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Ordinary Differential Equations: Analysis, Qualitative Theory and Control

Solutions to Exercises

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Chapter 1

Exercise 1.1
(a) Writing $g\colon z_1 \mapsto 1/(z_1+\sigma)$, then, in a sufficiently small neighbourhood of 0, g may be approximated by the first two terms of its Taylor expansion about 0:

$$\frac{1}{z_1+\sigma} = g(z_1) \approx g(0) + g'(0)z_1 = \frac{1}{\sigma} - \frac{z_1}{\sigma^2}.$$

Now consider the nonlinear terms in the satellite model. For $(z_1, z_2, z_3, z_4, w_1, w_2)$ in a sufficiently small neighbourhood of $0 \in \mathbb{R}^6$, expansion and keeping only constant and linear terms, gives

$$\begin{aligned}(z_1+\sigma)(z_4+\omega)^2 &= z_1z_4^2 + 2z_1z_4\omega + \omega^2 z_1 + \sigma z_4^2 + 2\omega\sigma z_4 + \sigma\omega^2 \\ &\approx z_1\omega^2 + 2\omega\sigma z_4 + \sigma\omega^2.\end{aligned}$$

$$\begin{aligned}\frac{1}{(z_1+\sigma)^2} &\approx \left(\frac{1}{\sigma} - \frac{z_1}{\sigma^2}\right)^2 \approx \frac{1}{\sigma^2} - \frac{2z_1}{\sigma^3}. \\ \frac{z_2(z_4+\omega)}{z_1+\sigma} &\approx z_2(z_4+\omega)\left(\frac{1}{\sigma} - \frac{z_1}{\sigma^2}\right) \approx \frac{\omega z_2}{\sigma} \\ \frac{w_2}{z_1+\sigma} &\approx w_2\left(\frac{1}{\sigma} - \frac{z_1}{\sigma^2}\right) \approx \frac{w_2}{\sigma}.\end{aligned}$$

Therefore, for $(z_1, z_2, z_3, z_4, w_1, w_2)$ in a sufficiently small neighbourhood of $0 \in \mathbb{R}^6$, we have

$$f_2(z_1, z_4, w_1) \approx \omega^2 z_1 + 2\sigma\omega z_4 + \sigma\omega^2 - \sigma^3\omega^2\left(\frac{1}{\sigma} - \frac{z_1}{\sigma^2}\right) + w_1 = 3\omega^2 z_1 + 2\omega\sigma z_4 + w_1$$

$$f_4(z_1, z_2, z_4, w_2) = -\frac{2\omega z_2}{\sigma} + \frac{w_2}{\sigma}.$$

(b) With $u = (u_1, u_2) = 0$, we have

$$f_u(t, z) = f_0(t, z_1, z_2, z_3, z_4) = \big(z_2, f_2(z_1, z_4, 0), z_3, f_4(z_1, z_2, z_4, 0)\big).$$

For the putative solution $x = (x_1, x_2, x_3, x_4)$, we have

$$\dot{x}_1(t) = 0 = x_2(t),\ \dot{x}_2(t) = 0 = \sigma_\varepsilon(\varepsilon+\omega)^2 - \frac{\sigma^3\omega^2}{\sigma_\varepsilon^2} = f_2(x_1(t), x_2(t), x_4(t), 0)$$

$$\dot{x}_3(t) = \varepsilon = x_4(t), \quad \dot{x}_4(t) = 0 = f_4(x_1(t), x_2(t), x_4(t), 0),$$

and so x is indeed a solution of the nonlinear system.
(c) Let $\delta > 0$ be arbitrary. Noting that

$$\sigma_\varepsilon = \sigma\left(\frac{\omega}{\omega+\varepsilon}\right)^{2/3},$$

we may choose $\varepsilon > 0$ sufficiently small so that

$$(\sigma_\varepsilon - \sigma)^2 + \varepsilon^2 < \delta^2.$$

Define $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) := (\sigma_\varepsilon - \sigma, 0, 0, \varepsilon)$. Then $\|\xi\| < \delta$ and, by part (b), $t \mapsto x(t) = (\sigma_\varepsilon - \sigma, 0, \varepsilon t, \varepsilon)$ is a solution with initial data $x(0) = \xi$. Moreover, $\|x(t)\| > \varepsilon t$ for all $t \geq 0$ and so $\|x(t)\| \to \infty$ as $t \to \infty$.

Exercise 1.2
(a) Let $x\colon I \to \mathbb{R}^N$ be a solution. By the chain rule (Proposition A.34), the derivative $(E \circ x)'$ of the composition $E \circ x$ satisfies

$$(E \circ x)'(t) = \langle (\nabla E)(x(t)), \dot{x}(t) \rangle = \langle (\nabla E)(x(t)), f(x(t)) \rangle = 0\,, \quad \forall\, t \in I\,.$$

Consequently, there exists $\gamma \in \mathbb{R}$ such that $E(x(t)) = (E \circ x)(t) = \gamma$ for all $t \in I$.

(b) $\langle (\nabla E)(z), f(z) \rangle = -g(z_1)z_2 + z_2 g(z_1) = 0$ for all $z \in \mathbb{R}^2$.

(c) Applying part (b) with g given by $g(s) = -b \sin s$ shows that

$$E(z) = E(z_1, z_2) = b \int_0^{z_1} \sin s \, \mathrm{d}s = b(1 - \cos z_1) + z_2^2/2$$

is a first integral.

(d) $(\nabla E)(z) = (\nabla E)(z_1, z_2) = \big(d - c/z_1\,,\, b - a/z_2\big)$ and so

$$\begin{aligned}
&\langle (\nabla E)(z_1, z_2), \big(z_1(-a + bz_2), z_2(c - dz_1)\big) \rangle \\
&\qquad = ac - bcz_2 - adz_1 + bdz_1z_2 - ac + adz_1 + bcz_2 - bdz_1z_2 \\
&\qquad = 0\,, \quad \forall (z_1, z_2) \in (0, \infty) \times (0, \infty)\,.
\end{aligned}$$

(e) Assume $E\colon G \to \mathbb{R}$ is a first integral for (1.12). We have seen that the image of any solution of (1.12) is contained in some level set of E. Therefore, in principle, a study of the level sets of a first integral can provide insight into the qualitative behaviour of solutions of (1.12). For any *constant* function E, trivially we have $\langle (\nabla E)(z), f(z) \rangle = 0$ for all $z \in G$ and, moreover, G is the only non-empty level set E. Therefore, if non-constancy is removed from the definition of a first integral, then every constant function is a first integral and the result in (a) above does not provide any useful information.

Exercise 1.3
In parts (a)-(d), it is assumed that $k(\xi) \neq 0$.

(a) $K'(z) = 1/k(z) \neq 0$ for all $z \in U$. Therefore, $K\colon U \to K(U)$ is strictly monotone and so has an inverse function $K^{-1}\colon K(U) \to U$. Moreover, $K(U)$ is an open interval containing 0 and $K^{-1}(0) = \xi$.

(b) Since H is continuous with $H(\tau) = 0$, there exists $\varepsilon > 0$ such that $I := (\tau - \varepsilon, \tau + \varepsilon)$ is contained in J and $H(I)$ is contained in $K(U)$.

(c) Differentiating the relation $K(x(t)) = H(t)$ for all $t \in I$ gives $K'(x(t))\dot{x}(t) = h(t)$ for all $t \in I$. Since $K' = 1/k$, we have $\dot{x}(t) = k(x(t))h(t)$ for all $t \in I$. Moreover, $x(\tau) = K^{-1}(H(\tau)) = K^{-1}(0) = \xi$ and so $x\colon I \to G$, $t \mapsto K^{-1}(H(t))$ is a solution of the initial-value problem. Assume $x_1, x_2\colon I \to G$ are two solutions of the initial-value problem. Then $K(x_1(t)) = H(t) = K(x_2(t))$ for all $t \in I$ and so $x_1(t) = K^{-1}(K(x_2(t))) = x_2(t)$ for all $t \in I$.

(d) Set $J := \mathbb{R}$, $I := (-1, 1)$, $U := (0, \infty)$, $k(x) = z^3$ for all $z \in U$ and $h(t) := t$ for all $t \in J$. Define $K\colon U \to K(U)$ by $K(z) := \int_1^z \mathrm{d}s/k(s) = (1 - z^{-2})/2$ for all $z \in U$ and define $H\colon J \to \mathbb{R}$ by $H(t) := \int_0^t h(s)\mathrm{d}s = t^2/2$ for all $t \in J$. Then $K(U) = (-\infty, 1/2)$, $H(I) = (-1/2, 1/2) \subset K(U)$ and $K^{-1}\colon K(U) \to U$ is given by $K^{-1}(z) = 1/\sqrt{1 - 2z}$. By parts (a)-(c), it follows that $x\colon I \to \mathbb{R}$, $t \mapsto K^{-1}(H(t)) = 1/\sqrt{1 - t^2}$, solves the initial-value problem. Moreover, since $x(t) \to \infty$ as $t \to \pm 1$, the solution x is maximal.

In parts (e) and (f) below, it is assumed that $k(\xi) = 0$.

(e) First, we prove that $x(t) = \xi$ for all $t \in I$ with $t \geq \tau$. Suppose that this claim is false. Then there exists $I^* = (\sigma, \rho) \subset I$ such that $\sigma \geq \tau$, $x(\sigma) = \xi$, $x(\rho) \neq \xi$ and $x(t) \in (\xi - \delta, \xi) \cup (\xi, \xi + \delta)$ for all $t \in I^*$. Set $c := (\rho - \sigma) \max_{t \in [\sigma, \rho]} |h(t)|$. Since, for all $t \in I^*$, $\dot{x}(t) = k(x(t))h(t)$ and $k(x(t)) \neq 0$, we have

$$c \geq \int_r^\rho |h(t)|\mathrm{d}t \geq \left| \int_r^\rho h(t)\mathrm{d}t \right| = \left| \int_r^\rho \frac{\dot{x}(t)}{k(x(t))}\mathrm{d}t \right| = \left| \int_{x(r)}^{x(\rho)} \frac{\mathrm{d}s}{k(s)} \right| \quad \forall\, r \in I^*.$$

Observe that either $x(t) \in (\xi, \xi + \delta)$ for all $t \in I^*$ or $x(t) \in (\xi - \delta, \xi)$ for all $t \in I^*$. If the former is the case, then $x(\rho) > \xi$ and passing to the limit $r \to \sigma$ (and so $x(r) \downarrow x(\sigma) = \xi$) yields a contradiction to the second of properties (1.16). If the latter is the case, then $x(\rho) < \xi$ and passing to the limit $r \to \sigma$ (and so $x(r) \uparrow x(\sigma) = \xi$) yields a contradiction to the first of properties (1.16). We may now conclude that $x(t) = \xi$ for all $t \in I$ with $t \geq \tau$.

The above argument applies *mutatis mutandis* to conclude that $x(t) = \xi$ for all $t \in I$ with $t \leq \tau$.

(f) The function k fails to satisfy properties (1.16).

Exercise 1.4

(a) Let $x\colon J \to \mathbb{R}$ be a solution of (1.18). We first show that $x(t) = 0$ for all $t \in J$ with $t \geq \tau$. Suppose otherwise, then there exists $I = (\sigma, \rho) \subset J$ with $\sigma \geq \tau$, $x(\sigma) = 0$ and $x(t) \neq 0$ for all $t \in I$. Define $\alpha := (\rho - \sigma) \max_{t \in [\sigma, \rho]} |a(t)|$. Observe that $(\mathrm{d}/\mathrm{d}t)(\ln |x(t)|) = \dot{x}(t)/x(t) = a(t)$ for all points $t \in I$ at which a is continuous. Therefore,

$$\big| \ln |x(\rho)/x(s)| \big| = \left| \int_s^\rho a(t)\mathrm{d}t \right| \leq \int_s^\rho |a(t)|\mathrm{d}t \leq \alpha \quad \forall\, s \in (\sigma, \rho)$$

which is impossible since, by choosing s sufficiently close to σ, $x(s)$ can be made arbitrarily close to $x(\sigma) = 0$ and so the term on the left can be made arbitrarily large. Therefore, $x(t) = 0$ for all $t \in J$ with $t \geq \tau$.

