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STABILITY RESULTS OF POPOV-TYPE FOR INFINITE-DIMENSIONAL SYSTEMS WITH APPLICATIONS TO INTEGRAL CONTROL

R. F. CURTAIN, H. LOGEMANN AND O. STAFFANS

1. *Introduction*

Absolute stability problems and their relations to positive-real conditions have played a prominent role in systems and control theory and have led to a number of important stability criteria for unity feedback controls applied to linear dynamical systems subject to static input or output non-linearities, see [**1**, **13**, **14**, **23**, **34**, **39**] for the finite-dimensional and [**3**, **6**, **9**, **18**, **22**, **34**, **38**] for the infinite-dimensional case, to mention just a few references. In this paper we study an absolute stability problem for the feedback system shown in Figure 1.

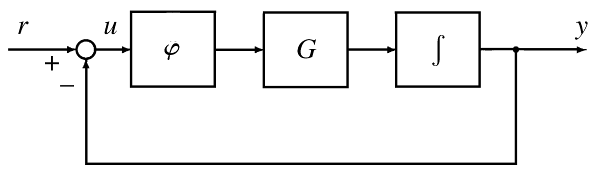

FIGURE 1.

The input-output operator G is linear, shift-invariant and bounded from $L^2(\mathbb{R}_+, U)$ into itself and $\varphi\colon U \to U$ is a locally Lipschitz non-linearity, where U is a real separable Hilbert space. It is well known that G can be represented by a transfer function $\mathbf{G}$ which is analytic and bounded on the open right-half of the complex plane. For simplicity we assume in the introduction that $\mathbf{G}$ admits an analytic extension to an open neighbourhood of 0 (this assumption will be weakened in §§ 2–4). In § 2 we show that if $r\colon \mathbb{R}_+ \to U$ is continuous, then the feedback system in Figure 1 has a unique continuous solution u which can be continued as long as it remains bounded. In fact, the existence and uniqueness result in § 2 (Lemma 2.1) is more general in the sense that it allows for unbounded G and time-varying φ. The main result in § 3 (Theorem 3.1) shows in particular that if $\mathbf{G}(0)$ is invertible and if there exist a linear bounded self-adjoint operator $P\colon U \to U$, a linear, bounded, invertible operator $Q\colon U \to U$ with $Q\mathbf{G}(0) = [Q\mathbf{G}(0)]^* \geqslant 0$ and numbers $q \geqslant 0$ and $\varepsilon > 0$ such that

$$P + \tfrac{1}{2}\left(q\mathbf{G}(i\omega) + \frac{1}{i\omega} Q\mathbf{G}(i\omega) + q\mathbf{G}^*(i\omega) - \frac{1}{i\omega}\mathbf{G}^*(i\omega)Q^*\right) \geqslant \varepsilon I, \quad \text{for a.a. } \omega \in \mathbb{R},$$

then for any r of the form $r = r_1 + r_2$ with $r_1 \in W^{1,2}(\mathbb{R}_+, U)$ and $r_2 \in U$ and for

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any locally Lipschitz gradient field $\varphi: U \to U$ corresponding to a non-negative potential and such that

$$\langle \varphi(v), Qv \rangle \geqslant \langle \varphi(v), P\varphi(v) \rangle, \quad \text{for all } v \in U,$$

the solution u of the feedback system shown in Figure 1 exists on $\mathbb{R}_+$ (no finite escape-time), $u, y \in L^\infty(\mathbb{R}_+, U)$, $\varphi \circ u \in L^2(\mathbb{R}_+, U)$, $\lim_{t\to\infty} \varphi(u(t)) = 0$ and, under certain extra assumptions, $u(t)$ and $y(t)$ converge as $t \to \infty$. Moreover, it is shown in § 3 that if the above positive-real condition for $\mathbf{G}$ holds with $q = 0$, then we can allow the non-linearity φ to be time-varying (Theorem 3.3). If $Q = I$ and $P = (1/a)I$ for some $a \in (0, \infty]$, then the above inequality involving φ is equivalent to the standard sector condition

$$\langle \varphi(v), \varphi(v) - av \rangle \leqslant 0, \quad \text{for all } v \in U,$$

which in turn is equivalent to

$$\| \varphi(v) - \tfrac{1}{2}av \| \leqslant \tfrac{1}{2}a\|v\|, \quad \text{for all } v \in U.$$

We emphasize that in contrast to previous results in the literature, the two main results in § 3 (Theorems 3.1 and 3.3) consider feedback systems, where the linear part contains an integrator (meaning in particular that the linear system is not input-output stable) and where at the same time the lower gain $\inf_{v\in U} \|\varphi(v)\|/\|v\|$ of the non-linearity φ is allowed to be equal to zero (which, for example, is the case for bounded non-linearities such as saturation). One of the motivations for studying this situation is its importance in § 4, where we use the absolute stability results from § 3 to develop an input-output theory of low-gain integral control in the presence of input non-linearities. The low-gain integral control problem has its roots in control engineering, where it is often required that the output y of a system tracks a constant reference signal ρ, that is, the error $e(t) := y(t) - \rho$ should be small in some sense for large t. It is well known that for exponentially stable time-invariant single-input single-output systems with positive steady-state gain (that is, $\mathbf{G}(0) > 0$) this can be achieved by feeding the error into an integrator with sufficiently small positive gain parameter and then closing the feedback loop. In § 4 we develop generalizations of this result to linear systems subject to input and/or output non-linearities.

It is assumed in § 4 that G, $\mathbf{G}$ and φ are as before, but in a single-input–single-output context (that is, $U = \mathbb{R}$). Consider the feedback system shown in Figure 2, where $\rho \in \mathbb{R}$ is a constant, $k \in \mathbb{R}$ is a gain parameter and $g: \mathbb{R}_+ \to \mathbb{R}$ is an output disturbance signal.

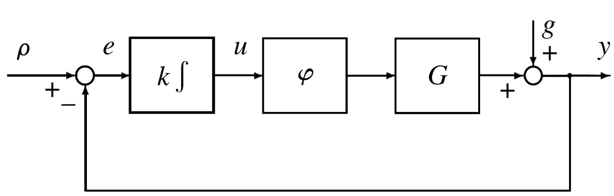

FIGURE 2.

Setting $r(t) := u(0) + k\int_0^t (\rho - g(\tau))\,d\tau$ and $z := k(y - g)$, one sees clearly that Figure 2 and Figure 3 are equivalent in the sense that the signal u solves the feedback system in Figure 2 if and only if u solves the feedback system in

Figure 3. This shows that the feedback system in Figure 2 can be reduced to that in Figure 1.

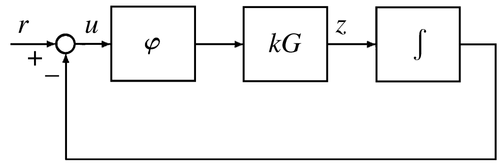

FIGURE 3.

The objective is to find a constant $k^* \in (0, \infty]$ in terms of the 'system data' G and φ such that for all $k \in (0, k^*)$ the tracking error $e(t)$ becomes small in some sense as $t \to \infty$. One of the main results in § 4 (Theorem 4.1) shows that if $\mathbf{G}(0) > 0$, $g \in L^2(\mathbb{R}_+, \mathbb{R})$ with $t \mapsto \int_0^t g(\tau)\, d\tau \in L^2(\mathbb{R}_+, \mathbb{R}) + \mathbb{R}$, φ is non-decreasing and globally Lipschitz with Lipschitz constant $\lambda > 0$ and $\rho/\mathbf{G}(0) \in \operatorname{im}\varphi$, then the limit $\lim_{t\to\infty} u(t) =: u^\infty$ exists and $\varphi \circ u - \varphi(u^\infty), e \in L^2(\mathbb{R}_+, \mathbb{R})$, provided that $k \in (0, 1/|\lambda f(G)|)$, where

$$f(G) := \sup_{q \geqslant 0}\{\operatorname*{ess\,inf}_{\omega \in \mathbb{R}} \operatorname{Re}[(q + 1/i\omega)\mathbf{G}(i\omega)]\}.$$

Under mild extra assumptions, the tracking error $e(t)$ converges to 0 as $t \to \infty$. Furthermore, in the other main contribution of § 4 (Theorem 4.4), we prove that a similar result holds for a generalized version of the feedback scheme in Figure 2 which allows for a time-varying gain and non-linearities in the input as well as in the output. Finally, § 5 is devoted to applications of the input-output results in §§ 3 and 4 to the class of well-posed state-space systems which are documented, for example, in [**7**, **26**, **27**, **29**, **30**, **31**, **32**, **35**, **36**]. We remark that the class of well-posed, linear, infinite-dimensional systems is rather general: it includes most distributed parameter systems and all time-delay systems (retarded and neutral) which are of interest in applications.

NOTATION. Let X be a real or complex Banach space; if X is real, then its complexification is denoted by X_c; if X is a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, then $\langle \cdot, \cdot \rangle$ extends in a natural way to a (complex) inner product on X_c and we shall use the same symbol $\langle \cdot, \cdot \rangle$ for the original inner product and its extension. A set $S \subset X$ is called a sphere centred at $z \in X$ if there exists $\eta \geqslant 0$ such that $S = \{x \in X \mid \|x - z\| = \eta\}$. For $\tau \geqslant 0$, $\mathbf{S}_\tau$ denotes the operator of the right-shift by τ on $L^p_{\text{loc}}(\mathbb{R}_+, X)$, where $\mathbb{R}_+ := [0, \infty)$. For $0 < \tau < \tau^* \leqslant \infty$, the truncation operator $\mathbf{P}_\tau$: $L^p_{\text{loc}}([0, \tau^*), X) \to L^p(\mathbb{R}_+, X)$ is given by $(\mathbf{P}_\tau u)(t) = u(t)$ if $t \in [0, \tau]$ and $(\mathbf{P}_\tau u)(t) = 0$ if $t > \tau$. For $\alpha \in \mathbb{R}$, we define the exponentially weighted L^p-space $L^p_\alpha(\mathbb{R}_+, X) := \{f \in L^p_{\text{loc}}(\mathbb{R}_+, X) \mid f(\cdot)\exp(-\alpha\,\cdot) \in L^p(\mathbb{R}_+, X)\}$ and endow it with the norm $\|f\|_{p,\alpha} := (\int_0^\infty \|e^{-\alpha t} f(t)\|^p\, dt)^{1/p}$. For an arbitrary interval $J \subset \mathbb{R}_+$, $C(J, X)$ denotes the space of all continuous functions defined on J with values in X; $W^{1,2}(\mathbb{R}_+, X)$ denotes the space of all functions $f \in L^2(\mathbb{R}_+, X)$ for which there exists $g \in L^2(\mathbb{R}_+, X)$ such that $f(t) - f(0) = \int_0^t g(s)\, ds$ for all $t \in \mathbb{R}_+$. For a function f: $\mathbb{R}_+ \to X$ and a subset $Z \subset X$, we say that $f(t)$ approaches Z as $t \to \infty$ if

$$\operatorname{dist}(f(t), Z) = \inf_{z \in Z} \|f(t) - z\| \to 0 \quad \text{as } t \to \infty.$$

If $\lim_{t\to\infty} f(t)$ exists, we denote this limit by f^∞, that is, $f^\infty := \lim_{t\to\infty} f(t)$. We say that a (strongly) measurable function $f\colon \mathbb{R}_+ \to X$ has an *essential limit* at ∞ if there exists $l \in X$ such that $\operatorname{ess\,sup}_{\tau\geqslant t}\|f(\tau) - l\|$ tends to 0 as $t\to\infty$ and we write $\operatorname{ess\,lim}_{t\to\infty} f(t) = l$ (a routine exercise shows that $\operatorname{ess\,lim}_{t\to\infty} f(t) = l$ if and only if there exists a function $\widetilde{f}\colon \mathbb{R}_+ \to X$ such that $\widetilde{f}(t) = f(t)$ for almost all $t\in\mathbb{R}_+$ and $\lim_{t\to\infty}\widetilde{f}(t) = l$). For $\alpha\in\mathbb{R}$, let $H^2(\mathbb{C}_\alpha, X)$ denote the Hardy–Lebesgue space of square-integrable holomorphic functions defined in $\mathbb{C}_\alpha$ with values in X, where $\mathbb{C}_\alpha := \{s\in\mathbb{C} \mid \operatorname{Re} s > \alpha\}$; $H^\infty(\mathbb{C}_\alpha, X)$ denotes the space of bounded holomorphic functions defined on $\mathbb{C}_\alpha$ with values in X. We use $\mathscr{B}(X_1, X_2)$ to denote the space of bounded linear operators from a Banach space X_1 to a Banach space X_2; we write $\mathscr{B}(X)$ for $\mathscr{B}(X, X)$. Let $A\colon \operatorname{dom}(A)\subset X\to X$ be a densely defined linear operator, where $\operatorname{dom}(A)$ denotes the domain of A. The resolvent set of A is denoted by $\operatorname{res}(A)$; X_1 denotes the space $\operatorname{dom}(A)$ endowed with the graph norm of A, whilst X_{-1} denotes the completion of X with respect to the norm $\|x\|_{-1} = \|(\alpha I - A)^{-1}x\|$, where $\alpha\in\operatorname{res}(A)$ (different choices of α lead to equivalent norms) and $\|\cdot\|$ denotes the norm on X. Clearly, $X_1\subset X\subset X_{-1}$ and the canonical injections are bounded and dense. If A generates a strongly continuous semigroup $\mathbf{T} = (\mathbf{T}_t)_{t\geqslant 0}$ on X, then $\mathbf{T}$ restricts to a strongly continuous semigroup on X_1 and extends to a strongly continuous semigroup on X_{-1} with the exponential growth constant being the same on all three spaces. Correspondingly, A restricts to a generator on X_1 and extends to a generator on X_{-1}. We shall use the same symbol $\mathbf{T}$ (respectively, A) for the original semigroup (respectively, generator) and the associated restrictions and extensions: with this convention, we may write $A\in\mathscr{B}(X, X_{-1})$ (considered as a generator on X_{-1}, the domain of A is X). The Laplace transform is denoted by $\mathfrak{L}$.

2. *Existence and uniqueness of solutions to the feedback system*

Throughout this section, let U be a real Hilbert space and let

$$G\colon L^2_{\mathrm{loc}}(\mathbb{R}_+, U)\to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$$

be a linear, continuous and causal operator, where we regard the space $L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ as a Fréchet space with its topology given by the family of seminorms $u\mapsto\|\mathbf{P}_n u\|_{L^2}$, with $n\in\mathbb{N}$. Recall that G is called causal if $\mathbf{P}_\tau G\mathbf{P}_\tau = \mathbf{P}_\tau G$ for all $\tau\in\mathbb{R}_+$. Note that a linear operator $G\colon L^2_{\mathrm{loc}}(\mathbb{R}_+, U)\to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ is continuous and causal if and only if for every $\tau\in\mathbb{R}$ there exists a constant $\gamma_\tau\geqslant 0$ such that

$$\|\mathbf{P}_\tau G u\|_{L^2}\leqslant\gamma_\tau\|\mathbf{P}_\tau u\|_{L^2},\quad \text{for all } u\in L^2_{\mathrm{loc}}(\mathbb{R}_+, U).$$

Consider the Volterra equation

$$u(t) = r(t) - \int_0^t (G(\varphi\circ u))(\tau)\,d\tau,\quad \text{for } t\geqslant 0, \tag{2.1}$$

which describes the feedback system shown in Figure 1. In (2.1), $r\colon\mathbb{R}_+\to U$ is the input of the feedback system (or forcing function), $\varphi\colon\mathbb{R}_+\times U\to U$ is a time-dependent non-linearity, and $\varphi\circ u$ denotes the function $t\mapsto\varphi(t, u(t))$. Let J be a time-interval of the form $J = [0, T]$ (for $0\leqslant T<\infty$) or $J = [0, T)$ (for $0 < T\leqslant\infty$). In order to define the concept of a solution of (2.1) on J, we need to give a meaning to Gv for $v\in L^2_{\mathrm{loc}}([0, T), U)$ if T is finite (recall that G operates on L^2_{loc}-functions defined on the whole time-axis $\mathbb{R}_+$). This can be done as follows: we define an operator $G_T\colon L^2_{\text{loc}}([0,T),U) \to L^2_{\text{loc}}([0,T),U)$ by setting

$$(G_T v)(t) = (G\mathbf{P}_\tau v)(t), \quad \text{for } 0 \leqslant t \leqslant \tau < T.$$

Since G is causal, this definition does not depend on the choice of τ and so G_T is well defined. Note that $G_T(L^2([0,T),U)) \subset L^2([0,T),U)$ for finite T. In the following we will not distinguish between G and G_T and we drop the subscript T.

A function $u\colon J \to U$ is called a *solution* of (2.1) on J, if the function $t \mapsto \varphi(t, u(t))$ is in $L^2_{\text{loc}}(J, U)$ (so that $G(\varphi \circ u)$ is defined on the interval J) and (2.1) holds for all $t \in J$ (for almost all $t \in J$ if $r(t)$ is only defined for a.a. $t \geqslant 0$ or if we consider r as an equivalence class of functions coinciding almost everywhere). If r is continuous, then so is the right-hand side of (2.1), and hence any solution u of (2.1) is then necessarily continuous.

The following lemma shows that if r is continuous and φ satisfies certain standard regularity conditions (including a Lipschitz-type condition), then (2.1) has a unique continuous solution which can be continued as long as it remains bounded.

LEMMA 2.1. *Let $G\colon L^2_{\text{loc}}(\mathbb{R}_+, U) \to L^2_{\text{loc}}(\mathbb{R}_+, U)$ be a linear, continuous and causal operator, let $r \in C(\mathbb{R}_+, U)$, and let $\varphi\colon \mathbb{R}_+ \times U \to U$ be such that $t \mapsto \varphi(t, v)$ is measurable for every $v \in U$, $t \mapsto \varphi(t, 0)$ is in $L^2_{\text{loc}}(\mathbb{R}_+, U)$, and for every bounded set $V \subset U$ there exists $\lambda_V \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ such that*

$$\sup_{v, w \in V} \frac{\|\varphi(t,v) - \varphi(t,w)\|}{\|v - w\|} \leqslant \lambda_V(t), \quad \text{a.a. } t \geqslant 0. \tag{2.2}$$

Then the Volterra equation (2.1) *has a unique continuous solution defined on a maximal interval of existence* $[0, T)$*, where* $0 < T \leqslant \infty$*. If* $T < \infty$*, then* $\limsup_{t \to T} \|u(t)\| = \infty$.

Proof. We will be brief: for more details we refer the reader to the proofs of similar results in [**12**, § 12.2; **19**, Appendix].

Step 1. Existence, uniqueness and extension on small intervals.

