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Low-Gain Integral Control of Well-Posed Linear Infinite-Dimensional Systems with Input and Output Nonlinearities[1]

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Time-varying low-gain integral control strategies are presented for asymptotic tracking of constant reference signals in the context of exponentially stable, well-posed, linear, infinite-dimensional, single-input–single-output, systems—subject to globally Lipschitz, nondecreasing input and output nonlinearities. It is shown that applying error feedback using an integral controller ensures that the tracking error is small in a certain sense, provided that (a) the steady-state gain of the linear part of the system is positive, (b) the reference value r is feasible in an entirely natural sense, and (c) the positive gain function $t \mapsto k(t)$ is ultimately sufficiently small and not of class L^1. Under a weak restriction on the initial data it is shown that (a), (b), and (c) ensure asymptotic tracking. If, additionally, the impulse response of the linear part of the system is a finite signed Borel measure, the global Lipschitz assumption on the output nonlinearity may be considerably relaxed. © 2001 Academic Press

Key Words: infinite-dimensional well-posed systems; input–output nonlinearities; integral control; robust control; saturation; time-varying control; tracking.

1. INTRODUCTION

This present paper extends the line of work on low-gain control of infinite-dimensional systems initiated by recent contributions [9–16]. Consider the feedback configuration shown in Figure 1, where $\mathbf{\Sigma}$ denotes

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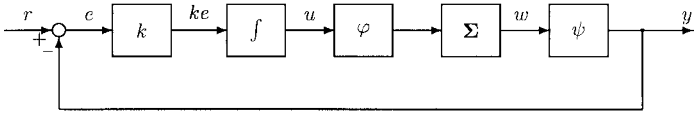

FIG. 1. Low-gain control with input and output nonlinearities.

an exponentially stable, time-invariant, single-input–single-output system with transfer function $\mathbf{G}$, φ and ψ denote static nonlinearities, and r denotes the reference value. In [6] the authors developed a theory of low-gain integral control for the feedback system in Figure 1 under the assumption that $\boldsymbol{\Sigma}$ is finite dimensional. In particular, the following principle was established in [6]. If $\mathbf{G}(0) > 0$ and if the reference value r is feasible in an entirely natural sense, then there exists a number $k^* > 0$ such that, for all nondecreasing, globally Lipschitz input nonlinearities φ and all nondecreasing, locally Lipschitz and affinely sector-bounded output nonlinearities ψ (the sector-bound assumption on ψ can be removed if φ is bounded), the following statement holds: For all positive, bounded, and continuous integrator gains $k(\cdot)$ (thus, in particular, for positive constant gains), the output $y(t)$ of the closed-loop system depicted in Figure 1 converges to r as $t \to \infty$ provided that $\limsup_{t\to\infty} k(t) < k^*$ and k is not of class L^1.

The number k^* is closely related to the supremum κ^* of the set of all numbers $\kappa > 0$ such that the function

$$s \mapsto 1 + \kappa \frac{\mathbf{G}(s)}{s}$$

is positive real: for example, if both φ and ψ are globally Lipschitz with Lipschitz constants $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively, then $k^* = \kappa^*/(\lambda_1\lambda_2)$. Formulae for the computation of κ^* have appeared in [13] for various classes of systems.

Under the above assumptions on $\boldsymbol{\Sigma}$, φ, and ψ, the problem of tracking feasible constant reference signals r by low-gain integral control reduces to that of appropriately choosing the gain function k. In a purely linear and finite-dimensional context, such a controller design approach ("tuning regulator theory"; see [5, 17, 21]) is well known and has been successfully applied in process control (see, for example, [3, 18]).

In the present paper we prove that, with suitable modifications, the above principle for finite-dimensional $\boldsymbol{\Sigma}$ continues to hold when $\boldsymbol{\Sigma}$ is an exponentially stable, well-posed, infinite-dimensional system. We remark that the class of well-posed, linear, infinite-dimensional systems is the largest class of distributed parameter systems for which satisfactory state-space and frequency-domain theories are currently available (see Curtain and Weiss [4], Salamon [23, 24], Staffans [25], and Weiss [27–30]). Well-posed systems allow for considerable unboundedness of the control and observation operators, and include most distributed parameter systems and all time-delay systems (retarded and neutral) that are of interest in applications. The essence of our approach is to invoke a particular coordinate transformation and perform a Lyapunov-type analysis of the transformed systems in which an infinite-dimensional generalization of the Kalman–Yakubovich lemma plays a central role. We emphasize that the finite-dimensional Lyapunov argument developed in [6] does not generalize to the infinite-dimensional case and so the stability analysis given here is fundamentally different from that in [6].

Finally, we mention that [6] contains applications of the above principle to the problem of tuning the integrator gain adaptively (for other contributions to adaptive tuning regulator theory, see [2, 11, 15, 16, 19, 20]). Generalizations of the finite-dimensional adaptive results in [6] to the infinite-dimensional case are currently under investigation and will be reported elsewhere.

2. PRELIMINARIES ON WELL-POSED LINEAR SYSTEMS

First, some notation. For a Hilbert space H and $\tau \geq 0$, $\mathbf{R}_\tau$ denotes the operator of right shift by τ on $L^p_{\mathrm{loc}}(\mathbb{R}_+, H)$, where $\mathbb{R}_+ := [0, \infty)$; the truncation operator $\mathbf{P}_\tau : L^p_{\mathrm{loc}}(\mathbb{R}_+, H) \to L^p(\mathbb{R}_+, H)$ is given by $(\mathbf{P}_\tau u)(t) = u(t)$ if $t \in [0, \tau]$ and $(\mathbf{P}_\tau u)(t) = 0$ otherwise; for $\alpha \in \mathbb{R}$, we define the exponentially weighted L^p space $L^p_\alpha(\mathbb{R}_+, H) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}_+, H) \mid f(\cdot)\exp(-\alpha\cdot) \in L^p(\mathbb{R}_+, H)\}$; for $t_1 < t_2$, $W^{1,2}([t_1, t_2], H)$ denotes the space of all functions $f : [t_1, t_2] \to H$ for which there exists $g \in L^2([t_1, t_2], H)$ such that $f(t) = f(t_1) + \int_{t_1}^t g(s)\, ds$ for all $t \in [t_1, t_2]$; for an unbounded interval $J \subset \mathbb{R}$, $W^{1,2}_{\mathrm{loc}}(J, H)$ denotes the space of all functions $f : J \to H$ such that for all $t_1, t_2 \in J$ with $t_1 < t_2$, the restriction of f to $[t_1, t_2]$ belongs to $W^{1,2}([t_1, t_2], H)$; $\mathscr{B}(H_1, H_2)$ denotes the space of bounded linear operators from a Hilbert space H_1 to a Hilbert space H_2; for $\alpha \in \mathbb{R}$, $\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha\}$; the Laplace transform is denoted by $\mathfrak{L}$; the Fourier transform is denoted by $\mathfrak{F}$; the space of finite signed Borel measures on $\mathbb{R}_+$ is denoted by $\mathfrak{M}$; $H^2(\mathbb{C}_\alpha)$ denotes the Hardy–Lebesgue space of square-integrable holomorphic functions defined on $\mathbb{C}_\alpha$ and $H^\infty(\mathbb{C}_\alpha)$ denotes the space of bounded and holomorphic functions on $\mathbb{C}_\alpha$; finally a.a. (respectively, a.e.) is the abbreviation for almost all (respectively, almost everywhere).

Remark 2.1. Each element of $L^1_{\mathrm{loc}}(\mathbb{R}_+, H)$ is an equivalence class of locally integrable functions that coincide almost everywhere on $\mathbb{R}_+$. Therefore, if $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+, H)$, it is not clear what we mean when we say that $f(t)$ converges to $h \in H$ as $t \to \infty$. In this paper, we adopt the following convention: for $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+, H)$ and $h \in H$ we say that $\lim_{t\to\infty} f(t) = h$ if there exists a representative $\tilde{f}: \mathbb{R}_+ \to H$ of f such that $\lim_{t\to\infty} \tilde{f}(t) = h$ in the usual sense. This convention guarantees that a given $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+, H)$ has at most one limit as $t \to \infty$ and the usual algebra of limits is valid.

The concept of a well-posed, linear, infinite-dimensional system was first defined by Salamon [23, 24]. The following equivalent definition is due to Weiss [30] (see also the forthcoming monograph by Staffans [25]).

DEFINITION 2.2. Let U, X, and Y be real Hilbert spaces. A *well-posed* linear system with *state space* X, *input space* U, and *output space* Y is a quadruple $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$, where

$\mathbf{T} = (\mathbf{T}_t)_{t\geq 0}$ is a C_0 semigroup of bounded linear operators on X;

$\mathbf{\Phi} = (\mathbf{\Phi}_t)_{t\geq 0}$ is a family of bounded linear operators from $L^2(\mathbb{R}_+, U)$ to X such that, for all τ, $t \geq 0$,

$$\mathbf{\Phi}_{\tau+t}(\mathbf{P}_\tau u + \mathbf{R}_\tau v) = \mathbf{T}_t \mathbf{\Phi}_\tau u + \mathbf{\Phi}_t v, \qquad \forall\, u, v \in L^2(\mathbb{R}_+, U);$$

$\mathbf{\Psi} = (\mathbf{\Psi}_t)_{t\geq 0}$ is a family of bounded linear operators from X to $L^2(\mathbb{R}_+, Y)$ such that $\mathbf{\Psi}_0 = 0$ and, for all τ, $t \geq 0$,

$$\mathbf{\Psi}_{\tau+t} x^0 = \mathbf{P}_\tau \mathbf{\Psi}_\tau x^0 + \mathbf{R}_\tau \mathbf{\Psi}_t \mathbf{T}_\tau x^0, \qquad \forall\, x^0 \in X;$$

$\mathbf{F} = (\mathbf{F}_t)_{t\geq 0}$ is a family of bounded linear operators from $L^2(\mathbb{R}_+, U)$ to $L^2(\mathbb{R}_+, Y)$ such that $\mathbf{F}_0 = 0$ and, for all τ, $t \geq 0$,

$$\mathbf{F}_{\tau+t}(\mathbf{P}_\tau u + \mathbf{R}_\tau v) = \mathbf{P}_\tau \mathbf{F}_\tau u + \mathbf{R}_\tau(\mathbf{\Psi}_t \mathbf{\Phi}_\tau u + \mathbf{F}_t v), \qquad \forall\, u, v \in L^2(\mathbb{R}_+, U).$$

For an input $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ and initial state $x^0 \in X$, the associated state function $x \in C(\mathbb{R}_+, X)$ and output $w \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ of $\mathbf{\Sigma}$ are given by

$$x(t) = \mathbf{T}_t x^0 + \mathbf{\Phi}_t \mathbf{P}_t u \quad \text{for all } t \geq 0, \tag{1a}$$

$$\mathbf{P}_t w = \mathbf{\Psi}_t x^0 + \mathbf{F}_t \mathbf{P}_t u \quad \text{for a.a. } t \geq 0. \tag{1b}$$

$\mathbf{\Sigma}$ is said to be *exponentially stable* if the semigroup $\mathbf{T}$ is exponentially stable:

$$\omega(\mathbf{T}) := \lim_{t\to\infty} \frac{1}{t} \ln\|\mathbf{T}_t\| < 0.$$

$\mathbf{\Psi}_\infty$ and $\mathbf{F}_\infty$ denote the unique operators from X to $L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ and from $L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ to $L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$, respectively, satisfying

$$\mathbf{\Psi}_\tau = \mathbf{P}_\tau \mathbf{\Psi}_\infty, \qquad \mathbf{F}_\tau = \mathbf{P}_\tau \mathbf{F}_\infty, \qquad \forall\, \tau \geq 0. \tag{2}$$

If $\mathbf{\Sigma}$ is exponentially stable, then the operators $\mathbf{\Phi}_t$ and $\mathbf{\Psi}_t$ are uniformly bounded, $\mathbf{\Psi}_\infty$ is a bounded operator from X to $L^2(\mathbb{R}_+, Y)$, and $\mathbf{F}_\infty$ maps $L^2(\mathbb{R}_+, U)$ boundedly to $L^2(\mathbb{R}_+, Y)$. Since $\mathbf{P}_\tau \mathbf{F}_\infty = \mathbf{P}_\tau \mathbf{F}_\infty \mathbf{P}_\tau$ for all $\tau \geq 0$, $\mathbf{F}_\infty$ is a *causal* operator.