The above argument applies *mutatis mutandis* to conclude that $x(t) = 0$ for all $t \in J$ with $t \leq \tau$.

(b) Clearly, $x(\tau) = \xi$ and, invoking Theorem A.30, we have $\dot{x}(t) = a(t)x(t)$ at all points t of continuity of a. Therefore, x is a solution. Suppose $y\colon J \to \mathbb{R}$ is also a solution, and write $z := x - y$. Then $z(\tau) = \xi - \xi = 0$ and $\dot{z}(t) = \dot{x}(t) - \dot{y}(t) = a(t)(x(t) - y(t)) = a(t)z(t)$. By the result in (a), the zero function is the only solution on J of the initial-value problem: $\dot{z}(t) = a(t)z(t)$, $z(0) = 0$. Therefore, $y(t) = x(t)$ for all $t \in J$ and so x is the unique maximal solution.

(c) By properties of the exponential function, sufficiency of the condition is clear. We proceed to prove necessity and argue by contraposition. Assume that $\int_\tau^t a(s)\mathrm{d}s \not\to -\infty$ as $t \to \infty$. Then there exist $\alpha \in \mathbb{R}$ and a sequence (t_n) in $\mathbb{R}$, with $t_n \to \infty$ as $n \to \infty$, such that $\int_\tau^{t_n} a(s)\mathrm{d}s \geq \alpha$ for all $n \in \mathbb{N}$. Therefore, $|x(t_n)| \geq e^\alpha |\xi| > 0$ for all $n \in \mathbb{N}$ and so, for $\xi \neq 0$, $x(t) \not\to 0$ as $t \to \infty$.

(d) Define $A := \int_0^T a(s)\mathrm{d}s$, $B := \int_0^T |a(s)|\mathrm{d}s$ and $C := \big|\int_0^\tau a(s)\mathrm{d}s\big|$. Observe that, for every integer m,

$$\int_0^{mT} a(s)\mathrm{d}s = mA \quad \text{and} \quad \int_{mT}^{(m+1)T} |a(s)|\mathrm{d}s = B.$$

Let (t_n) be any sequence in $\mathbb{R}$ with $t_n \to \infty$ as $n \to \infty$. For each $n \in \mathbb{N}$, there exists a unique integer m_n such that $m_nT \leq t_n < (m_n+1)T$. Clearly, $m_n \to \infty$ as $n \to \infty$. Now

$$\int_\tau^{t_n} a(s)\mathrm{d}s = \left(\int_\tau^0 + \int_0^{m_nT} + \int_{m_nT}^{t_n}\right) a(s)\mathrm{d}s$$

Therefore, $m_nA - B - C \leq \int_\tau^{t_n} a(s)\mathrm{d}s \leq m_nA + B + C$ for all $n \in \mathbb{N}$ and so $\int_\tau^{t_n} a(s)\mathrm{d}s \to -\infty$ as $n \to \infty$ if, and only if, $A < 0$. We may now infer that $\int_\tau^t a(s)\mathrm{d}s \to -\infty$, as $t \to \infty$ if, and only if, $A < 0$. Invoking the result in (c) completes the proof.

Exercise 1.5
Let $K \subset J$ be a compact interval containing τ and let $E \subset K$ be the finite set of points $t \in K$ at which either a or b (possibly both) fails to be differentiable. Write $\hat{K} := K\backslash E$. Multiplying both sides of the differential equation by μ, we have

$$(\mu x)'(t) = \mu(t)\dot{x}(t)+\dot{\mu}(t)x(t) = \mu(t)\big(a(t)x(t)+b(t)\big)-\mu(t)a(t)x(t) = \mu(t)b(t) \ \ \forall\, t \in \hat{K},$$

which, on integration and imposing the condition $x(\tau) = \xi$, gives

$$\int_\tau^t \mu(s)b(s)\mathrm{d}s = \mu(t)x(t) - \mu(\tau)x(\tau) = \mu(t)x(t) - \xi \ \ \forall\, t \in K.$$

Thus, we arrive at a candidate solution $x\colon K \to \mathbb{R}$ of the initial-value problem:

$$x(t) = \frac{1}{\mu(t)}\left(\xi + \int_\tau^t \mu(s)b(s)\mathrm{d}s\right) \quad \forall\, t \in K.$$

To verify that x is indeed a solution, simply note that $x(\tau) = \xi$ and, invoking the (generalized) fundamental theorem of calculus,

$$\dot{x}(t) = \frac{a(t)}{\mu(t)}\left(\xi + \int_\tau^t \mu(s)b(s)\mathrm{d}s\right) + \frac{\mu(t)b(t)}{\mu(t)} = a(t)x(t) + b(t) \ \ \forall\, t \in \hat{K}.$$

Since $K \subset J$ is arbitrary, we may conclude that the function

$$J \to \mathbb{R}, \quad t \mapsto x(t) := a(t)\mu^{-1}(t)\left(\xi + \int_\tau^t \mu^{-1}(s)b(s)\mathrm{d}s\right)$$

solves the initial-value problem. Assume $y\colon J \to \mathbb{R}$ is also a solution of the initial-value problem. Write $e = x - y$ and so e solves the problem

$$\dot{e}(t) = a(t)e(t), \quad e(\tau) = 0.$$

By part (a) of Exercise 1.4, we may infer that $e = 0$ and so $y = x$. Therefore, x is the unique solution on J of the initial-value problem.

Exercise 1.6
Write $w(0) = w^0$ and $q(0) = q^0$. On $[0, t_s)$, we have

$$\dot{w}(t) = (a-\mu)w(t), \quad \dot{q}(t) = -\nu q(t), \quad (w(0), q(0)) = (w^0, q^0),$$

and so $w(t) = e^{(a-\mu)t}w^0$ and $q(t) = e^{-\nu t}q^0$ for all $t \in [0, t_s)$. Write $w^* := e^{(a-\mu)t_s}w^0$ and $q^* := e^{-\nu t_s}q^0$. Then, on $[t_s, 1]$, we have

$$\dot{w}(t) = -\mu w(t), \quad \dot{q}(t) = -\nu q(t) + bw(t), \quad (w(t_s), q(t_s)) = (w^*, q^*),$$

and so, for all $t \in [t_s, 1]$, we have

$$w(t) = e^{-mu(t-t_s)}w^* = e^{-\mu t}e^{at_s}w^0,$$

$$\begin{aligned} q(t) &= e^{-\nu(t-t_s)}q^* + \int_{t_s}^{t} e^{-\nu(t-s)}be^{-\mu s}e^{at_s}w^0\mathrm{d}s \\ &= e^{-\nu t}\left(q^0 + \frac{bw^0e^{at_s}}{\mu-\nu}\left(e^{-(\mu-\nu)t_s} - e^{-(\mu-\nu)t}\right)\right) \end{aligned}$$

The optimal control maximizes $q(1)$ and so (noting that $bw^0/(\mu-\nu) > 0$) the parameter t_s should be such that the function

$$g\colon (0,1) \to (0,\infty),\ \tau \mapsto e^{a\tau}\left(e^{-(\mu-\nu)\tau} - e^{-(\mu-\nu)}\right)$$

attains its maximum at $\tau = t_s$. A straightforward calculation reveals that the first derivative $g'(\tau)$ is zero if, and only if,

$$\tau = t_s := 1 - \frac{\ln(a/(a+\mu-\nu))}{\mu-\nu}.$$

Moreover, the second derivative g'' is negative valued. Therefore, g attains its maximum at $\tau = t_s$.

Chapter 2

Exercise 2.1
Set $\xi = 1$ and define $A\colon [-1,1] \to \mathbb{R}$ by

$$A(t) := \begin{cases} 0, & -1 \le t < 0, \\ 1, & 0 \le t \le 1. \end{cases}$$

Exercise 2.2
Let $J = \mathbb{R}$ and $N = 2$. Let $a\colon \mathbb{R} \to \mathbb{F}$ be any piecewise continuous function with the property that the set E of points at which it fails to be continuous is non-empty. Define $A\colon \mathbb{R} \to \mathbb{F}^{2\times 2}$ and $\xi \in \mathbb{F}^2$ by

$$A(t) := \begin{pmatrix} 0 & 1 \\ 0 & a(t) \end{pmatrix}, \qquad \xi := \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then the initial-value problem $\dot{x}(t) = A(t)x(t)$, $x(0) = \xi$ has constant solution $x\colon \mathbb{R} \to \mathbb{F}^2$, $t \mapsto x(t) = \xi$, whilst A fails to be continuous at each $\sigma \in E$.

Exercise 2.3
Let $x\colon J_x \to \mathbb{F}^N$ be a solution of $\dot{x}(t) = A(t)x(t)$. Then there exists $\tau \in J_x$ such that

$$x(t) - x(\tau) = \int_\tau^t A(\sigma)x(\sigma)\mathrm{d}\sigma \quad \forall\, t \in J_x.$$

Let $t_1, t_2 \in J_x$ be arbitrary. Then

$$\begin{aligned} x(t_2) - x(t_1) = x(t_2) - x(\tau) - \big(x(t_1) - x(\tau)\big) &= \left(\int_\tau^{t_2} - \int_\tau^{t_1}\right) A(\sigma)x(\sigma)\mathrm{d}\sigma \\ &= \int_{t_1}^{t_2} A(\sigma)x(\sigma)\mathrm{d}\sigma. \end{aligned}$$

Exercise 2.4
Observe that $M_2(t,s) - M_1(t,s) = \int_s^t A(\sigma)\mathrm{d}\sigma$ for all $(t,s) \in J \times J$ and, for all $n \in \mathbb{N}$,

$$M_{n+2}(t,s) - M_{n+1}(t,s) = \int_s^t A(\sigma)\big[M_{n+1}(\sigma,s) - M_n(\sigma,s)\big]\mathrm{d}\sigma \quad \forall\, (t,s) \in J \times J.$$

The result (2.3) follows by induction.
Assume that for, some $n \in \mathbb{N}$, the equality in (2.4) holds for all $(t,s) \in J \times J$. Then

$$\begin{aligned} \int_s^t \int_s^{\sigma_1} \cdots \int_s^{\sigma_n} \mathrm{d}\sigma_{n+1} \cdots \mathrm{d}\sigma_2 \mathrm{d}\sigma_1 &= \int_s^t \frac{(\sigma_1 - s)^n}{n!}\,\mathrm{d}\sigma_1 \\ &= \frac{1}{n!}\int_0^{t-s} \sigma^n \mathrm{d}\sigma = \frac{(t-s)^{n+1}}{(n+1)!} \quad \forall\, (t,s) \in J \times J. \end{aligned}$$

Since $\int_s^t \mathrm{d}\sigma_1 = t - s$ for all $(t,s) \in J \times J$, (2.4) follows by induction.