We use the standard 'method of steps' which is described in quite some detail in [**12**, § 12.2]. The crucial part of this step of the proof is an argument which shows that every continuous solution u on a closed finite interval $[0, \tau]$ can be extended to a continuous solution defined on $[0, \tau + \varepsilon]$ for some $\varepsilon > 0$. Let $\tau \geqslant 0$, and let $u \in C([0, \tau], U)$ be a solution of (2.1) on $[0, \tau]$ (if $\tau = 0$, we simply take $u(0) = r(0)$). Let $\varepsilon > 0$, let $\eta := \|u(\tau)\| + 1$, and set

$$\mathscr{C}_\varepsilon := \{ v \in C([0, \tau + \varepsilon], U) \mid v(t) = u(t) \text{ for } t \in [0, \tau] \text{ and } \|v(t)\| \leqslant \eta \text{ for } t \in [\tau, \tau + \varepsilon] \}.$$

Endowed with the metric $(u_1, u_2) \mapsto \sup_{t \in [\tau, \tau+\varepsilon]} \|u_1(t) - u_2(t)\|$, $\mathscr{C}_\varepsilon$ is a complete metric space. Using the continuity of r, the fact that $t \mapsto \varphi(t, 0)$ is in $L^2_{\text{loc}}(\mathbb{R}_+, U)$, the Lipschitz-type condition (2.2), and the linearity, continuity and causality of G, it is a routine exercise to show that there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$ the operator defined by the right-hand side of (2.1) is a strict contraction, mapping $\mathscr{C}_\varepsilon$ into itself. Hence, for $\varepsilon \in (0, \varepsilon^*]$, this contraction has a unique fixed point $v_\varepsilon \in \mathscr{C}_\varepsilon$, and this means, in particular, that $v := v_{\varepsilon^*}$ is a continuous solution which extends the solution u defined on $[0, \tau]$ to the larger interval $[0, \tau + \varepsilon^*]$. Moreover, if $w\colon [0, \tau + \varepsilon^*] \to U$ is another function with this property, then w is continuous and so there exists $\varepsilon \in (0, \varepsilon^*]$ such that $w|_{[0,\tau+\varepsilon]} \in \mathscr{C}_\varepsilon$. By the uniqueness of the fixed point in $\mathscr{C}_\varepsilon$, it follows that

$$w(t) = v_\varepsilon(t) = v(t), \quad \text{for all } t \in [0, \tau + \varepsilon].$$

Step 2. Extended uniqueness.

Let u_1 and u_2 be two continuous solutions of (2.1) on $[0, \tau_1]$ and $[0, \tau_2]$, where $0 < \tau_1 \leqslant \tau_2$. We claim that $u_1(t) = u_2(t)$ for all $t \in [0, \tau_1]$. Seeking a contradiction, suppose that there exists $t \in [0, \tau_1]$ such that $u_1(t) \neq u_2(t)$. Defining

$$t^* = \inf\{t \in [0, \tau_1] \mid u_1(t) \neq u_2(t)\},$$

we find from an application of Step 1 (with $\tau = 0$) that $t^* > 0$. Furthermore, $u_1(t^*) = u_2(t^*)$ (by the continuity of u_1 and u_2) and so, by supposition, $t^* < \tau_1$. Applying Step 1 with τ replaced by t^* shows that there exists $\varepsilon > 0$ such that $u_1(t) = u_2(t)$ for all $t \in [0, t^* + \varepsilon]$, contradicting the definition of t^*.

Step 3. Maximal interval of existence.

Let $\mathscr{T} \subset \mathbb{R}_+$ be the set of all $\tau > 0$ such that there exists a solution u^τ of (2.1) on the interval $[0, \tau]$. Set $T := \sup \mathscr{T}$ and define a function u: $[0, T) \to \mathbb{R}$ by setting

$$u(t) = u^\tau(t), \quad \text{for } t \in [0, \tau], \text{ where } \tau \in \mathscr{T}.$$

By Step 2 the function u is well defined and is the unique continuous solution of (2.1) on the interval $[0, T)$. If $T < \infty$, then u must be unbounded, because otherwise the limit of the right-hand side of (2.1) as $t \uparrow T$ would be finite and u could be extended to a continuous solution of (2.1) on the closed interval $[0, T]$, which by Step 1 could be extended beyond T, contradicting the definition of T. □

3. *Stability results of Popov-type*

Throughout this section, let U be a real separable Hilbert space and let $G \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ be a shift-invariant operator, that is, $\mathbf{S}_\tau G = G\mathbf{S}_\tau$ for all $\tau \geqslant 0$. Since the operator G is shift-invariant, it is causal, and so G can be extended to a linear, continuous, shift-invariant (and hence causal) operator mapping $L^2_{\text{loc}}(\mathbb{R}_+, U)$ into itself. We shall use the same symbol G to denote the original operator on $L^2(\mathbb{R}_+, U)$ and its shift-invariant extension to $L^2_{\text{loc}}(\mathbb{R}_+, U)$. As is well known, a shift-invariant operator $G \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ has a *transfer function* $\mathbf{G} \in H^\infty(\mathbb{C}_0, \mathscr{B}(U_c))$ in the sense that

$$(\mathfrak{L}(Gu))(s) = \mathbf{G}(s)\mathfrak{L}(u)(s), \quad \text{for all } u \in L^2(\mathbb{R}_+, U), \text{ and all } s \in \mathbb{C}_0.$$

Since U is separable, $\mathbf{G}$ has strong non-tangential limits at almost every point $i\omega$ on the imaginary axis (see [**25**, Theorem B, p. 85]) and this limit is denoted by $\mathbf{G}(i\omega)$ (whenever it exists). We introduce the following assumption.

(A) The limit $\mathbf{G}(0) := \lim_{s \to 0, s \in \mathbb{C}_0} \mathbf{G}(s)$ exists and

$$\limsup_{s \to 0, s \in \mathbb{C}_0} \left\| \frac{1}{s} (\mathbf{G}(s) - \mathbf{G}(0)) \right\| < \infty.$$

We first consider the feedback system shown in Figure 1 for a class of time-independent non-linearities φ, so-called gradient fields, a concept which we now define. For a C^1-function Φ: $U \to \mathbb{R}$, let Φ': $U \to U^*$ denote the derivative of Φ.

Using the Riesz representation theorem, we define the gradient $\nabla\Phi\colon U \to U$ of Φ by

$$\langle (\nabla\Phi)(v), w\rangle = [(\Phi')(v)](w), \quad \text{for all } w \in U.$$

For all $v \in U$ we have $\|(\nabla\Phi)(v)\| = \|(\Phi'(v)\|$, and therefore, by the C^1-property of Φ, the gradient $\nabla\Phi$ is continuous. As usual, a function $\varphi\colon U \to U$ is called a gradient field if there exists a C^1-function $\Phi\colon U \to \mathbb{R}$ (sometimes called the potential) such that $\varphi = \nabla\Phi$.

Before we state the main result of this section, a stability criterion of Popov-type in an input-output context, it is useful to note that if $r \in W^{1,2}(\mathbb{R}_+, U) + U$, then $r^\infty = \lim_{t\to\infty} r(t)$ exists and $r - r^\infty \in W^{1,2}(\mathbb{R}_+, U)$ (since $r - v \in W^{1,2}(\mathbb{R}_+, U)$ for some $v \in U$, and so the function $r - v$ and its derivative are in $L^2(\mathbb{R}_+, U)$, implying that $\lim_{t\to\infty} r(t) = v = r^\infty$).

THEOREM 3.1. *Let $G \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ be a shift-invariant operator with transfer function $\mathbf{G}$ satisfying Assumption* (A) *with $\mathbf{G}(0)$ invertible, and let $\varphi\colon U \to U$ be a locally Lipschitz continuous gradient of a non-negative C^1-function $\Phi\colon U \to \mathbb{R}$. Assume that there exist self-adjoint $P \in \mathscr{B}(U)$, invertible $Q \in \mathscr{B}(U)$ with $Q\mathbf{G}(0) = [Q\mathbf{G}(0)]^* \geqslant 0$, and numbers $q \geqslant 0$ and $\varepsilon > 0$ such that*

$$\langle \varphi(v), Qv\rangle \geqslant \langle \varphi(v), P\varphi(v)\rangle, \quad \textit{for all } v \in U, \tag{3.1}$$

and

$$P + \tfrac{1}{2}\left(q\mathbf{G}(i\omega) + \frac{1}{i\omega}Q\mathbf{G}(i\omega) + q\mathbf{G}^*(i\omega) - \frac{1}{i\omega}\mathbf{G}^*(i\omega)Q^*\right) \geqslant \varepsilon I, \quad \textit{for a.a. } \omega \in \mathbb{R}. \tag{3.2}$$

Moreover, let $r \in W^{1,2}(\mathbb{R}_+, U) + U$. Then the following statements hold.

(1) *The solution u of* (2.1) *exists on $\mathbb{R}_+$* (*no finite escape-time*) *and there exists a constant $K \geqslant 0$* (*which depends only on $\mathbf{G}(0)$, Q, q and ε, but not on r*) *such that*

$$\|u\|_{L^\infty} + \|\dot u\|_{L^2} + \|\varphi\circ u\|_{L^2} + (\|\langle(\varphi\circ u), Qu\rangle\|_{L_1})^{1/2} + \sup_{t\geqslant 0}\left\|\int_0^t (\varphi\circ u)(\tau)\,d\tau\right\| \leqslant K\eta, \tag{3.3}$$

where

$$\eta := \sqrt{\Phi(r(0))} + \|r^\infty\| + \|r - r^\infty\|_{L^2} + \|\dot r\|_{L^2}. \tag{3.4}$$

(2) *The following limits exist:*

$$\begin{aligned}
&\lim_{t\to\infty} (\varphi\circ u)(t) = 0, \quad \lim_{t\to\infty}\left(u(t) + \mathbf{G}(0)\int_0^t (\varphi\circ u)(\tau)\,d\tau\right) = r^\infty,\\
&\lim_{t\to\infty} (\Phi\circ u)(t), \qquad \lim_{t\to\infty}\int_0^t \langle(\varphi\circ u)(\tau), Qu(\tau)\rangle\,d\tau;
\end{aligned} \tag{3.5}$$

in particular, $u^\infty := \lim_{t\to\infty} u(t)$ exists if and only if $\lim_{t\to\infty}\int_0^t(\varphi\circ u)(\tau)\,d\tau$ exists, in which case $\varphi(u^\infty) = 0$ and

$$\mathbf{G}(0)\lim_{t\to\infty}\int_0^t (\varphi\circ u)(\tau)\,d\tau = r^\infty - u^\infty. \tag{3.6}$$

(3) *There exists a sphere* $S \subset U$ *centred at* 0, *such that the solution* u *of* (2.1) *satisfies*

$$\lim_{t\to\infty} \operatorname{dist}(u(t), \mathbf{G}(0)[Q\mathbf{G}(0)]^{-1/2}S) = 0; \tag{3.7}$$

in particular, if $\dim U = 1$, *then* $\lim_{t\to\infty} u(t)$ *and* $\lim_{t\to\infty} \int_0^t (\varphi \circ u)(\tau)\, d\tau$ *exist.*

(4) *Under the extra assumptions*

(B) *every closed and bounded subset of* $\varphi^{-1}(\{0\})$ *is compact,*

(C) $\varphi^{-1}(\{0\}) \cap \mathbf{G}(0)[Q\mathbf{G}(0)]^{-1/2}S$ *is a finite set for any sphere* $S \subset U$ *centred at zero,*

(D) $\inf_{v\in V} \|\varphi(v)\| > 0$ *for any bounded, closed and non-empty set* $V \subset U$ *which does not intersect* $\varphi^{-1}(\{0\})$,

the existence of the limits $\lim_{t\to\infty} u(t)$ *and* $\lim_{t\to\infty} \int_0^t (\varphi \circ u)(\tau)\, d\tau$ *is guaranteed.*

(5) *Under the extra assumption*

(E) $\operatorname{ess\,lim}_{t\to\infty} (Gv)(t) = 0$ *for all* $v \in C(\mathbb{R}_+, U) \cap L^2(\mathbb{R}_+, U)$ *with* $v(t) \to 0$ *as* $t \to \infty$,

we have $\operatorname{ess\,lim}_{t\to\infty} (G(\varphi \circ u))(t) = 0$ *and* $\operatorname{ess\,lim}_{t\to\infty} (\dot{u}(t) - \dot{r}(t)) = 0$.

Note that Assumption (C) is always true if $\dim U = 1$, and that Assumptions (B) and (D) always hold if $\dim U < \infty$. In particular, if $\dim U < \infty$, then it follows from statement (4) that $\lim_{t\to\infty} u(t)$ exists, provided every bounded set $V \subset U$ contains at most finitely many zeros of φ. If $\dim U < \infty$, a sufficient condition for Assumption (E) to hold is that the convolution kernel of G is a bounded matrix-valued measure.

Before proving Theorem 3.1, we state a slightly simplified version of this result (where P and Q are scalars and $P \geqslant 0$) in the form of a corollary which is convenient in the context of applications of Theorem 3.1 to integral control. To this end, for $a \in \mathbb{R}_+ \cup \{\infty\}$, let $\mathscr{S}(a)$ denote the set of all functions $\varphi\colon U \to U$ satisfying the sector condition

$$\langle \varphi(v), v \rangle \geqslant \frac{1}{a} \|\varphi(v)\|^2, \quad \text{for all } v \in U, \tag{3.8}$$

where $1/\infty := 0$. We claim that for any gradient field $\varphi\colon U \to U$ belonging to $\mathscr{S}(a)$ there exists a *non-negative* C^1-potential *vanishing at zero*. To prove this claim, let $\Phi\colon U \to \mathbb{R}$ be the unique C^1-function with $\Phi(0) = 0$ such that $\varphi = \nabla\Phi$. For $v \in U$, define $f\colon [0, 1] \to U$ by $f(t) = tv$. Then

$$\frac{d}{dt}(\Phi \circ f)(t) = \langle \nabla\Phi(f(t)), \dot{f}(t) \rangle = \langle \varphi(tv), v \rangle, \quad \text{for all } t \in [0, 1],$$

and integration from 0 to 1 yields

$$\Phi(v) = \int_0^1 \langle \varphi(tv), v \rangle\, dt \geqslant 0,$$

where the non-negativity follows from the sector condition on φ.

The following result is now an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let* $G \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ *be a shift-invariant operator with transfer function* $\mathbf{G}$ *satisfying Assumption* (A) *with* $\mathbf{G}(0)$ *invertible and* $\mathbf{G}(0) = \mathbf{G}^*(0) \geqslant 0$, *and let* $\varphi\colon U \to U$ *be a locally Lipschitz continuous gradient* *field which belongs to $\mathscr{S}(a)$ for some $a \in (0, \infty]$. If there exist $q \geqslant 0$ and $\varepsilon > 0$ such that*

$$\frac{1}{a}I + \frac{1}{2}\left[\left(q + \frac{1}{i\omega}\right)\mathbf{G}(i\omega) + \left(q - \frac{1}{i\omega}\right)\mathbf{G}^*(i\omega)\right] \geqslant \varepsilon I, \quad \textit{for a.a. } \omega \in \mathbb{R}, \tag{3.9}$$

then, for all $r \in W^{1,2}(\mathbb{R}_+, U) + U$, the conclusions of Theorem 3.1 *hold with $P = (1/a)I$, $Q = I$ and $\Phi(v) := \int_0^1 \langle \varphi(tv), v \rangle \, dt$.*

Proof of Theorem 3.1. By Lemma 2.1, the Volterra equation (2.1) has a unique solution u defined on a maximal interval of existence $[0, T)$, where $0 < T \leqslant \infty$. Hence

$$u(t) + \int_0^t (G(\varphi \circ u))(\tau) \, d\tau = r(t), \quad \text{for all } t \in [0, T), \tag{3.10}$$

or, equivalently,

$$\dot{u}(t) + (G(\varphi \circ u))(t) = \dot{r}(t), \text{ for a.a. } t \in [0, T); \quad u(0) = r(0). \tag{3.11}$$

Let us rewrite (3.10) in a slightly more convenient form, namely

$$u(t) + (H(\varphi \circ u))(t) + \mathbf{G}(0)\int_0^t (\varphi \circ u)(\tau) \, d\tau = r(t), \quad \text{for all } t \in [0, T), \tag{3.12}$$

where we have defined the operator H by

$$(Hv)(t) := \int_0^t (Gv)(\tau) \, d\tau - \mathbf{G}(0)\int_0^t v(\tau) \, d\tau,$$
$$\text{for all } v \in L^2_{\text{loc}}(\mathbb{R}_+, U) \text{ and all } t \in \mathbb{R}_+. \tag{3.13}$$

It is not difficult to show that this operator is shift-invariant and that its transfer function is given by

$$\mathbf{H}(s) = (\mathbf{G}(s) - \mathbf{G}(0))/s, \quad \text{for } s \in \mathbb{C}_0. \tag{3.14}$$

From Assumption (A) and from the fact that $\mathbf{G} \in H^\infty(\mathbb{C}_0, \mathscr{B}(U_c))$, we conclude that $\mathbf{H} \in H^\infty(\mathbb{C}_0, \mathscr{B}(U_c))$, and hence $H \in \mathscr{B}(L^2(\mathbb{R}_+, U))$.

