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Algebraic properties of systems of nuclear type

Ruth F. CURTAIN

Mathematics Institute, Rijksuniversiteit Groningen, Postbus 800, 9700 AV Groningen, The Netherlands

Hartmut LOGEMANN

Institut für Dynamische Systeme, Universität Bremen, Postfach 330440, 2800 Bremen 33, West Germany

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Abstract: Systems of nuclear type have an impulse response in $L_1 \cap L_2$ and a nuclear Hankel operator and are known to have excellent approximation properties. This class is shown to be closed under parallel cascade and feedback configurations. Similar properties are shown to hold for systems of extended nuclear type: those which are the sum of a system of nuclear type and a totally unstable finite-dimensional transfer function.

Keywords: Nuclear systems; infinite-dimensional systems; interconnection of systems.

Notation

For $A \in \mathbb{C}^{p \times m}$ let $\bar{\sigma}(A)$ denote the largest singular value of A.

For

$$f\colon [0, \infty) \to \mathbb{R}^{p \times m}, \quad t \mapsto \left(f_{ij}(t)\right)_{\substack{1 \le i \le p,\\ 1 \le j \le m}}$$

and $q > 0$ define

$$\| f \|_q = \left(\int_0^\infty | f(t) |_e^q \, \mathrm{d}t \right)^{1/q},$$

where $| f(t) |_e := (\Sigma_{i,j} | f_{ij}(t) |^2)^{1/2}$. It is clear that for all $q > 0$,

$$L_q(0, \infty; \mathbb{R}^{p \times m}) := \left\{ f : [0, \infty) \to \mathbb{R}^{p \times m} \mid \| f \|_q < \infty \right\}$$

is a Banach space.

By H_2 we denote the usual Hardy–Lebesgue space on the right-half plane.

Let $H_\infty(\mathbb{C}^{p \times m})$ denote the space of all $\mathbb{C}^{p \times m}$-valued functions which are holomorphic and bounded in $\mathrm{Re}(s) > 0$. Provided with the norm

$$\| H \|_\infty = \sup_{\mathrm{Re}(s) > 0} \bar{\sigma}(H(s)),$$

$H_\infty(\mathbb{C}^{p \times m})$ becomes a Banach space. It is a Banach algebra if $p = m$.

The subalgebra H_∞^- of $H_\infty(\mathbb{C})$ is defined in the following way:

$$H_\infty^- := \left\{ f \in H_\infty \mid \exists \alpha < 0 : f \text{ extends to a bounded holomorphic function on } \mathrm{Re}(s) > \alpha \right\}.$$

1. Introduction

Infinite-dimensional linear systems which have a nuclear Hankel operator have been studied in [4], [5] and [8]. We recall that nuclearity means that the sum of the singular values of the Hankel operator is finite. In [8] important realisation and approximation properties were established for systems of nuclear type: these are (stable) systems with an impulse response in $L_1 \cap L_2(0, \infty; \mathbb{R}^{p \times m})$ and which have a nuclear Hankel operator. It is the last property which explains the good approximation behaviour of systems of nuclear type. They have optimal Hankel norm approximations G_k of MacMillan degree k which satisfy

$$\| G - G_k \|_\infty \le \sum_{i=k+1}^{\infty} G_i$$

and the truncated balanced approximation G_k^{b} of degree k satisfies

$$\left\| G - G_k^{\mathrm{b}} \right\| \le 2 \sum_{i=k+1}^{\infty} G_i$$

(cf. [5,8]). These excellent approximation properties have important implications in the design of robust finite-dimensional controllers (cf. [6,7]) and so it is natural to ask whether or not this nuclearity property is retained under parallel, cascade and feedback configurations. This question is answered in the affirmative in Section 2.

Nuclear systems are of necessity stable and so it is useful to consider unstable infinite-dimensional systems which have good L_∞-approximation properties. Here we consider systems which are the sum of a transfer function of nuclear type and a totally unstable strictly proper rational one; we call this class systems of extended nuclear type. This is motivated by the fact that if $G(s)$ is of extended nuclear type, then $G(s+a)$ will be of nuclear type for some real positive a. In Section 3 we show that this class is also closed under the algebraic operations of addition and multiplication. Moreover we construct algebras of transfer functions of extended nuclear type which are closed under certain feedback configurations.