Exercise 2.5
The result follows from the Peano-Baker series (2.6) if it can be shown that, for all $n \in \mathbb{N}$,

$$\begin{aligned} &\frac{1}{n!}\left(\int_\tau^t A(\sigma)\mathrm{d}\sigma\right)^n = \\ &\qquad \int_\tau^t A(\sigma_1)\int_\tau^{\sigma_2} A(\sigma_2) \cdots \int_\tau^{\sigma_{n-1}} A(\sigma_n)\mathrm{d}\sigma_n \cdots \mathrm{d}\sigma_2 \mathrm{d}\sigma_1 \quad \forall\, t, \tau \in \mathbb{R}. \qquad (*) \end{aligned}$$

Clearly, $(*)$ holds for $n=1$. Let $n \in \mathbb{N}$ and assume that $(*)$ holds. Observe that commutativity of $A(t)$ and $A(\sigma)$ for all t, σ implies commutativity of $A(t)$ and $\int_\tau^t A(\sigma)\mathrm{d}\sigma$ which, in conjunction with the product rule for differentiation and Theorem A.30, gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\tau^t A(\sigma)\mathrm{d}\sigma\right)^{n+1} = (n+1)A(t)\left(\int_\tau^t A(\sigma)\mathrm{d}\sigma\right)^n$$

at all points t of continuity of A. Integrating and dividing by $(n+1)!$, we have

$$\begin{aligned}\frac{1}{(n+1)!}\left(\int_\tau^t A(\sigma)\mathrm{d}\sigma\right)^{n+1} &= \int_\tau^t A(\sigma)\frac{1}{n!}\left(\int_\tau^\sigma A(\rho)\mathrm{d}\rho\right)^n \mathrm{d}\sigma \\ &= \int_\tau^t A(\sigma)\int_\tau^\sigma A(\sigma_1)\cdots\int_\tau^{\sigma_{n-1}} A(\sigma_n)\mathrm{d}\sigma_n\cdots\mathrm{d}\sigma_1\mathrm{d}\sigma.\end{aligned}$$

By induction, it follows that $(*)$ holds for all $n \in \mathbb{N}$.

Exercise 2.6
We have

$$\big(G(s)\exp\big(-H(s)\big)\big)' = \big(G'(s) - H'(s)G(s)\big)\exp\big(-H(s)\big) \geq 0 \quad \forall\, s \in [t, \tau]$$

which, on integration, gives

$$c = G(\tau) \geq G(t)\exp\big(-H(t)\big).$$

Hence, we arrive at the requisite inequality

$$g(t) \leq G(t) \leq c\exp\big(H(t)\big) = c\exp\left(\int_t^\tau h(s)\mathrm{d}s\right) = c\exp\left(\left|\int_\tau^t h(s)\mathrm{d}s\right|\right).$$

Exercise 2.7
Let $\tau \in J$ be arbitrary. Consider the initial-value problems $\dot{x}(t) = A(t)x(t)$, $x(\tau) = \xi$, and $\dot{\tilde{x}}(t) = \tilde{A}(t)\tilde{x}(t)$, $\tilde{x}(\tau) = \tilde{\xi}$. The unique solutions on J are given, respectively, by $x(t) = \Phi(t,\tau)\xi$ and $\tilde{x}(t) = \tilde{\Phi}(t,\tau)\tilde{\xi}$ for all $t \in J$. Now,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \tilde{x}(t), x(t)\rangle = \langle \tilde{A}(t)\tilde{x}(t), x(t)\rangle + \langle \tilde{x}(t), A(t)x(t)\rangle = \langle \big(\tilde{A}(t) + A^*(t)\big)\tilde{x}(t), x(t)\rangle = 0$$

for all points $t \in J$ at which A is continuous. Therefore, $\langle \tilde{x}(t), x(t)\rangle = \langle \tilde{\xi}, \xi\rangle$ for all $t \in J$ and so

$$\langle \tilde{\xi}, \xi\rangle = \langle \tilde{\Phi}(t,\tau)\tilde{\xi}, \Phi(t,\tau)\xi\rangle = \langle \Phi^*(t,\tau)\tilde{\Phi}(t,\tau)\tilde{\xi}, \xi\rangle \quad \forall\, t \in J.$$

Since $\tilde{\xi}, \xi \in \mathbb{F}^N$ and $\tau \in J$ are arbitrary, we may now infer that $\Phi^*(t,\tau)\tilde{\Phi}(t,\tau) = I$ for all $(t,\tau) \in J \times J$. Therefore, $\Phi^*(t,\tau) = \tilde{\Phi}^{-1}(t,\tau) = \tilde{\Phi}(\tau,t)$ for all $(t,\tau) \in J \times J$, whence the required result.

Exercise 2.8
Let $\mathbb{F} = \mathbb{R}$, $N = 2$, $J = [0,1]$ and define $y_1, y_2 \in C(J, \mathbb{F}^N)$ by

$$y_1(t) := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\; \forall\, t \in J, \quad y_2(t) := \begin{pmatrix} 1+t \\ 0 \end{pmatrix} \;\; \forall\, t \in J.$$

Then y_1 and y_2 are linearly independent. However, $y_2(t) = (1+t)y_1(t)$ for all $t \in J$ and so the vectors $y_1(t), y_2(t) \in \mathbb{R}^2$ fail to be linearly independent for all $t \in J$.

Exercise 2.9
By inspection, we see that $\psi_1\colon t \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a solution. Written componentwise, the system of differential equations is: $\dot{x}_1(t) = x_2(t)$, $\dot{x}_2(t) = 2tx_2(t)$. By separation of variables, we find that $t \mapsto e^{t^2}$ satisfies the second equation and so

$$\psi_2\colon t \mapsto \begin{pmatrix} \int_0^t e^{s^2} \mathrm{d}s \\ e^{t^2} \end{pmatrix}$$

is a solution. Evidently ψ_1 and ψ_2 are linearly independent. Writing $\Psi = \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix}$, the transition matrix function Φ is given by

$$\begin{aligned} \Phi(t,\tau) = \Psi(t)\Psi^{-1}(\tau) &= \begin{pmatrix} 1 & \int_0^t e^{s^2} \mathrm{d}s \\ 0 & e^{t^2} \end{pmatrix} \begin{pmatrix} 1 & -e^{-\tau^2} \int_0^\tau e^{s^2} \mathrm{d}s \\ 0 & e^{-\tau^2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & e^{-\tau^2} \int_\tau^t e^{s^2} \mathrm{d}s \\ 0 & e^{t^2-\tau^2} \end{pmatrix} \quad \forall\, (t,\tau) \in \mathbb{R} \times \mathbb{R}. \end{aligned}$$

Exercise 2.10
(1) For all $k \in \mathbb{N}$, $P^k = \mathrm{diag}(p_1^k, \ldots, p_N^k)$ and so

$$\exp(P) = \sum_{k=0}^{\infty} P^k/k! = \mathrm{diag}\left(\sum_{k=0}^{\infty} p_1^k/k!, \ldots, \sum_{k=0}^{\infty} p_N^k/k!\right) = \mathrm{diag}\big(e^{p_1}, \ldots, e^{p_N}\big).$$

(2) $\qquad (\exp(P))^* = \big(\sum_{k=0}^{\infty} P^k/k!\big)^* = \sum_{k=0}^{\infty} (P^*)^k/k! = \exp(P^*).$

(3) By Corollary 2.3, $(\mathrm{d}/\mathrm{d}t)\exp(Pt) = P\exp(Pt)$. Moreover,

$$P\exp(Pt) = P\sum_{k=0}^{\infty} (Pt)^k/k! = \left(\sum_{k=0}^{\infty} (Pt)^k/k!\right) P = \exp(Pt)P.$$

Exercise 2.11
Let $\mathbb{F} = \mathbb{R}$, $N = 2$ and consider the non-commuting matrices

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In this case,

$$P + Q = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{with} \quad (P+Q)^n = P + Q \quad \forall\, n \in \mathbb{N},$$

and so

$$\begin{aligned} \exp(P+Q) &= \sum_{k=0}^{\infty} \frac{1}{k!}(P+Q)^k = I + \left(\sum_{k=1}^{\infty} \frac{1}{k!}\right)(P+Q) = I + (e-1)(P+Q) \\ &= \begin{pmatrix} e & e-1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\exp(P)\exp(Q) = \exp(P)(I+Q) = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e & e \\ 0 & 1 \end{pmatrix} \neq \exp(P+Q).$$

Exercise 2.12
Let $\{v_1, \ldots, v_K\}$ be a basis of V. Since V is closed under complex conjugation, it follows that $\{\bar{v}_1, \ldots, \bar{v}_K\}$ is also a basis of V. Therefore,

$$V = \operatorname{span}\{v_1, \ldots, v_K, \bar{v}_1, \ldots, \bar{v}_K\} = \operatorname{span}\{\operatorname{Re} v_1, \ldots, \operatorname{Re} v_K, \operatorname{Im} v_1, \ldots, \operatorname{Im} v_K\}$$

and so the family $\{\operatorname{Re} v_1, \ldots, \operatorname{Re} v_K, \operatorname{Im} v_1, \ldots, \operatorname{Im} v_K\}$ of vectors in $\mathbb{R}^N$ contains a basis.

Exercise 2.13
Let $x\colon J_x \to \mathbb{F}^N$ be a solution of $\dot{x}(t) = A(t)x(t) + b(t)$. Then there exists $\tau \in J_x$ such that

$$x(t) - x(\tau) = \int_\tau^t \big(A(\sigma)x(\sigma) + b(\sigma)\big)\mathrm{d}\sigma \quad \forall\, t \in J_x.$$

Let $t_1, t_2 \in J_x$ be arbitrary. Then

$$\begin{aligned} x(t_2) - x(t_1) = x(t_2) - x(\tau) - \big(x(t_1) - x(\tau)\big) &= \left(\int_\tau^{t_2} - \int_\tau^{t_1}\right) \big(A(\sigma)x(\sigma) + b(\sigma)\big)\mathrm{d}\sigma \\ &= \int_{t_1}^{t_2} \big(A(\sigma)x(\sigma) + b(\sigma)\big)\mathrm{d}\sigma. \end{aligned}$$

Exercise 2.14
Let $\mathcal{S}_{\mathrm{ih}}$ denote the set of all solutions of $\dot{x}(t) = A(t)x(t) + b(t)$ and let $y \in \mathcal{S}_{\mathrm{ih}}$. Assume $z \in \mathcal{S}_{\mathrm{ih}}$ and write $x := z - y$. Then $\dot{x}(t) = \dot{z}(t) - \dot{y}(t) = A(t)z(t) + b(t) - A(t)y(t) - b(t) = A(t)(z(t) - y(t)) = A(t)x(t)$ at every $t \in J$ which is not a point of discontinuity of A or b. Therefore, $x \in \mathcal{S}_{\mathrm{hom}}$ and so $z \in y + \mathcal{S}_{\mathrm{hom}}$. This establishes the inclusion $\mathcal{S}_{\mathrm{ih}} \subset y + \mathcal{S}_{\mathrm{hom}}$. To establish the reverse inclusion, assume $z \in y + \mathcal{S}_{\mathrm{hom}}$. Then $z = y + x$ for some $x \in \mathcal{S}_{\mathrm{hom}}$ and so $\dot{z}(t) = A(t)y(t) + b(t) + A(t)x(t) = A(t)z(t) + b(t)$ at every $t \in J$ which is not a point of discontinuity of A or b. Therefore, $z \in \mathcal{S}_{\mathrm{ih}}$.

Exercise 2.15
Let $\mathcal{P}(n)$ denote the statement

$$\text{“}\Phi(t + np, \tau) = \Phi(t, 0)\Phi^n(p.0)\Phi(0, \tau)\ \forall\,(t, \tau) \in \mathbb{R} \times \mathbb{R}\text{”}$$

We already know that $\Phi(t + p, \tau) = \Phi(t, 0)\Phi(p, 0)\Phi(0, \tau)$ for all $(t, \tau) \in \mathbb{R} \times \mathbb{R}$, and so $\mathcal{P}(1)$ is a true statement. Assume $n \in \mathbb{N}$ and $\mathcal{P}(n)$ true. Then

$$\begin{aligned} \Phi(t + (n+1)p, \tau) &= \Phi(t + p + np, \tau) = \Phi(t + p, 0)\Phi^n(p, 0)\Phi(0, \tau) \\ &= \Phi(t, 0)\Phi(p, 0)\Phi(0, 0)\Phi^n(p, 0)\Phi(0, \tau) \\ &= \Phi(t, 0)\Phi^{n+1}(p, 0)\Phi(0, \tau)\ \forall\,(t, \tau) \in \mathbb{R} \times \mathbb{R} \end{aligned}$$

and so $\mathcal{P}(n+1)$ is true. By induction, it follows that $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.