We multiply (3.11) by q, multiply (3.12) by Q, and add the results to obtain

$$q\dot{u}(t) + Qu(t) + G_{q,Q}(\varphi \circ u)(t) + Q\mathbf{G}(0)\int_0^t (\varphi \circ u)(t) \, d\tau$$
$$= q\dot{r}(t) + Qr(t), \quad \text{for a.a. } t \in [0, T), \tag{3.15}$$

where we have defined the operator $G_{q,Q}$ by

$$G_{q,Q} := qG + QH.$$

Invoking (3.14), we see clearly that the transfer function $\mathbf{G}_{q,Q}$ of $G_{q,Q}$ is given by

$$\mathbf{G}_{q,Q}(s) := q\mathbf{G}(s) + Q(\mathbf{G}(s) - \mathbf{G}(0))/s.$$

Note that, by the self-adjointness of $Q\mathbf{G}(0)$, the 'positive-real' condition (3.2) can be expressed as

$$P + \tfrac{1}{2}(\mathbf{G}_{q,Q}(i\omega) + \mathbf{G}^*_{q,Q}(i\omega)) \geqslant \varepsilon I, \quad \text{for a.a. } \omega \in \mathbb{R}. \tag{3.16}$$

The most central ingredient in this proof is the 'energy balance' equation obtained by replacing t in (3.15) by τ, taking the inner product with $(\varphi \circ u)(\tau)$, and integrating with respect to τ from 0 to $t \in [0, T)$ (here we are following the general procedure described in [**12**, § 18.2]). The first term on the left-hand side of this energy balance equation is $q\int_0^t \langle (\varphi \circ u)(\tau), \dot{u}(\tau)\rangle \, d\tau$. Since φ is the gradient of Φ and $u(0) = r(0)$, we conclude that

$$q\int_0^t \langle (\varphi \circ u)(\tau), \dot{u}(\tau)\rangle \, d\tau = q(\Phi \circ u)(t) - q\Phi(r(0)).$$

Using the fact that $Q\mathbf{G}(0) = [Q\mathbf{G}(0)]^* \geqslant 0$, we obtain for the last term on the left-hand side of the energy balance equation

$$\int_0^t \left\langle (\varphi \circ u)(\tau), Q\mathbf{G}(0) \int_0^\tau \varphi \circ u \right\rangle d\tau = \tfrac{1}{2}\langle \varphi_u(t), Q\mathbf{G}(0)\varphi_u(t)\rangle,$$

where we have introduced the abbreviation

$$\varphi_u(t) := \int_0^t (\varphi \circ u)(\tau)\, d\tau. \tag{3.17}$$

Keeping all the other terms in their original form, we note that the energy balance equation becomes

$$\begin{aligned} q(\Phi \circ u)(t) &+ \int_0^t \langle (\varphi \circ u)(\tau), Qu(\tau)\rangle \, d\tau \\ &+ \int_0^t \langle (\varphi \circ u)(\tau), G_{q,Q}(\varphi \circ u)(\tau)\rangle \, d\tau + \tfrac{1}{2}\langle \varphi_u(t), Q\mathbf{G}(0)\varphi_u(t)\rangle \\ &= q\Phi(r(0)) + \int_0^t \langle (\varphi \circ u)(\tau), q\dot{r}(\tau) + Qr(\tau)\rangle \, d\tau, \quad \text{for all } t \in [0, T). \end{aligned} \tag{3.18}$$

Invoking Parseval's theorem, we see from (3.16) that

$$\begin{aligned} &\int_0^t \langle (\varphi \circ u)(\tau), G_{q,Q}(\varphi \circ u)(\tau)\rangle \, d\tau \\ &\quad \geqslant \varepsilon \int_0^t \|(\varphi \circ u)(\tau)\|^2 \, d\tau - \int_0^t \langle (\varphi \circ u)(\tau), P(\varphi \circ u)(\tau)\rangle \, d\tau, \quad \text{for all } t \in [0, T). \end{aligned}$$

Combining this estimate with (3.18), we obtain

$$\begin{aligned} q(\Phi \circ u)(t) &+ \varepsilon \int_0^t \|(\varphi \circ u)(\tau)\|^2 \, d\tau + \tfrac{1}{2}\langle \varphi_u(t), Q\mathbf{G}(0)\varphi_u(t)\rangle \\ &+ \int_0^t \langle (\varphi \circ u)(\tau), Qu(\tau) - P(\varphi \circ u)(\tau)\rangle \, d\tau \\ &\leqslant q\Phi(r(0)) + \int_0^t \langle (\varphi \circ u)(\tau), q\dot{r}(\tau) + Qr(\tau)\rangle \, d\tau, \quad \text{for all } t \in [0, T). \end{aligned} \tag{3.19}$$

The inequality (3.19) can be further simplified. Since $r \in W^{1,2}(\mathbb{R}_+, U) + U$, $r^\infty := \lim_{t\to\infty} r(t)$ exists and $r - r^\infty \in W^{1,2}(\mathbb{R}_+, U)$. Writing the integral on the right-hand side of (3.19) in the form

$$\int_0^t \langle(\varphi \circ u)(\tau), q\dot{r}(\tau) + Q(r(\tau) - r^\infty)\rangle \, d\tau + \langle \varphi_u(t), Qr^\infty\rangle,$$

we can combine the term $\langle \varphi_u(t), Qr^\infty\rangle$ with the third term on the left-hand side of (3.19) by completing the square, whilst the remaining term can be estimated as follows (since $ab \leqslant \frac{1}{2}\varepsilon a^2 + b^2/(2\varepsilon)$ for non-negative numbers a and b)

$$\begin{aligned}\int_0^t |\langle(\varphi \circ u)(\tau), q\dot{r}(\tau) + Q(r(\tau) - r^\infty)\rangle| \, d\tau \\ \leqslant \tfrac{1}{2}\varepsilon \int_0^t \|(\varphi \circ u)(\tau)\|^2 \, d\tau \\ + \frac{1}{2\varepsilon}\int_0^t \|q\dot{r}(\tau) + Q(r(\tau) - r^\infty)\|^2 \, d\tau, \quad \text{for all } t \in [0, T).\end{aligned}$$

With these further changes, (3.19) becomes

$$\begin{aligned}q(\Phi \circ u)(t) + \tfrac{1}{2}\varepsilon \int_0^t \|(\varphi \circ u)(\tau)\|^2 \, d\tau + \tfrac{1}{2}\|[Q\mathbf{G}(0)]^{1/2}(\varphi_u(t) - [\mathbf{G}(0)]^{-1}r^\infty)\|^2 \\ + \int_0^t \langle(\varphi \circ u)(\tau), Qu(\tau) - P(\varphi \circ u)(\tau)\rangle \, d\tau \\ \leqslant q\Phi(r(0)) + \frac{1}{2\varepsilon}\int_0^t \|q\dot{r}(\tau) + Q(r(\tau) - r^\infty)\|^2 \, d\tau \\ + \tfrac{1}{2}\|[Q\mathbf{G}(0)]^{1/2}[\mathbf{G}(0)]^{-1}r^\infty\|^2, \quad \text{for all } t \in [0, T).\end{aligned} \tag{3.20}$$

By the non-negativity assumption on Φ and by the 'sector condition' (3.1), all terms on the left-hand side of this inequality are non-negative. Moreover, note that the right-hand side does not depend on u. Inequality (3.20) is the key estimate from which we shall derive the theorem.

Proof of statement (1). Inspecting (3.20), we immediately observe that for some constant $K > 0$ (which can be explicitly computed, and which depends only on ε, q, Q and $\mathbf{G}(0)$),

$$\left(\int_0^T \|(\varphi \circ u)(\tau)\|^2 \, d\tau\right)^{1/2} + \sup_{0 \leqslant t < T}\left\|\int_0^t (\varphi \circ u)(\tau) \, d\tau\right\| \leqslant K\eta, \tag{3.21}$$

$$\int_0^T |\langle(\varphi \circ u)(\tau), Qu(\tau) - P(\varphi \circ u)(\tau)\rangle| \, d\tau \leqslant K\eta^2, \tag{3.22}$$

where η is defined by (3.4). Invoking (3.1), we see that

$$\begin{aligned}\langle(\varphi \circ u), P(\varphi \circ u)\rangle &\leqslant \langle(\varphi \circ u), Qu\rangle \\ &= \langle(\varphi \circ u), Qu - P(\varphi \circ u)\rangle + \langle(\varphi \circ u), P(\varphi \circ u)\rangle,\end{aligned}$$

and so it follows from (3.21) and (3.22) that (for a suitably enlarged constant K)

$$\int_0^T \langle(\varphi \circ u)(\tau), Qu(\tau)\rangle \, d\tau \leqslant K\eta^2. \tag{3.23}$$

By (3.13),

$$\int_0^t (G(\varphi \circ u))(\tau)\, d\tau = (H(\varphi \circ u))(t) + \mathbf{G}(0) \int_0^t (\varphi \circ u)(\tau)\, d\tau, \quad \text{for all } t \in [0, T).$$

Setting $f := H(\varphi \circ u)$, we have $f(0) = 0$, and so

$$\|(H(\varphi \circ u))(t)\|^2 = \|f(t)\|^2 \leqslant 2 \int_0^T |\langle f(\tau), \dot{f}(\tau)\rangle|\, d\tau, \quad \text{for all } t \in [0, T).$$

Now $\dot{f} = G(\varphi \circ u) - \mathbf{G}(0)(\varphi \circ u)$ and therefore

$$\|(H(\varphi \circ u))(t)\|^2 \leqslant 2\|H\|(\|G\| + \|\mathbf{G}(0)\|) \int_0^T \|(\varphi \circ u)(\tau)\|^2\, d\tau, \text{ for all } t \in [0, T). \tag{3.24}$$

Moreover, we observe that

$$\begin{aligned}\|r^\infty - r(t)\|^2 &= 2\left|\int_t^\infty \langle \dot{r}(\tau), r(\tau) - r^\infty\rangle\, d\tau\right| \\ &\leqslant 2\|r - r^\infty\|_{L^2}\|\dot{r}\|_{L^2} \leqslant \eta^2, \quad \text{for all } t \in \mathbb{R}_+,\end{aligned}$$

with η given by (3.4). It follows that

$$\|r\|_{L^\infty} \leqslant 2\eta. \tag{3.25}$$

Combining (3.11), (3.12), (3.21), (3.24) and (3.25) shows that (for a suitably enlarged constant K)

$$\sup_{0 \leqslant t < T} \|u(t)\| + \left(\int_0^t \|\dot{u}(\tau)\|^2\, d\tau\right)^{1/2} \leqslant K\eta. \tag{3.26}$$

However, this combined with Lemma 2.1 implies that $T = \infty$. It now follows from (3.21), (3.22), (3.23) and (3.26) that (3.3) holds.

Proof of statement (2). By (3.3), $u \in L^\infty(\mathbb{R}_+, U)$ and $\dot{u} \in L^2(\mathbb{R}_+, U)$, which means that u is bounded and uniformly continuous. Since φ is locally Lipschitz continuous, $\varphi \circ u$ is also uniformly continuous. In addition, $\varphi \circ u \in L^2(\mathbb{R}_+, U)$ (see (3.3)), and it follows from Barbalat's lemma that $\lim_{t\to\infty}(\varphi \circ u)(t) = 0$.

By (3.12), the second limit in (3.5) is equivalent to the claim that $\lim_{t\to\infty}(H(\varphi \circ u))(t) = 0$. The latter follows from the fact that the function $H(\varphi \circ u)$ belongs to L^2 (since H is bounded and $\varphi \circ u \in L^2(\mathbb{R}_+, U)$), as does its derivative $G(\varphi \circ u) - \mathbf{G}(0)(\varphi \circ u)$. The limit $\lim_{t\to\infty}(\Phi \circ u)(t)$ exists, since, by (3.3), the derivative $\langle \varphi \circ u, \dot{u}\rangle$ of $\Phi \circ u$ is integrable. Furthermore, the existence of the limit $\lim_{t\to\infty} \int_0^t \langle(\varphi \circ u)(\tau), Qu(\tau)\rangle\, d\tau$ follows from the integrability of the integrand (see (3.3)).

By the second limit in (3.5) it is clear that $u^\infty := \lim_{t\to\infty} u(t)$ exists if and only if $\lim_{t\to\infty} \int_0^t (\varphi \circ u)(\tau)\, d\tau$ exists, in which case it is obvious that $\varphi(u^\infty) = 0$ (since $\varphi(u(t)) \to 0$ as $t \to \infty$) and, moreover, (3.6) holds trivially.

Proof of statement (3). Replacing t by τ in (3.11), taking the inner product with $(\varphi \circ u)(\tau)$, and integrating from 0 to t yields

$$(\Phi \circ u)(t) + \int_0^t \langle(\varphi \circ u)(\tau), G(\varphi \circ u)(\tau)\rangle\, d\tau = \Phi(r(0)) + \int_0^t \langle(\varphi \circ u)(\tau), \dot{r}(\tau)\rangle\, d\tau.$$

Multiplying this identity by q, subtracting the result from (3.18) and invoking (3.17) gives

$$\begin{aligned}\int_0^t \langle(\varphi\circ u)(\tau), Qu(\tau)\rangle\, d\tau \\ + \int_0^t \langle(\varphi\circ u)(\tau), QH(\varphi\circ u)(\tau)\rangle\, d\tau + \tfrac{1}{2}\langle\varphi_u(t), Q\mathbf{G}(0)\varphi_u(t)\rangle \\ = \int_0^t \langle(\varphi\circ u)(\tau), Q(r(\tau)-r^\infty)\rangle\, d\tau + \langle\varphi_u(t), Qr^\infty\rangle.\end{aligned}$$

Combining the last terms on the left-hand and right-hand sides by completing the square gives

$$\begin{aligned}&\tfrac{1}{2}\|[Q\mathbf{G}(0)]^{1/2}(\varphi_u(t)-[\mathbf{G}(0)]^{-1}r^\infty)\|^2 \\ &\quad = \tfrac{1}{2}\|[Q\mathbf{G}(0)]^{1/2}[\mathbf{G}(0)]^{-1}r^\infty\|^2 \\ &\quad - \int_0^t \langle(\varphi\circ u)(\tau), Qu(\tau)\rangle\, d\tau - \int_0^t \langle(\varphi\circ u)(\tau), QH(\varphi\circ u)(\tau)\rangle\, d\tau \\ &\quad + \int_0^t \langle(\varphi\circ u)(\tau), Q(r(\tau)-r^\infty)\rangle\, d\tau.\end{aligned}$$

All the integrands on the right-hand side are in $L^1(\mathbb{R}_+, \mathbb{R})$, and so the right-hand side has a finite limit as $t\to\infty$. Therefore $\lim_{t\to\infty}\|[Q\mathbf{G}(0)]^{1/2}(\varphi_u(t)-[\mathbf{G}(0)]^{-1}r^\infty)\|$ exists. This, combined with the existence of the second limit in (3.5), implies (3.7). Finally, if $\dim U = 1$, then the sphere S consists of just one or two points, and hence, by continuity, $u^\infty = \lim_{t\to\infty} u(t)$ exists.

Proof of statement (4). Since u is bounded, there exists a closed bounded ball $W\subset U$ such that $u(t)\in W$ for all $t\geqslant 0$. By Assumption (B), $\varphi^{-1}\{0\}\cap W$ is compact. Let $\delta>0$. Then, by the compactness property, $\varphi^{-1}\{0\}\cap W$ is contained in a finite union of open balls with radius δ, each ball centred at some point in $\varphi^{-1}\{0\}\cap W$. Call this union W_δ. We claim that $u(t)\in W_\delta$ for all sufficiently large t. This is trivially true if $W\subset W_\delta$. If not, then the set $V := W\setminus W_\delta$ is non-empty. Moreover, V is bounded and closed with $\varphi^{-1}(\{0\})\cap V=\emptyset$, and so, by Assumption (D), $\inf_{v\in V}\|\varphi(v)\|>0$. We know from (3.5) that $\lim_{t\to\infty}(\varphi\circ u)(t)=0$, and so, also in this case, $u(t)\in W_\delta$ for all sufficiently large $t\geqslant 0$. This implies that $u(t)$ approaches $\varphi^{-1}\{0\}\cap W$ as $t\to\infty$. Using the compactness of $\varphi^{-1}\{0\}\cap W$ and the continuity of u, a routine argument shows that the trajectory $\{u(t)\mid t\in\mathbb{R}_+\}$ of u is precompact. Therefore, by a standard result, the ω-limit set of u,

$$\Omega_u := \{l\in U\mid u(t_k)\to l \text{ for some sequence } t_k\to\infty\},$$

is non-empty, compact, connected and is approached by $u(t)$ as $t\to\infty$. Consequently, $\Omega_u\subset\varphi^{-1}\{0\}\cap W$. Furthermore, by (3.7), it is also true that $\Omega_u\subset\mathbf{G}(0)[Q\mathbf{G}(0)]^{-1/2}S$ for some $S\subset U$ centred at 0. Therefore, by Condition (C), Ω_u is finite. Being both finite and connected, Ω_u must consist of exactly one point u^∞, and we conclude that $u(t)\to u^\infty$ as $t\to\infty$.

Proof of statement (5). The proof of statement (5) is obvious and is therefore left to the reader. □

If the positive real condition (3.9) holds with $q=0$, then we can allow the non-linearity φ to be time-varying. We use the notation $\varphi \circ u$ for the function $t \mapsto \varphi(t, u(t))$ in this case. In the following, we want to allow for discontinuous and even locally unbounded functions r in the Volterra equation (2.1). Recall that for a given forcing function $r\colon \mathbb{R}_+ \to U$, a function $u\colon \mathbb{R}_+ \to U$ is called a solution of (2.1) if $\varphi \circ u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ (guaranteeing that $G(\varphi \circ u)$ in (2.1) is well defined) and if (2.1) holds for all $t \in \mathbb{R}_+$ (for a.a. $t \in \mathbb{R}_+$ if $r(t)$ is only defined for a.a. $t \in \mathbb{R}_+$ of if we consider r as an equivalence class of functions coinciding a.e.). Moreover, although we allow r and u to be very irregular, it follows from (2.1) that if $u\colon \mathbb{R}_+ \to U$ is a solution, then the difference $u(t) - r(t)$ is equal to a continuous function for a.a. $t \geqslant 0$. For this reason, we shall consider $u(t) - r(t)$ to be everywhere defined (and equal to $\int_0^t G(\varphi \circ u)(\tau)\, d\tau$).

We are now in a position to state the following result, a sufficient closed-loop stability condition of circle-criterion-type in an input-output context.

THEOREM 3.3. *Let $G \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ be a shift-invariant operator with transfer function $\mathbf{G}$ satisfying Assumption* (A) *with $\mathbf{G}(0)$ invertible, and let $\varphi\colon \mathbb{R}_+ \times U \to U$ be a time-varying non-linearity. Assume that there exist self-adjoint $P \in \mathscr{B}(U)$, invertible $Q \in \mathscr{B}(U)$ with $Q\mathbf{G}(0) = [Q\mathbf{G}(0)]^* \geqslant 0$ and a number $\varepsilon > 0$ such that*

$$\langle \varphi(t, v), Qv \rangle \geqslant \langle \varphi(t, v), P\varphi(t, v) \rangle, \quad \text{for all } t \in \mathbb{R}_+ \text{ and all } v \in U, \tag{3.27}$$

and

$$P + (Q\mathbf{G}(i\omega) - \mathbf{G}^*(i\omega)Q^*)/(2i\omega) \geqslant \varepsilon I, \quad \text{for a.a. } \omega \in \mathbb{R}. \tag{3.28}$$

Let $r \in L^2(\mathbb{R}_+, U) + U$, that is, $r = r_1 + r_2$ with $r_1 \in L^2(\mathbb{R}_+, U)$ and $r_2 \in U$, and let $u\colon \mathbb{R}_+ \to U$ be a solution of (2.1)*. Then the following statements hold.*

(1) *There exists a constant K* (*which depends only on ε, Q and $\mathbf{G}(0)$, but not on r*) *such that*

$$\|u - r\|_{L^\infty} + \|\varphi \circ u\|_{L^2} + (\|\langle (\varphi \circ u), Qu \rangle\|_{L_1})^{1/2} + \sup_{t \geqslant 0} \left\| \int_0^t (\varphi \circ u)(\tau)\, d\tau \right\| \leqslant K\eta, \tag{3.29}$$

where

$$\eta = \|r_1\|_{L^2} + \|r_2\|.$$

(2) *The following limits exist:*

$$\lim_{t \to \infty} \left(u(t) - r(t) + \mathbf{G}(0) \int_0^t (\varphi \circ u)(\tau)\, d\tau \right) = 0, \quad \lim_{t \to \infty} \int_0^t \langle (\varphi \circ u)(\tau), Qu(\tau) \rangle\, d\tau;$$

in particular, $\lim_{t \to \infty}(u(t) - r(t))$ exists if and only if $\lim_{t \to \infty} \int_0^t (\varphi \circ u)(\tau)\, d\tau$ exists, in which case

$$\lim_{t \to \infty} (u(t) - r(t)) = -\mathbf{G}(0) \lim_{t \to \infty} \int_0^t (\varphi \circ u)(\tau)\, d\tau.$$

(3) *There exists a sphere $S \subset U$ centred at* 0*, such that*

$$\lim_{t \to \infty} \operatorname{dist}(u(t) - r_1(t), \mathbf{G}(0)[Q\mathbf{G}(0)]^{-1/2} S) = 0;$$

in particular, if $\dim U = 1$, then $\lim_{t \to \infty}(u(t) - r(t))$ and $\lim_{t \to \infty} \int_0^t (\varphi \circ u)(\tau)\, d\tau$ exist.