2. Systems of nuclear type

This class of systems was introduced in [8] and was found to have good approximation properties. The aim of this section is to prove that the class of systems of nuclear type is closed under cascade, parallel and feedback configurations.

Definition 2.1. *Systems of nuclear type.* Consider the class of linear infinite-dimensional systems, which have an impulse response $h \in L_1 \cap L_2(0, \infty; \mathbb{R}^{p\times m})$. The input–output relationship is well-defined by the following bounded map from $L_2(0, \infty; \mathbb{R}^m)$ to $L_2(0, \infty, \mathbb{R}^p)$:

$$y(t) = \int_0^t h(t-s)u(s)\,\mathrm{d}s. \tag{2.1}$$

Its Laplace transform, $G(s) = \hat{h}(s)$, is analytic and bounded in $\mathrm{Re}(s) > 0$ and its Hankel operator, Γ_h, is a bounded compact operator from $L_2(0, \infty; \mathbb{R}^m)$ to $L_2(0, \infty; \mathbb{R}^p)$ given by

$$(\Gamma_h u)(t) = \int_0^\infty h(t+s)u(s)\,\mathrm{d}s. \tag{2.2}$$

If Γ_h is nuclear, we say that the system is of nuclear type.

Recall that a linear bounded operator $T\colon H_1 \to H_2$ (where H_1 and H_2 are Hilbert spaces) is called nuclear if the sum of its singular values σ_i is finite:

$$\| T \|_{\mathrm{N}} := \sum_{i=1}^{\infty} \sigma_i < \infty. \tag{2.3}$$

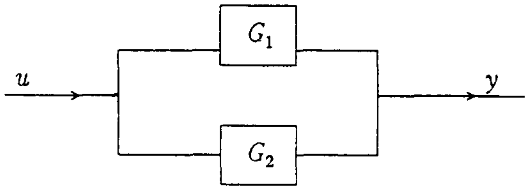

Fig. 1. Parallel configuration of Corollary 2.3.

Let us recall some standard facts about nuclear operators (cf. e.g. [10]).

Lemma 2.2. (a) *If Γ_{h_1} and Γ_{h_2} defined as in* (2.2) *are both nuclear, then so is $\alpha\Gamma_{h_1} + \beta\Gamma_{h_2}$ for any scalars α and β. Moreover*

$$\| \alpha\Gamma_{h_1} + \beta\Gamma_{h_2} \|_{\mathrm{N}} \le |\alpha|\, \| \Gamma_{h_1} \|_{\mathrm{N}} + |\beta|\, \| \Gamma_{h_2} \|_{\mathrm{N}}. \tag{2.4}$$

(b) *If H_1 and H_2 are two Hilbert spaces and $K \in \mathscr{L}(H_1, H_2)$ is nuclear, then so is KB and AK for any $A \in \mathscr{L}(H_2)$ and $B \in \mathscr{L}(H_1)$. Moreover*

$$\begin{aligned} \| AK \|_{\mathrm{N}} &\le \| A \|\, \| K \|_{\mathrm{N}}, \\ \| KB \|_{\mathrm{N}} &\le \| K \|_{\mathrm{N}} \| B \|. \end{aligned} \tag{2.5}$$

An immediate consequence of Lemma 2.2 is the following corollary.

Corollary 2.3. *Consider the parallel configuration in Figure 1 where $y = (G_1 + G_2)u$.*

Then if G_1 and G_2 are of nuclear type, so is $G_1 + G_2$.

Lemma 2.4. *Consider the cascade configuration in Figure 2 where the transfer matrices G_1 and G_2 are of size $p \times q$ and $q \times m$, respectively, and $y = G_1G_2u$. Then if G_1 and G_2 are of nuclear type so is G_1G_2. Moreover denoting the inverse Laplace transforms of G_1 and G_2 (i.e. the impulse responses corresponding to G_1 and G_2) by h_1 and h_2, respectively, and setting*

$$h(t) := (h_1 * h_2)(t) = \int_0^t h_1(t-s)h_2(s)\,\mathrm{d}s, \tag{2.6}$$

the following inequality holds:

$$\| \Gamma_h \|_{\mathrm{N}} \le \| \Gamma_{h_1} \|_{\mathrm{N}} \| h_2 \|_1 + \| \Gamma_{h_2} \|_{\mathrm{N}} \| h_1 \|_1. \tag{2.7}$$

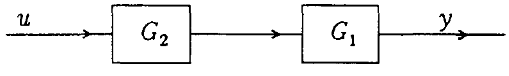

Fig. 2. Cascade configuration of Lemma 2.4.