Exercise 2.16
Note initially that, since $\Phi(p, 0)$ is invertible, $0 \notin \sigma(\Phi(p, 0))$ and so, for the function $f\colon z \mapsto z^n$, we have $f'(\mu) \neq 0$ for all $\mu \in \sigma(\Phi(p, 0))$. Therefore, by the spectral mapping theorem (Theorem 2.19),

$$\ker(\Phi^n(p, 0) - I) = \ker(\Phi(p, 0) - \lambda I).$$

(a) Let $x\colon \mathbb{R} \to \mathbb{F}^N$ be a non-zero solution (and so, in particular, $x(0) \neq 0$). Assume $x(0) \in \ker(\Phi(p, 0) - \lambda I)$. Then $\Phi^n(p, 0)x(0) = \lambda^n x(0) = x(0)$ and so, invoking (2.32), we have, for all $t \in \mathbb{R}$,

$$x(t + np) = \Phi(t + np, 0)x(0) = \Phi(t, 0)\Phi^n(p, 0)x(0) = \Phi(t, 0)x(0) = x(t).$$

Therefore, x is np-periodic.
Conversely, assume that x is np-periodic. Then $x(np) = \Phi(np,0)x(0) = x(0)$. By (2.32), we have $\Phi(np,0) = \Phi^n(p,0)$. Therefore, $(\Phi^n(p,0) - I)x(0) = 0$ and so $x(0) \in \ker(\Phi^n(p,0) - I) = \ker(\Phi(p,0) - \lambda I)$.

(b) That $\mathcal{S}_{np}$ is a vector space is clear. Let $\mathcal{B}$ be a basis of $\ker(\Phi(p,0) - I)$. For $z \in \mathcal{B}$, let x_z denote the np-periodic solution $t \mapsto \Phi(t,0)z$. By part (a), the set $\{x_z\colon\ z \in \mathcal{B}\}$ is a basis for $\mathcal{S}_{np}$, whence the result.

Exercise 2.17
Sufficiency. Assume that λ is an eigenvalue of $\Phi(p,0)$ and $\lambda^n = \mu$. Let $v \in \mathbb{C}^N$ be an associated eigenvector and so $\Phi^n(p,0)v = \lambda^n v = \mu v$. Define x by $x(t) := \Phi(t,0)v$ for all $t \in \mathbb{R}$. Invoking (2.32), with $\tau = 0$, gives

$$x(t+np) = \Phi(t+np,0)v = \Phi(t,0)\Phi^n(p,0)v = \mu\Phi(t,0)v = \mu x(t) \quad \forall\, t \in \mathbb{R}.$$

Necessity. Assume that x is a non-zero solution of (2.30), with the property $x(t+np) = \mu x(t)$ for all $t \in \mathbb{R}$. Write $v := x(0) \neq 0$. Invoking (2.32), with $\tau = 0$, we have

$$\mu\Phi(t,0)v = \mu x(t) = x(t+np) = \Phi(t+np,0)v = \Phi(t,0)\Phi^n(p,0)v,$$

and thus, $\Phi(t,0)\big(\Phi^n(p,0) - \mu I\big)v = 0$. Consequently $\big(\Phi^n(p,0) - \mu I\big)v = 0$ and so μ is an eigenvalue of $\Phi^n(p,0)$. By Theorem 2.19 (with $f(z) = z^n$),

$$\sigma(\Phi^n(p,0)) = \big\{\lambda^n \colon \lambda \in \sigma(\Phi(p,0))\big\}.$$

Therefore, $\Phi(p,0)$ has an eigenvalue λ with the property that $\lambda^n = \mu$.

Exercise 2.18
For convenience, write

$$A(t) = \begin{pmatrix} A_1 & 0 \\ 0 & a(t) \end{pmatrix} \quad \text{with} \quad A_1 := \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix} \quad \text{and} \quad a(t) = 1 + \sin t.$$

It is straightforward to verify that

$$\exp(A_1 t) = \begin{pmatrix} \cos(t/2) & \sin(t/2) \\ -\sin(t/2) & \cos(t/2) \end{pmatrix} \quad \forall t \in \mathbb{R}.$$

Moreover, from Example 2.18, we know that the transition function φ generated by a is such that $\varphi(t,0) = \exp(1 - \cos t + t)$ for all $t \in \mathbb{R}$. Therefore,

$$\Phi(t,0) = \begin{pmatrix} \exp(A_1 t) & 0 \\ 0 & \varphi(t,0) \end{pmatrix} = \begin{pmatrix} \cos(t/2) & \sin(t/2) & 0 \\ -\sin(t/2) & \cos(t/2) & 0 \\ 0 & 0 & \exp(1 - \cos t + t) \end{pmatrix}.$$

For $p = 2\pi$, we immediately see that the spectrum of $\Phi(p,0)$ is $\{-1, e^p\}$. Also,

$$\Phi(p,0) + I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^p + 1 \end{pmatrix}$$

Let ξ be any non-zero vector in $\ker\big(\Phi(p,0) + I\big)$, then $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ 0 \end{pmatrix}$ with ξ_1 and ξ_2 not both zero. Then the solution of the initial-value problem $\dot{x}(t) = A(t)x(t)$, $x(0) = \xi$, is

$$t \mapsto \Phi(t,0)\xi = \begin{pmatrix} \cos(t/2)\xi_1 + \sin(t/2)\xi_2 \\ -\sin(t/2)\xi_1 + \cos(t/2)\xi_2 \\ 0 \end{pmatrix}$$

which is evidently non-constant and of period π.

Exercise 2.19
Necessity. Assume that (2.33) has a p-periodic solution x. Write $\xi := x(0)$. Then

$$\xi = x(p) = \Phi(p,0)\xi + \int_0^p \Phi(p,s)b(s)\mathrm{d}s = \Phi(p,0)\xi + \eta$$

whence $\eta = \big(I - \Phi(p,0)\big)\xi$ and so $\eta \in \operatorname{im}\big(I - \Phi(p,0)\big)$.
Sufficiency. Assume that $\eta \in \operatorname{im}\big(I - \Phi(p,0)\big)$. Let $\xi \in \mathbb{F}^N$ be such that $\eta = \big(I - \Phi(p,0)\big)\xi$ and define $x\colon \mathbb{R} \to \mathbb{F}^N$ by

$$x(t) = \Phi(t,0)\xi + \int_0^t \Phi(t,s)b(s)\mathrm{d}s \quad \forall\, t \in \mathbb{R}.$$

Clearly, x is a solution of (2.33). We will show that x is p-periodic. Invoking (2.31), (2.32) and periodicity of b, we have $\Phi(t+p,s) = \Phi(t,0)\Phi(p,s)$ and $\Phi(t+p,s+p)b(s+p) = \Phi(t,s)b(s)$ for all $t,s \in \mathbb{R}$. Therefore,

$$\begin{aligned}
x(t+p) &= \Phi(t+p,0)\xi + \int_0^{t+p} \Phi(t+p,s)b(s)\mathrm{d}s \\
&= \Phi(t,0)\left(\Phi(p,0)\xi + \int_0^p \Phi(p,s)b(s)\mathrm{d}s\right) + \int_p^{t+p} \Phi(t+p,s)b(s)\mathrm{d}s \\
&= \Phi(t,0)\big(\Phi(p,0)\xi + \eta\big) + \int_0^t \Phi(t+p,s+p)b(s+p)\mathrm{d}s \\
&= \Phi(t,0)\xi + \int_0^t \Phi(t,s)b(s)\mathrm{d}s = x(t) \quad \forall\, t \in \mathbb{R}
\end{aligned}$$

and so x is p-periodic.

Exercise 2.20 Let $H = I \in \mathbb{C}^{2\times 2}$, the 2×2 identity matris. Then $\sigma(H) = \{1\}$. The eigenvalue 1 has algebraic multiplicity 2, coincident with its geometric multiplicity. The matrix $G = \begin{pmatrix} 0 & 0 \\ 0 & e^{2\pi i}\end{pmatrix}$ is a logarithm (but not the principal logarithm) of H, with $\sigma(G) = \{0, e^{2\pi i}\}$. The eigenvalue $\lambda = 1$ of H has principal logarithm $\operatorname{Log}\lambda = 0 \in \sigma(G)$. However, the latter eigenvalue of G has algebraic multiplicity 1, coincident with its geometric multiplicity.

Exercise 2.21
We first show that φ_1 is periodic of period 2. The function φ_1 is the unique solution of $\ddot{y}(t) + a(t)y(t) = 0$ with initial data $y(0) = 1$, $\dot{y}(0) = 0$. Therefore, on $[0,\tau]$, we have

$$(\varphi_1(t), \dot{\varphi}_1(t)) = \big(\cos(\omega t), -\omega\sin(\omega t)\big) = \big(\cos(\pi t/\tau), (\pi/\tau)\sin(\pi t/\tau)\big).$$

Thus, $\big(\varphi_1(\tau), \dot{\varphi}_1(\tau)\big) = (-1,0)$. On $[\tau, 1]$, we have $(\varphi_1(t), \dot{\varphi}_1(t)) = (-1,0)$. In particular, $(\varphi_1(1), \dot{\varphi}_1(1)) = (-1,0)$. An analogous calculation on the interval $[1,2]$ gives

$$\big(\varphi_1(2), \dot{\varphi}_1(2)\big) = (1,0) = \big(\varphi_1(0), \dot{\varphi}_1(0)\big).$$

We may now conclude that φ_1 is periodic of period 2.

Now consider the function φ_2, which is the unique solution of $\ddot{y}(t)+a(t)y(t)=0$ with initial data $y(0)=0$, $\dot{y}(0)=1$. We claim that

$$\big(\varphi_2(n),\dot{\varphi}_2(n)\big)=\big((-1)^n n(1-\tau)\,,\,(-1)^n\big)\quad\forall\,n\in\mathbb{N}.$$

On $[0,\tau]$, we have

$$\big(\varphi_2(t),\dot{\varphi}_2(t)\big)=\big(\sin(\omega t)/\omega\,,\,\cos(\omega t)\big)=\big(\tau\sin(\pi t/\tau)/\pi\,,\,\cos(\pi t/\tau)\big).$$

Thus, $\big(\varphi_2(\tau),\dot{\varphi}_2(\tau)\big)=(0,-1)$ and so

$$\big(\varphi_2(t)\,,\,\dot{\varphi}_2(t)\big)=\big(\varphi_2(\tau)+\dot{\varphi}_2(\tau)(t-\tau)\,,\,\dot{\varphi}_2(\tau)\big)=\big(-(t-\tau)\,,\,-1\big)\quad\forall\,t\in[\tau,1].$$

In particular, $\big(\varphi_1(1),\dot{\varphi}_2(1)\big)=\big(-(1-\tau),-1\big)$ and so the claim holds for $n=1$. Assume $m\in\mathbb{N}$ and the claim holds with $n=m$. On $[m,m+\tau]$, we have

$$\begin{aligned}\varphi_2(t)&=\varphi_2(m)\cos(\omega(t-m))+(\dot{\varphi}_2(m)/\omega)\sin(\omega(t-m))\\ \dot{\varphi}_2(t)&=-\omega\varphi_2(m)\sin(\omega(t-m))\,,\,\dot{\varphi}_1(m)\cos(\omega(t-m).\end{aligned}$$

Thus, $\big(\varphi_2(m+\tau)\,,\,\dot{\varphi}_2(m+\tau)\big)=\big(-\varphi_2(m)\,,\,-\dot{\varphi}_2(m)\big)=\big((-1)^{m+1}m(1-\tau)\,,\,(-1)^{m+1}\big)$ and so, for all $t\in[m+\tau,m+1]$.

$$\begin{aligned}\varphi_2(t)&=\varphi_2(m+\tau)+\dot{\varphi}_2(m+\tau)(t-m-\tau)=(-1)^{m+1}\big(m(1-\tau)+(t-m-\tau)\big)\\ \dot{\varphi}_2(t)&=\dot{\varphi}_2(m+\tau)=(-1)^{m+1}.\end{aligned}$$

In particular,

$$\big(\varphi_2(m+1),\dot{\varphi}_2(m+1)\big)=\big((-1)^{m+1}(m+1)(1-\tau)\,,\,(-1)^{m+1}\big)$$

and so the claim holds with $n=m+1$. By induction, it follows that the claim holds for all $n\in\mathbb{N}$ and so φ_2 is unbounded with $\varphi_2(n)=(-1)^n(1-\tau)$ for all $n\in\mathbb{N}$.

