$ dist $(u(t), \varphi^{-1}(\varphi^r)) = 0$,
(v) if $\Psi^r_c \cap \bar{\Phi} = \Psi^r_c \cap \mathrm{int}(\Phi)$, then $u(\cdot)$ is bounded,
(vi) if $\Psi^r_c \cap \bar{\Phi} = \Psi^r_c \cap \Phi$, $\Psi^r_c \cap \mathscr{C} = \emptyset$ and $r \notin \mathscr{C}(\psi)$, then the convergence in (i) to (iii) is of order $\exp(-\alpha K(t))$ for some $\alpha > 0$, where $K: \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $K(t) = K_n$ for $t \in [n\tau, (n+1)\tau)$.

(B) Let $\lambda_1 > 0$ be a Lipschitz constant for φ. If $\Pi_{(p^*+\delta)}\psi \in \mathscr{M}_L(\lambda_2)$ for some $\delta > 0$ and $\lambda_2 > 0$, then statement (A) holds with $k^* = \kappa^*/(\lambda_1\lambda_2)$, where κ^* is given by (5).

Remarks 4.2

(i) An immediate consequence of Theorem 4.1 is the following: if $(k_n) \in \mathscr{G}$ is chosen such that, as $n \to \infty$, k_n tends to zero sufficiently slowly in the sense that $(k_n) \notin l^1(\mathbb{R}_+)$, then the tracking objective is achieved.

(ii) Let $\lambda_1 > 0$ be a Lipschitz constant for φ. If $\Pi_{(p^*+\delta)}\psi \in \mathscr{M}_L(\lambda_2)$ for some $\delta > 0$ and some $\lambda_2 > 0$, we may infer that the tracking objective is achievable by a *constant* gain sequence (k_n) with $k_n = k \in (0, \kappa^*/(\lambda_1\lambda_2))$ for all $n \in \mathbb{N}_0$ and, moreover, if the extra assumptions of statement (vi) are satisfied, the convergence is of exponential order.

Proof of Theorem 4.1. (A) It follows from the previous subsection that x_n, y_n (given by (27)) and u_n (generated by (30c)) satisfy (28) with (A, B, C, D) given by (29). Moreover, A is power stable and $G(1) = G_c(0) > 0$. Hence $\Sigma = (A, B, C, D) \in \mathscr{S}$. Since by assumption $K_n := \sum_{j=0}^n k_j \to \infty$ as $n \to \infty$, part (B) of Theorem 3.3 furnishes the existence of a number $k^* > 0$ such that statements (a1)–(a5) of Theorem 3.3 hold for $\Sigma = (A, B, C, D)$, whenever $\limsup_{n\to\infty} k_n < k^*$. In particular, $\lim_{n\to\infty}\varphi(u_n) = \varphi^r \in \Psi^r \cap \bar{\Phi} = \Psi^r_c \cap \bar{\Phi}$. Since the continuous-time signal u is given by (30a), assertions (i), (iv) and (v) follow. Assertion (ii) is a consequence of (i) and the Hurwitz property of A_c. Finally, assertion (iii) follows from (i), (ii) and continuity of ψ.

To show (vi), note that by Theorem 3.3(b2), there exist $M > 0$ and $\rho > 1$ such that

$$|\varphi(u_n) - \varphi^r| \leqslant M\rho^{-K_n} \quad \text{and} \quad \|x_n - x^r\| \leqslant M\rho^{-K_n}$$

for all $n \in \mathbb{N}_0$, where $x^r = -A_c^{-1}B_c\varphi^r$. For all $t \in [n\tau, (n+1)\tau)$, $\varphi(u(t)) = \varphi(u_n)$ and so $|\varphi(u(t)) - \varphi^r| \leqslant M\rho^{-K_n} = M\exp(-\alpha K_n)$, where $\alpha = \ln\rho > 0$. Therefore, the convergence in (i) is of order $\exp(-\alpha K(t))$. Define $\zeta(t) := x(t) - x^r$. Then $\zeta(\cdot)$ satisfies

$$\dot{\zeta}(t) = A_c\zeta(t) + B_c(\varphi(u(t)) - \varphi^r)$$

and a routine estimate involving the variations of parameters formula shows that

$$\|\zeta(t)\| \leqslant \max_{0\leqslant s\leqslant\tau}\|e^{A_c s}\| M e^{-\alpha K_n} + \tau \max_{0\leqslant s\leqslant r}\|e^{A_c s}\|\,\|B_c\| M e^{-\alpha K_n}, \quad t \in [n\tau, (n+1)\tau)$$

which proves that convergence in (ii) is of order $\exp(-\alpha K(t))$. The claim that convergence in (iii) is also of order $\exp(-\alpha K(t))$ follows by an argument analogous to that used in the proof of the latter part of (b2) in Theorem 3.3.