Weiss [27] established that if $\alpha > \omega(T)$ and $u \in L^2_\alpha(\mathbb{R}_+, U)$, then $\mathbf{F}_\infty u \in L^2_\alpha(\mathbb{R}_+, Y)$ and there exists a unique holomorphic $\mathbf{G} : \mathbb{C}_{\omega(\mathbf{T})} \to \mathscr{B}(U, Y)$ such that

$$\mathbf{G}(s)(\mathfrak{L}u)(s) = [\mathfrak{L}(\mathbf{F}_\infty u)](s), \qquad \forall\, s \in \mathbb{C}_\alpha, \tag{3}$$

where $\mathfrak{L}$ denotes Laplace transform. In particular, $\mathbf{G}$ is bounded on $\mathbb{C}_\alpha$ for all $\alpha > \omega(\mathbf{T})$. The function $\mathbf{G}$ is called the *transfer function* of $\boldsymbol{\Sigma}$.

$\boldsymbol{\Sigma}$ and its transfer function $\mathbf{G}$ are said to be *regular* if there exists a linear operator D such that

$$\lim_{\substack{s\to\infty \\ s\in\mathbb{R}}} \mathbf{G}(s)u = Du, \qquad \forall\, u \in U,$$

in which case, by the principle of uniform boundedness, it follows that $D \in \mathscr{B}(U, Y)$. The operator D is called the *feedthrough operator* of $\boldsymbol{\Sigma}$.

The generator of $\mathbf{T}$ is denoted by A with domain $\operatorname{dom}(A)$. Let X_1 be the space $\operatorname{dom}(A)$ endowed with the graph norm. The norm on X is denoted by $\|\cdot\|$, whilst $\|\cdot\|_1$ denotes the graph norm. Let X_{-1} be the completion of X with respect to the norm $\|x\|_{-1} = \|(\lambda I - A)^{-1}x\|$, where $\lambda \in \varrho(A)$ is any fixed element of the resolvent set $\varrho(A)$ of A. Then $X_1 \subset X \subset X_{-1}$ and the canonical injections are bounded and dense. The semigroup $\mathbf{T}$ can be restricted to a C_0 semigroup on X_1 and extended to a C_0 semigroup on X_{-1}. The exponential growth constant is the same on all three spaces. The generator on X_{-1} is an extension of A to X (which is bounded as an operator from X to X_{-1}). We use the same symbol $\mathbf{T}$ (respectively, A) for the original semigroup (respectively, its generator) and the associated restrictions and extensions. With this convention, we may write $A \in \mathscr{B}(X, X_{-1})$. Considered as a generator on X_{-1}, the domain of A is X.

By a representation theorem due to Salamon [23] (see also Weiss [28, 29]), there exist unique operators $B \in \mathscr{B}(U, X_{-1})$ and $C \in \mathscr{B}(X_{\mathbf{1}}, Y)$ (the *control operator* and the *observation operator* of $\boldsymbol{\Sigma}$, respectively) such that for all $t \geq 0$, $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, and $x^0 \in X_1$,

$$\boldsymbol{\Phi}_t \mathbf{P}_t u = \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau \quad \text{and} \quad (\boldsymbol{\Psi}_\infty x^0)(t) = C\mathbf{T}_t x^0.$$

B is said to be *bounded* if it is so as a map from the input space U to the state space X; otherwise, B is said to be *unbounded*. C is said to be *bounded* if it can be extended continuously to X; otherwise, C is said to be *unbounded*. The operators A, B, C (and D, in the regular case) are called the *generating operators* of $\boldsymbol{\Sigma}$.

The *Lebesgue extension* of C was adopted in [29] and is defined by

$$C_L x^0 = \lim_{t\to 0} C \frac{1}{t} \int_0^t \mathbf{T}_\tau x^0 d\tau, \tag{4}$$

where $\mathrm{dom}(C_L)$ is the set of all those $x^0 \in X$ for which the above limit exists. Clearly $X_1 \subset \mathrm{dom}(C_L) \subset X$. Furthermore, for any $x^0 \in X$, we have that $\mathbf{T}_t x^0 \in \mathrm{dom}(C_L)$ for almost all $t \geq 0$ and

$$(\boldsymbol{\Psi}_\infty x^0)(t) = C_L \mathbf{T}_t x^0 \quad \text{for a.a. } t \geq 0.$$

It is known (see [4] and [23]) that for all s, $s_0 \in \varrho(A)$ with $s \neq s_0$, the transfer function $\mathbf{G}$ and the generating operators are related in the manner

$$\frac{1}{s - s_0}(\mathbf{G}(s) - \mathbf{G}(s_0)) = -C(sI - A)^{-1}(s_0 I - A)^{-1}B. \tag{5}$$

This identity shows that the operators A, B, and C determine $\mathbf{G}$ only up to an additive constant. Under the assumption that $\boldsymbol{\Sigma}$ is regular (with feedthrough operator D), it was shown in [27] that $(sI - A)^{-1}BU \subset \mathrm{dom}(C_L)$ for all $s \in \varrho(A)$ and the transfer function $\mathbf{G}$ can be expressed as

$$\mathbf{G}(s) = C_L(sI - A)^{-1}B + D, \qquad \forall\, s \in \mathbb{C}_{\omega(\mathbf{T})},$$

which is familiar from finite-dimensional systems theory.

If $\boldsymbol{\Sigma}$ is a well-posed linear system with generating operators (A, B, C), then for any $x^0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, the function x given by (1a) is the unique solution of the initial-value problem

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x^0, \tag{6}$$

where the derivative on the left-hand side of (6) has, of course, to be understood in X_{-1}. In other words, if we consider the initial-value problem (6) in the space X_{-1}, then for any $x^0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, (6) has a unique strong solution (in the sense of Pazy [22, p. 109]) given by the variation of parameters formula

$$t \mapsto x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} Bu(\tau)\, d\tau. \tag{7}$$

In the case where the input function u has additional smoothness, we have the following result (cf. Lemma 2.5 in Salamon [24]).

LEMMA 2.3. *Let $\boldsymbol{\Sigma} = (\mathbf{T}, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \mathbf{F})$ be a well-posed system with state space X, input space U, and generating operators (A, B, C). If $x^0 \in X$ and $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+, U)$ are such that $Ax^0 + Bu(0) \in X$, then the unique solution* (7) *of the initial-value problem* (6) *is continuously differentiable in X.*

The following lemma is essentially contained in [14]. The proof of statement (a) is implicit in the argument establishing Lemma 2.2 of [14] and statement (b) is part of Lemma 2.2 of [14].

LEMMA 2.4. *Let $\boldsymbol{\Sigma} = (\mathbf{T}, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \mathbf{F})$ be an exponentially stable well-posed system with state space X, input space U, and generating operators (A, B, C).*

(a) *For all* $(x^0, u) \in X \times L^\infty(\mathbb{R}_+, U)$, *the function* x *given by* (7) *satisfies*

$$x \in L^\infty(\mathbb{R}_+, X).$$

(b) *If* $u \in L^\infty(\mathbb{R}_+, U)$ *and* $\lim_{t\to\infty} u(t) = u_\infty$ *exists, then, for all* $x^0 \in X$, *the function* x *given by* (7) *satisfies*

$$\lim_{t\to\infty} \|x(t) + A^{-1}Bu_\infty\| = 0.$$

If $\boldsymbol{\Sigma}$ is a well-posed system with state space X, input space U, output space Y, and generating operators (A, B, C), then for all $x^0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, the corresponding state function x of $\boldsymbol{\Sigma}$ satisfies (6) in X_{-1} and the corresponding output function $w \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ is given by

$$w(t) = C_L \mathbf{T}_t x^0 + (\mathbf{F}_\infty u)(t) \quad \text{for a.a. } t \geq 0. \tag{8}$$

If $x^0 \in X_1$ and $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+, U)$ with $u(0) = 0$, it was shown in [23] that

$$w(t) = C(x(t) - (s_0 I - A)^{-1}Bu(t)) + \mathbf{G}(s_0)u(t), \qquad \forall\, t \geq 0, \tag{9}$$

where s_0 is an arbitrary element of $\varrho(A)$ and $\mathbf{G}$ is the transfer function of $\boldsymbol{\Sigma}$. If $\boldsymbol{\Sigma}$ is regular (with feedthrough operator D), then the state function x defined by (7) satisfies $x(t) \in \mathrm{dom}(C_L)$ for almost all $t \geq 0$ and the output w given by (8) can be written in the familiar form

$$w(t) = C_L x(t) + Du(t) \quad \text{for a.a. } t \geq 0.$$

LEMMA 2.5. *Let* $\boldsymbol{\Sigma} = (\mathbf{T}, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \mathbf{F})$ *be a well-posed system with state space* X, *input space* U, *transfer function* $\mathbf{G}$, *and generating operators* (A, B, C), *and let* $s_0 \in \varrho(A)$. *For* $x^0 \in X$ *and* $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, *let* x *and* w *denote the state and output functions of* $\boldsymbol{\Sigma}$. *Then*

$$\begin{aligned} &C(A - s_0 I)^{-1}(x(t) - x(t_0)) \\ &\quad = \int_{t_0}^{t} (s_0 C(A - s_0 I)^{-1}x(\tau) + w(\tau) - \mathbf{G}(s_0)u(\tau))\, d\tau, \qquad \forall\, t \geq t_0 \geq 0. \end{aligned}$$

In particular, if $0 \in \varrho(A)$, *then*

$$CA^{-1}(x(t) - x(t_0)) = \int_{t_0}^{t} (w(\tau) - \mathbf{G}(0)u(\tau))\, d\tau, \qquad \forall\, t \geq t_0 \geq 0.$$

Proof. Let $t_1 > t_0 \geq 0$ and choose sequences $(x_n^0) \subset X_1$ and $(u_n) \subset W^{1,2}([0, t_1], U)$, with $u_n(0) = 0$ for all n, such that

$$\lim_{n\to\infty} \|x^0 - x_n^0\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|u - u_n\|_{L^2(0,\, t_1;\, U)} = 0. \tag{10}$$

Moreover, define x_n to be the solution of (6) for initial value x_n^0 and input u_n, and let w_n be the corresponding output. Since $Ax_n^0 + Bu_n(0) = Ax_n^0 \in X$, x_n is, by Lemma 2.3, continuously differentiable in X and satisfies

$$\dot{x}_n(t) = Ax_n(t) + Bu_n(t) \quad (\text{in } X)\ \forall\, t \geq 0.$$

It follows that

$$\begin{aligned} C(A - s_0 I)^{-1}\dot{x}_n(t) &= C(A - s_0 I)^{-1}(Ax_n(t) + Bu_n(t)) \\ &= C[x_n(t) + s_0(A - s_0 I)^{-1}x_n(t) - (s_0 I - A)^{-1}Bu_n(t)], \end{aligned}$$

and hence, by (9),

$$C(A - s_0 I)^{-1}\dot{x}_n(t) = s_0 C(A - s_0 I)^{-1}x_n(t) + w_n(t) - \mathbf{G}(s_0)u_n(t).$$

Thus, for all $t \in [t_0, t_1]$,

$$\begin{aligned} &C(A - s_0 I)^{-1}(x_n(t) - x_n(t_0)) \\ &\quad = \int_{t_0}^{t} (s_0 C(A - s_0 I)^{-1}x_n(\tau) + w_n(\tau) - \mathbf{G}(s_0)u_n(\tau))\, d\tau. \end{aligned}$$

By (10) and the continuity properties of a well-posed system we have that $\lim_{n\to\infty} \|x(t) - x_n(t)\| = 0$ for all $t \in [0, t_1]$ and $\lim_{n\to\infty} \|w - w_n\|_{L^2(0,t_1;Y)} = 0$. Letting $n \to \infty$, we conclude that

$$\begin{aligned} &C(A - s_0 I)^{-1}(x(t) - x(t_0)) \\ &\quad = \int_{t_0}^{t} (s_0 C(A - s_0 I)^{-1}x(\tau) + w(\tau) - \mathbf{G}(s_0)u(\tau))\, d\tau, \qquad \forall\, t \in [t_0, t_1]. \end{aligned}$$

Since t_1 was arbitrary, the claim follows. ∎

For notational simplicity we assume in the rest of this section that $\boldsymbol{\Sigma}$ is a single-input–single-output system, i.e., $U = Y = \mathbb{R}$. An inspection of the proofs shows that the results below (Lemma 2.6 and Corollary 2.7) generalize in an obvious manner to well-posed systems with finite-dimensional input and output spaces $U = \mathbb{R}^m$ and $Y = \mathbb{R}^p$.