Proof. Since

$$\| h_1 * h_2 \|_i \le \| h_1 \|_1 \| h_2 \|_i, \quad i = 1, 2, \tag{2.8}$$

we have $h_1 * h_2 \in L_1 \cap L_2(0, \infty; \mathbb{R}^{p \times m})$. It remains to show the nuclearity of Γ_h:

$$\begin{aligned}
(\Gamma_h u)(t) &= \int_0^\infty \int_0^{t+s} h_1(t+s-\alpha) h_2(\alpha)\, \mathrm{d}\alpha\, u(s)\, \mathrm{d}s \\
&= \int_0^\infty \int_0^{t+s} h_1(\alpha) h_2(t+s-\alpha) u(s)\, \mathrm{d}\alpha\, \mathrm{d}s \\
&= \int_t^\infty h_1(\alpha) \int_{\alpha-t}^\infty h_2(t+s-\alpha) u(s)\, \mathrm{d}s\, \mathrm{d}\alpha \\
&\quad + \int_0^t h_1(\alpha) \int_0^\infty h_2(t+s-\alpha) u(s)\, \mathrm{d}s\, \mathrm{d}\alpha \\
&= \int_0^\infty h_1(t+\beta) \int_0^\infty h_2(s) u(s+\beta)\, \mathrm{d}s\, \mathrm{d}\beta \\
&\quad + \int_0^t h_1(t-\beta) \int_0^\infty h_2(s+\beta) u(s)\, \mathrm{d}s\, \mathrm{d}\beta \\
&= \left(\Gamma_{h_1}(M_{h_2} u)\right)(t) + \left(K_{h_1}(\Gamma_{h_2} u)\right)(t)
\end{aligned} \tag{2.9}$$

where

$$(M_{h_2} u)(t) = \int_0^\infty h_2(s) u(s+t)\, \mathrm{d}s \tag{2.10}$$

and

$$(K_{h_1} u)(t) = \int_0^t h_1(t-s) u(s)\, \mathrm{d}s. \tag{2.11}$$

Realizing that

$$\| M_{h_2} u \|_2 \le \| h_2 \|_1 \| u \|_2 \tag{2.12}$$

and

$$\| K_{h_1} u \|_2 \le \| h_1 \|_1 \| u \|_2 \tag{2.13}$$

and using Lemma 2.2, we obtain from (2.9) that Γ_h is nuclear and the inequality (2.7) holds. □

The feedback configuration is harder to analyze and so we first consider a special case.

Theorem 2.5. *Consider a nuclear system with m inputs and m outputs and denote its transfer function by Q and its impulse response by q. Then if* $\| q \|_1 < 1$, *the closed loop transfer function*

$$F = (I - Q)^{-1} Q$$

is in $H_\infty(\mathbb{C}^{m \times m})$ *and is of nuclear type. Moreover its impulse response f is given by*

$$f(t) = \sum_{r=1}^\infty q_r(t) \tag{2.14}$$

with

$$q_{r+1}(t) = \int_0^t q(t-s) q_r(s)\, \mathrm{d}s, \quad r = 2, 3, 4, \ldots, \tag{2.15a}$$

$$q_1(t) = q(t). \tag{2.15b}$$

Proof. Trivially Q is holomorphic in $\mathrm{Re}(s) > 0$ and it follows from

$$\| Q \|_\infty \le \| q \|_1 < 1 \tag{2.16}$$

that $Q \in H_\infty(\mathbb{C}^{m \times m})$. Moreover, since $H_\infty(\mathbb{C}^{m \times m})$ is a Banach algebra, we obtain from (2.16) that $(I - Q)^{-1} \in H_\infty(\mathbb{C}^{m \times m})$ and hence