(4) *If we relax condition* (3.27) *and only require that, for some* $t_0 > 0$,

$$\langle \varphi(t, v), Qv \rangle \geqslant \langle \varphi(t, v), P\varphi(t, v) \rangle, \quad \text{for all } t \geqslant t_0 \text{ and all } v \in U, \tag{3.30}$$

then the left-hand side of (3.29) *is still finite* (*but no longer bounded in terms of* η) *and statements* (2) *and* (3) *remain valid.*

By taking $Q = I$ and $P = 1/a$ (where $a \in (0, \infty]$) in the above theorem we obtain an analogue of Corollary 3.2.

Proof of Theorem 3.3. Taking $q = 0$ and replacing r^∞ by r_2 in (3.20) and multiplying the resulting inequality by 2 gives

$$\begin{aligned}
\varepsilon \int_0^t \|(\varphi \circ u)(\tau)\|^2 \, d\tau &+ \|[Q\mathbf{G}(0)]^{1/2}(\varphi_u(t) - [\mathbf{G}(0)]^{-1} r_2)\|^2 \\
&+ 2 \int_0^t \langle (\varphi \circ u)(\tau), Qu(\tau) - P(\varphi \circ u)(\tau) \rangle \, d\tau \\
&\leqslant \frac{1}{\varepsilon} \int_0^t \|Q(r(\tau) - r_2)\|^2 \, d\tau + \|[Q\mathbf{G}(0)]^{1/2} [\mathbf{G}(0)]^{-1} r_2\|^2, \quad \text{for all } t \in \mathbb{R}_+.
\end{aligned} \tag{3.31}$$

Statements (1)–(3) can be derived from (3.31) by arguments identical to those used to prove the corresponding statements in Theorem 3.1.

It remains to prove statement (4). The estimate (3.31) is still valid, but the last term on the left-hand side could have either sign. However, setting

$$L := 2 \int_0^{t_0} |\langle (\varphi \circ u)(s), Qu(s) - P(\varphi \circ u)(s) \rangle| \, ds,$$

we see from (3.31) that

$$\begin{aligned}
\varepsilon \int_0^t \|(\varphi \circ u)(\tau)\|^2 \, d\tau &+ \|[Q\mathbf{G}(0)]^{1/2}(\varphi_u(t) - [\mathbf{G}(0)]^{-1} r_2)\|^2 \\
&+ 2 \int_{t_0}^t \langle (\varphi \circ u)(\tau), Qu(\tau) - P(\varphi \circ u)(\tau) \rangle \, d\tau \\
&\leqslant L + \frac{1}{\varepsilon} \int_0^t \|Q(r(\tau) - r_2)\|^2 \, d\tau \\
&\quad + \|[Q\mathbf{G}(0)]^{1/2} [\mathbf{G}(0)]^{-1} r_2\|^2, \quad \text{for all } t \in [t_0, \infty).
\end{aligned}$$

By (3.27), the integral $\int_{t_0}^t \langle (\varphi \circ u)(s), Qu(s) - P(\varphi \circ u)(s) \rangle \, ds$ is non-negative. The rest of the proof carries over with no further changes, except that the right-hand side of (3.29) is now bounded in terms of η and L (where the latter constant depends on the restriction of u to $[0, t_0]$). □

REMARK 3.4. In Theorem 3.3 the existence of a global solution is assumed. If r is continuous and φ satisfies the assumptions in Lemma 2.1, then, by the same lemma, there exists a unique continuous solution u of (2.1) on a maximal interval $[0, T)$. An inspection of the proof of Theorem 3.3 shows that if (3.27) and (3.28) hold, then $T = \infty$: simply note that (3.31) holds for all $t \in [0, T)$, implying, in particular, that u is bounded on $[0, T)$, and hence $T = \infty$, by Lemma 2.1.

Let us make some comments on how the results that we have proved in this section are related to the existing literature. Obviously, the development of frequency-domain criteria for the stability of the feedback interconnection of a linear time-invariant dynamical system and a static (possibly time-depending) non-linearity is a classical theme in control theory, known as absolute stability theory. In the finite-dimensional case there are a large number of results available in the literature, many of which have been obtained by Lyapunov techniques applied to state-space models with the so-called Kalman–Yakubovich–Popov (or positive-real) lemma playing a crucial role; see, for example, [**1**, **13**, **14**, **15**, **16**, **23**, **28**, **34**, **39**] and the references therein. In the infinite-dimensional case the literature on absolute stability problems is dominated by input-output approaches; see, for example, [**8**], which gives the first treatment of the Popov criterion in a distributed parameter context, and see the relevant chapters in the books [**4**, **9**, **16**, **21**, **22**, **28**, **34**] and the references therein. The number of absolute stability results in an infinite-dimensional state-space setting is fairly limited (see [**2**, **3**, **6**, **11**, **17**, **18**, **37**, **38**]). Some of the references, namely [**2**, **4**, **37**] and parts of [**16**] (see also the forthcoming paper [**5**] by the authors), consider the frequency-domain condition (3.2) (or a variation thereof) with $\varepsilon = 0$, and, not surprisingly, the corresponding conclusions are weaker than in Theorem 3.1: typically, under suitable regularity assumptions on r and the impulse response of the linear system, boundedness of u is shown, but not, for example, square-integrability of $\varphi \circ u$. These results cannot be applied to obtain any interesting applications to low-gain integral control (see § 4) and hence they are, in a sense, not relevant in the context of the present paper. A significant difference is that Theorems 3.1 and 3.3 consider feedback systems where the linear part contains an integrator (that is, we are considering a so-called critical case in the terminology of [**1**, **16**], meaning, in particular, that the linear system is not input-output stable) and where at the same time the lower gain $\inf_{v \in U} \|\varphi(v)\|/\|v\|$ of the non-linearity φ is allowed to be equal to zero (which, for example, is the case for bounded non-linearities such as saturation). In fact, one of the motivations for studying this situation is its importance in the application to the low-gain integral control problem in the presence of input non-linearities of saturation type (see § 4). By contrast, in most (if not all, if we ignore the contributions which consider (3.2) with $\varepsilon = 0$) of the input-output absolute stability results available in the literature, the lower gain of the non-linearity is either assumed to be positive, or, if the lower gain is allowed to be zero, the linear part is assumed to be input-output stable (in particular, it is not allowed to contain an integrator); see, for example, [**8**; **9**, pp. 186–189; **16**, pp. 70–71; **22**, pp. 115–116; **28**, pp. 162–167; **34**, pp. 344–356]. Moreover, in all but two of these references (the exceptions being [**8**] and [**16**, Theorem 1.17.4, p. 70]) it is assumed that the input signal $r \in W^{1,1}(\mathbb{R}_+, U)$ in the Popov criterion (that is, $r^\infty = 0$ in Theorem 3.1) and $r \in L^2(\mathbb{R}_+, U)$ in the circle criterion (that is, $r_2 = 0$ in Theorem 3.3). As a consequence, these results are not applicable to the low-gain control tracking problem nor to strongly or exponentially stable well-posed state-space systems; see §§ 4 and 5. Furthermore, we remark that in the input-output absolute stability results available in the literature the regularity assumptions imposed on the impulse response of the linear system (where it is usually assumed that the convolution kernel of G is a function or a bounded measure without singular part) are more restrictive than in our results. As a relatively minor point we mention that in our results the input space U may be infinite-dimensional, whereas in most of the previous contributions (to the best of our knowledge, the only exception being [**22**]) it is assumed that $\dim U < \infty$ or, in many cases, even $\dim U = 1$. Summarizing, we believe it is fair to say that Theorems 3.1 and 3.3 represent a substantial improvement on related input-output results in the control literature.

More relevant in our context are the frequency-domain results developed in the stability theory of integral equations. To the best of our knowledge, the most general of such results are those given in [**12**], so we only compare our results to those found in [**12**, §§ 17.5 and 18.2] (and refer the reader to [**12**] for further historical comments). There are, of course, numerous differences in terminology, and it is not always a trivial matter to spot the relevance of the results in [**12**, §§ 17.5 and 18.2] to absolute stability problems in infinite-dimensional control theory. Moreover, there are some obvious differences of a technical nature: in our results we do allow U to be infinite-dimensional, whereas none of those in [**12**] does. In Theorems 3.1 and 3.3 we do not require that G maps a locally bounded function into a locally bounded function (this requirement is equivalent to the standing assumption used in [**12**] that G can be written as a convolution operator with a locally finite measure kernel). On the other hand, our assumptions that $G \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ and that (A) holds are not imposed in [**12**]. From a control-theoretic point of view in general and for applications to low-gain integral control in particular, our assumptions that $G \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ and that (A) holds seem to be more natural. (A small technical difference which may confuse the reader is the fact that we include the constant ε in (3.2), whereas the same constant ε in [**12**] appears in (3.1): to get comparable results one should take our operator P to be equal to the operator $A - \varepsilon I$ in [**12**]). The most significant difference between our results and those given in [**12**] is that in [**12**] it is assumed (as in the absolute stability literature) that $r^\infty = 0$ (in Theorem 3.1) and $r_2 = 0$ (in Theorem 3.3), excluding applications to the low-gain control tracking problem and to strongly or exponentially stable well-posed state-space systems; see §§ 4 and 5. If we ignore the differences mentioned above (and a number of other details), then parts of statement (1) of Theorem 3.1, namely the facts that u is bounded and $\varphi \circ u \in L^2(\mathbb{R}_+, U)$, are reminiscent of [**12**, Corollary 18.2.4] (see also [**12**, Theorem 18.2.1]). A similar comment applies to the relation of Theorem 3.3 and [**12**, Theorem 17.5.1]. As we have already mentioned, the energy balance analysis in the proofs of Theorems 3.1 and 3.3 is strongly inspired by the general methodology described in [**12**, § 18.2].

4. *Application to integral control in the presence of input/output non-linearities*

In this section we assume that $\dim U = 1$, that is, $U = \mathbb{R}$. We will apply Corollary 3.2 and Theorem 3.3 to derive results on low-gain integral control.

4.1. *Integral control in the presence of input non-linearities*

Consider the feedback system shown in Figure 2, where $\rho \in \mathbb{R}$ is a constant, $k \in \mathbb{R}$ is a gain parameter, $\varphi: \mathbb{R} \to \mathbb{R}$ is a static input non-linearity and $G \in \mathscr{B}(L^2(\mathbb{R}_+, \mathbb{R}))$ is a shift-invariant operator with transfer function denoted by $\mathbf{G}$. Mathematically, Figure 2 is described by the abstract Volterra integro-differential equation

$$\dot{u} = k[\rho - (g + G(\varphi \circ u))], \quad u(0) = u^0 \in \mathbb{R}, \tag{4.1}$$

where the function g models the effect of non-zero initial conditions of the system with input-output operator G. Of course, (4.1) is equivalent to the integral equation

$$u(t) = u^0 + k\int_0^t (\rho - g(\tau))\,d\tau - k\int_0^t (G(\varphi \circ u))(\tau)\,d\tau, \quad \text{for } t \geqslant 0.$$

The aim in this subsection is to choose the gain parameter k such that the tracking error

$$e(t) := \rho - (g + G(\varphi \circ u)) = \dot{u}(t)/k \tag{4.2}$$

becomes small in a certain sense as $t \to \infty$. For example, we might want to achieve 'tracking in measure', that is, for all $\varepsilon > 0$, the Lebesgue measure of the set $\{\tau \geqslant t \,|\, |e(\tau)| \geqslant \varepsilon\}$ tends to 0 as $t \to \infty$, or 'essential asymptotic tracking', that is, $\operatorname{ess\,lim}_{t\to\infty} e(t) = 0$, or the aim might be 'asymptotic tracking', that is, $\lim_{t\to\infty} e(t) = 0$. Trivially, tracking in measure is guaranteed if $e \in L^p(\mathbb{R}_+, \mathbb{R})$ for some $p \in (0, \infty)$. Obviously, by (4.2), statements about tracking can be regarded as statements about the asymptotic behavior of the derivative $\dot{u}$ of the solution u of (4.1).

In order to state the main result of this subsection we introduce the following assumption.

(A$'$) $\mathbf{G}$ is differentiable at zero in the sense that the limits

$$\mathbf{G}(0) := \lim_{s\to 0, s\in\mathbb{C}_0} \mathbf{G}(s)$$

and

$$\mathbf{G}'(0) := \lim_{s\to 0, s\in\mathbb{C}_0} (\mathbf{G}(s) - \mathbf{G}(0))/s$$

exist, and, in addition, the function $s \mapsto (\mathbf{G}(s) - \mathbf{G}(0) - s\mathbf{G}'(0))/s^2$ belongs to $H^2(\mathbb{C}_0, \mathbb{C})$.

Clearly, Assumption (A$'$) is stronger than (A). A sufficient condition for (A$'$) to hold is that $\mathbf{G}$ admits an analytic extension to a neighbourhood of 0. If condition (A$'$) holds, then the constant

$$f(G) := \sup_{q\geqslant 0}\{\operatorname*{ess\,inf}_{\omega\in\mathbb{R}} \operatorname{Re}[(q + 1/i\omega)\mathbf{G}(i\omega)]\}$$

satisfies $-\infty < f(G) \leqslant \infty$.

Recall the definition of the class $\mathscr{S}(a)$ of sector-bounded functions; see (3.8). Note that if $a < \infty$ and $\dim U = 1$, then a function $\varphi\colon \mathbb{R} \to \mathbb{R}$ belongs to $\mathscr{S}(a)$ if and only if

$$0 \leqslant \varphi(v)v \leqslant av^2, \quad \text{for all } v \in \mathbb{R}. \tag{4.3}$$

THEOREM 4.1. *Let* $G \in \mathscr{B}(L^2(\mathbb{R}_+, \mathbb{R}))$ *be a shift-invariant operator with transfer function* $\mathbf{G}$. *Assume that Assumption* (A$'$) *holds with* $\mathbf{G}(0) > 0$ *and that* $g \in L^2(\mathbb{R}_+, \mathbb{R})$ *with* $t \mapsto \int_0^t g(\tau)\,d\tau \in L^2(\mathbb{R}_+, \mathbb{R}) + \mathbb{R}$. *Let* $\varphi\colon \mathbb{R} \to \mathbb{R}$ *be locally Lipschitz continuous and non-decreasing. Let* $\rho \in \mathbb{R}$ *and assume that* $\rho/\mathbf{G}(0) \in \operatorname{im}\varphi$. *Under these conditions the following statements hold.*

(1) *Assume that* $\varphi - \varphi(0) \in \mathscr{S}(a)$ *for some* $a \in (0, \infty)$. *Then there exists a constant* $k^* \in (0, \infty]$ (*depending on* G, φ *and* ρ) *such that for all* $k \in (0, k^*)$, *the unique solution* u *of* (4.1) *is defined on* $\mathbb{R}_+$ (*no finite escape-time*), *the limit*

$\lim_{t\to\infty} u(t) =: u^\infty$ *exists and satisfies* $\varphi(u^\infty) = \rho/\mathbf{G}(0)$,

$$e = \dot{u}/k \in L^2(\mathbb{R}_+, \mathbb{R}) \quad \textit{and} \quad \varphi \circ u - \varphi(u^\infty) \in L^2(\mathbb{R}_+, \mathbb{R}).$$

Moreover,

$$\operatorname*{ess\,lim}_{t\to\infty} e(t) = 0,$$

provided that $\operatorname{ess\,lim}_{t\to\infty} g(t) = 0$ *and G satisfies Property* (E) *in Theorem* 3.1. *If* $f(G) = 0$, *then the above conclusions are valid with* $k^* = \infty$.

(2) *Assume that* φ *is globally Lipschitz continuous with Lipschitz constant* $\lambda > 0$. *Then the conclusions of statement* (1) *are valid with* $k^* = 1/|\lambda f(G)|$, *where* $1/0 := \infty$.

(3) *Under the assumption that* $f(G) > 0$, *the conclusions of statement* (1) *are valid with* $k^* = \infty$.

REMARK 4.2. (1) Theorem 4.1 ensures that the tracking error is square-integrable and hence we have tracking in measure. If $\operatorname{ess\,lim}_{t\to\infty} g(t) = 0$ and G satisfies Property (E) in Theorem 3.1, then Theorem 4.1 guarantees essential asymptotic tracking. Clearly, if $\lim_{t\to\infty} g(t) = 0$, G satisfies Property (E) and Gv is continuous whenever v is, then we have asymptotic tracking.

(2) Note that in statement (2) of Theorem 4.1 the constant k^* depends only on G and the Lipschitz constant of φ, but not on ρ.

(3) If $g \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$, then $g \in L^2(\mathbb{R}_+, \mathbb{R})$ and $\int_0^t g(\tau)\,d\tau$ converges exponentially fast to $\int_0^\infty g(\tau)\,d\tau$ as $t \to \infty$. Thus, a sufficient condition for g to satisfy the assumption in Theorem 4.1 is that $g \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$.

Theorem 4.1 gives an input-output point of view of the low-gain integral control problem with input non-linearities. The papers [**18**] and [**19**] contain related results in a state-space setting (see [**18**, Theorem 4.1; **19**, Theorem 3.3]). We emphasize that Theorem 4.1 considerably improves the latter results as follows.

(i) The range of gains guaranteed to achieve tracking (specified by k^*) is larger than in [**18**] and [**19**], where, in the case of globally Lipschitz φ with Lipschitz constant $\lambda > 0$, the maximal value for the gain is given by $k^* = 1/|\lambda f_0(G)|$ with $f_0(G) = \operatorname{ess\,inf}_{\omega\in\mathbb{R}} \operatorname{Re}[\mathbf{G}(i\omega)/i\omega]$. In many situations, the difference is substantial: as a simple example, consider the operator G given by $(Gv)(t) = \int_0^t e^{-(t-\tau)} v(\tau)\,d\tau$, for which $f(G) = 0$ and $f_0(G) = -1$, leading to $k^* = \infty$ in Theorem 4.1 and to $k^* = 1/\lambda$ in [**18**] and [**19**].

(ii) Theorem 4.1 applies to strongly stable well-posed state-space systems (see § 5), whilst in [**18**] and [**19**] the underlying linear well-posed system is assumed to be exponentially stable.