The next lemma provides a useful formula for the output (1b) if $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$.

LEMMA 2.6. *Let $\boldsymbol{\Sigma} = (\mathbf{T}, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \mathbf{F})$ be a single-input–single-output, well-posed system with state space X, transfer function $\mathbf{G}$, and generating operators (A, B, C). Let $s_0 \in \varrho(A)$ and set $\mathbf{H}(s) := -C(sI - A)^{-1}(s_0 I - A)^{-1}B$. Let $x^0 \in X$, $u^0 \in \mathbb{R}$, and $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ with $u(0) = u^0$ and denote the corresponding output of $\boldsymbol{\Sigma}$ by w. Then*

$$\begin{aligned} w(t) = {}& C_L \mathbf{T}_t(x^0 - (s_0 I - A)^{-1}Bu^0) + (\mathfrak{L}^{-1}(\mathbf{H}) \star (\dot{u} - s_0 u))(t) \\ &+ \mathbf{G}(s_0)u(t) \quad \text{for a.a. } t \in \mathbb{R}_+. \end{aligned}$$

In particular, if $\mathbf{T}_{t_0}(Ax^0 + Bu^0) \in X$ for some $t_0 \geq 0$, then $w \in W^{1,2}_{\mathrm{loc}}([t_0, \infty), \mathbb{R})$.

Proof. Let $x^0 \in X$, $u \in W^{1,2}_{\rm loc}(\mathbb{R}_+, \mathbb{R})$, and $\tau > 0$. Define

$$u_\tau(t) := \begin{cases} u(t), & 0 \le t \le \tau, \\ u(\tau), & t > \tau, \end{cases}$$

and note that $\dot{u}_\tau \in L^2(\mathbb{R}_+, \mathbb{R})$. It is sufficient to show that

$$w(t) = C_L \mathbf{T}_t(x^0 - (s_0 I - A)^{-1} B u^0) + (\mathfrak{L}^{-1}(\mathbf{H}) \star (\dot{u}_\tau - s_0 u_\tau))(t) \\ + \mathbf{G}(s_0) u_\tau(t) \quad \text{for a.a. } t \in [0, \tau]. \tag{11}$$

Let $\alpha > \max(0, \operatorname{Re} s_0, \omega(\mathbf{T}))$. Taking the Laplace transform of the right-hand side of (11), we obtain for all $s \in \mathbb{C}_\alpha$, using (5),

$$C(sI - A)^{-1} x^0 + \mathbf{H}(s) u^0 + \mathbf{H}(s)((s - s_0)\hat{u}_\tau(s) - u^0) + \mathbf{G}(s_0)\hat{u}_\tau(s) \\ = C(sI - A)^{-1} x^0 + \mathbf{G}(s)\hat{u}_\tau(s).$$

Inverse Laplace transform, using (3), gives (11). Moreover, assume that $\mathbf{T}_{t_0}(Ax^0 + Bu^0) \in X$ for some $t_0 \ge 0$. To prove that $w \in W^{1,2}_{\rm loc}([t_0, \infty), \mathbb{R})$, it is sufficient to show that the function given by the right-hand side of (11) is in $W^{1,2}_{\rm loc}([t_0, \infty), \mathbb{R})$. Since $\mathbf{T}_{t_0}(Ax^0 + Bu^0) \in X$, we have that $\mathbf{T}_{t_0}(x^0 - (s_0 I - A)^{-1} B u^0) \in X_1$. Consequently, the function

$$t \mapsto C_L \mathbf{T}_t(x^0 - (s_0 I - A)^{-1} B u^0)$$

is continuously differentiable on $[t_0, \infty)$ and hence in $W^{1,2}_{\rm loc}([t_0, \infty), \mathbb{C})$. Therefore, it is sufficient to show that the function $\tilde{w} : \mathbb{R} \to \mathbb{C}$ given by

$$\tilde{w}(t) = \begin{cases} e^{-\alpha t}(\mathfrak{L}^{-1}(\mathbf{H}) \star \tilde{u})(t) & \text{if } t \ge 0, \\ 0 & \text{if } t < 0, \end{cases}$$

where $\tilde{u} = \mathbf{P}_\tau(\dot{u}_\tau - s_0 u_\tau)$, is in $W^{1,2}(\mathbb{R}, \mathbb{C})$. To this end, note that the Fourier transform $\mathfrak{F}(\tilde{w})$ of $\tilde{w}$ is given by

$$(\mathfrak{F}\tilde{w})(\omega) = (\mathfrak{L}\tilde{w})(i\omega) = \frac{\mathbf{G}(\alpha + i\omega) - \mathbf{G}(s_0)}{\alpha + i\omega - s_0} (\mathfrak{L}\tilde{u})(\alpha + i\omega). \tag{12}$$

Since $\tilde{u} \in L^2(\mathbb{R}_+, \mathbb{C})$, it follows that the function $\omega \mapsto (\mathfrak{L}\tilde{u})(\alpha + i\omega)$ is in $L^2(\mathbb{R}, \mathbb{C})$. Note also that the function $\omega \mapsto \mathbf{G}(\alpha + i\omega)$ is bounded. Hence, by (12), $\tilde{w} \in L^2(\mathbb{R}, \mathbb{C})$ and, moreover, the function $\omega \mapsto \omega(\mathfrak{F}\tilde{w})(\omega)$ is in $L^2(\mathbb{R}, \mathbb{C})$. It follows from the Fourier characterization of $W^{1,2}(\mathbb{R}, \mathbb{C})$ (see, for example, [8, p. 165]) that $\tilde{w} \in W^{1,2}(\mathbb{R}, \mathbb{C})$. ∎

The following corollary is an easy consequence of Lemma 2.6.

COROLLARY 2.7. *Let* $\boldsymbol{\Sigma} = (\mathbf{T}, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \mathbf{F})$ *be an exponentially stable, single-input–single-output, well-posed system with state space* X, *transfer function* $\mathbf{G}$, *and generating operators* (A, B, C). *Let* $x^0 \in X$, $u^0 \in \mathbb{R}$, *and* $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ *with* $u(0) = u^0$, *and denote the corresponding output of* $\boldsymbol{\Sigma}$ *by* w. *If* $\dot{u} \in L^2(\mathbb{R}_+, \mathbb{R})$ *and* $\lim_{t\to\infty} u(t) =: u_\infty$ *exists and is finite, then*

$$\lim_{t\to\infty} [w(t) - C_L \mathbf{T}_t(x^0 + A^{-1}Bu^0)] = \mathbf{G}(0)u_\infty.$$

In particular, under the additional assumption that $\mathbf{T}_{t_0}(Ax^0 + Bu^0) \in X$ *for some* $t_0 \geq 0$,

$$\lim_{t\to\infty} w(t) = \mathbf{G}(0)u_\infty.$$

Remark 2.8. Corollary 2.7 shows that, under certain smoothness assumptions on the input and initial data, the final-value theorem holds for a general exponentially stable, single-input–single-output, well-posed system $\boldsymbol{\Sigma}$. If $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathfrak{M}$, the latter smoothness assumptions may be relaxed: if the impulse response of $\boldsymbol{\Sigma}$ is a finite signed Borel measure on $\mathbb{R}_+$, then for any input $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ with $\lim_{t\to\infty} u(t) = u_\infty$ and any $x^0 \in X$,

$$\lim_{t\to\infty} [w(t) - C_L \mathbf{T}_t x^0] = \mathbf{G}(0)u_\infty.$$

Under the additional assumption that $\mathbf{T}_{t_0} x^0 \in X_1$ for some $t_0 \geq 0$, $\lim_{t\to\infty} w(t) = \mathbf{G}(0)u_\infty$.

Proof of Corollary 2.7. By exponential stability, $0 \in \varrho(A)$, and an application of Lemma 2.6 with $s_0 = 0$ leads to

$$\begin{aligned} w(t) = C_L \mathbf{T}_t(x^0 + A^{-1}Bu^0) + (\mathfrak{L}^{-1}(\mathbf{H}) \star \dot{u})(t) \\ + \mathbf{G}(0)u(t) \quad \text{for a.a. } t \in \mathbb{R}_+, \end{aligned} \tag{13}$$

where $\mathbf{H}(s) = (\mathbf{G}(s) - \mathbf{G}(0))/s$. Since $\mathbf{G} \in H^\infty(\mathbb{C}_\alpha)$ for some $\alpha < 0$, it follows that $\mathbf{H} \in H^2(\mathbb{C}_\alpha)$ and hence, by the Paley–Wiener theorem, $\mathfrak{L}^{-1}(\mathbf{H}) \in L^2_\alpha(\mathbb{R}_+, \mathbb{R}) \subset L^2(\mathbb{R}_+, \mathbb{R})$. Since by assumption $\dot{u} \in L^2(\mathbb{R}_+, \mathbb{R})$, we may infer that

$$\lim_{t\to\infty} (\mathfrak{L}^{-1}(\mathbf{H}) \star \dot{u})(t) = 0.$$

Combining this with (13) yields the claim. ∎

3. PROBLEM FORMULATION

The problem of tracking constant reference signals r will be addressed in a context of uncertain single-input–single-output well-posed linear systems $\mathbf{\Sigma}$ with state space X, transfer function $\mathbf{G}$, and generating operators (A, B, C), having a nonlinearity φ in the input channel and a nonlinearity ψ in the output channel:

$$\dot{x}(t) = Ax(t) + B\varphi(u(t)) \quad \text{for a.a. } t \in \mathbb{R}_+,\ x(0) = x^0 \in X, \tag{14a}$$

$$w(t) = C_L \mathbf{T}_t x^0 + (\mathbf{F}_\infty(\varphi \circ u))(t) \quad \text{for a.a. } t \in \mathbb{R}_+, \tag{14b}$$

$$y(t) = \psi(w(t)) \quad \text{for a.a } t \in \mathbb{R}_+. \tag{14c}$$

3.1. *The Class $\mathcal{S}$ of Linear Systems*

Let $\mathcal{S}$ denote the class of single-input–single-output well-posed linear systems $\mathbf{\Sigma}$ with state space X and transfer function $\mathbf{G}$ satisfying

(a) $\mathbf{\Sigma}$ is exponentially stable; (b) $\mathbf{G}(0) > 0$.

If $\mathbf{G}$ is the transfer function of a system $\mathbf{\Sigma} \in \mathcal{S}$, then it is readily shown that the positive-real condition

$$1 + \kappa \mathrm{Re} \frac{\mathbf{G}(s)}{s} \geq 0, \qquad \forall\, s \in \mathbb{C}_0, \tag{15}$$

holds for all sufficiently small $\kappa > 0$; see [16, Lemma 3.10]. Define

$$\kappa^* := \sup\{\kappa > 0 | 1 + \kappa \mathrm{Re}(\mathbf{G}(s)/s) \geq 0,\ \forall\, s \in \mathbb{C}_0\}. \tag{16}$$

Lower bounds and formulae for κ^* may be found in [13]. The following lemma will be invoked in a later stability analysis.

LEMMA 3.1. *Let $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F}) \in \mathcal{S}$ with generating operators (A, B, C) and let $\Delta > 1/\kappa^*$. Then there exist $P \in \mathcal{B}(X)$, $P = P^* \geq 0$, and $M \in \mathcal{B}(X_1, \mathbb{R})$ such that*

$$\begin{aligned} \langle Ax_1, Px_2 \rangle + \langle Px_1, Ax_2 \rangle &= -\langle Mx_1, Mx_2 \rangle, \qquad &\forall\, x_1, x_2 \in X_1, \\ (A^{-1}B)^* Px &= Cx - \sqrt{2\Delta} Mx, \qquad &\forall\, x \in X_1, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes inner product in X.