Exercies 2.22 The functions φ_1 and φ_2 are the unique solutions of $\ddot{y}(t)=-a(t)y(t)$ with respective initial data $\varphi_1(0)=1$, $\dot{\varphi}_1(0)=0$ and $\varphi_2(0)=0$, $\dot{\varphi}_2(0)=1$. Define ψ_1 and ψ_2 by $\psi_1(t):=\varphi_1(-t)$ and $\psi_2(t):=\varphi_2(-t)$ for all $t\in\mathbb{R}$. Then, since a is even, we have $\ddot{\psi}(t)=\ddot{\varphi}(-t)=-a(-t)\varphi_1(-t)=-a(t)\psi_1(t)$ and, similarly, $\ddot{\psi}_2(t)=-a(t)\psi(t)$. Therefore, φ_1 and ψ_2 are the unique solutions of $\ddot{y}(t)=-a(t)y(t)$ with respective initial data $\psi(0)=1$, $\dot{\psi}_1(0)=0$ and $\psi_2(0)=0$, $\dot{\psi}_2(0)=-1$. We may now infer that, for all $t\in\mathbb{R}$, $\varphi_1(t)=\psi_1(t)=\varphi_1(-t)$ and $\varphi_2(t)=-\psi_2(t)=-\varphi_2(-t)$. It follows that

$$\Phi(-p,0)=\begin{pmatrix}\varphi_1(-p)&\varphi_2(-p)\\ \dot{\varphi}_1(-p)&\dot{\varphi}_2(-p)\end{pmatrix}=\begin{pmatrix}\varphi_1(p)&-\varphi_2(p)\\ -\dot{\varphi}(p)&\dot{\varphi}_2(p)\end{pmatrix}.$$

Also, since $\Phi(t,\tau)=\Phi(t+p,\tau_p)$ for all $t,\tau\in\mathbb{R}$ and setting $(t,\tau)=(-p,0)$, we have $\Phi(-p,0)=\Phi(0,p)=\Phi^{-1}(p,0)$. Recalling that $\det\Phi(p,0)=1$, we may conclude that

$$\begin{aligned}\begin{pmatrix}\varphi_1(p)&\varphi_2(p)\\ \dot{\varphi}_1(p)&\dot{\varphi}_2(p)\end{pmatrix}&=\Phi(p,0)=\Phi^{-1}(-p,0)\\ &=\begin{pmatrix}\varphi_1(p)&-\varphi_2(p)\\ -\dot{\varphi}_1(p)&\dot{\varphi}_2(p)\end{pmatrix}^{-1}=\begin{pmatrix}\dot{\varphi}_2(p)&\varphi_2(p)\\ \dot{\varphi}_1(p)&\varphi_1(p)\end{pmatrix}\end{aligned}$$

and so $\varphi_1(p)=\dot{\varphi}_2(p)$. Therefore, $\gamma=\big(\varphi_1(p)+\dot{\varphi}_2(p)\big)/2=\varphi_1(p)$.

Exercise 2.23

$$\int_0^{2\pi} \operatorname{tr} A(s)\mathrm{d}s = \int_0^{2\pi} \big(2 + \sin s - \cos s\big)\mathrm{d}s = 4\pi > 0.$$

An application of Corollary 2.33 (with $p = 2\pi$) shows that there exists a solution which is unbounded on $\mathbb{R}_+$.

Exercise 2.24

For the putative solution x we find $\dot{x}(t) = e^{t/2}\begin{pmatrix}\sin t - \cos t/2\\ \cos t + \sin t/2\end{pmatrix}$ for all t and

$$\begin{aligned} A(t)x(t) &= e^{t/2}\begin{pmatrix} -1 + 3(\cos t)^2/2 & 1 - 3\sin t\cos t/2 \\ -1 - 3\sin t\cos t/2 & -1 + 3(\sin t)^2/2 \end{pmatrix}\begin{pmatrix} -\cos t \\ \sin t\end{pmatrix} \\ &= e^{t/2}\begin{pmatrix}\sin t - \cos t/2\\ \cos t + \sin t/2\end{pmatrix} = \dot{x}(t) \quad \forall\, t. \end{aligned}$$

Therefore, x is indeed a solution and $\|x(t)\| = e^{t/2} \to \infty$ as $t \to \infty$.

Exercise 2.25

The identity (2.49) clearly holds for $k = 1$. Assume that (2.49) holds for some $k \in \mathbb{N}$. Then

$$\begin{aligned} X^{k+1} - Y^{k+1} &= (X+Y)(X^k - Y^k) + XY^k - YX^k = X(X^k - Y^k) + (X-Y)Y^k \\ &= (X-Y)Y^k + \sum_{j=1}^{k} X^{k+1-j}(X-Y)Y^{j-1} = \sum_{j=1}^{k+1} X^{k+1-j}(X-Y)Y^{j-1} \end{aligned}$$

and so the identity holds for $k+1$. The result follows by induction.

Chapter 3

Exercise 3.1
(a) Let $T > 0$ and $\xi_1, \xi_2 \in \mathbb{R}$ be arbitrary. With input u of the form $t \mapsto \alpha + \beta t$, with parameters $\alpha, \beta \in \mathbb{R}$, the solution of the initial-value problem is given by

$$x_1(t) = \xi_1 + \xi_2 t + \alpha t^2/2 + \beta t^3/6, \quad x_2(t) = \xi_2 + \alpha t + \beta t^2/2.$$

Imposing the requisite soft-landing conditions, $x_1(T) = 0 = x_2(T)$, gives a pair of simultaneous equations for the parameters α and β:

$$\xi_1 + \xi_2 T + \alpha T^2/2 + \beta T^3/6 = 0 = \xi_2 + \alpha T + \beta T^2/2$$

which may be expressed in the equivalent form

$$\begin{pmatrix} 2T & T^2 \\ 3T & T^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -6 & -6T \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

and which has unique solution given by

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\frac{1}{T^2} \begin{pmatrix} 6T & 6T^2 - 2T \\ -12 & 6 - 12T \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

(b) Let $\xi > 0$ be arbitrary. Under the control u,

$$t \mapsto \left\{ \begin{array}{ll} -g, & 0 \le t \le S \\ \alpha, & S \le t \le T \end{array} \right. ,$$

patameterized by $T > 0$ and $S \in (0, T)$, we find

$$x_2(S) = -gS, \;\; x_1(S) = \xi - gS^2/2$$

and

$$x_2(T) = -g(S) + \alpha(T - S), \;\; x_1(T) = \xi_1 - gS^2/2 - gS(T - S) + \alpha(T_S)^2/2.$$

Imposing the soft-landing condition $x_1(T) = 0 = x_2(T)$, yields the unique solution

$$S = \sqrt{\frac{2\alpha\xi_1}{g(g+\alpha)}}, \;\; T = S + \sqrt{\frac{2g\xi_1}{\alpha(g+\alpha)}}$$

Exercise 3.2
(a) Noting that

$$(B, AB) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -2\omega & 0 \end{pmatrix}$$

has non-zero determinant and, since $R = \operatorname{im} \mathcal{C}(A, B) \supset \operatorname{im}(B, AB)$, we may conclude that $R = \mathbb{R}^4$, that is, all states are reachable from 0.

(b) In this case,

$$\mathcal{C}(A, B_1) = (B_1, AB_1, A^2B_1, A^3B_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2\omega & 0 \\ 0 & -2\omega & 0 & 0 \end{pmatrix}$$

and so the set of states reachable from 0 is a three-dimensional subspace:

$$R = \operatorname{span}\left\{ \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\-2\omega \end{pmatrix}, \begin{pmatrix} 0\\0\\2\omega\\0 \end{pmatrix} \right\}.$$

Exercise 3.3
For notational convenience, write $\alpha := -(M+m)g/Ml$ and $\beta = mg/M$. The reachability matrix is

$$\mathcal{C}(A,B) := (B, AB, A^2B, A^3B) = \frac{1}{Ml}\begin{pmatrix} 0 & -1 & 0 & \alpha\\ -1 & 0 & \alpha & 0\\ 0 & l & 0 & \beta\\ l & 0 & \beta & 0 \end{pmatrix}$$

with determinant $(\alpha l + \beta)^2/(Ml)^4 = 1/(Ml)^4 > 0$. Therefore, $\operatorname{rk}\mathcal{C}(A,B) = 4 = N$ and so the system is controllable.

Exercise 3.4
By Proposition 3.8, $\operatorname{im}\mathcal{C}(A,B)$ is A-invariant. It immediately follows that $\operatorname{im}\mathcal{C}(A,B)$ is A^k-invariant for all $k \in \mathbb{N}$. Let $v \in \operatorname{im}\mathcal{C}(A,B)$ and $t \in \mathbb{R}$. Then $(t^k/k!)A^kv \in \operatorname{im}\mathcal{C}(A,B)$ for all $k \in \mathbb{N}$. Since $\operatorname{im}\mathcal{C}(A,B)$ is a subspace of $\mathbb{R}^N$, $\operatorname{im}\mathcal{C}(A,B)$ is closed and so $\exp(At)v = \sum_{k=0}^{\infty}(t^k/k!)A^kv$ is in $\operatorname{im}\mathcal{C}(A,B)$.

Exercise 3.5
Let $\xi \in \operatorname{im}\mathcal{C}(A,B)$ be arbitrary. By e^{AT}-invariance of the subspace $\operatorname{im}\mathcal{C}(A,B)$ (Exercise 3.4), $-\exp(AT)\xi \in \operatorname{im}\mathcal{C}(A,B)$ and so there exists $u \in PC([0,T],\mathbb{R}^M)$ such that

$$-\exp(AT)\xi = x(T;0,u) = \int_0^T \exp(A(T-t))Bu(t)\mathrm{d}t,$$

Therefore $0 = \exp(AT)\xi + \int_0^T \exp(A(T-t))Bu(t)\mathrm{d}t = x(T;\xi,u)$ and so $\xi \in D_T$. Since $\xi \in \operatorname{im}\mathcal{C}(A,B)$ is arbitrary, it follows that $\operatorname{im}\mathcal{C}(A,B) \subset D_T$.
Now let $\xi \in D_T$ be arbitrary. Then there exists $u \in PC([0,T],\mathbb{R}^M)$ such that

$$0 = x(T;\xi,u) = \exp(AT)\xi + \int_0^T \exp(A(T-t))Bu(t)\mathrm{d}t$$

and so $-\exp(AT)\xi = x(T;0,u)$. Therefore, $-\exp(AT)\xi \in \operatorname{im}\mathcal{C}(A,B)$ and, by e^{AT}-invariance of the latter, we have $\xi \in \operatorname{im}\mathcal{C}(A,B)$. Since $\xi \in D_T$ is arbitrary, it follows that $D_T \subset \operatorname{im}\mathcal{C}(A,B)$. We may now conclude that $D_T = \operatorname{im}\mathcal{C}(A,B)$.