$$F = (I - Q)^{-1} Q \in H_\infty(\mathbb{C}^{m \times m}).$$

The operator $K \in \mathscr{L}(L_2(0, \infty; \mathbb{R}^m))$ defined by

$$(Ku)(t) = \int_0^t q(t-s) u(s)\, \mathrm{d}s \tag{2.17}$$

is the time-domain map associated to the transfer matrix Q. The input–output relationship in the time-domain for the closed–loop system is given by

$$y = (I - K)^{-1} K u. \tag{2.18}$$

If we notice that $\| K \| = \| Q \|_\infty$, where $\| K \|$ denotes the induced norm from $L_2(0, \infty; \mathbb{R}^m)$ (cf. e.g. [9]), then it follows from (2.16) that $(I - K)^{-1} K$ exists as an operator in $\mathscr{L}(L_2(0, \infty; \mathbb{R}^m))$.

The Neumann series for $(I - K)^{-1}$ gives

$$K_{\mathrm{cl}} := (I - K)^{-1} K = \sum_{r=1}^\infty K^r \tag{2.19}$$

and it is clear that the series in (2.19) converges in the uniform operator norm. Moreover it is easy to show that the iterates K^r of K are given by

$$(K^r u)(t) = \int_0^r q_r(t-s) u(s)\, \mathrm{d}s, \tag{2.20}$$

where the q_r are defined as in (2.15). Thus the kernel corresponding to K_{cl} is

$$f(t) = \sum_{r=1}^\infty q_r(t). \tag{2.21}$$

That $f \in L_1 \cap L_2(0, \infty; \mathbb{R}^{m\times m})$ follows from the fact that $q \in L_1 \cap L_2(0, \infty, \mathbb{R}^{m\times m})$ and $\| q \|_1 < 1$:

$$\| f \|_1 \leq \sum_{r=1}^{\infty} \| q_r \|_1 \leq \sum_{r=1}^{\infty} \| q \|_1^r < \infty, \tag{2.22}$$

$$\| f \|_2 \leq \| q \|_2 + \sum_{r=2}^{\infty} \| q_r \|_2$$

$$\leq \| q \|_2 + \| q \|_2 \sum_{r=2}^{\infty} \| q \|_1^{r-1} < \infty. \tag{2.23}$$

It now remains to establish the nuclearity of the Hankel operator associated with f. From (2.9) in the proof of Lemma 2.4, it follows that

$$\Gamma_{q_{r+1}} = \Gamma_{q_r} M_q + K^r \Gamma_q, \quad r = 1, 2, 3, \ldots, \tag{2.24}$$

where

$$(M_q u)(t) := \int_0^{\infty} q(s) u(t+s)\, \mathrm{d}s \tag{2.25}$$

and $\| M_q \| \leq \| q \|_1 < 1$ from (2.12)

Thus $I - M_q$ has a bounded inverse and (2.21), (2.24) yield the following

$$\Gamma_f = \sum_{r=1}^{\infty} \Gamma_{q_r} = \left(I + \sum_{r=1}^{\infty} K^r \right) \Gamma_q (I - M_q)^{-1}$$

$$= (I + K_{\mathrm{cl}}) \Gamma_q (I - M_q)^{-1} \tag{2.26}$$

Lemma 2.2 (b) now completes the proof. □

By combining the results of Corollary 2.3, Lemma 2.4 and Theorem 2.5 we deduce that systems of nuclear type are closed under more general feedback configurations.

Corollary 2.6. *Let G and K be transfer matrices of size $m \times p$ and $p \times m$ and with impulse responses g and k, respectively. If G and K are of nuclear type and $\| g * k \|_1 < 1$, then the following closed-loop transfer matrices are of nuclear type*:

$$(I - GK)^{-1} GK = G(I - KG)^{-1} K,$$

$$(I - GK)^{-1} G = G(I - KG)^{-1},$$

$$K(I - GK)^{-1} = (I - KG)^{-1} K.$$

Proof. GK is square and of nuclear type by Lemma 2.4 and so by Theorem 2.5, if $\| g * k \|_1 < 1$, then $(I - GK)^{-1} GK$ is also of nuclear type. Postmultiplying the identity

$$(I - GK)^{-1} = I + (I - GK)^{-1} GK \tag{2.27}$$

with G shows that $(I - GK)^{-1} G$ is of nuclear type. Similarly premultiplying (2.27) with K yields that $K(I - GK)^{-1}$ is nuclear. □

3. Extended nuclear systems

To allow for unstable systems we introduce the following class of transfer functions.