Proof. Let $\mathbf{G}$ be the transfer function of $\mathbf{\Sigma}$. By (5),

$$\mathbf{H}(s) := C(sI - A)^{-1} A^{-1} B = \begin{cases} \frac{\mathbf{G}(s) - \mathbf{G}(0)}{s}, & s \in \varrho(A),\ s \neq 0, \\ \mathbf{G}'(0), & s \in \varrho(A),\ s = 0. \end{cases} \tag{17}$$

By exponential stability of $\mathbf{\Sigma}$, $\mathbf{H}$ is holomorphic in $\mathbb{C}_\alpha$ for some $\alpha < 0$. Since $\Delta > 1/\kappa^*$, the positive-real condition (15) guarantees the existence of a constant $\varepsilon > 0$ such that

$$\Delta + \operatorname{Re} \mathbf{H}(i\omega) \geq \varepsilon, \qquad \forall\, \omega \in \mathbb{R}.$$

By a result in van Keulen [26] (see Theorem 3.10 and Remark 3.14 in [26]), there exists $\widetilde{P} \in \mathscr{B}(X)$, $\widetilde{P} = \widetilde{P}^*$ such that

$$\begin{aligned} &\langle Ax_1, \widetilde{P}x_2\rangle + \langle \widetilde{P}x_1, Ax_2\rangle \\ &\quad = 1/(2\Delta)\langle [(A^{-1}B)^*\widetilde{P} + C]x_1, [(A^{-1}B)^*\widetilde{P} + C]x_2\rangle, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \forall\, x_1, x_2 \in X_1. \end{aligned} \tag{18}$$

Setting

$$P := -\widetilde{P} \in \mathscr{B}(X), \qquad M := \sqrt{1/(2\Delta)}[C - (A^{-1}B)^*P] \in \mathscr{B}(X_1, \mathbb{R}),$$

we obtain, using (18),

$$\langle Ax_1, Px_2\rangle + \langle Px_1, Ax_2\rangle = -\langle Mx_1, Mx_2\rangle, \qquad \forall\, x_1, x_2 \in X_1, \tag{19a}$$

$$(A^{-1}B)^*Px = Cx - \sqrt{2\Delta}Mx, \qquad \forall\, x \in X_1. \tag{19b}$$

By a routine argument it follows from (19a) and exponential stability of $\mathbf{\Sigma}$ that $P \geq 0$. ∎

3.2. *Input/Output Nonlinearities*

In this section we introduce certain sets of nonlinearities that will play a role in the sequel. Moreover, some definitions and auxiliary results are collected. The following sets of monotone, nondecreasing nonlinearities are first introduced:

$$\begin{aligned} \mathscr{M} &:= \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ locally Lipschitz and nondecreasing}\}, \\ \mathscr{M}(\lambda) &:= \{f \in \mathscr{M} \mid 0 \leq (f(\xi) - f(0))\xi \leq \lambda\xi^2, \forall\, \xi \in \mathbb{R}\}, \\ \mathscr{M}_L(\lambda) &:= \{f \in \mathscr{M} \mid f \text{ is globally Lipschitz with Lipschitz constant } \lambda\}. \end{aligned}$$

Clearly, $\mathscr{M}_L(\lambda) \subset \mathscr{M}(\lambda) \subset \mathscr{M}$.

Remark 3.2. (i) Let $f \in \mathscr{M}(\lambda)$. Then $f(\xi) = f(0) + \gamma(\xi)\xi$ for all ξ, where

$$\gamma : \xi \mapsto \begin{cases} (f(\xi) - f(0))/\xi, & \xi \neq 0, \\ \lambda, & \xi = 0, \end{cases}$$

is upper semicontinuous and $\gamma(\xi) \in [0, \lambda]$ for all $\xi \in \mathbb{R}$.

(ii) If $f \in \mathcal{M}(\lambda)$, then for each $\nu \in \mathbb{R}$, there exists $\tilde{\lambda} \geq 0$ such that the function $\xi \mapsto f(\xi+\nu) - f(\nu)$ is of class $\mathcal{M}(\tilde{\lambda})$.

(iii) If $f \in \mathcal{M}_L(\lambda)$, then for each $\nu \in \mathbb{R}$, $\xi \mapsto f(\xi+\nu) - f(\nu)$ is also of class $\mathcal{M}_L(\lambda)$.

We say that a pair of nonlinearities (φ, ψ) is in $\mathcal{N}$ if $\varphi \in \mathcal{M}_L(\lambda_1)$ for some $\lambda_1 > 0$, $\psi \in \mathcal{M}$, and at least one of the following conditions holds:

$$\text{(i) } \varphi \text{ is bounded} \quad \text{or} \quad \text{(ii) } \psi \in \mathcal{M}(\lambda_2) \text{ for some } \lambda_2 > 0.$$

Equivalently,

$$\mathcal{N} := \{(\varphi, \psi) \in \mathcal{M}_L(\lambda_1) \times \mathcal{M} \mid \lambda_1 > 0, \varphi \text{ unbounded} \\ \Rightarrow \quad \psi \in \mathcal{M}(\lambda_2) \text{ for some } \lambda_2 > 0\}.$$

The proof of the following lemma is given in [14].

LEMMA 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz and let (ε_n) be any sequence with $\varepsilon_n > 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$. Define the function $f^\diamond : \mathbb{R} \to \mathbb{R}$ by*

$$f^\diamond(\xi) := \limsup_{n\to\infty} \frac{f(\xi+\varepsilon_n) - f(\xi)}{\varepsilon_n}. \tag{20}$$

Then $f^\diamond$ is Borel measurable and $f^\diamond \in L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R})$. If $g : \mathbb{R}_+ \to \mathbb{R}$ is absolutely continuous, then $f \circ g$ is absolutely continuous and

$$\frac{d}{dt}(f \circ g)(t) = f^\diamond(g(t))\dot{g}(t) \quad \textit{for a.a. } t \in \mathbb{R}_+.$$

If $f \in \mathcal{M}_L(\lambda)$ for some $\lambda \geq 0$, then

$$0 \leq f^\diamond(\xi) \leq \lambda, \qquad \forall\, \xi \in \mathbb{R}. \tag{21}$$

3.3. *The Tracking Objective and Feasibility*

Given $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F}) \in \mathcal{S}$ and static nonlinearities $\varphi, \psi : \mathbb{R} \to \mathbb{R}$, the tracking objective is to determine, by feedback, an input u such that, for given $r \in \mathbb{R}$, the output y of (14) has the property $y(t) \to r$ as $t \to \infty$ (cf. Remark 2.1). Clearly, if this objective is achievable, then r is necessarily in the closure of $\operatorname{im} \psi$. We impose a stronger condition, namely,

$$\Psi^r \cap \overline{\Phi} \neq \emptyset, \quad \text{where } \Psi^r := \{v \in \mathbb{R} \mid \psi(\mathbf{G}(0)v) = r\}, \\ \Phi := \operatorname{im} \varphi, \ \overline{\Phi} := \operatorname{clos}(\Phi)$$

and refer to the set

$$\mathcal{R} := \{r \in \mathbb{R} \mid \Psi^r \cap \overline{\Phi} \neq \emptyset\}$$

as the set of *feasible reference values*. The next proposition shows that if ψ is continuous and monotone, then $r \in \mathcal{R}$ is close to being a necessary condition for tracking insofar as if tracking of r is achievable whilst maintaining boundedness of $\varphi \circ u$ together with ultimate continuity and ultimate boundedness of w, then $r \in \mathcal{R}$.

PROPOSITION 3.4. *Let* $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F}) \in \mathcal{S}$ *and* $x^0 \in X$. *Let* $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ *and let* ψ *be continuous and monotone. Let* $u : \mathbb{R}_+ \to \mathbb{R}$ *be such that* $\varphi \circ u \in L^\infty(\mathbb{R}_+, \mathbb{R})$. *If* $w : \mathbb{R}_+ \to \mathbb{R}$ *given by* (14*b*) *is such that for some* $\tau \geq 0$, w *restricted to* $[\tau, \infty)$ *is continuous and bounded with*

$$\lim_{t \to \infty} \psi(w(t)) = r, \tag{22}$$

then $r \in \mathcal{R}$.

Proof. By assumption, $\varphi \circ u \in L^\infty(\mathbb{R}_+, \mathbb{R})$ and so, by Lemma 2.4(a), the solution x of the initial-value problem (14a) is bounded. Since by assumption the function w is ultimately continuous and bounded, it has nonempty, compact, connected ω-limit set Ω. By continuity of ψ and (22), it follows that

$$\psi(\omega) = r, \qquad \forall\, \omega \in \Omega. \tag{23}$$

It suffices to show that $\Omega \cap \mathbf{G}(0)\overline{\Phi} \neq \varnothing$. Indeed, if there exist $\omega \in \Omega$ and $v \in \overline{\Phi}$ such that $\omega = \mathbf{G}(0)v$, then, by (23), $\psi(\mathbf{G}(0)v) = \psi(\omega) = r$, showing that $\Psi^r \cap \overline{\Phi}$ is nonempty.

Seeking a contradiction, suppose that $\Omega \cap \mathbf{G}(0)\overline{\Phi} = \varnothing$. Then Ω and $\mathbf{G}(0)\overline{\Phi}$ are closed disjoint intervals and so there exist $\varepsilon > 0$ and $\beta = \pm 1$ such that

$$\inf\{\beta\omega \mid \omega \in \Omega\} - \sup\{\beta\mathbf{G}(0)v \mid v \in \overline{\Phi}\} = 2\varepsilon.$$

Moreover, since $w(t)$ approaches its compact ω-limit set Ω as $t \to \infty$, there exists $T > 0$ such that

$$\inf \Omega - \varepsilon < w(t) < \sup \Omega + \varepsilon, \qquad \forall\, t > T,$$

and so $\beta w(t) > \inf\{\beta\omega \mid \omega \in \Omega\} - \varepsilon$ for all $t > T$. Using Lemma 2.5 (with $s_0 = 0$), it follows that

$$\begin{aligned}
&\beta C A^{-1}(x(t) - x(T)) \\
&\quad = \beta \int_T^t (w(\tau) - \mathbf{G}(0)\varphi(u(\tau)))\, d\tau \\
&\quad \geq \int_T^t (\beta w(\tau) - \sup\{\beta\mathbf{G}(0)v \mid v \in \overline{\Phi}\})\, d\tau \\
&\quad > \int_T^t (\inf\{\beta\omega | \omega \in \Omega\} - \varepsilon - \sup\{\beta\mathbf{G}(0)v \mid v \in \overline{\Phi}\})\, d\tau \\
&\quad = \varepsilon(t - T), \qquad \forall\, t > T,
\end{aligned}$$

contradicting the boundedness of x. Therefore, $\Omega \cap \mathbf{G}(0)\overline{\Phi} \neq \varnothing$. ∎

4. INTEGRAL CONTROL

Let $\mathbf{\Sigma} \in \mathcal{S}$, $(\varphi, \psi) \in \mathcal{M}_L(\lambda_1) \times \mathcal{M}$, $k \in L^\infty(\mathbb{R}_+, \mathbb{R})$, and $r \in \mathbb{R}$. We investigate integral control action

$$u(t) = u^0 + \int_0^t k(\tau)[r - y(\tau)]\,d\tau \tag{24}$$

with prescribed time-varying gain $k(\cdot)$. An application of the control (24) to $\mathbf{\Sigma}$ leads to the following system of nonlinear nonautonomous differential equations:

$$\dot{x}(t) = Ax(t) + B\varphi(u(t)), \quad x(0) = x^0 \in X, \tag{25a}$$

$$\dot{u}(t) = k(t)[r - \psi(C_L \mathbf{T}_t x^0 + (\mathbf{F}_\infty(\varphi \circ u))(t))], \quad u(0) = u^0 \in \mathbb{R}. \tag{25b}$$

For $a \in (0, \infty]$, a continuous function

$$(x, u) : [0, a) \to X \times \mathbb{R}$$

is a *solution* of (25) if (x, u) is absolutely continuous as an $(X_{-1} \times \mathbb{R})$-valued function, $(x(0), u(0)) = (x^0, u^0)$ and (25) is satisfied almost everywhere on $[0, a)$, where the derivative in (25a) should be interpreted in the space X_{-1}.[2]

4.1. *Globally Lipschitz Output Nonlinearities*

If $(\varphi, \psi) \in \mathcal{M}_L(\lambda_1) \times \mathcal{M}_L(\lambda_2)$, we have the following existence and uniqueness result.