Proof of Theorem 4.1. Choose some $u^\rho \in \mathbb{R}$ such that $\varphi(u^\rho) = \rho/\mathbf{G}(0)$ (such a u^ρ exists, since, by assumption, $\rho/\mathbf{G}(0) \in \operatorname{im}\varphi$). For any $k \in \mathbb{R}$, it follows, from Lemma 2.1, that (4.1) has a unique solution u defined on a maximal interval of existence $[0, T)$, where $0 < T \leqslant \infty$. We define a function v: $[0, T) \to \mathbb{R}$ by setting $v := u - u^\rho$. Moreover, we define $\widetilde{\varphi}$: $\mathbb{R} \to \mathbb{R}$ by

$$\widetilde{\varphi}(w) := \varphi(w + u^\rho) - \varphi(u^\rho), \quad \text{for all } w \in \mathbb{R}.$$

Then a straightforward calculation shows that v satisfies

$$v(t) = r(t) - k \int_0^t (G(\widetilde{\varphi} \circ v))(\tau)\, d\tau, \quad \text{for all } t \in [0, T), \tag{4.4}$$

where the function r is given by

$$r(t) = u^0 - u^\rho - k \int_0^t [g(\tau) + \varphi(u^\rho)(G\theta)(\tau) - \rho]\, d\tau \tag{4.5}$$

(here θ denotes the unit-step function). In order to apply Corollary 3.2 to equation (4.4), we first show that

$$r \in W^{1,2}(\mathbb{R}_+, \mathbb{R}) + \mathbb{R},$$

that is, r satisfies the relevant assumption in Corollary 3.2. By the assumption on g, the function $t \mapsto u^0 - u^\rho - k \int_0^t g(\tau)\, d\tau$ is in $W^{1,2}(\mathbb{R}_+, \mathbb{R}) + \mathbb{R}$. Therefore, by (4.5), it is sufficient to show that the function

$$t \mapsto \int_0^t [\varphi(u^\rho)(G\theta)(\tau) - \rho]\, d\tau,$$

belongs to $W^{1,2}(\mathbb{R}_+, \mathbb{R}) + \mathbb{R}$. Recalling that $\varphi(u^\rho) = \rho/\mathbf{G}(0)$, we may write

$$\begin{aligned}\int_0^t [\varphi(u^\rho)(G\theta)(\tau) - \rho]\, d\tau &= \left(\int_0^t (G\theta)(\tau)\, d\tau - t\mathbf{G}(0) \right) \varphi(u^\rho) \\ &= h(t)\varphi(u^\rho),\end{aligned} \tag{4.6}$$

where the function h is defined by

$$h(t) := \int_0^t (G\theta)(\tau)\, d\tau - t\mathbf{G}(0), \quad \text{for all } t \in \mathbb{R}_+. \tag{4.7}$$

Hence it is sufficient to prove that $h \in W^{1,2}(\mathbb{R}_+, \mathbb{R}) + \mathbb{R}$. We mention that $h = H\theta$, where H is the shift-invariant operator defined by (3.13); that is, h is the step-response of H and, correspondingly, $\dot{h}$ is the impulse response (or convolution kernel) of H. The Laplace transforms of $\dot{h}$ and $h - \mathbf{G}'(0)$ are given by the functions

$$s \mapsto (\mathbf{G}(s) - \mathbf{G}(0))/s \quad \text{and} \quad s \mapsto (\mathbf{G}(s) - \mathbf{G}(0) - s\mathbf{G}'(0))/s^2,$$

respectively, both of which, by Assumption (A′), belong to $H^2(\mathbb{C}_0, \mathbb{C})$. Consequently $\dot{h}, h - \mathbf{G}'(0) \in L^2(\mathbb{R}_+, \mathbb{R})$, showing that $h \in W^{1,2}(\mathbb{R}_+, \mathbb{R}) + \mathbb{R}$.

Proof of statement (1). Since φ is non-decreasing and $\varphi - \varphi(0) \in \mathscr{S}(a)$ for some $a \in (0, \infty)$, a routine argument invoking (4.3) shows that there exists $b \in (0, \infty)$ such that $\widetilde{\varphi} \in \mathscr{S}(b)$. Define $k^* := 1/|bf(G)|$, where as usual $1/0 := \infty$. We may assume that $f(G) < \infty$ because the case $f(G) = \infty$ is included in statement (3). Therefore, $k^* \in (0, \infty]$. Let $k \in (0, k^*)$ and set $\varepsilon := \frac{1}{2}(1/b + kf(G))$. Then $\varepsilon > 0$, and we can choose some $q > 0$ such that

$$\operatorname*{ess\,inf}_{\omega \in \mathbb{R}} \operatorname{Re}[(q + 1/i\omega)\mathbf{G}(i\omega)] \geqslant f(G) - \varepsilon/k = (\varepsilon - 1/b)/k.$$

Thus

$$1/b + \operatorname{Re}[(q + 1/i\omega)k\mathbf{G}(i\omega)] \geqslant \varepsilon, \quad \text{for a.a. } \omega \in \mathbb{R},$$

and (3.9) holds with a replaced by b and $\mathbf{G}$ replaced by $k\mathbf{G}$. An application of

Corollary 3.2 to (4.4) yields the facts that the solution v of (4.4) exists on $\mathbb{R}_+$, $\lim_{t\to\infty} v(t)$ exists and is finite, $\lim_{t\to\infty}(\widetilde{\varphi}\circ v)(t)=0$, $\widetilde{\varphi}\circ v\in L^2(\mathbb{R}_+,\mathbb{R})$, and $\dot v\in L^2(\mathbb{R}_+,\mathbb{R})$. Consequently, $\lim_{t\to\infty} u(t)=: u^\infty$ exists and is finite,

$$\varphi(u^\infty)=\varphi(u^\rho)=\rho/\mathbf{G}(0),$$

and

$$\varphi\circ u-\varphi(u^\infty)\in L^2(\mathbb{R}_+,\mathbb{R}),\quad e=\dot u/k=\dot v/k\in L^2(\mathbb{R}_+,\mathbb{R}).$$

Assume now that $\operatorname{ess\,lim}_{t\to\infty} g(t)=0$ and that G satisfies Condition (E). To complete the proof of statement (1), it remains to show that $\operatorname{ess\,lim}_{t\to\infty} e(t)=0$. By Corollary 3.2, $\operatorname{ess\,lim}_{t\to\infty} e(t)=\operatorname{ess\,lim}_{t\to\infty}\dot v(t)/k=0$ if and only if $\operatorname{ess\,lim}_{t\to\infty}\dot r(t)=0$. Thus it is sufficient to prove that $\operatorname{ess\,lim}_{t\to\infty}\dot r(t)=0$, which in turn, by (4.5)–(4.7), is equivalent to showing that

$$\operatorname*{ess\,lim}_{t\to\infty}\dot h(t)=0. \tag{4.8}$$

To prove the latter, write the unit step function θ as a sum $\theta=\theta_1+\theta_2$, where both of these functions are continuously differentiable, $\theta_1(0)=0$ and $\theta_1(t)=1$ for $t\geqslant 1$. Using the operator H defined in (3.13) and the shift-invariance of G (which guarantees that G commutes with integration), we may write $\dot h$ in the form

$$\dot h(t)=(G\theta)(t)-\mathbf{G}(0)=(H\dot\theta_1)(t)+(G\theta_2)(t),\quad \text{for all } t\geqslant 1.$$

Since $\dot\theta_1\in L^2(\mathbb{R}_+,\mathbb{R})$ and the convolution kernel $\dot h$ of H is in $L^2(\mathbb{R}_+,\mathbb{R})$, it follows that $(H\dot\theta_1)(t)\to 0$ as $t\to\infty$. Moreover, by Condition (E), $\operatorname{ess\,lim}_{t\to\infty}(G\theta_2)(t)=0$, yielding (4.8).

Proof of statement (2). Since φ is non-decreasing and globally Lipschitz with Lipschitz constant $\lambda>0$, it is easy to show that $\widetilde{\varphi}\in\mathscr{S}(\lambda)$. Now the arguments in the proof of statement (1) apply with b replaced by λ.

Proof of statement (3). By assumption, φ is non-decreasing, and hence it is clear that $\widetilde{\varphi}\in\mathscr{S}(\infty)$. Since $f(G)>0$, we can choose some $q\geqslant 0$ such that $\operatorname{ess\,inf}_{\omega\in\mathbb{R}}\operatorname{Re}[(q+1/i\omega)\mathbf{G}(i\omega)]\geqslant\frac{1}{2}f(G)$. This implies that, for any $k>0$,

$$\operatorname{Re}[(q+1/i\omega)k\mathbf{G}(i\omega)]\geqslant\tfrac{1}{2}kf(G)>0,\quad \text{for a.a. } \omega\in\mathbb{R}.$$

Therefore, (3.9) holds with $a=\infty$, $\mathbf{G}$ replaced by $k\mathbf{G}$, and ε replaced by $\frac{1}{2}kf(G)$. As in the proof of statement (1), the claim now follows from Corollary 3.2. $\square$

4.2. *Integral control in the presence of input and output non-linearities*

In this subsection we generalize the feedback scheme in § 4.1 to allow for a time-varying gain and non-linearities in the input as well as in the output. Consider the feedback system shown in Figure 4, where $\rho\in\mathbb{R}$ is a constant, $\kappa\colon\mathbb{R}_+\to\mathbb{R}$ is a time-varying gain, the operator $G\in\mathscr{B}(L^2(\mathbb{R}_+,\mathbb{R}))$ is shift-invariant with transfer function denoted by $\mathbf{G}$, and $\varphi\colon\mathbb{R}\to\mathbb{R}$ and $\psi\colon\mathbb{R}\to\mathbb{R}$ are static input and output non-linearities, respectively.

The feedback system shown in Figure 4 is described by the abstract Volterra integro-differential equation

$$\dot u=\kappa[\rho-\psi(g+G(\varphi\circ u))],\quad u(0)=u^0\in\mathbb{R}, \tag{4.9}$$

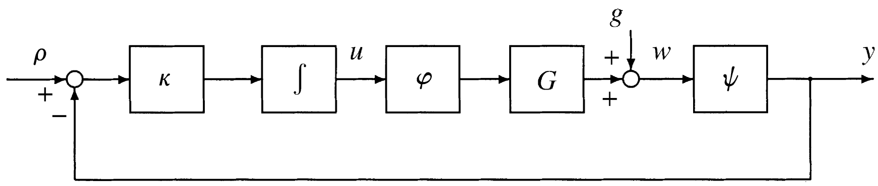

FIGURE 4.

where the function g models the effect of non-zero initial conditions of the system with input-output operator G. An absolutely continuous function $u\colon [0,T) \to \mathbb{R}$, where $0 < T \leqslant \infty$, is called a solution of (4.9) on $[0,T)$ if $u(0) = u^0$ and u satisfies the differential equation in (4.9) a.e. on $[0,T)$.

The following lemma, the proof of which can be found in the appendix of [**10**], shows that under certain conditions the initial-value problem (4.9) has a unique solution defined on $\mathbb{R}_+$.

LEMMA 4.3. *Assume that $G \in \mathscr{B}(L^2(\mathbb{R}_+, \mathbb{R}))$ is shift-invariant, φ and ψ are globally Lipschitz continuous, $\kappa \in L^\infty(\mathbb{R}_+, \mathbb{R})$, $g \in L^2(\mathbb{R}_+, \mathbb{R})$, and $\rho \in \mathbb{R}$. Then, for each $u^0 \in \mathbb{R}$, the initial-value problem* (4.9) *has a unique solution defined on $\mathbb{R}_+$.*

The objective in this subsection is to determine gain functions κ such that the tracking error

$$e(t) := \rho - y(t) = \rho - \psi(g(t) + (G(\varphi \circ u))(t)) \tag{4.10}$$

becomes small in a certain sense as $t \to \infty$. We introduce the set of feasible reference values

$$\mathscr{R}(G, \varphi, \psi) := \{\psi(\mathbf{G}(0)v) \mid v \in \overline{\operatorname{im} \varphi}\}.$$

It is clear that $\mathscr{R}(G, \varphi, \psi)$ is an interval, provided that φ and ψ are continuous. The motivation for the introduction of $\mathscr{R}(G, \varphi, \psi)$ is as follows. If asymptotic tracking occurs, we would expect that $(\varphi \circ u)^\infty := \lim_{t\to\infty}(\varphi \circ u)(t)$ exists. Assuming that $(\varphi \circ u)^\infty$ is finite and that the final-value theorem holds for the linear system with input-output operator G, we may conclude that $\lim_{t\to\infty}(G(\varphi \circ u))(t) = \mathbf{G}(0)(\varphi \circ u)^\infty$. If additionally, $\lim_{t\to\infty} g(t) = 0$, it follows from (4.10) that $\rho = \psi(\mathbf{G}(0)(\varphi \circ u)^\infty) \in \mathscr{R}(G, \varphi, \psi)$. In fact, it has been shown in [**10**] that if ψ is continuous and monotone, then $\rho \in \mathscr{R}(G, \varphi, \psi)$ is close to being a necessary condition for asymptotic tracking insofar as, if asymptotic tracking of ρ is achievable, whilst maintaining boundedness of $\varphi \circ u$ together with ultimate continuity and ultimate boundedness of $w = g + G(\varphi \circ u)$, then $\rho \in \mathscr{R}(G, \varphi, \psi)$.

Setting

$$f_0(G) := \operatorname*{ess\,inf}_{\omega\in\mathbb{R}} \operatorname{Re}[\mathbf{G}(i\omega)/i\omega],$$

one sees clearly that if the transfer function $\mathbf{G}$ satisfies Assumption (A), then $-\infty < f_0(G) \leqslant 0$.

We are now in a position to state the main result of this subsection.

THEOREM 4.4. *Let $G \in \mathscr{B}(L^2(\mathbb{R}_+, \mathbb{R}))$ be a shift-invariant operator with transfer function $\mathbf{G}$. Assume that Assumption* (A) *holds with $\mathbf{G}(0) > 0$ and $g \in L^2(\mathbb{R}_+, \mathbb{R})$, that $\varphi\colon \mathbb{R} \to \mathbb{R}$ and $\psi\colon \mathbb{R} \to \mathbb{R}$ are non-decreasing and globally*

Lipschitz continuous with Lipschitz constants $\lambda_1 > 0$ *and* $\lambda_2 > 0$, $\rho \in \mathscr{R}(G, \varphi, \psi)$, *and* $\kappa\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is measurable and bounded with*

$$\limsup_{t\to\infty} \kappa(t) < 1/|\lambda_1 \lambda_2 f_0(G)|, \tag{4.11}$$

where $1/0 := \infty$. *Let* $u\colon \mathbb{R}_+ \to \mathbb{R}$ *be the unique solution of* (4.9) (*which exists by Lemma* 4.3). *Then the following statements hold.*

(1) *The limit* $(\varphi \circ u)^\infty := \lim_{t\to\infty} \varphi(u(t))$ *exists and is finite and*

$$(\varphi \circ u)' \in L^2(\mathbb{R}_+, \mathbb{R}).$$

(2) *The signals* $w = g + G(\varphi \circ u)$ *and* $y = \psi \circ w$ (*see Figure* 4) *can be split into* $w = w_1 + w_2$ *and* $y = y_1 + y_2$, *where* w_1 *and* y_1 *are continuous and have finite limits satisfying*

$$\lim_{t\to\infty} w_1(t) = \mathbf{G}(0)(\varphi \circ u)^\infty, \quad \lim_{t\to\infty} y_1(t) = \psi(\mathbf{G}(0)(\varphi \circ u)^\infty),$$

and $w_2, y_2 \in L^2(\mathbb{R}_+, \mathbb{R})$. *Under the additional assumptions that* $\operatorname{ess\,lim}_{t\to\infty} g(t) = 0$ *and G satisfies Condition* (E) *in Theorem* 3.1, *we have*

$$\operatorname*{ess\,lim}_{t\to\infty} w_2(t) = 0, \quad \operatorname*{ess\,lim}_{t\to\infty} y_2(t) = 0.$$

(3) *If* $\kappa \notin L^1(\mathbb{R}_+, \mathbb{R})$, *then* $\lim_{t\to\infty} y_1(t) = \rho$ *and the error signal e can be split into* $e = e_1 + e_2$, *where* e_1 *is continuous with* $\lim_{t\to\infty} e_1(t) = 0$ *and* $e_2 \in L^2(\mathbb{R}_+, \mathbb{R})$. *Under the additional assumptions that* $\operatorname{ess\,lim}_{t\to\infty} g(t) = 0$ *and G satisfies Condition* (E) *in Theorem* 3.1, *we have*

$$\operatorname*{ess\,lim}_{t\to\infty} e(t) = 0.$$

(4) *If* ρ *is an interior point of* $\mathscr{R}(G, \varphi, \psi)$, *then u is bounded.*

REMARK 4.5. (1) Statement (3) of Theorem 4.4 implies tracking in measure. Under the assumption that $\operatorname{ess\,lim}_{t\to\infty} g(t) = 0$ and G satisfies Property (E) in Theorem 3.1, statement (3) of Theorem 4.4 guarantees essential asymptotic tracking. Moreover, if $\lim_{t\to\infty} g(t) = 0$, G satisfies Property (E) and Gv is continuous whenever v is, then we have asymptotic tracking.

(2) Note that it is not necessary to know $f_0(G)$ or the Lipschitz constants λ_1 and λ_2 in order to apply Theorem 4.4. If κ is chosen such that $\kappa(t) \to 0$ and $\kappa \notin L^1(\mathbb{R}_+, \mathbb{R})$ (for example, $\kappa(t) = (1+t)^{-p}$ with $p \in (0, 1]$), then the conclusions of statement (3) hold. However, from a practical point of view, gain functions κ with $\lim_{t\to\infty} \kappa(t) = 0$ might not be appropriate, since the system essentially operates in open loop as $t \to \infty$. In [**20**] it has been shown how $|f_0(G)|$ (or upper bounds for $|f_0(G)|$) can be obtained from frequency-response experiments performed on the linear part of the plant.

(3) Under certain conditions on G, g and φ the global Lipschitz condition on ψ can be relaxed. More precisely, under the extra assumptions that $G \in \mathscr{B}(L^\infty(\mathbb{R}_+, \mathbb{R}))$ (or equivalently, that G is a convolution operator with a finite measure kernel), that $g \in L^\infty(\mathbb{R}_+, \mathbb{R})$ and that φ is bounded (this is the case if φ is of saturation type), then the conclusions of Theorem 4.4 remain valid for all non-decreasing locally Lipschitz continuous ψ with the global Lipschitz constant λ_2 in the

statement of Theorem 4.4 replaced by

$$\sup_{v,w\in[-K,K]} \frac{|\psi(v)-\psi(w)|}{|v-w|} < \infty,$$

where $K := \|g\|_\infty + \|G\|_{\mathscr{B}(L^\infty(\mathbb{R}_+,\mathbb{R}))}\|\varphi\|_\infty$.

(4) An inspection of the last part of the proof of statement (4) shows that if (4.11) holds and if $\kappa \in L^1(\mathbb{R}_+,\mathbb{R})$, then the limit $u^\infty := \lim_{t\to\infty} u(t)$ exists and is finite. In this case $\psi(\mathbf{G}(0)(\varphi\circ u)^\infty) = \psi(\mathbf{G}(0)\varphi(u^\infty))$, but in general, $\psi(\mathbf{G}(0)\varphi(u^\infty)) \neq \rho$ (trivial example $\kappa \equiv 0$) and tracking is not guaranteed.

(5) If $\rho \in \mathscr{R}(G,\varphi,\psi)$ is not an interior point, then u might be unbounded. A trivial example is given by $\varphi = \arctan$, $\psi = \mathrm{id}$, $\rho = \frac{1}{2}\mathbf{G}(0)\pi$ and $\kappa \notin L^1(\mathbb{R}_+,\mathbb{R})$, in which case it follows from statements (2) and (3) that $(\varphi\circ u)^\infty = \frac{1}{2}\pi$ and hence $\lim_{t\to\infty} u(t) = \infty$.