Definition 3.1. A transfer function G is said to be of extended nuclear type if it admits a decomposition of the form $G = G_{\mathrm{n}} + G_{\mathrm{f}}$, where G_{n} is a transfer function of nuclear type and G_{f} is a strictly proper rational function with all its poles in $\mathrm{Re}(s) > 0$.

It should be mentioned that a transfer function $G = G_{\mathrm{n}} + G_{\mathrm{f}}$ of extended nuclear type does not necessarily belong to the Callier–Desoer class of transfer functions (cf. [1,2]), because there might fail to exist a number $\alpha < 0$ such that G_{n} has an analytic continuation to $\mathrm{Re}(s) > \alpha$.

We remark that if $G(s)$ is of extended nuclear type, then $G(s + a)$ will be of nuclear type for all real positive a satisfying

$$a > \max\{\mathrm{Re}(\lambda) \mid \lambda \text{ is a pole of } G\}.$$

This follows from the important characterization of nuclearity by Coifman and Rochberg [3]:

A transfer matrix $G(s)$ with entries in H_2 is of nuclear type if and only if it posseses a uniformly convergent expansion valid in $\mathrm{Re}(s) > 0$ of the form

$$G(s) = \sum_{i=1}^{\infty} \frac{A_i}{s + \lambda_i} \tag{3.1}$$

for some complex number λ_i in $\mathrm{Re}(s) > 0$ and some complex matrices A_i which satisy

$$\sum_{i=1}^{\infty} \frac{| A_i |}{\mathrm{Re}(\lambda_i)} < \infty.$$

We remark that the λ_i and A_i are not unique.

We show that the class of extended nuclear systems is closed under parallel and cascade connections (i.e. the set of all $(m \times m)$-transfer matrices of extended nuclear type forms an algebra).

Lemma 3.2. *Suppose that G and F are systems of extended nuclear type. Then*
(a) *$G + F$ is of extended nuclear type.*
(b) *GF is of extended nuclear type.*

Proof. (a) This follows from Corollary 2.3.
(b) Let G and F have decompositions $G = G_n + G_f$ and $F = F_n + F_f$ according to Definition 3.1. Then by Lemma 2.4 it remains to prove that the product $G_n F_f$ is of extended nuclear type, where G_n is of nuclear type and F_f is a strictly proper rational function with all its poles in $\mathrm{Re}(s) > 0$. Appealing to the representation (3.1), we examine $(s-a)^{-1} G_n(s)$ for some a with $\mathrm{Re}(a) > 0$. We have

$$\begin{aligned}\frac{1}{s-a} G_n(s) &= \frac{1}{s-a} \sum_{i=1}^{\infty} \frac{A_i}{s+\lambda_i} \\ &= \sum_{i=1}^{\infty} \frac{A_i}{(s-a)(s+\lambda_i)} \\ &= -\sum_{i=1}^{\infty} \frac{A_i}{(s+\lambda_i)(a+\lambda_i)} \\ &\quad + \frac{1}{s-a} \sum_{i=1}^{\infty} \frac{A_i}{a+\lambda_i} \end{aligned} \tag{3.2}$$

Realizing that $0 < \mathrm{Re}(a) < |\lambda_i + a|$ and $0 < \mathrm{Re}(\lambda_i) < |\lambda_i + a|$ for $i = 1, 2, 3, \ldots$, we obtain from the convergence of $\sum_{i=1}^{\infty} |A_i| / \mathrm{Re}(\lambda_i)$ that

$$\sum_{i=1}^{\infty} \frac{|A_i|}{\mathrm{Re}(\lambda_i)\, |a+\lambda_i|} < \frac{1}{\mathrm{Re}(a)} \sum_{i=1}^{\infty} \frac{|A_i|}{\mathrm{Re}(\lambda_i)} < \infty$$

and

$$\sum_{i=1}^{\infty} \frac{|A_i|}{|a+\lambda_i|} < \sum_{i=1}^{\infty} \frac{|A_i|}{\mathrm{Re}(\lambda_i)} < \infty.$$