LEMMA 4.1. *Let* $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F}) \in \mathcal{S}$ *with generating operators* (A, B, C). *Let* $(\varphi, \psi) \in \mathcal{M}_L(\lambda_1) \times \mathcal{M}_L(\lambda_2)$, $r \in \mathbb{R}$, *and* $k \in L^\infty(\mathbb{R}_+, \mathbb{R})$. *For each* $(x^0, u^0) \in X \times \mathbb{R}$, *the initial-value problem* (25) *has a unique solution* $(x, u) : \mathbb{R}_+ \to X \times \mathbb{R}$.

The proof of Lemma 4.1 can be found in the Appendix.

Henceforth, we assume that the gain function $k : \mathbb{R}_+ \to (0, \infty)$ satisfies

$$k \in \mathcal{G} := \{g | g : \mathbb{R}_+ \to (0, \infty),\ g \text{ is measurable and bounded}\}.$$

The next theorem forms the core of the paper.

[2]Being a Hilbert space, $X_{-1} \times \mathbb{R}$ is reflexive, and hence any absolutely continuous $(X_{-1} \times \mathbb{R})$-valued function is a.e. differentiable and can be recovered from its derivative by integration; see [1, Theorem 3.1, p. 10].

THEOREM 4.2. *Let* $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F}) \in \mathcal{S}$, $(\varphi, \psi) \in \mathcal{M}_L(\lambda_1) \times \mathcal{M}_L(\lambda_2)$, $r \in \mathcal{R}$, *and* $k \in \mathcal{G}$. *Let* (A, B, C) *be the generating operators of* $\mathbf{\Sigma}$. *For* $(x^0, u^0) \in X \times \mathbb{R}$, *let* $(x, u) : \mathbb{R}_+ \to X \times \mathbb{R}$ *be the unique solution of the initial-value problem* (25). *Let* $k^* = \kappa^*/(\lambda_1\lambda_2)$, *where* κ^* *is given by* (16). *Assume that* $\limsup_{t\to\infty} k(t) < k^*$. *Then the following hold.*

A. $\lim_{t\to\infty} \varphi(u(t))$ *exists and is finite.*

B. *If* $K(t) := \int_0^t k \to \infty$ *as* $t \to \infty$, *then*

(b1) $\lim_{t\to\infty} \varphi(u(t)) =: \varphi^r \in \Psi^r \cap \overline{\Phi}$,

(b2) $\lim_{t\to\infty} \|x(t) + A^{-1}B\varphi^r\| = 0$,

(b3) $|r - y(t)| \le e_0(t) + e_1(t)$, *where* $y(t) = \psi(C_L \mathbf{T}_t x^0 + (\mathbf{F}_\infty(\varphi \circ u))(t))$, $e_0 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R}_+)$ *for some* $\alpha < 0$, *and* $e_1 : \mathbb{R}_+ \to \mathbb{R}_+$ *is bounded with* $\lim_{t\to\infty} e_1(t) = 0$,

(b4) *if* $\mathbf{T}_{t_0}(Ax^0 + B\varphi(u^0)) \in X$ *for some* $t_0 \ge 0$, *then* $\lim_{t\to\infty} y(t) = r$,

(b5) *if* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, *then* $\lim_{t\to\infty} \operatorname{dist}(u(t), \varphi^{-1}(\varphi^r)) = 0$,

(b6) *if* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \operatorname{int}(\Phi)$, *then* $u(\cdot)$ *is bounded.*

Remark 4.3. (i) Statement (b3) says that if $k \in \mathcal{G}$ is *not* of class L^1 and $\limsup_{t\to\infty} k(t) < k^*$, then the tracking error $e(t) = r - y(t)$ that results from an application of the integral control (24) is small in the sense that $|e| \le e_0 + e_1$, where $e_0 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R}_+)$ for some $\alpha < 0$ and $e_1 \in L^\infty(\mathbb{R}_+, \mathbb{R}_+)$ with $\lim_{t\to\infty} e_1(t) = 0$. This means in particular that for any $\varepsilon > 0$,

$$\lim_{\tau\to\infty} \mu_L(\{t \ge \tau \,|\, |e(t)| \ge \varepsilon\}) = 0,$$

where μ_L denotes Lebesgue measure on $\mathbb{R}_+$.

(ii) In (b4) an additional assumption on the initial data is identified such that asymptotic tracking is ensured. An inspection of the proof below shows that if C is bounded, then asymptotic tracking occurs for arbitrary initial data.

(iii) An immediate consequence of part B of Theorem 4.2 is the following: if $k \in \mathcal{G}$ is chosen such that $k(t)$ tends to zero sufficiently slowly as $t \to \infty$ (in the sense that $k \notin L^1$), then (b1)–(b6) hold.

(iv) Simple examples show that the assertions (b1)–(b6) of part B of Theorem 4.2 are in general not true if the additional assumption $k \notin L^1$ is violated.

(v) Statements (b5) and (b6) capture aspects of the qualitative behaviour of the function u. For example, if $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, then (b5) implies that $u(t)$ converges as $t \to \infty$, provided that the set $\varphi^{-1}(\Psi^r \cap \Phi)$ is a singleton.

Proof of Theorem 4.2. Let $\mathbf{G}$ denote the transfer function of $\mathbf{\Sigma}$.

A. By assumption, $r \in \mathscr{R}$, and so $\Psi^r \cap \overline{\Phi} \neq \emptyset$. Therefore, there exists $\varphi^* \in \Psi^r \cap \overline{\Phi}$. Since $\psi \in \mathscr{M}_L(\lambda_2)$ for some $\lambda_2 > 0$, the function

$$\tilde{\psi} : \mathbb{R} \to \mathbb{R}, \qquad \xi \mapsto \psi(\xi + \mathbf{G}(0)\varphi^*) - r \tag{26}$$

is in $\mathscr{M}_L(\lambda_2)$ [cf. Remark 3.2(iii)] and, hence, in $\mathscr{M}(\lambda_2)$. Therefore, by Remark 3.2(i),

$$\tilde{\psi}(\xi) = \gamma(\xi)\xi, \qquad \forall\, \xi \in \mathbb{R}, \tag{27}$$

where the function γ is defined by

$$\gamma(\xi) = \begin{cases} \tilde{\psi}(\xi)/\xi, & \xi \neq 0, \\ \lambda_2, & \xi = 0, \end{cases} \tag{28}$$

and satisfies $0 \leq \gamma(\xi) \leq \lambda_2$ for all $\xi \in \mathbb{R}$. Introduce continuous functions z and v and a function $\tilde{w} \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ given by

$$z(t) := x(t) + A^{-1}B\varphi(u(t)), \quad v(t) := \varphi(u(t)) - \varphi^*, \qquad \forall\, t \geq 0, \tag{29}$$

and

$$\tilde{w}(t) := w(t) - \mathbf{G}(0)\varphi^* \quad \text{for a.a. } t \geq 0, \tag{30}$$

where, as in (14b), $w(t) = C_L \mathbf{T}_t x^0 + (\mathbf{F}_\infty(\varphi \circ u))(t)$ for almost all $t \geq 0$. By Lemma 3.3, $\dot{v}(t) = \varphi^\diamond(u(t))\dot{u}(t)$ for almost all $t \in [0, \infty)$. Since (z, v) is absolutely continuous as an $(X_{-1} \times \mathbb{R})$-valued function, we obtain by direct calculation, using (26),

$$\dot{z}(t) = Az(t) - k(t)\varphi^\diamond(u(t))A^{-1}B\tilde{\psi}(\tilde{w}(t)) \quad \text{for a.a. } t \geq 0, \tag{31a}$$

$$\dot{v}(t) = -k(t)\varphi^\diamond(u(t))\tilde{\psi}(\tilde{w}(t)) \quad \text{for a.a. } t \geq 0, \tag{31b}$$

with initial values $z^0 := x^0 + A^{-1}B\varphi(u^0)$ and $v^0 := \varphi(u^0) - \varphi^*$. Invoking (27),

$$\dot{z}(t) = Az(t) + A^{-1}B\zeta(t) \quad \text{for a.a. } t \geq 0, \tag{32a}$$

$$\dot{v}(t) = \zeta(t) \quad \text{for a.a. } t \geq 0, \tag{32b}$$

where

$$\begin{aligned} \zeta(t) :&= \frac{d}{dt}\varphi(u(t)) = \dot{v}(t) \\ &= -k(t)\eta(t)\tilde{w}(t) \quad \text{with } \eta(t) := \varphi^\diamond(u(t))\gamma(\tilde{w}(t)). \end{aligned} \tag{33}$$

Note that

$$0 \leq \eta(t) \leq \lambda_1\lambda_2 =: \lambda, \qquad \forall\, t \geq 0. \tag{34}$$

By assumption, $\limsup_{t\to\infty} k(t) < k^*$. Hence there exists $k_* > 0$ such that

$$\limsup_{t\to\infty} k(t) < k_* < k^*. \tag{35}$$

Since $1/(k_*\lambda) > 1/\kappa^*$, it follows from Lemma 3.1 that there exist $P \in \mathscr{B}(X)$, $P = P^* \geq 0$, and $M \in \mathscr{B}(X_1, \mathbb{R})$ such that

$$\langle Ax_1, Px_2\rangle + \langle Px_1, Ax_2\rangle = -\langle Mx_1, Mx_2\rangle, \qquad \forall\, x_1, x_2 \in X_1, \tag{36a}$$

$$(A^{-1}B)^*Px = Cx - \sqrt{2/(k_*\lambda)}Mx, \qquad \forall\, x \in X_1. \tag{36b}$$

We investigate asymptotic properties of (z, v) using a Lyapunov approach. To make use of Eqs. (36) (which hold in X_1), we have to use an approximation argument. For this purpose, choose $(z_n^0) \subset X_1$ such that

$$\lim_{n\to\infty} \|z_n^0 - z^0\| = 0.$$

For all n, consider the initial-value problem

$$\dot{\xi} = A\xi + A^{-1}B\zeta, \qquad \xi(0) = z_n^0 \in X_1,$$

and define $z_n : \mathbb{R}_+ \to X$ by

$$z_n(t) := \mathbf{T}_t z_n^0 + A^{-1}\int_0^t \mathbf{T}_{t-\tau}B\zeta(\tau)\,d\tau.$$

Clearly, $z_n(t) \in X_1$ for all $t \geq 0$ and $Az_n \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$. Hence, by a standard result on abstract Cauchy problems (see Pazy [22, p. 109]), z_n is absolutely continuous as a X-valued function and

$$\dot{z}_n(t) = Az_n(t) + A^{-1}B\zeta(t) \quad \text{in } X, \text{ for a.a. } t \in \mathbb{R}_+. \tag{37}$$

Define the absolutely continuous function

$$V_n : \mathbb{R}_+ \to \mathbb{R}_+, \qquad t \mapsto \langle z_n(t), Pz_n(t)\rangle + \mathbf{G}(0)v^2(t).$$

Differentiating V_n and using (32b), (33), (36), and (37), we have for almost all $t \in \mathbb{R}_+$,

$$\begin{aligned}\dot{V}_n(t) &= \langle Az_n(t), Pz_n(t)\rangle + \langle Pz_n(t), Az_n(t)\rangle \\ &\quad + 2\langle \zeta(t), (A^{-1}B)^*Pz_n(t)\rangle + 2\mathbf{G}(0)v(t)\zeta(t) \\ &= -\langle Mz_n(t), Mz_n(t)\rangle + 2\Big\langle \zeta(t), Cz_n(t) - \sqrt{2/(k_*\lambda)}Mz_n(t)\Big\rangle \\ &\quad + 2\mathbf{G}(0)v(t)\zeta(t) \\ &= -(Mz_n(t))^2 - 2k(t)\eta(t) \\ &\quad \times \Big(\Big\langle \tilde{w}(t), Cz_n(t) - \sqrt{2/(k_*\lambda)}Mz_n(t)\Big\rangle + \mathbf{G}(0)v(t)\tilde{w}(t)\Big).\end{aligned} \tag{38}$$