Theorem 4.4 gives an input-output point of view of the low-gain integral control problem with input non-linearities. The paper [**10**] contains a related result in a state-space setting (see [**10**, Theorem 4.2]). We emphasize that Theorem 4.4 considerably improves the latter result in the sense that it applies to strongly stable well-posed state-space systems (see § 5), whilst in [**10**] it is a crucial assumption that the underlying linear well-posed system is exponentially stable.

In order to prove Theorem 4.4, we need the following technical lemma.

LEMMA 4.6. *Let $f: \mathbb{R}\to\mathbb{R}$ be locally Lipschitz continuous. Define the function $f^\nabla: \mathbb{R}\to\mathbb{R}$ by*

$$f^\nabla(\xi) := \limsup_{n\to\infty} \frac{f(\xi+1/n)-f(\xi)}{1/n}.$$

Then f^∇ is Borel measurable and $f^\nabla \in L^\infty_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$. If $v: \mathbb{R}_+\to\mathbb{R}$ is absolutely continuous, then $f\circ v$ is absolutely continuous and

$$\frac{d}{dt}(f\circ v)(t) = f^\nabla(v(t))\dot v(t), \quad \text{for a.a. } t\in\mathbb{R}_+.$$

If f is non-decreasing and globally Lipschitz continuous with Lipschitz constant $\lambda \geqslant 0$, then

$$0 \leqslant f^\nabla(\xi) \leqslant \lambda, \quad \text{for all } \xi\in\mathbb{R}.$$

The proof of Lemma 4.6 can be found in [**19**].

Proof of Theorem 4.4. Let $u: \mathbb{R}_+\to\mathbb{R}$ be the unique solution of (4.9) (which exists by Lemma 4.3). We shall prove Theorem 4.4 by applying Theorem 3.3 to the equation satisfied by the input signal

$$w = g + G(\varphi\circ u)$$

of the output non-linearity ψ (see Figure 4), modified with an offset which depends on ρ. Since $\rho\in\mathscr{R}(G,\varphi,\psi)$, there exists $\varphi^\rho \in \overline{\mathrm{im}\,\varphi}$ satisfying

$$\psi(\mathbf{G}(0)\varphi^\rho) = \rho.$$

We define

$$\begin{aligned} \tilde w &:= w - \mathbf{G}(0)\varphi^\rho = g + G(\varphi\circ u) - \mathbf{G}(0)\varphi^\rho, \\ \tilde\psi(\xi) &:= \psi(\xi + \mathbf{G}(0)\varphi^\rho) - \rho, \quad \text{for all } \xi\in\mathbb{R}. \end{aligned} \tag{4.12}$$

Note that $\widetilde{\psi}(0) = 0$, and so, since ψ is non-decreasing and globally Lipschitz with Lipschitz constant λ_2,

$$0 \leqslant \widetilde{\psi}(\xi)\xi \leqslant \lambda_2 \xi^2, \quad \text{for all } \xi \in \mathbb{R}. \tag{4.13}$$

Using (4.9), we may write

$$\dot{u} = -\kappa(\widetilde{\psi} \circ \widetilde{w}).$$

It follows from Lemma 4.6 that

$$\begin{aligned}(\varphi \circ u)'(t) &= \varphi^{\nabla}(u(t))\dot{u}(t) = -\varphi^{\nabla}(u(t))\kappa(t)\widetilde{\psi}(\widetilde{w}(t)) \\ &= -(N \circ \widetilde{w})(t), \quad \text{for a.a. } t \in \mathbb{R}_+,\end{aligned} \tag{4.14}$$

where the function $N\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is defined by

$$N(t, \xi) := \varphi^{\nabla}(u(t))\kappa(t)\widetilde{\psi}(\xi), \quad \text{for all } (t, \xi) \in \mathbb{R}_+ \times \mathbb{R}.$$

By Lemma 4.6, $0 \leqslant \varphi^{\nabla}(\xi) \leqslant \lambda_1$ for all $\xi \in \mathbb{R}_+$, and combining this with (4.13) yields

$$0 \leqslant N(t, \xi)\xi \leqslant \lambda_1 \lambda_2 \kappa(t)\xi^2, \quad \text{for all } (t, \xi) \in \mathbb{R}_+ \times \mathbb{R}. \tag{4.15}$$

By (4.14),

$$(\varphi \circ u)(t) = \varphi(u^0) - \int_0^t (N \circ \widetilde{w})(\tau)\, d\tau, \quad \text{for all } t \in \mathbb{R}_+.$$

We apply G to this equation and use the fact that, by shift-invariance, G commutes with the integration operator to obtain

$$(G(\varphi \circ u))(t) = \varphi(u^0)(G\theta)(t) - \int_0^t (G(N \circ \widetilde{w}))(\tau)\, d\tau, \quad \text{for a.a. } t \in \mathbb{R}_+,$$

where θ denotes the unit-step function. Invoking (4.12) yields

$$\widetilde{w}(t) = r(t) - \int_0^t (G(N \circ \widetilde{w}))(\tau)\, d\tau, \quad \text{for a.a. } t \in \mathbb{R}_+, \tag{4.16}$$

where

$$r := g - \mathbf{G}(0)\varphi^{\rho} + \varphi(u^0)G\theta.$$

Proof of statement (1). Clearly, (4.16) is of the form (2.1), and so we may apply Theorem 3.3, provided the relevant assumptions are satisfied. By (4.11), there exists $a > 0$ satisfying

$$\lambda_1 \lambda_2 \limsup_{t \to \infty} \kappa(t) < a < 1/|f_0(G)|.$$

Clearly, by the definition of $f_0(G)$, there exists $\varepsilon > 0$ such that

$$1/a + \operatorname{Re}[\mathbf{G}(i\omega)/i\omega] \geqslant \varepsilon, \quad \text{for a.a. } \omega \in \mathbb{R}.$$

Moreover, it follows from (4.15) that there exists $t_0 \geqslant 0$ such that

$$0 \leqslant N(t, \xi)\xi \leqslant a\xi^2, \quad \text{for all } (t, \xi) \in [t_0, \infty) \times \mathbb{R}.$$

The above two inequalities show that (3.28) and (3.30) hold with u, φ, Q, and P replaced by $\widetilde{w}$, N, I, and $1/a$, respectively. In order to apply Theorem 3.3, it remains to verify that $r \in L^2(\mathbb{R}_+, \mathbb{R}) + \mathbb{R}$. To this end note that

$$r = g + \varphi(u^0)\dot{h} + \mathbf{G}(0)(\varphi(u^0) - \varphi^{\rho}), \tag{4.17}$$

where h is the function defined in (4.7). In the proof of Theorem 4.1 it was shown that $\dot{h} \in L^2(\mathbb{R}_+, \mathbb{R})$. Since, by assumption, $g \in L^2(\mathbb{R}_+, \mathbb{R})$, it follows from (4.17) that $r \in L^2(\mathbb{R}_+, \mathbb{R}) + \mathbb{R}$. An application of part (4) of Theorem 3.3 to (4.16) now yields the facts that

$$(\varphi \circ u)' = -(N \circ \widetilde{w}) \in L^2(\mathbb{R}_+, \mathbb{R}),$$

and the limit

$$\lim_{t \to \infty} (\varphi \circ u)(t) = \varphi(u^0) - \lim_{t \to \infty} \int_0^t (N \circ \widetilde{w})(\tau)\, d\tau$$

exists and is finite, completing the proof of statement (1).

Proof of statement (2). Using the shift-invariant operator H defined in (3.13) and the function h defined in (4.7), a straightforward calculation yields

$$w = g + G(\varphi \circ u) = w_1 + w_2,$$

where

$$w_1 := H((\varphi \circ u)') + \mathbf{G}(0)(\varphi \circ u), \quad w_2 := g + \varphi(u^0)\dot{h}.$$

By assumption $g \in L^2(\mathbb{R}_+, \mathbb{R})$ and, as was pointed out in the proof of statement (1), we also know that $\dot{h} \in L^2(\mathbb{R}_+, \mathbb{R})$. Consequently, $w_2 \in L^2(\mathbb{R}_+, \mathbb{R})$. Moreover, H is a convolution operator with kernel given by $\dot{h}$ and $(\varphi \circ u)' \in L^2(\mathbb{R}_+, \mathbb{R})$; hence $(H(\varphi \circ u)')(t)$ is continuous and tends to zero as $t \to \infty$. Combining this with the continuity of $\varphi \circ u$ shows that w_1 is continuous and, by statement (1), $\lim_{t \to \infty} w_1(t) = \mathbf{G}(0)(\varphi \circ u)^\infty$. We obtain the required splitting of y by defining $y_1 := \psi \circ w_1$ and $y_2 := y - y_1 = \psi \circ w - \psi \circ w_1$. Then y_1 is continuous with $\lim_{t \to \infty} y_1(t) = \psi(\mathbf{G}(0)(\varphi \circ u)^\infty)$. Furthermore, $y_2 \in L^2(\mathbb{R}_+, \mathbb{R})$ because of the global Lipschitz continuity of ψ and the fact that $w_2 \in L^2(\mathbb{R}_+, \mathbb{R})$. Under the additional assumptions that $\operatorname{ess\,lim}_{t \to \infty} g(t) = 0$ and that G satisfies Condition (E), we know (by (4.8)) that $\operatorname{ess\,lim}_{t \to \infty} \dot{h}(t) = 0$, and therefore, we may conclude that

$$\operatorname*{ess\,lim}_{t \to \infty} w_2(t) = \operatorname*{ess\,lim}_{t \to \infty} y_2(t) = 0,$$

completing the proof of statement (2).

Proof of statement (3). We use the splitting $y = y_1 + y_2$ given in statement (2). Seeking a contradiction, suppose that $y_1^\infty := \lim_{t \to \infty} y_1(t) < \rho$ (the case $y^\infty > \rho$ can be treated in an analogous way). Then $\varepsilon := \rho - y_1^\infty > 0$. We further split y_2 into $y_2 = y_3 + y_4$, where

$$y_3(t) := \min\{y_2(t), \tfrac{1}{3}\varepsilon\}, \quad y_4(t) := y_2(t) - y_3(t), \quad \text{for all } t \in \mathbb{R}_+.$$

Thus, $y_3(t) \leqslant \frac{1}{3}\varepsilon$ for all $t \in \mathbb{R}_+$ and $y_4 = \chi_E(y_2 - \frac{1}{3}\varepsilon)$, where χ_E is the indicator function of the set $E := \{t \in \mathbb{R}_+ \mid y_2(t) > \frac{1}{3}\varepsilon\}$. In particular, since $y_2 \in L^2(\mathbb{R}_+, \mathbb{R})$, the set E has finite Lebesgue measure, and so $y_4 \in L^1(\mathbb{R}_+, \mathbb{R})$. Since $y_3 \leqslant \frac{1}{3}\varepsilon$ and $\lim_{t \to \infty} y_1(t) = y_1^\infty$, by taking $t_0 \geqslant 0$ large enough, we have $\rho - y_1(t) - y_3(t) \geqslant \frac{1}{3}\varepsilon$ for all $t \geqslant t_0$. We substitute the decomposition $y = y_1 + y_3 + y_4$ into $\dot{u} = \kappa(\rho - y)$ to obtain

$$\dot{u}(\tau) = \kappa(\tau)[\rho - y_1(\tau) - y_3(\tau) - y_4(\tau)] \geqslant \tfrac{1}{3}\varepsilon\kappa(\tau) - \kappa(\tau)y_4(\tau), \quad \text{for a.a. } \tau \geqslant t_0.$$

Observe that $\kappa y_4 \in L^1(\mathbb{R}_+, \mathbb{R})$ since $\kappa \in L^\infty(\mathbb{R}_+, \mathbb{R})$ and $y_4 \in L^1(\mathbb{R}_+, \mathbb{R})$. Integrating the above inequality from t_0 to t gives

$$u(t) \geqslant u(t_0) + \tfrac{1}{3}\varepsilon \int_{t_0}^{t} \kappa(\tau)\, d\tau - \int_{t_0}^{\infty} \kappa(\tau) y_4(\tau)\, d\tau \to \infty \quad \text{as } t \to \infty.$$

Consequently, since φ is non-decreasing,

$$\overline{\varphi} := \sup_{v \in \mathbb{R}} \varphi(v) = (\varphi \circ u)^{\infty}.$$

Hence, by statement (2), $y_1^{\infty} = \psi(\mathbf{G}(0)\overline{\varphi})$. Since $\rho \in \mathscr{R}(G, \varphi, \psi)$ (by assumption) and using the fact that ψ is non-decreasing and $\mathbf{G}(0) > 0$, we obtain

$$\rho \leqslant \sup \mathscr{R}(G, \varphi, \psi) = \psi(\mathbf{G}(0)\overline{\varphi}) = y_1^{\infty},$$

contradicting the supposition that $y_1^{\infty} < \rho$. Setting $\underline{\varphi} := \inf_{v \in \mathbb{R}} \varphi(v)$, we see that an analogous argument shows that if $y^{\infty} > \rho$, then necessarily $y^{\infty} = \psi(\mathbf{G}(0)\underline{\varphi})$, which likewise leads to a contradiction since $\rho \geqslant \psi(\mathbf{G}(0)\underline{\varphi})$. Finally, defining $e_1 := \rho - y_1$ and $e_2 := -y_2$, we obtain the splitting $e = \rho - y = e_1 + e_2$. It follows immediately from the properties of y_1 and y_2 that e_1 is continuous with $\lim_{t \to \infty} e_1(t) = 0$ and $e_2 \in L^2(\mathbb{R}_+, \mathbb{R})$. Moreover, using statement (2), we see that if $\operatorname{ess\,lim}_{t \to \infty} g(t) = 0$ and G satisfies Condition (E), then $\operatorname{ess\,lim}_{t \to \infty} e_2(t) = 0$, implying that $\operatorname{ess\,lim}_{t \to \infty} e(t) = 0$.

Proof of statement (4). By statement (1), the limit $(\varphi \circ u)^{\infty} = \lim_{t \to \infty} (\varphi \circ u)(t)$ exists and is finite. If $\kappa \notin L^1(\mathbb{R}_+, \mathbb{R})$, then by statements (2) and (3), $\rho = \psi(\mathbf{G}(0)(\varphi \circ u)^{\infty})$. Unboundedness of u would imply that there exists a sequence (v_n) with $\lim_{n \to \infty} |v_n| = \infty$ and such that

$$\rho = \lim_{n \to \infty} \psi(\mathbf{G}(0)\varphi(v_n)).$$

Since the function $v \mapsto \psi(\mathbf{G}(0)\varphi(v))$ is non-decreasing, this would in turn yield $\rho = \sup \mathscr{R}(G, \varphi, \psi)$ or $\rho = \inf \mathscr{R}(G, \varphi, \psi)$, showing that u must be bounded if ρ is an interior point of $\mathscr{R}(G, \varphi, \psi)$. Finally, if $\kappa \in L^1(\mathbb{R}_+, \mathbb{R})$, consider the equation

$$\dot{u} = \kappa(\rho - y_1 - y_2),$$

where $y = y_1 + y_2$, with y_1 and y_2 as in statement (2). Since κ is also in $L^{\infty}(\mathbb{R}_+, \mathbb{R})$, it follows that $\kappa \in L^2(\mathbb{R}_+, \mathbb{R})$. Furthermore, y_1 is bounded and $y_2 \in L^2(\mathbb{R}_+, \mathbb{R})$. Therefore, $\kappa(\rho - y_1 - y_2) \in L^1(\mathbb{R}_+, \mathbb{R})$, showing that $\dot{u}$ is integrable and hence u is bounded. □

5. *Applications to well-posed state-space systems*

This section is devoted to applications of the results in §§ 3 and 4 to well-posed state-space systems. There are a number of equivalent definitions of well-posed systems; see [**7**, **26**, **27**, **29**, **30**, **31**, **32**, **35**, **36**]. We will be brief in the following and refer the reader to the above references for more details. Throughout this section, we shall be considering a well-posed system Σ with state-space X, input space U, and output space $Y = U$, generating operators (A, B, C), input-output operator G, and transfer function $\mathbf{G}$. Here X and U are real separable Hilbert spaces, A is the generator of a strongly continuous semigroup $\mathbf{T} = (\mathbf{T}_t)_{t \geqslant 0}$ on X, $B \in \mathscr{B}(U, X_{-1})$, and $C \in \mathscr{B}(X_1, U)$, where X_{-1} and X_1 are the usual extrapolation and interpolation spaces of X; see the subsection on notation at the end of the

Introduction. The norms on X, X_{-1} and X_1 are denoted by $\|\cdot\|$, $\|\cdot\|_{-1}$ and $\|\cdot\|_1$, respectively. Moreover, the operator B is an *admissible control operator* for $\mathbf{T}$, that is, for each $t \in \mathbb{R}_+$ there exists $\alpha_t \geqslant 0$ such that

$$\left\| \int_0^t \mathbf{T}_{t-\tau} B v(\tau)\, d\tau \right\| \leqslant \alpha_t \|v\|_{L^2([0,t],U)}, \quad \text{for all } v \in L^2([0,t],U);$$

the operator C is an *admissible observation operator* for $\mathbf{T}$, that is, for each $t \in \mathbb{R}_+$ there exists $\beta_t \geqslant 0$ such that

$$\left(\int_0^t \|C\mathbf{T}_\tau z\|^2\, d\tau \right)^{1/2} \leqslant \beta_t \|z\|, \quad \text{for all } z \in X_1.$$

The control operator B is said to be *bounded* if it is so as a map from the input space U to the state space X, otherwise B is said to be *unbounded*. The observation operator C is said to be *bounded* if it can be extended continuously to X, otherwise, C is said to be *unbounded.*

The so-called Λ-*extension* C_Λ of C is defined by

$$C_\Lambda z = \lim_{s\to\infty,\, s\in\mathbb{R}} Cs(sI-A)^{-1} z,$$

with $\operatorname{dom}(C_\Lambda)$ consisting of all $z \in X$ for which the above limit exists. For every $z \in X$, $\mathbf{T}_t z \in \operatorname{dom}(C_\Lambda)$ for a.a. $t \in \mathbb{R}_+$ and, if $\omega > \omega(\mathbf{T})$, then $C_\Lambda \mathbf{T} z \in L^2_\omega(\mathbb{R}_+, U)$, where

$$\omega(\mathbf{T}) := \lim_{t\to\infty} \frac{1}{t} \ln \|\mathbf{T}_t\|$$

denotes the exponential growth constant of $\mathbf{T}$. The transfer function $\mathbf{G}$ satisfies

$$\frac{1}{s-s_0}(\mathbf{G}(s) - \mathbf{G}(s_0)) = -C(sI-A)^{-1}(s_0 I - A)^{-1} B,$$
$$\text{for all } s, s_0 \in \mathbb{C}_{\omega(\mathbf{T})},\, s \neq s_0, \tag{5.1}$$

and $\mathbf{G} \in H^\infty(\mathbb{C}_\omega, \mathscr{B}(U_c))$ for every $\omega > \omega(\mathbf{T})$. Moreover, the input-output operator G: $L^2_{\text{loc}}(\mathbb{R}_+, U) \to L^2_{\text{loc}}(\mathbb{R}_+, U)$ is continuous and shift-invariant; for every $\omega > \omega(\mathbf{T})$, $G \in \mathscr{B}(L^2_\omega(\mathbb{R}_+, U))$ and

$$(\mathfrak{L}(Gv))(s) = \mathbf{G}(s)(\mathfrak{L}(v))(s), \quad \text{for all } s \in \mathbb{C}_\omega \text{ and all } v \in L^2_\omega(\mathbb{R}_+, U).$$