So it follows from (3.2) via the Coifman–Rochberg Theorem that $(s-a)^{-1} G_n(s)$ is of extended nuclear type, and this argument extends to

$$\frac{1}{(s-a)^p} G_n(s)$$

by induction ($p = 1, 2, 3, \ldots$). Using the partial fraction expansion of F_f we may conclude that $G_n F_f$ is of extended nuclear type. □

Remark 3.3. The proof of Lemma 3.2(b) shows that a scalar transfer function of nuclear type remains nuclear after cancelling finitely many of its zeros z_i in $\mathrm{Re}(s) > 0$ ($i = 1, 2, 3, \ldots$) by multiplication with factors of the form $(s - z_i)^{-n_i}$, where n_i denotes the multiplicity of the zeros z_i.

The following corollary gives a characterization of extended nuclearity in terms of coprime factorizations.

Corollary 3.4. *A transfer matrix G of size $p \times m$ is of extended nuclear type if and only if it admits a factorization of the form $G = ND^{-1}$, where N and D are transfer matrices (of size $p \times m$ and $m \times m$, respectively) satisfying:*
(a) *N is a transfer matrix of nuclear type.*
(b) *D is a proper stable rational matrix satisfying $\det(D(\infty)) \neq 0$ and $\det(D(s)) \neq 0$ if $\mathrm{Re}(s) \leq 0$.*
(c) *N and D are right coprime, i.e. there exist matrices X and Y such that $XN + YD = I$ in $\mathrm{Re}(s) \geq 0$ and the inverse Laplace transforms of the entries of X and Y belong to the convolution algebra $\mathbb{R}\delta + L_1(0, \infty)$ (where δ denotes the Dirac distribution).*

Of course, an analogous statement holds for left coprime factorizations.

Proof. 'If': follows from the proof of Lemma 3.2(b).
'Only if': Write G in the form $G = G_n + G_f$ according to Definition 3.1. It is well known that there exist proper stable rational matrices M, D, X and Z such that D satisfies the conditions in (b), $G_f = MD^{-1}$ and $XM + ZD = I$. Setting $N := G_n D + M$ and $Y := Z - XG_n$ it follows that $G = ND^{-1}$ and $XN + YD = I$ and it is clear that N, X and Y have the required properties. □

Finally we want to construct algebras of transfer functions of extended nuclear type which are closed under feedback configurations. We shall consider all scalar transfer functions G of the form

$$G = G_i + G_f \tag{3.3}$$

where G_i is a transfer function in H_∞^- and G_f is a strictly proper rational function with all its poles in $\text{Re}(s) > 0$. Let us denote the set of all transfer functions fo the form (3.3) by $\mathscr{N}$. Moreover we define for $k \geq 0$,

$$\mathscr{N}_k := \{ G \in \mathscr{N} \mid |G(s)| = O(|s|^{-k}) \text{ as } |s| \to \infty \text{ in } \text{Re}(s) > \alpha \text{ for some } \alpha < 0 \}.$$

Theorem 3.5. (a) *The set $\mathscr{N}$ forms an algebra and $\mathscr{N}_k$ is a subalgebra of $\mathscr{N}$ for all $k \geq 0$.*

(b) *Let $G \in \mathscr{N}_k^{m\times p}$ and $K \in \mathscr{N}_k^{p\times m}$; then if $(I - GK)^{-1}$ is well posed (i.e. $(I - GK)^{-1} \in \mathscr{N}^{m\times m}$) we have*

$$(I - GK)^{-1} GK \in \mathscr{N}_k^{m\times m},$$

$$(I - GK)^{-1} G \in \mathscr{N}_k^{m\times p},$$

$$K(I - GK)^{-1} \in \mathscr{N}_k^{p\times m}.$$

(c) *If $k > 2$ then all elements in $\mathscr{N}_k$ are of extended nuclear type.*

Remark 3.6. (a) Roughly speaking Theorem 3.5 says that for all $k > 2$, $\mathscr{N}_k$ is an algebra of transfer functions of extended nuclear type which is closed under feedback.