Invoking (34) and (35), we see that there exists $t_0 \geq 0$ such that

$$\sup_{t \geq t_0} k(t)\eta(t) < k_* \lambda. \tag{39}$$

Integrating (38) gives

$$\begin{aligned} &V_n(t) - V_n(t_0) \\ &\quad = -\int_{t_0}^{t} (Mz_n(\tau))^2 d\tau - 2\int_{t_0}^{t} k(\tau)\eta(\tau) \\ &\qquad \times \Big[\Big\langle \tilde{w}(\tau), Cz_n(\tau) - \sqrt{2/(k_*\lambda)} Mz_n(\tau) \Big\rangle + \mathbf{G}(0)v(\tau)\tilde{w}(\tau)\Big] d\tau. \end{aligned} \tag{40}$$

Observe that

$$\lim_{n\to\infty} \|z_n(t) - z(t)\| = 0, \quad \lim_{n\to\infty} \|z_n - z\|_{L^2(0,\, t;\, X)} = 0, \qquad \forall\, t \geq 0. \tag{41}$$

Moreover, defining M_L by (4) with C and C_L replaced by M and M_L, respectively, it follows from (36b) that

$$M_L = \sqrt{(k_*\lambda)/2}(C_L - (A^{-1}B)^* P)$$

with $\operatorname{dom}(M_L) = \operatorname{dom}(C_L)$. Since for almost all $t \geq 0$,

$$Cz_n(t) = C_L z(t) = C\mathbf{T}_t z_n^0 - C_L \mathbf{T}_t z^0, \quad Mz_n(t) - M_L z(t) = M\mathbf{T}_t z_n^0 - M_L \mathbf{T}_t z^0,$$

we have

$$\begin{aligned} &\lim_{n\to\infty} \|Mz_n - M_L z\|_{L^2(0,\, t;\, \mathbb{R})} = 0 \quad \text{and} \\ &\lim_{n\to\infty} \|Cz_n - C_L z\|_{L^2(0,\, t;\, \mathbb{R})} = 0, \qquad \forall\, t \geq 0. \end{aligned} \tag{42}$$

Hence, defining

$$V : \mathbb{R}_+ \to \mathbb{R}_+, \qquad t \mapsto \langle z(t), Pz(t)\rangle + \mathbf{G}(0)v^2(t),$$

taking limits in (40) as $n \to \infty$, and invoking (41) and (42) yields

$$\begin{aligned} V(t) - V(t_0) = -\int_{t_0}^{t} (M_L z(\tau))^2 d\tau - 2\int_{t_0}^{t} k(\tau)\eta(\tau)\Big[\Big\langle \tilde{w}(\tau), C_L z(\tau) \\ - \sqrt{2/(k_*\lambda)} M_L z(\tau)\Big\rangle + \mathbf{G}(0)v(\tau)\tilde{w}(\tau)\Big] d\tau. \end{aligned} \tag{43}$$

Moreover, by (37),

$$Cz_n(t) = C\mathbf{T}_t z_n^0 + (\mathfrak{L}^{-1}(\mathbf{H}) \star \zeta)(t), \qquad \forall\, t \in \mathbb{R}_+, \tag{44}$$

where $\mathbf{H}(s) = C(sI - A)^{-1}A^{-1}B$. Since $\zeta(t) = (d/dt)(\varphi \circ u)(t)$ and $z^0 = x^0 + A^{-1}B\varphi(u^0)$, taking limits in (44) as $n \to \infty$ gives

$$C_L z(t) = C_L \mathbf{T}_t(x^0 + A^{-1}B\varphi(u^0)) + \left(\mathfrak{L}^{-1}(\mathbf{H}) \star \frac{d}{dt}(\varphi \circ u)\right)(t) \quad \text{for a.a } t \in \mathbb{R}_+.$$

Therefore, by Lemma 2.6,

$$C_L z(t) = w(t) - \mathbf{G}(0)(\varphi \circ u)(t) = \tilde{w}(t) - \mathbf{G}(0)v(t) \quad \text{for a.a. } t \in \mathbb{R}_+.$$

Using this in (43) and completing squares, we obtain

$$V(t) = V(t_0) - \int_{t_0}^{t} \left\{ \left(M_L z(\tau) - k(\tau)\eta(\tau)\sqrt{2/(k_*\lambda)}\tilde{w}(\tau)\right)^2 + 2k(\tau)\eta(\tau)\left[1 - \frac{k(\tau)\eta(\tau)}{k_*\lambda}\right]\tilde{w}^2(\tau)\right\} d\tau. \quad (45)$$

Note that, by (39), the integrand in (45) is nonnegative, implying that V is nonincreasing on $[t_0, \infty)$. Since V is bounded below, $\lim_{t\to\infty} V(t)$ exists and is finite. Equation (45) combined with (39) moreover implies that

$$\zeta = -k\eta\tilde{w} \in L^2(\mathbb{R}_+, \mathbb{R}). \quad (46)$$

Using this in (32a) and appealing to the fact that the semigroup $\mathbf{T}$ is exponentially stable, we may conclude that $z(t) \to 0$ in X. Combining this with the convergence of V, we may conclude that $\lim_{t\to\infty} \mathbf{G}(0)v^2(t)$ exists and is finite. Hence $\lim_{t\to\infty} \varphi(u(t))$ exists and is finite which is statement A.

B. Define

$$w_0(t) := C_L \mathbf{T}_t(x^0 + A^{-1}B\varphi(u^0)),$$
$$w_1(t) := \left(\mathfrak{L}^{-1}(\mathbf{H}) \star \frac{d}{dt}(\varphi \circ u)\right)(t) + \mathbf{G}(0)\varphi(u(t)),$$

where $\mathbf{H}$ is given by (17). Clearly, $\varphi \circ u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ and hence an application of Lemma 2.6 shows that

$$y(t) = \psi(w_0(t) + w_1(t)) \quad \text{for a.a. } t \in \mathbb{R}_+.$$

Since, by (46), $(d/dt)(\varphi \circ u) = \zeta \in L^2(\mathbb{R}_+, \mathbb{R})$, and by part A, $\lim_{t\to\infty} \varphi(u(t)) =: \varphi^r$ exists and is finite, we may infer from Corollary 2.7 that

$$\lim_{t\to\infty} w_1(t) = \mathbf{G}(0)\varphi^r. \quad (47)$$

Evidently, $\varphi^r \in \overline{\Phi}$ and so, to establish (b1), it suffices to show that $\varphi^r \in \Psi^r$. Seeking a contradiction, suppose that $\varphi^r \notin \Psi^r$. This implies that

$$\theta := \frac{r - \psi(\mathbf{G}(0)\varphi^r)}{2} \neq 0.$$

Using continuity of ψ and (47), we obtain for sufficiently large $s > 0$,

$$|\psi(w_1(t)) - \psi(\mathbf{G}(0)\varphi^r)| \le |\theta|, \qquad \forall\, t \ge s,$$

which, together with

$$\dot{u}(t) = k(t)[r - y(t)] = k(t)[2\theta + \psi(\mathbf{G}(0)\varphi^r) - \psi(w_0(t) + w_1(t))],$$

implies that

$$\begin{aligned} &-k(t)(|\theta| + |\psi(w_1(t)) - \psi(w_0(t) + w_1(t))|) \\ &\qquad \le \dot{u}(t) - 2\theta k(t) \\ &\qquad \le k(t)(|\theta| + |\psi(w_1(t)) - \psi(w_0(t) + w_1(t))|), \qquad \forall\, t \ge s. \end{aligned}$$

Since $\theta \ne 0$, either $\theta > 0$ or $\theta < 0$. Assume $\theta > 0$. Then

$$\dot{u}(t) \ge \theta k(t) - k(t)|\psi(w_1(t)) - \psi(w_0(t) + w_1(t))|, \qquad \forall\, t \ge s,$$

which, on integration and recalling that $\psi \in \mathscr{M}_L(\lambda_2)$, yields

$$\begin{aligned} u(t) - u(s) &\ge \theta(K(t) - K(s)) - \int_s^t k(\tau)|\psi(w_1(\tau)) - \psi(w_0(\tau) + w_1(\tau))|\,d\tau \\ &\ge \theta(K(t) - K(s)) - \lambda_2 \int_s^t k(\tau)|w_0(\tau)|\,d\tau, \qquad \forall\, t \ge s. \end{aligned}$$

By exponential stability of (32a), $w_0 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$, and thus $w_0 \in L^1(\mathbb{R}_+, \mathbb{R})$, which in turn implies that $kw_0 \in L^1(\mathbb{R}_+, \mathbb{R})$. Since $K(t) \to \infty$ as $t \to \infty$, we conclude that $u(t) \to \infty$ as $t \to \infty$; hence,

$$\sup \overline{\Phi} = \lim_{t\to\infty} \varphi(u(t)) = \varphi^r. \tag{48}$$

Let $\varphi^* \in \Psi^r \cap \overline{\Phi}$. Since $\theta > 0$ and ψ is nondecreasing, it follows $\varphi^* > \varphi^r$ and, consequently, by (48), $\sup \overline{\Phi} < \varphi^*$, a contradiction. A similar argument shows that the assumption $\theta < 0$ also leads to a contradiction. Therefore, we may conclude $\varphi^r \in \Psi^r \cap \overline{\Phi}$, which is statement (b1). Statement (b2) follows from Lemma 2.4, part (b). To prove statement (b3) first note that

$$\begin{aligned} |r - y(t)| &= |r - \psi(w_0(t) + w_1(t))| = |\psi(\mathbf{G}(0)\varphi^r) - \psi(w_0(t) + w_1(t))| \\ &\le \lambda_2|\mathbf{G}(0)\varphi^r - w_0(t) - w_1(t)| \\ &\le \lambda_2|w_0(t)| + \lambda_2|\mathbf{G}(0)\varphi^r - w_1(t)|. \end{aligned} \tag{49}$$

Since, by exponential stability, $w_0 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$ and $w_1(t) \to \mathbf{G}(0)\varphi^r$ as $t \to \infty$, statement (b3) follows by setting $e_0(t) := \lambda_2|w_0(t)|$ and $e_1(t) := \lambda_2|\mathbf{G}(0)\varphi^r - w_1(t)|$. If $\mathbf{T}_{t_0}(Ax^0 + B\varphi(u^0)) \in X$ for some $t_0 \ge 0$, then $\mathbf{T}_{t_0}(x^0 + A^{-1}B\varphi(u^0)) \in X_1$. Hence $w_0(t)$ converges (exponentially) to 0 as $t \to \infty$ and (b4) follows from (47) and (49).

Next, we establish statement (b5). Assume $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, which, together with (b1), implies the existence of $\xi^* \in \mathbb{R}$ such that $\varphi^r = \varphi(\xi^*)$. Seeking a contradiction, suppose that $\operatorname{dist}(u(t), \varphi^{-1}(\varphi^r)) \not\to 0$ as $t \to \infty$. Then there exist $\varepsilon > 0$ and a sequence $(t_n) \in \mathbb{R}_+$ with $t_n \to \infty$ as $n \to \infty$, such that

$$\operatorname{dist}(u(t_n), \varphi^{-1}(\varphi^r)) \geq \varepsilon. \tag{50}$$

If the sequence $(u(t_n))$ is bounded, we may assume without loss of generality that it converges to a finite limit u_∞. By continuity, $\varphi(u_\infty) = \varphi^r$ and so $u_\infty \in \varphi^{-1}(\varphi^r)$. This contradicts (50). Therefore, we may assume that $(u(t_n))$ is unbounded. Extracting a subsequence if necessary, we may then assume that either $u(t_n) \to \infty$ or $u(t_n) \to -\infty$ as $n \to \infty$: if the former holds, then $u(t_n) > \xi^*$ for all n sufficiently large; if the latter holds, then $u(t_n) < \xi^*$ for all n sufficiently large. In either case, by monotonicity of φ it follows that $\varphi(u(t_n)) = \varphi(\xi^*) = \varphi^r$ for all n sufficiently large. Clearly, this contradicts (50) and so statement (b5) must hold.