Exercise 3.6
Assume (A,B) is controllable. Let $\tilde{\xi} \in \mathbb{R}^N$ be arbitrary and set $\xi := \S\tilde{\xi}$. By controllability of (A,B), there exists $u \in PC([0,T],\mathbb{R}^N)$ such that $0 = \exp(AT)\xi + \int_0^T \exp(A(T-t))B(u(t))\mathrm{d}t$. Left multiplication by S^{-1} gives

$$\begin{aligned} 0 &= S^{-1}\exp(AT)S\tilde{\xi} + \int_0^T S^{-1}\exp(A(T-t)SS^{-1}Bu(t)\mathrm{d}t \\ &= \exp(\tilde{A}T)\tilde{\xi} + \int_0^T \exp(\tilde{A}(T-t))\tilde{B}u(t)\mathrm{d}t. \end{aligned}$$

Therefore, $(\tilde{A}, \tilde{B})$ is controllable. An analogous argument establishes that, if $(\tilde{A}, \tilde{B})$ is controllable, then (A, B) is controllable (alternatively, simply note that this fact is subsumed by what we have just proved).

Exercise 3.7
Recall that $\tilde{A} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ and $\tilde{B} = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$. Hence, $\tilde{A}^k \tilde{B} = \begin{pmatrix} A_1^k B_1 \\ 0 \end{pmatrix}$ for all $k \in \mathbb{N}$. By the Cayley-Hamilton theorem,

$$\operatorname{rk} \mathcal{C}(A_1, B_1) = \operatorname{rk}(B_1, A_1 B_1, \dots, A_1^{K-1} B_1) = \operatorname{rk}(B_1, A_1 B_1, \dots, A_1^{N-1} B_1),$$

and thus,

$$\begin{aligned}
\operatorname{rk} \mathcal{C}(A_1, B_1) &= \operatorname{rk}\left(\begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \begin{pmatrix} A_1 B_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} A_1^{N-1} B_1 \\ 0 \end{pmatrix} \right) \\
&= \operatorname{rk}(\tilde{B}, \tilde{A}\tilde{B}, \dots, \tilde{A}^{N-1}\tilde{B}) = \operatorname{rk}(S^{-1}(B, AB, \dots, A^{N-1}B)) \\
&= \operatorname{rk}(B, AB, \dots, A^{N-1}B) = \operatorname{rk} \mathcal{C}(A, B) = K\,.
\end{aligned}$$

Therefore, by Theorem 3.6, (A_1, B_1) is controllable.

Exercise 3.8
For this system, we have

$$\big(sI - A, b\big) = \begin{pmatrix} s & -1 & 0 & 0 & 0 & 0 \\ 0 & s & -1 & 0 & 0 & -1 \\ 0 & 0 & s & 0 & -1 & 0 \\ -\alpha & 0 & 0 & s & -1 & \beta \\ 0 & 0 & 0 & 0 & s & 1 \end{pmatrix}.$$

For $s \neq 0$, it is straightforward to verify that columns 1,2,3,4 and 6 are linearly independent for all $\alpha, \beta \in \mathbb{R}$. For $s = 0$, columns 1,2,3,5 and 6 are linearly independent if, and only if, $\alpha \neq 0$. Therefore, $\operatorname{rk}(sI - A, B) = 5$ if, and only if, $\alpha \neq 0$ and so, by the Hautus criterion, (A, B) is controllable for all pairs (α, β) with $\alpha \neq 0$, and (A, B) is not controllable for all pairs $(0, \beta)$.

Exercise 3.9
The matrix $\mathcal{O}(C_1, A)$ comprises rows 1,3,5,7 of $\mathcal{O}(C, A)$ (as given in Exercise 3.19), that is,

$$\mathcal{O}(C_1, A) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & -\omega^2 & 0 & 0 \end{pmatrix}.$$

In this case and noting that the third column is zero, $\det \mathcal{O}(C_1, A) = 0$ and so the system fails to be observable.
The matrix $\mathcal{O}(C_2, A)$ comprises rows 2,4,6,8 of $\mathcal{O}(C, A)$, that is,

$$\mathcal{O}(C_2, A) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \\ -6\omega^3 & 0 & 0 & -4\omega^2 \end{pmatrix}.$$

In this case, $\det \mathcal{O}(C_2, A) = -12\omega^4 \neq 0$ and so the system is observable.

Exercise 3.10

$$\begin{pmatrix} sI - A \\ C \end{pmatrix} = \begin{pmatrix} s & -1 & 0 & 0 \\ -3\omega^2 & s & 0 & -2\omega \\ 0 & 0 & s & -1 \\ 0 & 2\omega & 0 & s \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Rows 1, 3, 4 and 5 are linearly independent for all $s \in \mathbb{C}$. Therefore, the system is observable.
Assume that only the radial measurement y_1 is available, in which case C is replaced by its first row $C_1 = (1\ 0\ 0\ 0)$. Then we have

$$\begin{pmatrix} sI - A \\ C_1 \end{pmatrix} = \begin{pmatrix} s & -1 & 0 & 0 \\ -3\omega^2 & s & 0 & -2\omega \\ 0 & 0 & s & -1 \\ 0 & 2\omega & 0 & s \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Noting column 3, it is clear that this matrix fails to have full rank for $s = 0$. Therefore, the system with radial measurement only is not observable.
Now, assume that only the angular measurement y_2 is available, in which case C is replaced by its second row $C_2 = (0\ 0\ 1\ 0)$. Then we have

$$\begin{pmatrix} sI - A \\ C_2 \end{pmatrix} = \begin{pmatrix} s & -1 & 0 & 0 \\ -3\omega^2 & s & 0 & -2\omega \\ 0 & 0 & s & -1 \\ 0 & 2\omega & 0 & s \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

If $s \neq 0$, then it is readily verified that rows 1, 3, 4 and 5 are linearly independent, whilst, if $s = 0$, then rows 1, 2, 3 and 5 are linearly independent. Therefore, the system with angular measurement only is observable.

Exercise 3.11
Noting that $\big(\mathcal{O}(C, A)\big)^* = \mathcal{C}(A^*, C^*)$ and applying the Kalman controllability decomposition lemma (Lemma 3.10) to the pair (A^*, C^*), we may infer the existence of $T \in GL(N, \mathbb{R})$ such that

$$T^{-1}A^*T = \begin{pmatrix} M_1 & M_2 \\ 0 & M_3 \end{pmatrix}, \quad T^{-1}C^* = \begin{pmatrix} M_4 \\ 0 \end{pmatrix}$$

with $M_1 \in \mathbb{R}^{K\times K}$, $M_4 \in \mathbb{R}^{K\times P}$ and (M_1, M_4) a controllable pair.
Writing $S := (T^*)^{-1}$, $A_1 := M_1^*$, $A_2 := M_2^*$, $A_3 := M_3^*$ and $C_1 := M_4^*$, we have

$$S^{-1}AS = \tilde{A} = \big(T^{-1}A^*T\big)^* = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix}, \quad CS = \big(T^{-1}C^*\big)^* = \big(C_1, 0\big)$$

and, by controllability of (M_1, M_4), we have observability of $(M_4^*, M_1^*) = (C_1, A_1)$.

Exercise 3.12
(a) By the Kalman controllability decomposition lemma (Lemma 3.10), there exists $S \in GL(N, \mathbb{R})$ such that the matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ have the requisite structure and the pair (A_1, B_1) is controllable. It remains to show that (C_1, A_1) is observable. Suppose

otherwise. Then there exists $v \neq 0$ such that $C_1 A_1^k v = 0$ for all $k \in \mathbb{N}_0$. A straightforward computation shows that $\tilde{C}\tilde{A}^k$ has the structure $\tilde{C}\tilde{A}^k = \begin{pmatrix} C_1 A_1^k, & * \end{pmatrix}$. Writing $\tilde{v} := \begin{pmatrix} v \\ 0 \end{pmatrix} \neq 0$, we have

$$\tilde{C}\tilde{A}^k \tilde{v} = C_1 A_1^k v = 0 \quad \forall\, k \in \mathbb{N}_0$$

which contradicts observability of the pair $(\tilde{C}, \tilde{A})$.

(b) By the Kalman observability decomposition lemma (Lemma 3.22), there exists $S \in GL(N, \mathbb{R})$ such that the matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ have the requisite structure and the pair (C_1, A_1) is observable. It remains to show that (A_1, B_1) is controllable. Suppose otherwise. Then there exists $v \neq 0$ such that $v^* B_1 A_1^k = 0$ for all $k \in \mathbb{N}_0$. A straightforward computation shows that $\tilde{A}^k \tilde{B}$ has the structure $\tilde{A}^k \tilde{B} = \begin{pmatrix} A_1^k B_1 \\ * \end{pmatrix}$. Writing $\tilde{v} := \begin{pmatrix} v \\ 0 \end{pmatrix} \neq 0$, we have

$$\tilde{v}^* \tilde{A}^k \tilde{B} = v^* A_1^k B_1 = 0 \quad \forall\, k \in \mathbb{N}_0$$

which contradicts controllability of the pair $(\tilde{A}, \tilde{B})$.

Exercise 3.13

We prove the theorem using contraposition. To this end, assume that $\operatorname{rk} \begin{pmatrix} \lambda I - A \\ c \end{pmatrix} < N$ for some $\lambda \in \mathbb{C}$ (an eigenvalue of A). Then there exists $z \in \mathbb{C}^N$, $z \neq 0$ such that $\begin{pmatrix} \lambda I - A, \\ c \end{pmatrix} z = 0$. Thus, $Az = \lambda z$ and $Cz = 0$. As a consequence,

$$A^k C z = \lambda^k C z = 0 \quad \forall\, k \in \mathbb{N}_0\,,$$

implying that $\mathcal{O}(C, A) z = 0$ Since $z \neq 0$, this shows that $\operatorname{rk} \mathcal{C}(A, B) < N$. Hence, by the rank condition for observability (Theorem 3.18), the pair (C, A) is not observable.

Conversely, assume that the pair (C, A) is not observable. If $C = 0$, then $\operatorname{rk} \begin{pmatrix} sI - A \\ C \end{pmatrix} = \operatorname{rk}(sI - A) < N$ for all $s \in \sigma(A)$. If $C \neq 0$, then it follows by Kalman observability decomposition (Lemma 3.22) that there exists $S \in GL(N, \mathbb{R})$ such that

$$\tilde{A} := S^{-1} A S = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix}, \quad \tilde{C} := CS = \begin{pmatrix} C_1\,, \, 0 \end{pmatrix}.$$

where $A_1 \in \mathbb{R}^{K \times K}$, $C_1 \in \mathbb{R}^{P \times K}$ and $K < N$. Let $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^{N-K}$ be an eigenvalue/eigenvector pair of A_3. Then

$$v \neq 0\,, \quad (\lambda I - A_3) v = 0.$$

Setting

$$w := \begin{pmatrix} 0 \\ v \end{pmatrix} \in \mathbb{C}^N,$$

it follows that

$$(\lambda I - \tilde{A}) w = \begin{pmatrix} \lambda I - A_1 & -A_2 \\ 0 & \bar{\lambda} I - A_3 \end{pmatrix} w = 0\,, \quad \tilde{C} w = \begin{pmatrix} C_1\,, \, 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = 0.$$

Hence, $z = Sw \neq 0$ satisfies

$$S^{-1}(\lambda I - A)z = (\lambda I - \tilde{A})w = 0\,, \quad Cz = \tilde{C}w = 0\,,$$

implying that

$$(\lambda I - A)z = 0\,, \quad Cz = 0\,.$$

Consequently, $\begin{pmatrix} \lambda I - A \\ B \end{pmatrix} z = 0$ and hence, $\mathrm{rk} \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} < N$.