In the following, let $s_0 \in \mathbb{C}_{\omega(\mathbf{T})}$ be fixed, but arbitrary. For $x^0 \in X$ and $v \in L^2_{\text{loc}}(\mathbb{R}_+, U)$, let x and y denote the state and output functions of Σ, respectively, corresponding to the initial condition $x(0) = x^0 \in X$ and the input function v. Then $x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} B v(\tau)\, d\tau$ for all $t \in \mathbb{R}_+$, $x(t) - (s_0 I - A)^{-1} B v(t) \in \operatorname{dom}(C_\Lambda)$ for a.a. $t \in \mathbb{R}_+$, and

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bv(t), \quad x(0) = x^0, \quad \text{for a.a. } t \in \mathbb{R}_+, \\ y(t) &= C_\Lambda(x(t) - (s_0 I - A)^{-1} B v(t)) + \mathbf{G}(s_0) v(t), \quad \text{for a.a. } t \geqslant 0. \end{aligned} \tag{5.2}$$

Of course, the differential equation in (5.2) has to be interpreted in X_{-1}. Note that the second equation in (5.2) yields the following formula for the input-output operator G:

$$(Gv)(t) = C_\Lambda \left[\int_0^t \mathbf{T}_{t-\tau} B v(\tau)\, d\tau - (s_0 I - A)^{-1} B v(t) \right] + \mathbf{G}(s_0) v(t),$$
$$\text{for all } v \in L^2_{\text{loc}}(\mathbb{R}_+, U), \text{ a.a. } t \in \mathbb{R}_+. \tag{5.3}$$

In the following, we identify Σ and (5.2) and refer to (5.2) as a well-posed system. The above formulas for the output, the input-output operator and the transfer function reduce to a more recognizable form for the subclass of regular systems. Recall that the well-posed system (5.2) is called *regular* if the strong limit

$$\lim_{s\to\infty,\, s\in\mathbb{R}} \mathbf{G}(s)w = Dw, \quad \text{for all } w\in U,$$

exists. In this case, $x(t)\in\operatorname{dom}(C_\Lambda)$ for a.a. $t\in\mathbb{R}_+$, the output equation in (5.2) and the formula (5.3) for the input-output operator simplify to

$$y(t) = C_\Lambda x(t) + Dv(t), \quad \text{for a.a. } t \geqslant 0,$$

and

$$(Gv)(t) = C_\Lambda \int_0^t \mathbf{T}_{t-\tau} Bv(\tau)\, d\tau + Dv(t), \quad \text{for all } v\in L^2_{\text{loc}}(\mathbb{R}_+, U), \text{ a.a. } t\in\mathbb{R}_+,$$

respectively; moreover, $(sI-A)^{-1}BU\subset\operatorname{dom}(C_\Lambda)$ for all $s\in\operatorname{res}(A)$ and we have

$$\mathbf{G}(s) = C_\Lambda(sI-A)^{-1}B + D, \quad \text{for all } s\in\mathbb{C}_{\omega(\mathbf{T})}.$$

The operator $D\in\mathscr{B}(U)$ is called the *feedthrough operator* of (5.2). It can be shown that if B is a bounded control operator or if C is a bounded observation operator, then (5.2) is regular.

The well-posed system (5.2) is called *strongly stable* if the following four conditions are satisfied:

(i) G is L^2-stable, that is, $G\in\mathscr{B}(L^2(\mathbb{R}_+, U))$, or, equivalently, $\mathbf{G}\in H^\infty(\mathbb{C}_0, \mathscr{B}(U_c))$;

(ii) $\mathbf{T}$ is strongly stable, that is, $\lim_{t\to\infty}\mathbf{T}_t z = 0$ for all $z\in X$;

(iii) B is an infinite-time admissible control operator, that is, there exists $\alpha\geqslant 0$ such that $\|\int_0^\infty \mathbf{T}_\tau Bv(\tau)\,d\tau\| \leqslant \alpha\|v\|_{L^2(\mathbb{R}_+,U)}$ for all $v\in L^2(\mathbb{R}_+, U)$;

(iv) C is an infinite-time admissible observation operator, that is, there exists $\beta\geqslant 0$ such that $(\int_0^\infty \|C\mathbf{T}_\tau z\|^2\,d\tau)^{1/2}\leqslant\beta\|z\|$ for all $z\in X_1$.

The system (5.2) is called *exponentially stable* if $\omega(\mathbf{T})<0$. Obviously, exponential stability implies strong stability. The converse is not true; for examples of partial differential equation systems which are strongly, but not exponentially, stable, see [**6**, **24**, **33**].

Let $\varphi\colon \mathbb{R}_+\times U\to U$ be a (time-dependent) static non-linearity, let $\rho\in U$, $k\in\mathbb{R}$ and consider the well-posed system (5.2), with input non-linearity $v=\varphi\circ u$, in feedback interconnection with the integrator $\dot u = k(\rho - y)$, that is,

$$\begin{aligned}
&\dot x = Ax + B(\varphi\circ u), \quad x(0)=x^0,\\
&\dot u = k[\rho - C_\Lambda(x-(s_0I-A)^{-1}B(\varphi\circ u)) - \mathbf{G}(s_0)(\varphi\circ u)], \quad u(0)=u^0\in U,
\end{aligned}\tag{5.4}$$

where $\varphi\circ u$ denotes the function $t\mapsto\varphi(t,u(t))$. A *solution* of (5.4) on the interval $[0,T)$ (where $0<T\leqslant\infty$) is a continuous function $[0,T)\to X\times U$, $t\mapsto(x(t),u(t))$ such that

$$\varphi\circ u\in L^2_{\text{loc}}([0,T),U), \quad x(t)-(s_0I-A)^{-1}B(\varphi\circ u)(t)\in\operatorname{dom}(C_\Lambda)$$

for a.a. $t\in[0,T)$,

$$C_\Lambda[x-(s_0I-A)^{-1}B(\varphi\circ u)]\in L^1_{\text{loc}}([0,T),U),$$

and, for all $t \in [0, T)$,

$$x(t) = x^0 + \int_0^t (Ax(\tau) + B(\varphi \circ u)(\tau))\, d\tau,$$
$$u(t) = u^0 + k \int_0^t [\rho - C_\Lambda(x(\tau) - (s_0 I - A)^{-1} B(\varphi \circ u)(\tau)) - \mathbf{G}(s_0)(\varphi \circ u)(\tau)]\, d\tau. \tag{5.5}$$

In order to derive existence and uniqueness results for (5.4), the following lemma is useful.

LEMMA 5.1. *Let* $0 < T \leqslant \infty$ *and define* $r \in C(\mathbb{R}_+, U)$ *by*

$$r(t) = u^0 + k\rho t - k \int_0^t C_\Lambda \mathbf{T}_\tau x^0\, d\tau.$$

A continuous function (x, u): $[0, T) \to X \times U$ *is a solution of* (5.4) *if and only if* $\varphi \circ u \in L^2_{\text{loc}}([0, T), U)$,

$$u(t) = r(t) - k \int_0^t (G(\varphi \circ u))(\tau)\, d\tau, \quad \textit{for all } t \in [0, T), \tag{5.6}$$

and

$$x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} B(\varphi \circ u)(\tau)\, d\tau, \quad \textit{for all } t \in [0, T). \tag{5.7}$$

Proof. Assume first that (x, u): $[0, T) \to X \times U$ is a solution of (5.4). Then, trivially, $\varphi \circ u \in L^2_{\text{loc}}([0, T), U)$, and, by standard properties of well-posed systems, it is clear that x satisfies (5.7). Hence $x(t) - (s_0 I - A)^{-1} B(\varphi \circ u)(t) \in \text{dom}(C_\Lambda)$ for a.a. $t \in [0, T)$ and, by (5.3),

$$C_\Lambda(x(t) - (s_0 I - A)^{-1} B(\varphi \circ u)(t)) + \mathbf{G}(s_0)(\varphi \circ u)(t) = C_\Lambda \mathbf{T}_t x^0 + (G(\varphi \circ u))(t), \quad \text{for a.a. } t \in [0, T). \tag{5.8}$$

Therefore, by the second equation in (5.4),

$$\dot{u}(t) = k(\rho - C_\Lambda \mathbf{T}_t x^0) - k(G(\varphi \circ u))(t), \quad \text{for all } t \in [0, T),$$

and hence (5.6) holds. Conversely, assume that $\varphi \circ u \in L^2_{\text{loc}}([0, T), U)$ and that (5.6) and (5.7) hold. Then, by standard properties of well-posed systems, x satisfies the first equation in (5.5),

$$x(t) - (s_0 I - A)^{-1} B(\varphi \circ u)(t) \in \text{dom}(C_\Lambda) \quad \text{for a.a. } t \in [0, T),$$

$C_\Lambda[x - (s_0 I - A)^{-1} B(\varphi \circ u)] \in L^2_{\text{loc}}([0, T), U)$ and (5.8) holds. It follows from (5.6) that u satisfies the second equation in (5.5). □

The following result is an immediate consequence of Lemmas 2.1 and 5.1.

COROLLARY 5.2. *Let* $\rho \in U$ *and* $k \in \mathbb{R}$, *and let* φ: $\mathbb{R}_+ \times U \to U$ *be such that* $t \mapsto \varphi(t, v)$ *is measurable for every* $v \in U$, $t \mapsto \varphi(t, 0)$ *is in* $L^2_{\text{loc}}(\mathbb{R}_+, U)$, *and for every bounded set* $V \subset U$ *there exists* $\lambda_V \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ *such that* (2.2) *holds. Then the initial-value problem* (5.4) *has a unique solution* (x, u) *defined on* *a maximal interval of existence* $[0, T)$, *where* $0 < T \leqslant \infty$. *If* $T < \infty$, *then* $\limsup_{t\to T} \|u(t)\| = \infty$.

Consider (5.4) with time-independent φ, $k = 1$ and $\rho = 0$, that is,

$$\begin{aligned} &\dot{x} = Ax + B(\varphi \circ u), \quad x(0) = x^0, \\ &\dot{u} = -C_\Lambda(x - (s_0 I - A)^{-1}B(\varphi \circ u)) - \mathbf{G}(s_0)(\varphi \circ u), \quad u(0) = u^0 \in U. \end{aligned} \tag{5.9}$$

If $\varphi(0) = 0$, then for $x^0 = 0$ and $u^0 = 0$, the trivial function $t \mapsto (0, 0)$ is the unique solution of (5.9), called the zero solution. It is not difficult to show that $\varphi(0) = 0$ if φ is the gradient of a non-negative C^1-potential Φ with $\Phi(0) = 0$. We call the zero solution of (5.9) *stable in the large* if:

(i) for all $(x^0, u^0) \in X \times U$ there exists a solution of (5.9) on $\mathbb{R}_+$ (that is, $T = \infty$ in Lemma 5.2); and

(ii) there exists a continuous, strictly increasing function $\nu: \mathbb{R}_+ \to \mathbb{R}_+$ with $\nu(0) = 0$ and such that if (x, u) is the solution of the initial-value problem (5.9) with initial data $(x^0, u^0) \in X \times U$, then

$$\|x(t)\| + \|u(t)\| \leqslant \nu(\|x^0\| + \|u^0\|), \quad \text{for all } t \in \mathbb{R}_+.$$

We call the zero solution of (5.9) *globally asymptotically stable in the large* if it is stable in the large and, for all initial data $(x^0, u^0) \in X \times U$, the solution (x, u) of (5.9) tends to $(0, 0)$ as $t \to \infty$.

The following theorem gives an absolute stability result for the system (5.9). Before stating it, we remark that if (5.2) is strongly stable, then $\mathbf{G} \in H^\infty(\mathbb{C}_0, \mathscr{B}(U_c))$ and hence $\mathbf{G}$ is analytic on $\mathbb{C}_0$. If additionally $0 \in \operatorname{res}(A)$, then $\mathbf{G}$ can be analytically extended to a neighbourhood of 0. Hence the evaluation $\mathbf{G}(0)$ of $\mathbf{G}(s)$ at $s = 0$ is meaningful, and (5.1) holds for $s_0 = 0$ and $s \in \mathbb{C}_{\omega(\mathbf{T})}$, that is,

$$(\mathbf{G}(s) - \mathbf{G}(0))/s = C(sI - A)^{-1}A^{-1}B, \quad \text{for all } s \in \mathbb{C}_{\omega(\mathbf{T})}. \tag{5.10}$$

THEOREM 5.3. *Assume that the well-posed system* (5.2) *is strongly stable,* $0 \in \operatorname{res}(A)$, *and* $\mathbf{G}(0)$ *is invertible. Let* $\varphi: U \to U$ *be a locally Lipschitz continuous gradient of a non-negative* C^1*-function* $\Phi: U \to \mathbb{R}$. *If there exist self-adjoint* $P \in \mathscr{B}(U)$, *invertible* $Q \in \mathscr{B}(U)$ *with* $Q\mathbf{G}(0) = [Q\mathbf{G}(0)]^* \geqslant 0$ *and numbers* $q \geqslant 0$ *and* $\varepsilon > 0$ *such that* (3.1) *and* (3.2) *hold, then the solution* (x, u) *of* (5.9) *exists on* $\mathbb{R}_+$ *(no finite escape-time),* $x \in L^\infty(\mathbb{R}_+, X)$, $u \in L^\infty(\mathbb{R}_+, U)$, $\varphi \circ u \in L^2(\mathbb{R}_+, U)$,

$$\lim_{t\to\infty} \|x(t)\| = 0, \quad \lim_{t\to\infty} (\varphi \circ u)(t) = 0,$$

and there exists a constant $K \geqslant 0$ (*which depends only on* (A, B, C), $\mathbf{G}(0)$, Q, q *and* ε, *but not on* u^0 *and* x^0) *such that*

$$\|x\|_{L^\infty(\mathbb{R}_+, X)} + \|u\|_{L^\infty} + \|\varphi \circ u\|_{L^2} \leqslant K(\|x^0\| + \|u^0\| + \sqrt{\Phi(u^0)}). \tag{5.11}$$

If $\Phi(0) = 0$, *then the zero solution of* (5.9) *is stable in the large. If* $\varphi^{-1}(\{0\}) = \{0\}$, φ *satisfies Condition* (D) *in Theorem* 3.1 *and* $\Phi(0) = 0$, *then the zero solution of* (5.9) *is globally asymptotically stable in the large. If* $\mathbf{T}_{t_0} x^0 \in X_1$ *for some* $t_0 \geqslant 0$ *and* G *satisfies Condition* (E) *in Theorem* 3.1, *then* $\operatorname{ess\,lim}_{t\to\infty} \dot{u}(t) = 0$. *Furthermore, if* $\mathbf{T}_{t_0}(Ax^0 + B\varphi(u^0)) \in X$ *for some* $t_0 \geqslant 0$ *and* φ *is globally Lipschitz, then* $\dot{u}$ *is continuous on* $[t_0, \infty)$ *and* $\lim_{t\to\infty} \dot{u}(t) = 0$.

Proof. Let (x, u) be the unique solution of (5.9) defined on the maximal interval of existence $[0, T)$, where $0 < T \leqslant \infty$. By Lemma 5.1, u satisfies

$$u(t) = r(t) - \int_0^t (G(\varphi \circ u))(\tau)\, d\tau, \quad \text{for all } t \in [0, T), \tag{5.12}$$

where $r(t) := u^0 - \int_0^t C_\Lambda \mathbf{T}_\tau x^0\, d\tau$. In order to apply Theorem 3.1 to (5.12), we need to verify the relevant assumptions. Clearly, by strong stability and the fact that $0 \in \operatorname{res}(A)$, it is clear that $\mathbf{G}$ satisfies Assumption (A). To show that $r \in W^{1,2}(\mathbb{R}_+, U) + U$, note that $\dot{r}(t) = -C_\Lambda \mathbf{T}_t x^0$, and therefore $\dot{r} \in L^2(\mathbb{R}_+, U)$, since C is infinite-time admissible. Furthermore, $\int_0^t C_\Lambda \mathbf{T}_\tau x^0\, d\tau = CA^{-1}(\mathbf{T}_t - I)x^0$ and so

$$r(t) = -C\mathbf{T}_t A^{-1} x^0 + u^0 + CA^{-1} x^0, \quad \text{for all } t \in \mathbb{R}_+.$$

Again, by the infinite-time admissibility of C, it follows that the function $t \mapsto C\mathbf{T}_t A^{-1} x^0$ is in $L^2(\mathbb{R}_+, U)$, and we may conclude that $r \in W^{1,2}(\mathbb{R}_+, U) + U$ with $r^\infty = \lim_{t\to\infty} r(t) = u^0 + CA^{-1}x^0$. An application of Theorem 3.1 and Lemma 5.1 shows that $T = \infty$. Furthermore, by Theorem 3.1, $u \in L^\infty(\mathbb{R}_+, U)$, $\varphi \circ u \in L^2(\mathbb{R}_+, U)$, $\lim_{t\to\infty}(\varphi \circ u)(t) = 0$ and there exists a constant $K_1 > 0$ (not depending on r) such that

$$\|u\|_{L^\infty} + \|\varphi \circ u\|_{L^2} \leqslant K_1 \eta, \tag{5.13}$$

where

$$\eta = \sqrt{\Phi(r(0))} + \|r^\infty\| + \|r - r^\infty\|_{L^2} + \|\dot{r}\|_{L^2}.$$

Now $x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} B(\varphi \circ u)(\tau)\, d\tau$, and thus, the strong stability of the semigroup $\mathbf{T}$, the infinite-time admissibility of B, and the fact that $\varphi \circ u \in L^2(\mathbb{R}_+, U)$ yield $x \in L^\infty(\mathbb{R}_+, X)$, $\lim_{t\to\infty} \|x(t)\| = 0$ and

$$\|x\|_{L^\infty(\mathbb{R}_+, X)} \leqslant K_2(\|x^0\| + \|\varphi \circ u\|_{L^2}) \tag{5.14}$$

for some suitable constant $K_2 > 0$ (not depending on x^0 and u^0). By the infinite-time admissibility of C, there exists a constant $K_3 > 0$ (not depending on x^0 and u^0) such that

$$\eta \leqslant K_3(\|x^0\| + \|u^0\| + \sqrt{\Phi(u^0)}). \tag{5.15}$$

Combining (5.13)–(5.15) shows that there exists a constant $K > 0$ (not depending on x^0 and u^0) such that (5.11) holds. If we additionally assume that $\Phi(0) = 0$, then it is obvious that the function

$$\nu\colon \mathbb{R}_+ \to \mathbb{R}_+, \quad l \mapsto K\big(l + \sup_{\|v\| \leqslant l} \sqrt{\Phi(v)}\big)$$

satisfies $\nu(0) = 0$ and is strictly increasing. Furthermore, it is a routine exercise to show that ν is continuous. Noting that

$$K(\|x^0\| + \|u^0\| + \sqrt{\Phi(u^0)}) \leqslant \nu(\|x^0\| + \|u^0\|),$$

we see from (5.11) that

$$\|x(t)\| + \|u(t)\| \leqslant \nu(\|x^0\| + \|u^0\|), \quad \text{for all } t \in \mathbb{R}_+,$$

thus establishing stability in the large. The proof of the assertion on global asymptotic stability in the large is obvious and is therefore left to the reader.