To prove (b6), assume that $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \operatorname{int}(\Phi)$ and, for contradiction, suppose that u is unbounded. Then there exists a sequence $(t_n) \subset (0, \infty)$ with $t_n \to \infty$ and $|u(t_n)| \to \infty$ as $n \to \infty$. By monotonicity of φ and (b1), it then follows that either $\varphi^r = \sup \Phi$ or $\varphi^r = \inf \Phi$, contradicting the fact that $\varphi^r \in \Psi^r \cap \operatorname{int}(\Phi) \subset \operatorname{int}(\Phi)$. Therefore, u is bounded. This completes the proof of part B. ∎

4.2. *Locally Lipschitz Output Nonlinearities*

In this subsection it is shown that the global Lipschitz assumption on ψ may be relaxed to the requirement $(\varphi, \psi) \in \mathcal{N}$, provided that $x^0 \in X_1 = \operatorname{dom}(A)$ and $\mu := \mathfrak{L}^{-1}(\mathbf{G}) \in \mathfrak{M}$, where $\mathcal{N}$ is the set of pairs of nonlinearities introduced in Section 3.2 and $\mathfrak{M}$ denotes the space of all finite signed Borel measures.

The proof of the following lemma can be found in the Appendix.

LEMMA 4.4. *Let $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F}) \in \mathcal{S}$ with transfer function $\mathbf{G}$ and generating operators (A, B, C). Let $r \in \mathbb{R}$ and $k \in L^\infty(\mathbb{R}_+, \mathbb{R})$. Assume that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathfrak{M}$, $(\varphi, \psi) \in \mathcal{N}$, and $(x^0, u^0) \in X_1 \times \mathbb{R}$. Then the initial-value problem* (25) *has a unique solution $(x, u) : \mathbb{R}_+ \to X \times \mathbb{R}$.*

Before presenting the main results of this subsection, we describe a convenient family of projection operators. Specifically, with each $p \in [0, \infty]$, we associate an operator $\Pi_p : \mathcal{M} \to \mathcal{M}$ with the property $\Pi_p \circ \Pi_p = \Pi_p$ (whence the terminology projection operator), defined as

$$\text{if } p < \infty, \text{ then } \Pi_p f : \xi \mapsto \begin{cases} f(-p), & \xi < -p, \\ f(\xi), & |\xi| \leq p, \\ f(p), & \xi > p. \end{cases}$$

if $p = \infty$, then $\Pi_p f = \Pi_\infty f := f$.

THEOREM 4.5. *Let $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F}) \in \mathscr{S}$ with transfer function $\mathbf{G}$ and generating operators (A, B, C). Let $r \in \mathscr{R}$ and $k \in \mathscr{G}$. Assume that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathfrak{M}$, $(\varphi, \psi) \in \mathscr{N}$, and $(x^0, u^0) \in X_1 \times \mathbb{R}$. Let $(x, u) : \mathbb{R}_+ \to X \times \mathbb{R}$ be the unique solution of the initial-value problem (25). Then there exists $k^* > 0$, independent of (x^0, u^0) and k, such that the following hold.*

A. *If $\limsup_{t\to\infty} k(t) < k^*$, then $\lim_{t\to\infty} \varphi(u(t))$ exists and is finite.*

B. *If $\limsup_{t\to\infty} k(t) < k^*$ and $K(t) := \int_0^t k \to \infty$ as $t \to \infty$, then*

   (b1) $\lim_{t\to\infty} \varphi(u(t)) =: \varphi^r \in \Psi^r \cap \overline{\Phi}$,

   (b2) $\lim_{t\to\infty} \|x(t) + A^{-1}B\varphi^r\| = 0$,

   (b3) $\lim_{t\to\infty} y(t) = r$, *where* $y(t) = \psi(C_L \mathbf{T}_t x^0 + (\mathbf{F}_\infty(\varphi \circ u))(t))$,

   (b4) *if* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, *then* $\lim_{t\to\infty} \operatorname{dist}(u(t), \varphi^{-1}(\varphi^r)) = 0$,

   (b5) *if* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \operatorname{int}(\Phi)$, *then* $u(\cdot)$ *is bounded.*

Proof. Set $\mu := \mathfrak{L}^{-1}(\mathbf{G})$ and define

$$p := |\mu|(\mathbb{R}_+) \sup_{\xi\in\Phi} |\xi| \in (0, \infty],$$

where $|\mu|$ denotes the total variation of μ.

A. Let $\varepsilon > 0$. Since $x^0 \in X_1$ it follows from the exponential stability of $\mathbf{T}$ that there exists $t_0 \geq 0$ such that

$$|w(t)| \leq |C_L \mathbf{T}_t x^0| + |(\mu \star (\varphi \circ u))(t)| \leq \varepsilon + |\mu|(\mathbb{R}_+) \sup_{\xi\in\Phi} |\xi|, \quad \forall\, t \geq t_0.$$

Therefore,

$$|w(t)| \leq p + \varepsilon =: q \in (0, \infty], \quad \forall\, t \geq t_0,$$

and hence

$$\psi(w(t)) = \Pi_q \psi(w(t)), \qquad \forall\, t \geq t_0. \tag{51}$$

Define $\hat{\Psi}^r := \{\xi \in \mathbb{R} | \Pi_q \psi(\mathbf{G}(0)\xi) = r\}$. We claim that

$$\hat{\Psi}^r \cap \overline{\Phi} = \Psi^r \cap \overline{\Phi}. \tag{52}$$

To see this, note that $\mathbf{G}(0) \leq |\mu|(\mathbb{R}_+)$. Hence, $\mathbf{G}(0)|\xi| \leq p$ for all $\xi \in \overline{\Phi}$ and, therefore, $\Pi_q \psi(\mathbf{G}(0)\xi) = \psi(\mathbf{G}(0)\xi)$ for all $\xi \in \overline{\Phi}$, which in turn implies (52).

Since $r \in \mathscr{R}$, we may conclude from (52) that $\hat{\Psi}^r \neq \emptyset$. Let $\varphi^* \in \hat{\Psi}^r$. Since $\Pi_q \psi \in \mathscr{M}(\lambda_2)$ for some $\lambda_2 > 0$, the function

$$\tilde{\psi} : \mathbb{R} \to \mathbb{R}, \qquad \xi \mapsto \Pi_q \psi(\xi + \mathbf{G}(0)\varphi^*) - r$$

is in $\mathcal{M}(\tilde{\lambda}_2)$ for some $\tilde{\lambda}_2 > 0$ [cf. Remark 3.2(ii)]. Therefore, by Remark 3.2(i),

$$\tilde{\psi}(\xi) = \gamma(\xi)\xi, \qquad \forall\, \xi \in \mathbb{R},$$

where the function γ is defined by

$$\gamma(\xi) = \begin{cases} \tilde{\psi}(\xi)/\xi, & \xi \neq 0, \\ \tilde{\lambda}_2, & \xi = 0, \end{cases}$$

and satisfies $0 \leq \gamma(\xi) \leq \tilde{\lambda}_2$ for all $\xi \in \mathbb{R}$. Let $\lambda_1 > 0$ be a Lipschitz constant for φ and define

$$\lambda := \lambda_1 \tilde{\lambda}_2, \qquad k^* := \kappa^*/\lambda,$$

where κ^* is given by (16). Defining z and v by (29) and $\tilde{w}$ by (30), it follows by a direct calculation using (51) that the differential equations (31a) and (31b) hold for almost all $t \geq t_0$. Part A now follows in exactly the same way as the corresponding part in the proof of Theorem 4.2.

B. By part A, $\varphi^r := \lim_{t\to\infty} \varphi(u(t))$ exists and is finite. Moreover, $x^0 \in X_1$ and, hence, by the exponential stability of $\mathbf{T}$ and by Theorem 6.1, part (ii) of [7, p. 96]

$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} C\mathbf{T}_t x^0 + \lim_{t\to\infty} (\mu \star (\varphi \circ u))(t) = \mathbf{G}(0)\varphi^r. \tag{53}$$

In particular, w is bounded. Combining this with the local Lipschitz property of ψ and invoking (53), part B can be derived in a similar way as the corresponding part of Theorem 4.2. ∎

Remark 4.6. (i) The assumption that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathfrak{M}$ implies that $\mathbf{\Sigma}$ is regular and might be difficult to verify for a given well-posed system. However, it seems that it holds for most (if not all) practical examples of exponentially stable well-posed systems. In particular, if B or C is bounded, then $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathfrak{M}$ (see Lemma 2.3 in [11]).

(ii) If φ is bounded, then for any number $q > |\mu|(\mathbb{R}_+) \sup_{\xi\in\Phi} |\xi|$, an inspection of the proof above combined with Remark 3.2(iii) shows that k^* can be chosen as $k^* = \kappa^*/(\lambda_1\lambda_2)$, where κ^* is given by (16), λ_1 is a global Lipschitz constant for φ, and λ_2 is a Lipschitz constant for ψ on $[-q, q]$. Under the extra assumption that ψ is continuously differentiable, this statement remains true for $q = |\mu|(\mathbb{R}_+) \sup_{\xi\in\Phi} |\xi|$.

(iii) Parts (iii)–(v) of Remark 4.3 apply to Theorem 4.5.

5. EXAMPLE: DIFFUSION PROCESS WITH OUTPUT DELAY

Consider a diffusion process (with diffusion coefficient $a > 0$ and with Dirichlet boundary conditions) on the one-dimensional spatial domain

$I = [0, 1]$, with scalar nonlinear pointwise control action (applied at point $x_b \in I$, via a nonlinearity φ with Lipschitz constant $\lambda_1 > 0$) and delayed (delay $h \geq 0$) pointwise scalar observation (output at point $x_c \in I$) with a nonlinearity ψ in the output channel. It is assumed that $x_b < x_c$. We formally write this single-input–single-output system as

$$z_t(t, x) = az_{xx}(t, x) + \delta(x - x_b)\varphi(u(t)),$$

$$w(t) = z(t - h, x_c), \qquad y(t) = \psi(w(t)),$$

$$z(t, 0) = 0 = z(t, 1), \qquad \forall\, t > 0.$$

For simplicity, we assume zero initial conditions:

$$z(t, x) = 0, \qquad \forall\, (t, x) \in [-h, 0] \times [0, 1].$$

With input $\varphi(u(\cdot))$ and output $w(\cdot)$, this example qualifies as a well-posed (even regular) linear system with transfer function given by

$$\mathbf{G}(s) = \frac{e^{-sh}\sinh(x_b\sqrt{(s/a)})\sinh((1 - x_c)\sqrt{(s/a)})}{a\sqrt{(s/a)}\sinh\sqrt{(s/a)}}.$$

Since $\mathbf{G}(0) = x_b(1 - x_c)/a$, we have for $r \in \mathbb{R}$,

$$\Psi^r := \{v \in \mathbb{R} \mid (\mathbf{G}(0)v)^3 = r\} = \left\{\frac{ar^{1/3}}{x_b(1 - x_c)}\right\}.$$

From [13] we know that

$$\kappa^* = \frac{1}{|\mathbf{G}'(0)|} = \frac{6a^2}{x_b(1 - x_c)(6ha + 1 - x_b^2 - (1 - x_c)^2)}.$$

Note the dependence of κ^* on the time-delay h: the larger is h, the smaller is κ^*.

In the sequel, we choose as output nonlinearity $\psi \in \mathcal{M}$ the cubic $\psi : w \mapsto w^3$ and an input nonlinearity $\varphi \in \mathcal{M}_L(1)$ of saturation type, defined as

$$u \mapsto \varphi(u) := \begin{cases} 1, & u \geq 1, \\ u, & u \in (0, 1), \\ 0, & u \leq 0, \end{cases}$$

in which case $\overline{\Phi} = [0, 1]$. Thus $\Psi^r \cap \overline{\Phi} \neq \emptyset$ if and only if $0 \leq r \leq (x_b(1 - x_c)/a)^3$ and so the set of feasible reference values $\mathcal{R}$ is given by $\mathcal{R} = [0, (x_b(1 - x_c)/a)^3]$.