Exercise 3.14
Let $z \in \mathcal{O}(C, A)$ be arbitrary. Then

$$0 = Cz = CAz = \cdots = CA^{N-1}z.$$

By the Cayley-Hamilton theorem, we also have $CA^N z = 0$. Therefore $z \in \ker C$ and $Az \in \ker \mathcal{O}(C, A)$. Since $z \in \ker(C, A)$ is arbitrary, it follows that $\mathcal{O}(C, A)$ is contained in $\ker C$ and is A-invariant. Finally, let $\mathcal{S} \subset \mathbb{R}^N$ be an A-invariant subspace contained in $\ker C$ and let $z \in \mathcal{S}$ be arbitrary. By A-invariance of $\mathcal{S}$, we have $Az, \ldots, A^{N-1}z \in \mathcal{S}$ and, since $\mathcal{S} \subset \ker C$, it follows that $0 = Cz = CAz = \cdots = CA^{N-1}z$. Therefore, $z \in \ker \mathcal{O}(C, A)$ and, since $z \in \mathcal{S}$ is arbitrary, we may conclude that $\mathcal{S} \subset \ker \mathcal{O}(C, A)$.

Exercise 3.15
Obviously, the transfer function $\hat{G}_K$ is given by $\hat{G}_K(s) = C(sI - (A - BKC))^{-1}B$. Now $sI-(A-BKC) = (sI-A)(I+(sI-A)^{-1}BKC) = (I+BKC(sI-A)^{-1})(sI-A)$, and so

$$\begin{aligned} \hat{G}_K(s) &= C(sI - A)^{-1}\big(I + BKC(sI - A)^{-1}\big)^{-1}B \\ &= C\big(I + (sI - A)^{-1}BKC\big)^{-1}(sI - A)^{-1}B. \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{G}_K(s)(I + K\hat{G}(s)) &= C(sI - A)^{-1}\big(I + BKC(sI - A)^{-1}\big)^{-1}B\big(I + KC(sI - A)^{-1}B\big) \\ &= C(sI - A)^{-1}\big(I + BKC(sI - A)^{-1}\big)^{-1}\big(I + BKC(sI - A)^{-1}\big)B \\ &= C(sI - A)^{-1}B = \hat{G}(s) \end{aligned}$$

and

$$\begin{aligned} (I + \hat{G}(s)K)\hat{G}_K(s) &= \big(I + C(sI - A)^{-1}BK\big)C\big(I + (sI - A)^{-1}BKC\big)^{-1}(sI - A)^{-1}B \\ &= C\big(I + (sI - A)^{-1}BKC\big)\big(I + (sI - A)^{-1}BKC\big)^{-1}(sI - A)^{-1}B \\ &= C(sI - A)^{-1}B = \hat{G}(s). \end{aligned}$$

Exercise 3.16
In this case, $\omega = 1$ and

$$\hat{g}(i\omega) = \hat{G}(i) = \frac{\alpha}{i + \beta} = \frac{\alpha(\beta - i)}{1 + \beta^2}.$$

Invoking Proposition 3.27, we see that $-\pi/4$ is the argument of $\hat{g}(i)$ in $[0, 2\pi)$ and so $\beta = 1$. Furthermore,

$$\sqrt{2} = |\hat{g}(i)| = \frac{\alpha}{\sqrt{1 + \beta^2}} = \frac{\alpha}{\sqrt{2}}$$

and so $\alpha = 2$.

Exercise 3.17
(a) Since $R(s)$ is not identically equal to the zero matrix, it follows that $B \neq 0$ and $C \neq 0$.
First consider the case that (A, B) is controllable and (C, A) is observable. Then there is nothing to show: the claim follows with $T = I$. (To identify the triple (A, B, C) with the block structure given in Exercise 3.17, in the latter simply disregard the last two block rows and the last two block columns in $\tilde{A}$ and the last two blocks in $\tilde{B}$ and $\tilde{C}$.)
Now consider the case wherein (C, A) is observable and (A, B) is not controllable. By the result in part (a) of Exercise 3.12, there exists $T \in GL(N, \mathbb{R})$ such that

$$T^{-1}AT = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad T^{-1}B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \quad CT = (C_1, C_2),$$

with (A_1, B_1) controllable and (C_1, A_1) observable, proving the claim in this case.(To identify the above structure with the block structure given in Exercise 3.17, in the latter simply disregard the third block row and third block column in $\tilde{A}$ and the third blocks in $\tilde{B}$ and $\tilde{C}$.)
Finally, consider the case wherein (C, A) is not observable. By the observability decomposition lemma (Lemma 3.22), there exists $S \in GL(N, \mathbb{R})$ such that

$$S^{-1}AS = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix}, \quad S^{-1}B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad CS = (C_1, 0),$$

with (C_1, A_1) observable. If the pair (A_1, B_1) is controllable, then the claim follows with $T = S$. If (A_1, B_1) is not controllable, then, by the result in part (a) of Exercise 3.12 applied in the context of the triple (A_1, B_1, C_1), there exists an invertible matrix S_1 such that

$$S_1^{-1}A_1S_1 = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad S_1^{-1}B_1 = \begin{pmatrix} B_{11} \\ 0 \end{pmatrix}, \quad C_1S_1 = (C_{11}, C_{12})$$

where (A_{11}, B_{11}) is controllable and (C_{11}, A_{11}) is observable. Defining

$$T := S\tilde{S}, \quad \text{where } \tilde{S} := \begin{pmatrix} S_1 & 0 \\ 0 & I \end{pmatrix},$$

and setting $(A_{31}, A_{32}) = A_2S_1$, $A_{33} = A_3$ and $B_{31} = B_2$, we have

$$T^{-1}AT = \begin{pmatrix} S_1^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ 0 & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

and

$$T^{-1}B = \begin{pmatrix} S_1^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} B_{11} \\ 0 \\ B_{31} \end{pmatrix}, \quad CT = (C_1, 0) \begin{pmatrix} S_1 & 0 \\ 0 & I \end{pmatrix} = \big(C_{11}, C_{12}, 0\big)$$

with (A_{11}, B_{11}) controllable and (C_{11}, A_{11}) observable.

(b) A straightforward calculation reveals that

$$CA^kB = (CT)(T^{-1}A^kT)(T^{-1}B) = C_{11}A_{11}^kB_{11} \quad \forall\, k \in \mathbb{N}_0.$$

Therefore,

$$C\exp(At)B = C_{11}\exp(A_{11}t)B_{11} \quad \forall\, t \in \mathbb{R},$$

and applying Laplace transform gives

$$R(s) = C(sI - A)^{-1}B = C_{11}(sI - A_{11})^{-1}B_{11}.$$

Therefore, (A_{11}, B_{11}, C_{11}) is a realization of R.

Exercise 3.18
If (A, B) is not controllable, then there exists $z \in \mathbb{R}^N$ such that $z \neq 0$ and $z^*\mathcal{C}(A, B) = 0$. Let $S^{-1} \in GL(N, \mathbb{R})$ be such that z^* is the N-th row of S^{-1}. Then

$$S^{-1}B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \quad \text{where } B_1 \in \mathbb{R}^{(N-1)\times M}.$$

Partition the matrices $S^{-1}AS$ and CS accordingly, that is,

$$S^{-1}AS = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad CS = \begin{pmatrix} C_1 & C_2 \end{pmatrix},$$

where $A_1 \in \mathbb{R}^{(N-1)\times(N-1)}$ and $C_1 \in \mathbb{R}^{P\times(N-1)}$. Since $z^*A^kB = 0$ for all $k \in \mathbb{N}_0$, it follows that the last row of $S^{-1}A^kB$ is equal to zero for all $k \in \mathbb{N}_0$. Combining this with a routine calculation then shows that

$$(S^{-1}AS)^k(S^{-1}B) = S^{-1}A^kB = \begin{pmatrix} A_1^kB_1 \\ 0 \end{pmatrix} \quad \forall\, k \in \mathbb{N}_0,$$

and $CA^kB = (CS)(S^{-1}AS)^k(S^{-1}B) = C_1A_1^kB_1$ for all $k \in \mathbb{N}_0$. This in turn leads to

$$Ce^{At}B = \sum_{k=0}^{\infty} \frac{t^k}{k!}CA^kB = \sum_{k=0}^{\infty} \frac{t^k}{k!}C_1A_1^kB_1 \quad \forall\, t \in \mathbb{R}.$$

Applying Laplace transform yields,

$$R(s) = C(sI - A)^{-1}B = C_1(sI - A_1)^{-1}B_1.$$

Thus (A_1, B_1, C_1) is a realization of R. The dimension of this realization is $N - 1$, showing that the realization (A, B, C) is not minimal.

Exercise 3.19
(a) The claim follows immediately from the relations

$$\dot{x}_1 = A_1x_1 + B_1C_2x_2, \quad \dot{x}_2 = A_2x_2 + B_2u, \quad y = C_1x_1.$$

(b) Note that the inverse of

$$sI - A = \begin{pmatrix} sI - A_1 & -B_1C_2 \\ 0 & sI - A_2 \end{pmatrix}$$

is given by

$$(sI - A)^{-1} = \begin{pmatrix} (sI - A_1)^{-1} & (sI - A_1)^{-1}B_1C_2(sI - A_2)^{-1} \\ 0 & (sI - A_2)^{-1} \end{pmatrix}.$$

Therefore,

$$\begin{aligned}\hat{G}(s) &= C(sI-A)^{-1}B\\ &= (C_1,0)\begin{pmatrix}(sI-A_1)^{-1} & (sI-A_1)^{-1}B_1C_2(sI-A_2)^{-1}\\ 0 & (sI-A_2)^{-1}\end{pmatrix}\begin{pmatrix}0\\ B_2\end{pmatrix}\\ &= C_1(sI-A_1)^{-1}B_1C_2(sI-A_2)^{-1}B_2 = \hat{G}_1(s)\hat{G}_2(s).\end{aligned}$$

(c) For $j=1,2$, write $\hat{G}_j = n_j/d_j$, where n_j and d_j are coprime polynomials. It follows from Proposition 3.29 and Theorem 3.30 that the degree of d_j is equal to N_j. Moreover, note that the dimension of the realization (A,B,C) of $\hat{G}_1\hat{G}_2$ is equal to N_1+N_2 and $\hat{G}_1\hat{G}_2 = n_1n_2/(d_1d_2)$.
If the realization (A,B,C) is minimal, then, by Proposition 3.29, n_1 and d_2 are coprime and, furthermore, n_2 and d_1 are coprime, or, equivalently, there is no pole/zero cancellation in the product $\hat{G}_1\hat{G}_2$.
Conversely, assume that there is no pole/zero cancellation in the product $\hat{G}_1\hat{G}_2$. Then, the polynomials n_1n_2 and d_1d_2 are coprime. Since the degree of d_1d_2 is equal to N_1+N_2, another application of Proposition 3.29 shows that the realization (A,B,C) is minimal.

Chapter 4

Exercise 4.1
Let $a, b \in I_z$ be arbitrary and, without loss of generality, assume $a \leq b$. To conclude that I_z is an interval it suffices to show that $[a, b] \subset I_z$. Since $I_z := \cup_{y \in \mathcal{T}} I_y$, there exist $y_a, y_b \in \mathcal{T}$ such that $a \in I_{y_a}$ and $b \in I_{y_b}$. Since $\mathcal{T}$ is totally ordered, either $y_a \preceq y_b$ or $y_b \preceq y_a$. In the former case, $I_{y_a} \subset I_{y_b}$ and so $[a, b] \subset I_{y_b} \subset I_z$. In the latter case, $I_{y_b} \subset I_{y_a}$ and so $[a, b] \subset I_{y_a} \subset I_z$.
We proceed to show that z is well defined. Let $t \in I_z$ be arbitrary. Then $t \in I_y$ for some $y \in \mathcal{T}$. Define $v := y(t)$. Assume $\hat{y} \in \mathcal{T}$ is such that $t \in I_{\hat{y}}$ and define $\hat{v} := \hat{y}(t)$. Since $\mathcal{T}$ is totally ordered, either $y \preceq \hat{y}$ or $\hat{y} \preceq y$. In each case, $y(t) = \hat{y}(t)$. Therefore, with each $t \in I_z$, we may associate a unique element $z(t)$ of G given by $z(t) = y(t)$, where y is any element of $\mathcal{T}$ such that $t \in I_y$. The function $z\colon I_z \to G$, so defined, has the property

$$z|_{I_y} = y \ \forall\, y \in \mathcal{T}$$

and is the only function with that property.