(b) $(I - GK)^{-1} \in \mathscr{N}^{m\times m}$ if and only if

$$\det(I - (GK)(j\omega)) \neq 0$$

for all $\omega \in \mathbb{R}$ and there exists a $\rho > 0$ such that

$$\inf_{\substack{|s| \geq \rho \\ \text{Re}(s) \geq 0}} |\det(I - (GK)(s))| > 0.$$

(c) If in the definition of extended nuclearity we allow for poles on the imaginary axis, then Theorem 3.5 remains true if $\mathscr{N}$ and $\mathscr{N}_k$ are replaced by

$$\mathscr{T} := \{ G = G_i + G_f \mid G_i \in H_\infty^- \text{ and } G_f \text{ is a strictly proper rational function with all its poles in } \text{Re}(s) \geq 0 \},$$

and

$$\mathscr{T}_k := \{ G \in \mathscr{T} \mid |G(s)| = O(|s|^{-k}) \text{ as } |s| \to \infty \text{ in } \text{Re}(s) > \alpha \text{ for some } \alpha < 0 \},$$

respectively (cf. [9] for the algebra $\mathscr{T}$).

Part (a) of Theorem 3.5 is obvious and part (b) follows from the trivial fact that $\mathscr{N}_k$ is a $\mathscr{N}$-module. Part (c) is a consequence of the following lemma.

Lemma 3.7. *If G is a holomorphic function on $\text{Re}(s) > \alpha$ ($\alpha < 0$) such that $|G(s)| = O(|s|^{-k})$ as $|s| \to \infty$ in $\text{Re}(s) \geq \beta$, where $\alpha \leq \beta < 0$ and $k > 2$, then G is of nuclear type.*

Proof. According to a theorem by Coifman and Rochberg proved in [3] nuclearity of G is equivalent to

$$\int_{-\infty}^{+\infty} \int_0^\infty |G''(x + jy)| \, dx \, dy < \infty. \tag{3.4}$$

Choose a real number γ satisfying $\beta < \gamma < 0$, pick any s in $\text{Re}\, s \geq 0$ and set $\rho := -\gamma$. It follows from Cauchy's inequality that

$$|G''(s)| \leq \frac{2}{\rho^2} \max_{|z-s|=\rho} |G(z)| \tag{3.5}$$

and by assumption there exists a number $M > 0$ such that

$$|G(z)| \leq \frac{M}{|z|^k} \quad \text{for all } z \neq 0 \text{ with } \text{Re}(z) \geq \beta. \tag{3.6}$$

Combining (3.5) and (3.6) yields for all s satisfying $\text{Re}(s) \geq 0$ and $|s| \geq 2\rho$,

$$\begin{aligned} |G''(s)| &\leq \frac{2M}{\rho^2} \max_{|z-s|=\rho} \frac{1}{|z|^k} \\ &\leq \frac{2M}{\rho^2} \frac{1}{(|s| - \rho)^k} \\ &\leq \frac{2M}{\rho^2} \frac{1}{|s|^k \left(1 - \frac{\rho}{|s|}\right)^k} \\ &\leq \frac{2^{k+1} M}{\rho^2} \frac{1}{|s|^k}. \end{aligned} \tag{3.7}$$

Using (3.7) we obtain

$$\begin{aligned} &\int_{-\infty}^{+\infty} \int_0^\infty |G''(x + jy)| \, dx \, dy \\ &= \int_0^\infty \int_{-\pi/2}^{\pi/2} |G''(r\, e^{j\phi})| \, r \, d\phi \, dr \\ &\leq \int_0^{2\rho} \int_{-\pi/2}^{\pi/2} |G''(r\, e^{j\phi})| \, r \, d\phi \, dr \\ &\quad + \frac{2^{k+1} M \pi}{\rho^2} \int_{2\rho}^\infty r^{1-k} \, dr \\ &< \infty \quad (\text{for } k > 2), \end{aligned}$$

which is (3.4). □

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