(B) This follows from part (C) of Theorem 3.3. □

4.3. Adaptive gain

In this subsection, our goal is to establish the efficacy of the adaptation law (21) to generate an appropriate gain sequence in the sampled-data setting. Hence, consider the following adaptive sampled-data low-gain controller for (25):

$$u(t) = u_n \quad \text{for } t \in [n\tau, (n+1)\tau),\ n \in \mathbb{N}_0 \tag{31a}$$

$$y_n = y(n\tau), \quad n \in \mathbb{N}_0 \tag{31b}$$

$$u_{n+1} = u_n + l_n^{-1}(r - y_n), \quad u_0 \in \mathbb{R} \tag{31c}$$

$$l_{n+1} = l_n + \chi(|r - y_n|), \quad l_0 \in (0, \infty) \tag{31d}$$

Theorem 4.3

Let $\Sigma_c = (A_c, B_c, C_c, D_c) \in \mathscr{S}_c$, $(\varphi, \psi) \in \mathscr{N}$, $\tau > 0$, $\chi \in \mathscr{L}$ and $r \in \mathscr{R}_c$. Assume furthermore that, if φ is unbounded, then there exists $\delta > 0$ such that $\chi(\xi) \geqslant \delta\xi$ for all $\xi \in \mathbb{R}_+$. For all $(x_0, u_0, l_0) \in \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$, the solution $(x(\cdot), u(\cdot), (l_n))$ of the closed-loop system given by (25) and (31) is such that statements (i)–(v) of Theorem 4.1 hold. Moreover, if $\Psi_c^r \cap \bar{\Phi} = \Psi_c^r \cap \Phi$ and $\Psi_c^r \cap \mathscr{C}(\varphi) = \emptyset$, then the monotone gain sequence $(k_n) = (l_n^{-1})$ converges to a positive value.

Proof. First recall that the discretization $\Sigma = (A, B, C, D)$ of $\Sigma_c = (A_c, B_c, C_c, D_c) \in \mathscr{S}_c$ is of class $\mathscr{S}$. Let $(x_0, u_0, l_0) \in \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$. It follows from Section 4.1 that x_n (given by (27)) and (u_n, l_n) (given by (31c), (31d)) satisfy (22). By Corollary 3.5, statements (a1)–(a5) of Theorem 3.3 hold for (x_n, u_n) and $y_n = \psi(Cx_n + D\varphi(u_n))$. From this it follows as in the proof of Theorem 4.1 that statements (i)–(v) of Theorem 4.1 hold. If, moreover, $\Psi_c^r \cap \bar{\Phi} = \Psi_c^r \cap \Phi$ and $\Psi_c^r \cap \mathscr{C}(\varphi) = \emptyset$, then, again as an immediate consequence of Corollary 3.5, we conclude that the gain sequence $(k_n) = (l_n^{-1})$ converges to a positive limit. □

4.4. Example

Consider the second-order system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -ax_2 - bx_1 + \varphi(u), \quad w = x_1, \quad y = \psi(w) = w^3$$

where $a, b > 0$ and the input nonlinearity $\varphi \in \mathscr{M}_L(1)$ is of saturation type, defined as follows:

$$u \mapsto \varphi(u) := \begin{cases} \operatorname{sgn}(u), & |u| > 1 \\ u, & |u| \leqslant 1 \end{cases}$$

and so $\bar{\Phi} = \Phi = [-1, 1]$. The transfer function G_c of the associated linear system Σ_c is given by

$$G_c(s) = \frac{1}{s^2 + as + b} \quad \text{with } G_c(0) = 1/b > 0$$

Since $\Psi_c^r = \{v \in \mathbb{R} \mid \psi(G_c(0)v) = r\} = \{br^{1/3}\}$, we have $\Psi_c^r \cap \bar{\Phi} \neq \emptyset$ if and only if $-b^{-3} \leqslant r \leqslant b^{-3}$. Thus, the set $\mathscr{R}_c$ of feasible reference values is given by $\mathscr{R}_c = [-b^{-3}, b^{-3}]$.

Let $\chi \in \mathscr{L}$. By Theorem 4.3, it follows that the adaptive sampled-data controller (31) achieves the tracking objective for each feasible reference value $r \in \mathscr{R}_c$. Moreover, if r is in the interior of $\mathscr{R}_c$ (so that $\Psi_c^r \cap \mathscr{C}(\varphi) = \emptyset$), then the adapting gain sequence (k_n) converges to a positive value. For purposes of illustration, let $a = 2$, $b = 1$ (in which case $\mathscr{R}_c = [-1, 1]$), $\tau = 1$ and let $\chi = \varphi|_{\mathbb{R}_+}$ (the restriction of the saturation function to $\mathbb{R}_+$). For initial data $(x_1(0), x_2(0), u_0, l_0) = (0, 0, 0, 1)$ and

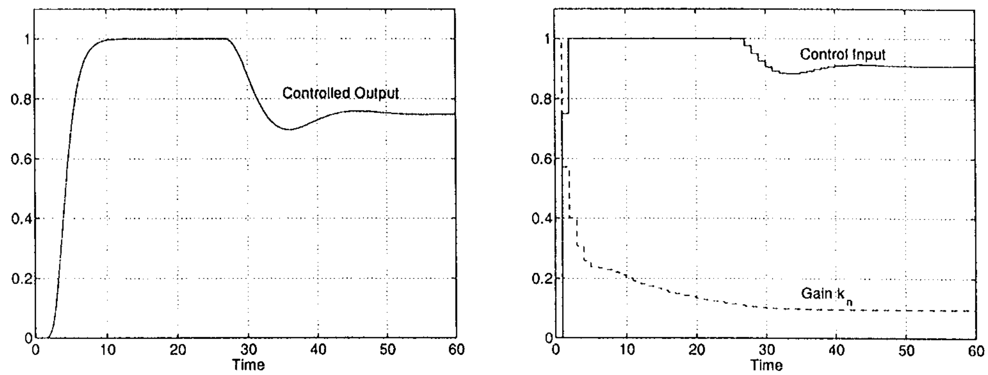

Figure 3. Performance under adaptive sampled-data control.

the feasible reference value $r = 0.75$, Figure 3 (generated using SIMULINK Simulation Software under MATLAB) depicts the system performance under adaptive sampled-data control. The convergence of the gain to a positive limiting value is evident.

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