It is not difficult to show that $\mu := \mathfrak{L}^{-1}(\mathbf{G}) \in \mathfrak{M}$. In fact, $\mu(E) = \int_E f(\tau)\, d\tau$ for some $f \in L^1(\mathbb{R}_+, \mathbb{R})$. Therefore, by Theorem 4.5, the integral control

$$u(t) = \int_0^t k(\tau)[r - y(\tau)]\, d\tau,$$

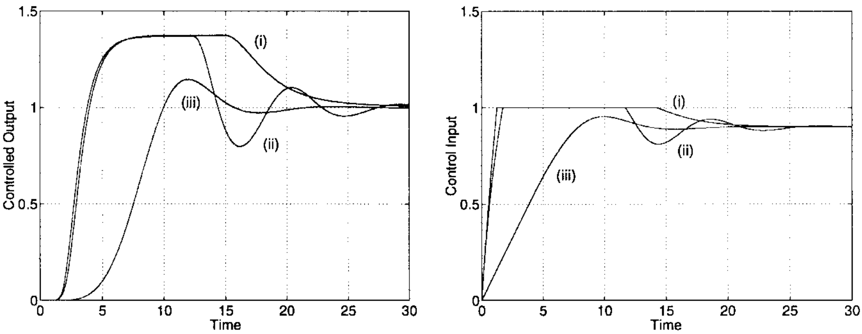

FIG. 2. Performance under low-gain control.

where the gain function k is chosen according to the assumptions of Theorem 4.5, guarantees asymptotic tracking of every constant reference values $r \in \mathcal{R}$.

Further analysis (invoking application of the maximum principle for the heat equation, which, for brevity, we omit here) confirms the physical intuition that f is nonnegative valued. This implies $\mathbf{G}(0) = \mu(\mathbb{R}_+) = |\mu|(\mathbb{R}_+)$. Thus, with reference to Remark 4.6(ii), k^* can be chosen as $k^* = \kappa^*/(\lambda_1\lambda_2)$, where $\lambda_1 = 1$ and $\lambda_2 = 3\mathbf{G}(0)^2$.

For purposes of illustration, we adopt the values

$$a = 0.1, \qquad x_b = \tfrac{1}{3}, \qquad x_c = \tfrac{2}{3}, \qquad h = 0.5,$$

such that $k^* = 6561/48500 \approx 0.135$. For unit reference signal $r = 1 \in \mathcal{R}$, Figure 2 depicts the behaviour of the system in each of the following three cases of controller gain functions k:

$$\text{(i) } k(t) = 1/(t+1); \qquad \text{(ii) } k(t) = \sqrt{1/(t+1)}; \qquad \text{(iii) } k(t) \equiv 0.13.$$

This figure was generated using SIMULINK simulation software within MATLAB wherein a truncated eigenfunction expansion of order 20 was adopted to model the diffusion process. Note that, while the constant gain case (iii) results in the best transient response, it requires structural information on the system to compute k^*, which may not be available in general. For gain functions $k \in \mathcal{G}$ with $k(t) \to 0$ as $t \to \infty$ and $k \notin L^1$ [as in cases (i) and (ii)], knowledge of k^* is not required.

APPENDIX: PROOF OF LEMMAS 4.1 AND 4.4

To prove Lemmas 4.1 and 4.4, we apply a result from [11] on the existence of a unique maximal solution u of the abstract Volterra integrodifferential equation,

$$\dot{u}(t) = (Fu)(t), \qquad t \geq t_0, \tag{54a}$$

$$u(t) = u_{t_0}(t), \qquad t \in [0, t_0], \tag{54b}$$

where $t_0 \geq 0$, $u_{t_0} \in C([0, t_0], \mathbb{R})$, and the operator $F : C(\mathbb{R}_+, \mathbb{R}) \to L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ is causal and *weakly Lipschitz* in the following sense:

For all $\alpha \geq 0$, $\delta > 0$, $\rho > 0$, and $\theta \in C([0, \alpha], \mathbb{R})$, there exists a continuous function $f : [0.\delta] \to \mathbb{R}_+$, with $f(0) = 0$, such that

$$\int_\alpha^{\alpha+\varepsilon} \|(Fv)(t) - (Fw)(t)\| dt \leq f(\varepsilon) \sup_{\alpha \leq t \leq \alpha+\varepsilon} \|v(t) - w(t)\|$$

for all $\varepsilon \in [0, \ \delta]$ and for all $v, w \in \mathscr{C}(\alpha, \delta, \rho, \theta)$, where

$$\mathscr{C}(\alpha, \delta, \rho, \theta) := \{w \in C([0, \alpha + \delta], \mathbb{R}) \mid w(t) = \theta(t) \ \forall \, t \in [0, \alpha], \\ \|w(t) - \theta(\alpha)\| \leq \rho \ \forall \, t \in [\alpha, \alpha + \delta]\}.$$

A *solution* of the initial-value problem (54) on an interval $[0, t_1)$, where $t_1 > t_0$, is a function $u \in C([0, t_1), \mathbb{R})$, with $u(t) = u_{t_0}(t)$ for all $t \in [0, t_0]$, such that u is absolutely continuous on $[t_0, t_1)$ and (54a) is satisfied for almost all $t \in [t_0, t_1)$.

Strictly speaking, to make sense of (54), we have to give a meaning to $(Fu)(t)$, $t \in [0, t_1)$, when u is a continuous function defined on a *finite* interval $[0, t_1)$ (recall that F operates on the space of continuous functions defined on the *infinite* interval $\mathbb{R}_+$). This can be easily done using causality of F: for all $t \in [0, t_1)$, $(Fu)(t) := (Fu^*)(t)$, where $u^* : \mathbb{R}_+ \to \mathbb{R}$ is any continuous function with $u^*(s) = u(s)$ for all $s \in [0, t]$.

The proof of the following result can be found in [11].

PROPOSITION A.1. *For every* $t_0 \geq 0$ *and every* $u_{t_0} \in C([0, t_0], \mathbb{R})$, *there exists a unique solution u of* (54) *defined on a maximal interval* $[0, t_{\max})$, *with* $t_{\max} > t_0$. *Moreover, if* $t_{\max} < \infty$, *then*

$$\limsup_{t \to t_{\max}} |u(t)| = \infty. \tag{55}$$

In the sequel we shall invoke Proposition A.1 only in the special case $t_0 = 0$.

For $(x^0, u^0) \in X \times \mathbb{R}$, define the causal map $F : C(\mathbb{R}_+, \mathbb{R}) \to L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ by

$$(Fu)(t) := k(t)[r - \psi((\boldsymbol{\Psi}_\infty x^0)(t) + (\mathbf{F}_\infty(\varphi \circ u))(t))]. \tag{56}$$

To proceed, we need the following lemma (see [11]).

LEMMA A.2. *For all* $\alpha \geq 0$, $v \in C([0, \alpha], \mathbb{R})$, $\delta > 0$, *and* $\rho > 0$, *there exists* $\gamma > 0$ *such that for all* $\varepsilon \in [0, \delta]$ *and* $u, w \in \mathscr{C}(\alpha, \delta, \rho, v)$,

$$\int_{\alpha}^{\alpha+\varepsilon} |(\mathbf{F}_\infty(\varphi \circ u))(\tau) - (\mathbf{F}_\infty(\varphi \circ w))(\tau)| d\tau \leq \varepsilon\gamma \sup_{\alpha \leq \tau \leq \alpha+\varepsilon} |u(\tau) - w(\tau)|. \quad (57)$$

It follows via a routine argument involving Lemma A.2 that, under the assumptions of Lemmas 4.1 and 4.4, respectively, the operator F as defined in (56) is weakly Lipschitz. Hence it follows from Proposition A.1 that the initial-value problems considered in Lemmas 4.1 and 4.4 have a unique solution u on a maximal interval of existence $[0, t_{\max})$.

To prove that $t_{\max} = \infty$, let $T \in \mathbb{R}_+$ be such that $T \leq t_{\max}$. Let κ be such that $0 \leq |k(t)| \leq \kappa$ for all $t \geq 0$. Multiplying (25b) by u, we obtain

$$\begin{aligned} u(\tau)\dot{u}(\tau) &= u(\tau)k(\tau)(r - \psi[(\boldsymbol{\Psi}_\infty x^0)(\tau) + (\mathbf{F}_\infty(\varphi \circ u))(\tau)]) \\ &\leq \kappa|u(\tau)|(|r - \psi(0)| + |\psi[(\boldsymbol{\Psi}_\infty x^0)(\tau) \\ &\quad + (\mathbf{F}_\infty(\varphi \circ u))(\tau)] - \psi(0)|). \end{aligned} \quad (58)$$

If the hypotheses of Lemma 4.1 hold, then $\psi \in \mathscr{M}_L(\lambda_2)$ for some $\lambda_2 \geq 0$ and

$$\begin{aligned} &|\psi[(\boldsymbol{\Psi}_\infty x^0)(\tau) + (\mathbf{F}_\infty(\varphi \circ u))(\tau)] - \psi(0)| \\ &\quad \leq \lambda_2|(\boldsymbol{\Psi}_\infty x^0)(\tau) + (\mathbf{F}_\infty(\varphi \circ u))(\tau)| \qquad \forall\, \tau \in [0, T). \end{aligned} \quad (59)$$

Next we show that, for some $\lambda_2 \geq 0$, (59) remains valid if the hypotheses of Lemma 4.1 are replaced by the hypotheses of Lemma 4.4. Let the latter hypotheses hold and note that $(\boldsymbol{\Psi}_\infty x^0)(\tau)$ is bounded on $\mathbb{R}_+$. If φ is bounded, then choose $\lambda_2 \geq 0$ to be a Lipschitz constant for ψ on the interval $[-a, a]$, where

$$a := \sup_{t \in \mathbb{R}_+} |(\boldsymbol{\Psi}_\infty x^0)(t)| + |\mu|(\mathbb{R}_+) \sup_{\xi \in \Phi} |\xi|$$

and (59) holds. If φ is unbounded, then $\psi \in \mathscr{M}(\lambda_2)$ for some $\lambda_2 \geq 0$, in which case

$$0 \leq (\psi(\xi) - \psi(0))\xi \leq \lambda_2 \xi^2, \qquad \forall\, \xi \in \mathbb{R},$$

and, again, (59) holds.

Combining (58) and (59), and estimating, we obtain that

$$\begin{aligned} u(\tau)\dot{u}(\tau) &\leq \kappa[(r - \psi(0))^2 + \lambda_2^2(\boldsymbol{\Psi}_\infty x^0)^2(\tau) + u^2(\tau) \\ &\quad + \lambda_2|(\mathbf{F}_\infty\varphi(u))(\tau)u(\tau)|], \qquad \forall\, \tau \in [0, T), \end{aligned} \quad (60)$$

for some $\lambda_2 \geq 0$. Integrating (60) and invoking the estimate

$$\int_0^t |(\mathbf{F}_\infty\varphi(u))u| \leq \int_0^t |\mathbf{F}_\infty(\varphi(u) - \varphi(0))||u| + \tfrac{1}{2}\left(\int_0^t (\mathbf{F}_\infty\varphi(0))^2 + \int_0^t u^2\right),$$

together with the Cauchy–Schwarz inequality and the global Lipschitz property of φ, it can be readily shown that there exist positive constants β_1 and β_2 such that

$$u^2(t) \le \beta_1 + \beta_2 \int_0^t u^2(\tau)\, d\tau, \qquad \forall\, t \in [0, T).$$

An application of Gronwall's lemma then shows that $u^2(t) \le \beta_1 e^{\beta_2 t}$ for all $t \in [0, T)$. Hence u is bounded on $[0, T)$. Since this holds for all $T \in \mathbb{R}_+$ with $T \le t_{\max}$, it follows by Proposition A.1 that $t_{\max} = \infty$.

Finally, to obtain a solution of (25), define

$$x(t) := \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} B\varphi(u(\tau))\, d\tau, \qquad \forall\, t \ge 0. \tag{61}$$

By well-posedness, x is a continuous X-valued function and, moreover, since A, considered as a generator on X_{-1}, is in $\mathscr{B}(X, X_{-1})$, the function $t \mapsto Ax(t)$ is a continuous X_{-1}-valued function. Consequently, by Pazy [22, Theorem 2.4, p. 107], in X_{-1} we have

$$\dot{x}(t) = Ax(t) + B\varphi(u(t)), \qquad \forall\, t \in \mathbb{R}_+.$$

It follows that (x, u) is the unique solution of (25) defined on $\mathbb{R}_+$. ∎

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