Under the assumption that $\mathbf{T}_{t_0} x^0 \in X_1$ for some $t_0 \geqslant 0$, it follows from the strong stability of $\mathbf{T}$ that

$$\lim_{t \to \infty} r(t) = -\lim_{t \to \infty} C\mathbf{T}_{t-t_0}\mathbf{T}_{t_0} x^0 = 0.$$

Hence, if $\mathbf{T}_{t_0} x^0 \in X_1$ for some $t_0 \geqslant 0$ and G satisfies Condition (E) in Theorem 3.1, we may conclude from statement (5) of Theorem 3.1 that $\operatorname{ess\,lim}_{t\to\infty} \dot{u}(t) = 0$. Finally, assume that $\mathbf{T}_{t_0}(Ax^0 + B\varphi(u^0)) \in X$ for some $t_0 \geqslant 0$ and that φ is globally Lipschitz with Lipschitz constant $\lambda > 0$. By the infinite-time admissibility of C and the fact that $\varphi \circ u \in L^2(\mathbb{R}_+, U)$, we have $\dot{u} \in L^2(\mathbb{R}_+, U)$. It is clear that $\varphi \circ u$ is absolutely continuous, and a routine argument using the global Lipschitz condition shows that $\|(\varphi \circ u)'(t)\| \leqslant \lambda \|\dot{u}(t)\|$ for a.a. $t \in \mathbb{R}_+$. Therefore

$$(\varphi \circ u)' \in L^2(\mathbb{R}_+, U). \tag{5.16}$$

We claim that it is sufficient to show that

$$\begin{aligned}\dot{u}(t) = &-C_\Lambda \mathbf{T}_t(x^0 + A^{-1}B\varphi(u^0)) - (H((\varphi \circ u)'))(t)\\ &- \mathbf{G}(0)(\varphi \circ u)(t), \quad \text{for a.a. } t \in \mathbb{R}_+,\end{aligned} \tag{5.17}$$

where H is the bounded shift-invariant operator defined in (3.14). Indeed, noting that

$$C_\Lambda \mathbf{T}_t(x^0 + A^{-1}B\varphi(u^0)) = CA^{-1}\mathbf{T}_{t-t_0}[\mathbf{T}_{t_0}(Ax^0 + B\varphi(u^0))], \quad \text{for all } t \in [t_0, \infty),$$

we find that the function $t \mapsto C_\Lambda \mathbf{T}_t(x^0 + A^{-1}B\varphi(u^0))$ is continuous on $[t_0, \infty)$ and, by the strong stability of $\mathbf{T}$, converges to 0 as $t \to \infty$. Moreover, the function $H((\varphi \circ u)') + \mathbf{G}(0)(\varphi \circ u)$ is continuous and, using (5.16), this function and its derivative are in $L^2(\mathbb{R}_+, U)$, showing that it converges to 0 as $t \to \infty$. It remains to show that (5.17) holds. Now, from (5.12), $\dot{u}(t) = -C_\Lambda \mathbf{T}_t x^0 - (G(\varphi \circ u))(t)$ and a routine calculation shows that

$$\dot{u} = f - H((\varphi \circ u)') - \mathbf{G}(0)(\varphi \circ u), \tag{5.18}$$

where $f(t) := -C_\Lambda \mathbf{T}_t x^0 - \varphi(u^0)((G\theta)(t) - \mathbf{G}(0))$ (as usual, θ denotes the unit-step function). Taking the Laplace transform of f and using (5.10), we obtain, for all $s \in \mathbb{C}_{\omega(\mathbf{T})}$,

$$\begin{aligned}(\mathfrak{L}(f))(s) &= -C(sI - A)^{-1}x^0 - \varphi(u^0)(\mathbf{G}(s) - \mathbf{G}(0))/s\\ &= -C(sI - A)^{-1}(x^0 + A^{-1}B\varphi(u^0)).\end{aligned}$$

Consequently, $f(t) = -C_\Lambda \mathbf{T}_t(x^0 + A^{-1}B\varphi(u^0))$ for a.a. $t \in \mathbb{R}_+$, and (5.17) follows from (5.18). □

Absolute stability questions for state-space systems of the form (5.9) have been addressed in [**2**, **3**, **6**, **18**, **37**, **38**]. In [**2**, **3**, **37**, **38**] single-input–single-output systems are considered; in [**2**, **3**] it is assumed that $\mathbf{T}$ is holomorphic, C is bounded, but B is not necessarily admissible, whilst in [**6**, **37**, **38**] both operators, B and C, are assumed to be bounded. In [**6**] the underlying linear system (5.2) is assumed to be strongly stable, whilst exponential stability of the linear system is a crucial assumption in [**3**, **18**, **38**]. The two papers [**2**, **37**] deal with the case that the positive-real condition (3.2) holds with $\varepsilon = 0$ and are therefore somewhat irrelevant in the context of this paper (see also the comments on the literature at the end of § 3). We emphasize that the positive-real condition imposed in [**18**] is identical to (3.2) with $q = 0$ and hence is considerably more restrictive than (3.2). In [**3**, **6**, **38**] it is assumed that $\mathbf{G}$ satisfies $\mathbf{G}(i\omega) + \mathbf{G}^*(i\omega) \geqslant \delta I > 0$ for a.a. $\omega \in \mathbb{R}$, which again is considerably more restrictive than (3.2). To see this, note that if $\mathbf{G}$ is analytic in a neighbourhood of 0 and if $Q\mathbf{G}(0)$ is self-adjoint, then the function

$$\omega \mapsto \frac{1}{i\omega}(Q\mathbf{G}(i\omega) - \mathbf{G}^*(i\omega)Q^*) = \frac{1}{i\omega}Q(\mathbf{G}(i\omega) - \mathbf{G}(0)) - \frac{1}{i\omega}(\mathbf{G}^*(i\omega) - \mathbf{G}^*(0))Q^*$$

is uniformly bounded. Consequently, if $\mathbf{G}(i\omega) + \mathbf{G}^*(i\omega) \geqslant \delta I > 0$ for a.a. $\omega \in \mathbb{R}$, then, given any self-adjoint $P \in \mathscr{B}(U)$, (3.2) holds for sufficiently large $q \geqslant 0$. Summarizing the above comments, it is clear that Theorem 5.3 is a significant improvement of the relevant results in [**3**, **6**, **18**, **38**].

We next present an 'internal' version of the result on tracking by low-gain integral control given in Theorem 4.1.

THEOREM 5.4. *Assume that the well-posed system* (5.2) *is strongly stable,* $\dim U = 1$, $0 \in \operatorname{res}(A)$, *and* $\mathbf{G}(0) > 0$. *Let* $\varphi\colon \mathbb{R} \to \mathbb{R}$ *be locally Lipschitz continuous and non-decreasing. Let* $\rho \in \mathbb{R}$ *and assume that* $\rho/\mathbf{G}(0) \in \operatorname{im}\varphi$. *Then the following statements hold.*

(1) *Assume that* $\varphi - \varphi(0) \in \mathscr{S}(a)$ *for some* $a \in (0, \infty)$. *Then there exists a constant* $k^* \in (0, \infty]$ (*depending on* G, φ *and* ρ) *such that for all* $k \in (0, k^*)$, *the unique solution* (x, u) *of* (5.4) *is defined on* $\mathbb{R}_+$ (*no finite escape-time*), *the limits* $\lim_{t\to\infty} x(t) =: x^\infty$ (*in* X) *and* $\lim_{t\to\infty} u(t) =: u^\infty$ *exist and satisfy* $x^\infty = x^\rho := -(\rho/\mathbf{G}(0))A^{-1}B$ *and* $\varphi(u^\infty) = \rho/\mathbf{G}(0)$, *respectively,*

$$e = k(\rho - y) \in L^2(\mathbb{R}_+, \mathbb{R}) \quad \textit{and} \quad \varphi \circ u - \varphi(u^\infty) \in L^2(\mathbb{R}_+, \mathbb{R}).$$

For every $k \in (0, k^*)$ *and every* $u^\rho \in \mathbb{R}$ *with* $\varphi(u^\rho) = \rho/\mathbf{G}(0)$, *there exists a strictly increasing continuous function* $\nu\colon \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\nu(0) = 0$ *and such that, for every* $(x^0, u^0) \in X \times U$,

$$\|x(t) - x^\rho\| + \|u(t) - u^\rho\| \leqslant \nu(\|x^0 - x^\rho\| + \|u^0 - u^\rho\|), \quad \textit{for all } t \in \mathbb{R}_+. \tag{5.19}$$

Moreover,

$$\operatorname*{ess\,lim}_{t\to\infty} e(t) = 0,$$

provided that $\mathbf{T}_{t_0}x^0 \in X_1$ *for some* $t_0 \geqslant 0$ *and that* G *satisfies Property* (E) *in Theorem* 3.1. *If* $f(G) = 0$, *the above conclusions are valid with* $k^* = \infty$.

(2) *Assume that* φ *is globally Lipschitz continuous with Lipschitz constant* $\lambda > 0$. *Then the conclusions of statement* (1) *are valid with* $k^* = 1/|\lambda f(G)|$ (*where* $1/0 := \infty$) *and there exists a constant* $K > 0$ *such that* (5.19) *holds with* $\nu = K\,\mathrm{id}$. *If* $k \in (0, 1/|\lambda f(G)|)$ *and* $\mathbf{T}_{t_0}(Ax^0 + B\varphi(u^0)) \in X$ *for some* $t_0 \geqslant 0$, *then the error* e *is continuous on* $[t_0, \infty)$ *and* $\lim_{t\to\infty} e(t) = 0$.

(3) *Under the assumption that* $f(G) > 0$, *the conclusions of statement* (1) *are valid with* $k^* = \infty$.

Proof. Let (x, u) be the unique solution of (5.4) defined on the maximal interval of existence $[0, T)$, where $0 < T \leqslant \infty$. By Lemma 5.1, u satisfies

$$\dot{u}(t) = k[\rho - (g + (G(\varphi \circ u))(t))], \quad \text{for all } t \in [0, T), \tag{5.20}$$

where $g(t) := C_\Lambda \mathbf{T}_t x^0$.

Proof of statement (1). In order to apply Theorem 4.1 to (5.20), we need to verify the relevant assumptions. Clearly, by the strong stability and the fact that $0 \in \operatorname{res}(A)$, it is clear that $\mathbf{G}$ satisfies Assumption (A$'$). By the infinite-time admissibility of C, $g \in L^2(\mathbb{R}_+, U)$, and, by an argument identical to that in the proof of Theorem 5.3, the function $t \mapsto \int_0^t g(\tau)\, d\tau$ is in $L^2(\mathbb{R}_+, U) + U$. An application of Theorem 4.1 and Lemma 5.1 shows that there exists $k^* > 0$ (not depending on x^0 and u^0) such that if $k \in (0, k^*)$, then $T = \infty$, $\lim_{t\to\infty} u(t) =: u^\infty$ exists and satisfies $\varphi(u^\infty) = \rho/\mathbf{G}(0)$, $e \in L^2(\mathbb{R}_+, U)$ and $\varphi \circ u - u^\infty \in L^2(\mathbb{R}_+, U)$. For the rest of the proof of statement (1) let $k \in (0, k^*)$. If $\mathbf{T}_{t_0} x^0 \in X_1$ for some $t_0 \geqslant 0$, then

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} C\mathbf{T}_{t-t_0}(\mathbf{T}_{t_0} x^0) = 0.$$

Hence, if $\mathbf{T}_{t_0} x^0 \in X_1$ for some $t_0 \geqslant 0$ and G satisfies Condition (E), then Theorem 4.1 guarantees that $\operatorname{ess\,lim}_{t\to\infty} e(t) = 0$. To prove the remaining assertions in statement (1), let $u^\rho \in \mathbb{R}$ be such that $\varphi(u^\rho) = \rho/\mathbf{G}(0)$, define $z(\cdot) := x(\cdot) - x^\rho$ and $v(\cdot) := u(\cdot) - u^\rho$ and set

$$\widetilde{\varphi}(w) := \varphi(w + u^\rho) - \varphi(u^\rho), \quad \text{for all } w \in \mathbb{R}.$$

We claim that

$$\begin{aligned} \dot{z} &= Az + B(\widetilde{\varphi} \circ v), \\ \dot{v} &= -k[C_\Lambda(z - (s_0 I - A)^{-1} B(\widetilde{\varphi} \circ v)) + \mathbf{G}(s_0)(\widetilde{\varphi} \circ v)], \end{aligned} \tag{5.21}$$

where $s_0 \in \mathbb{C}_{\omega(\mathbf{T})}$. The first equation in (5.21) (where, as usual, the derivative on the left-hand side has to be interpreted in X_{-1}) follows easily from the first equation in (5.4). Now $\dot{v} = \dot{u}$ and hence, by the second equation in (5.4),

$$\dot{v} = k[\widetilde{\rho} - C_\Lambda(z - (s_0 I - A)^{-1} B(\widetilde{\varphi} \circ v)) - \mathbf{G}(s_0)(\widetilde{\varphi} \circ v)], \tag{5.22}$$

where

$$\widetilde{\rho} = \rho - C_\Lambda(x^\rho - (s_0 I - A)^{-1} B\varphi(u^\rho)) - \mathbf{G}(s_0)\varphi(u^\rho).$$

Using the definition of x^ρ, the fact that $\varphi(u^\rho) = \rho/\mathbf{G}(0)$, the resolvent equation and (5.10) gives

$$\begin{aligned} \widetilde{\rho} &= \rho + \varphi(u^\rho)(C(s_0(s_0 I - A)^{-1} A^{-1} B) - \mathbf{G}(s_0)) \\ &= \rho + \varphi(u^\rho)(\mathbf{G}(s_0) - \mathbf{G}(0) - \mathbf{G}(s_0)) = \rho - \rho = 0. \end{aligned}$$

Combined with (5.22), this shows that the second equation in (5.21) holds. Since

$$\widetilde{\varphi} \circ v = \varphi \circ u - \varphi(u^\rho) = \varphi \circ u - \varphi(u^\infty) \in L^2(\mathbb{R}_+, U),$$

it follows from the strong stability of $\mathbf{T}$, the infinite-time admissibility of B, and the first equation in (5.21) that $\lim_{t\to\infty} \|z(t)\| = 0$, showing that $x(t)$ converges (in X) to x^ρ as $t \to \infty$. Note that

$$\widetilde{\Phi}\colon \mathbb{R} \to \mathbb{R}_+, \quad w \mapsto \int_0^w \widetilde{\varphi}(\xi)\, d\xi,$$

is a non-negative potential for $\widetilde{\varphi}$ with $\widetilde{\Phi}(0)=0$, and that (5.21) is of the form (5.9). In order to apply Theorem 5.3 to (5.21), we need to verify the relevant assumptions. To this end note that, exactly as in the proof of Theorem 4.1, it can be shown that there exist $b \in (0, \infty)$, $q \geqslant 0$ and $\varepsilon > 0$ such that

$$\widetilde{\varphi}(w)w \geqslant \widetilde{\varphi}^2(w)/b, \quad \text{for all } w \in \mathbb{R}$$

and

$$1/b + \mathrm{Re}[(q + 1/i\omega)k\mathbf{G}(i\omega)] \geqslant \varepsilon, \quad \text{for a.a. } \omega \in \mathbb{R},$$

showing that (3.1) and (3.2) hold with P, Q and $\mathbf{G}$ replaced by $1/b$, 1 and $k\mathbf{G}$, respectively. Hence, an application of Theorem 5.3 to (5.21) shows that there exists a strictly increasing continuous function ν: $\mathbb{R}_+ \to \mathbb{R}_+$ with $\nu(0)=0$, not depending on $z^0 := x^0 - x^\rho$ and $v^0 := u^0 - u^\rho$, and such that

$$\|z(t)\| + \|v(t)\| \leqslant \nu(\|z^0\| + \|v^0\|), \quad \text{for all } t \in \mathbb{R}_+,$$

yielding (5.19).

Proof of statement (2). It follows immediately from Theorem 4.1 that conclusions of statement (1) are valid with $k^* = 1/|\lambda f(G)|$. Applying Theorem 5.3 to (5.21) shows that there exists a constant $L > 0$ (not depending on z^0 and v^0) such that

$$\|z(t)\| + \|v(t)\| \leqslant L\Big(\|z^0\| + \|v^0\| + \sqrt{\widetilde{\Phi}(v^0)}\Big), \quad \text{for all } t \in \mathbb{R}_+.$$

By the global Lipschitz condition,

$$|\widetilde{\Phi}(v^0)| \leqslant |v^0| \sup_{|\xi| \leqslant |v^0|} |\widetilde{\varphi}(\xi)| \leqslant \lambda |v^0|^2,$$

and it follows that (5.19) holds with $\nu = K\,\mathrm{id}$, for some suitable constant $K > 0$. Assume now that $\mathbf{T}_{t_0}(Ax^0 + B\varphi(u^0)) \in X$ for some $t_0 \geqslant 0$. Since

$$Ax^0 + B\varphi(u^0) = Az^0 + B\widetilde{\varphi}(v^0),$$

we have $\mathbf{T}_{t_0}(Az^0 + B\widetilde{\varphi}(v^0)) \in X$. Therefore, if $k \in (0, k^*)$, it follows from an application of Theorem 5.3 to (5.21) that $e = \dot{u}/k = \dot{v}/k$ is continuous on $[t_0, \infty)$ and $\lim_{t\to\infty} e(t) = 0$. The remaining assertions in statement (2) follow as in the proof of statement (1) with b replaced by λ.

Proof of statement (3). Statement (3) follows from Theorem 4.1 and arguments similar to those in the proof of statement (1). □

Tracking of constant reference signals by low-gain integral control in the context of system (5.4) has been considered in [**18**, **19**] (in [**19**] it is assumed that the well-posed system (5.2) is regular). What we mentioned in § 4 in a comment relating to Theorem 4.1 (see the paragraph after Remark 4.2) also applies to Theorem 5.4: as compared to the relevant results in [**18**, **19**], Theorem 5.4 provides a considerably larger range of gains achieving tracking and does not require exponential stability of the underlying linear system.

Theorems 5.3 and 5.4 give state-space versions of Theorems 3.1 and 4.1, respectively. Similarly, state-space versions of Theorems 3.3 and 4.4 for well-posed linear systems can be obtained; the details are left to the reader.

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R. F. Curtain
Mathematics Institute
University of Groningen
P.O. Box 800
9700 AV Groningen
The Netherlands
R.F.Curtain@math.rug.nl

H. Logemann
Department of Mathematical Sciences
University of Bath
Claverton Down
Bath BA2 7AY
hl@maths.bath.ac.uk

O. Staffans
Department of Mathematics
Åbo Akademi University
FIN-20500 Åbo
Finland
olof.staffans@abo.fi
http://www.abo.fi/~staffans/