Exercise 4.2
For $\xi \neq 0$, separation of variables yields

$$\int_\xi^x \frac{ds}{s^2} = \int_\tau^t ds \quad \Longrightarrow \quad \left[-\frac{1}{s}\right]_\xi^x = t - \tau \quad \Longrightarrow \quad \frac{1}{x} = \frac{1}{\xi} + \tau - t\,.$$

(i) For $(\tau, \xi) = (0, 1)$, we obtain

$$x(t) = \frac{1}{1-t}\,,$$

with maximal interval of existence $(-\infty, 1)$.
(iii) For $(\tau, \xi) = (1, 1)$, we obtain

$$x(t) = \frac{1}{2-t}\,,$$

with maximal interval of existence $(-\infty, 2)$.
(ii) Clearly, in this case, $\mathbb{R} \to \mathbb{R}, t \mapsto 0$ is a maximal solution, with maximal interval of existence equal to $\mathbb{R}$.

Exercise 4.3
For $\xi \neq 0$, separation of variables gives

$$\int_\xi^{x(t)} \frac{\mathrm{d}s}{s^2} = \int_\tau^t s^3 \mathrm{d}s \quad \Longrightarrow \quad \left[-\frac{1}{s}\right]_\xi^{x(t)} = \left[\frac{1}{4}s^4\right]_\tau^t \quad \Longrightarrow \quad \frac{1}{x(t)} - \frac{1}{\xi} = \frac{1}{4}(\tau^4 - t^4)\,.$$

Consequently,

$$\frac{1}{x(t)} = \frac{4 + \xi(\tau^4 - t^4)}{4\xi} \quad \Longrightarrow \quad x(t) = \frac{4\xi}{4 + \xi(\tau^4 - t^4)}\,.$$

(a) For $(\tau, \xi) \in \mathbb{R} \times (0, \infty)$, the maximal interval of existence is bounded and is given by

$$\big(-(\tau^4 + 4/\xi)^{1/4}, (\tau^4 + 4/\xi)^{1/4}\big).$$

(b) For (τ, ξ) such that $\xi \in (-4/\tau^4, 0)$, the maximal interval of existence is $\mathbb{R}$.

Exercise 4.4
Seeking a contradiction, suppose that $f(x^\infty) \neq 0$. Setting $\lambda := f(x^\infty)$, it follows that λ has at least one component, λ_j say, which is not equal to zero: $\lambda_j \neq 0$. Since

$$\lim_{t\to\infty} \dot{x}(t) = \lim_{t\to\infty} f(x(t)) = \lambda\,,$$

we have $\lim_{t\to\infty} \dot{x}_j(t) = \lambda_j \neq 0$. Hence

$$\lim_{t\to\infty} x_j(t) = \begin{cases} \infty\,, & \text{if } \lambda_j > 0 \\ -\infty\,, & \text{if } \lambda_j < 0\,, \end{cases}$$

contradicting the assumption that $\lim_{t\to\infty} x(t) = x^\infty$.

Exercise 4.5
Let $I \subset J$ be an interval with $\tau \in I$ and let $x\colon I \to G$ be a solution of the non-autonomous initial-value problem, that is,

$$\dot{x}(t) = f(t, x(t)) \;\;\forall\, t \in I, \quad x(\tau) = \xi\,.$$

Set $I^- := I - \tau = \{t - \tau\colon t \in I\}$ and define $y\colon I^- \to I \times G \subset \mathbb{R}^{N+1}$ by

$$y(t) := (t + \tau, x(t + \tau))\,.$$

Note that $0 \in I^-$, since $\tau \in I$. Differentiation of y gives

$$\dot{y}(t) = (1, \dot{x}(t+\tau)) = (1, f(t+\tau, x(t+\tau))) = g(t+\tau, x(t+\tau))) = g(y(t))\,.$$

Moreover, $y(0) = (\tau, x(\tau)) = (\tau, \xi)$. We conclude that that y satisfies the autonomous initial-value problem, that is,

$$\dot{y}(t) = g(y(t)) \;\;\forall\, t \in I^-, \quad y(0) = (\tau, \xi).$$

Conversely, let $I \subset \mathbb{R}$ be an interval with $0 \in I$ and let $y\colon I \to J \times G$ be a solution of the autonomous initial-value problem, that is,

$$\dot{y}(t) = g(y(t)) \;\;\forall\, t \in I, \quad y(0) = (\tau, \xi)\,.$$

Writing $y(t) = (y_1(t), y_2(t)) \in \mathbb{R} \times \mathbb{R}^N$, we have that

$$\dot{y}_1(t) = 1 \;\;\forall\, t \in I, \quad y_1(0) = \tau\,.$$

Hence, $y_1(t) = t + \tau$ for all $t \in I$. Set $I^+ := I + \tau = \{t + \tau : t \in I\} \subset J$ and define $x\colon I^+ \to G \subset \mathbb{R}^N$ by $x(t) := y_2(t - \tau)$. Then, for all $t \in I^+$,

$$\dot{x}(t) = \dot{y}_2(t - \tau) = f(y_1(t-\tau), y_2(t-\tau)) = f(t, x(t))\,.$$

Finally, $x(\tau) = y_2(0) = \xi$. Thus, we may conclude that x solves the non-autonomous initial-value problem.

Exercise 4.6
Observe that, if $\{\inf I, \sup I\} \cap (J\backslash I) \neq \emptyset$, then $J\backslash I \neq \emptyset$ and so $I \neq J$. Conversely, assume $I \neq J$. Then, $J\backslash I \neq \emptyset$ and so there exists $\gamma \in J$ with $\gamma \neq I$. Write $\alpha := \inf I$ and $\omega := \sup I$. Then, either (i) $\gamma \geq \omega$ or (ii) $\gamma \leq \alpha$. If (i) holds with $\gamma = \omega$, then $\omega \notin I$ and so $\omega \in J\backslash I$. If (i) holds with $\gamma > \omega$, then, since I is relatively open in J, we again have $\omega \notin I$ and so $\omega \in J\backslash I$. If (ii) holds, then analogous reasoning shows that $\alpha \in J\backslash I$. Therefore, $\{\alpha, \omega\} \cap (J\backslash I) \neq \emptyset$.

Exercise 4.7
Let $x, y \in \mathbb{R}^N$. Let $\varepsilon > 0$ be arbitrary. Then there exists $v \in V$ such that $\mathrm{dist}(y, V) \geq \|y - v\| - \varepsilon$. Therefore,

$$\mathrm{dist}(x, V) \leq \|x - v\| \leq \|x - y\| + \|y - v\| \leq \|x - y\| + \mathrm{dist}(y, V) + \varepsilon$$

and so, since $\varepsilon > 0$ is arbitrary, $\mathrm{dist}(x, V) - \mathrm{dist}(y, V) \leq \|x - y\|$. Repeating this argument, with the roles of x and y interchanged, yields the second requisite inequality

$$\mathrm{dist}(y, V) - \mathrm{dist}(x, V) \leq \|x - y\|.$$

Exercise 4.8
(a) Let $\tau \in I$. Then, by the variation of parameters formula,

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_\tau^t e^{A(t-s)}b(s, x(s))\mathrm{d}s\,, \quad \forall\, t \in I\,.$$

Let $\alpha, \beta \in \mathbb{R}$ be such that $\tau \in (\alpha, \beta) \subset I$ (since $\tau \in I$ and I, as a maximal interval of existence, is open, such α and β exist). Then, setting

$$K := \max\{\|e^{A\sigma}\| : \alpha - \beta \leq \sigma \leq \beta - \alpha\} < \infty\,,$$

we obtain

$$\begin{aligned} \|x(t)\| &\leq \|e^{A(t-\tau)}\|\|x(\tau)\| + \left|\int_\tau^t \|e^{A(t-s)}\|\|b(s, x(s))\|\mathrm{d}s\right| \\ &\leq K\|x(\tau)\| + \left|\int_\tau^t K\gamma(s)\|x(s)\|\mathrm{d}s\right|\,, \quad \forall\, t \in (\alpha, \beta)\,. \end{aligned}$$

Setting $c := K\|x(\tau)\|$, an application of Gronwall's lemma yields

$$\|x(t)\| \leq c\exp\left(K\left|\int_\tau^t \gamma(s)\mathrm{d}s\right|\right) \leq c\exp\left(K\int_\alpha^\beta \gamma(s)\mathrm{d}s\right) < \infty\,, \quad \forall\, t \in (\alpha, \beta)\,.$$

Setting $\alpha^* := \inf I$, $\beta^* := \sup I$, it follows that $\alpha^* = -\infty$ and $\beta^* = \infty$, because otherwise, if, for example, $\beta^* < \infty$, the above argument would apply with $\beta = \beta^*$ and so x would be bounded on (τ, β^*), which, by Theorem 4.11, is impossible.
(b) By the variation of parameters formula,

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}b(s, x(s))\mathrm{d}s\,, \quad \forall\, t \geq 0\,,$$

and so

$$\|x(t)\| \leq Me^{\mu t}\|x(0)\| + \int_0^t Me^{\mu(t-s)}\gamma(s)\|x(s)\|\mathrm{d}s \quad \forall\, t \geq 0.$$

Therefore,

$$\|x(t)\|e^{-\mu t} \leq M\|x(0)\| + \int_0^t M\gamma(s)\|x(s)\|e^{-\mu s}\mathrm{d}s \quad \forall\, t \geq 0\,.$$

By Gronwall's lemma,

$$\|x(t)\|e^{-\mu t} \leq M\|x(0)\|\exp\left(M\int_0^t \gamma(s)\mathrm{d}s\right)\,, \quad \forall\, t \geq 0\,,$$

and so,

$$\|x(t)\| \leq M\|x(0)\| \exp\left(\mu t + M\int_0^t \gamma(s)\mathrm{d}s\right), \quad \forall\, t \geq 0\,. \tag{$*$}$$

(c) If $\mu < 0$ and there exists $T > 0$ such that

$$\sup_{t\geq T}\left(\frac{1}{t}\int_0^t \gamma(s)\mathrm{d}s\right) < \frac{|\mu|}{M}\,, \tag{$**$}$$

then

$$\mu t + M\int_0^t \gamma(s)ds = t\left(\mu + M\frac{1}{t}\int_0^t \gamma(s)\mathrm{d}s\right) \to -\infty \quad \text{as} \quad t \to \infty\,,$$

and thus $x(t) \to 0$ as $t \to \infty$, by $(*)$. The existence of a number $T > 0$ such that $(**)$ holds, is guaranteed, for example, if the improper Riemann integral $\int_0^\infty \gamma(s)ds$ of γ converges or if $\sup_{t\geq t^*}\gamma(t) < |\mu|/M$ for some $t^* > 0$.

Exercise 4.9
Let $\varepsilon > 0$.
(a) For $z \in (0, \varepsilon)$,

$$\frac{|g(z) - g(0)|}{|z - 0|} = \frac{\sqrt{z}}{z} = \frac{1}{\sqrt{z}} \to \infty \quad \text{as} \quad z \downarrow 0\,.$$

It follows that the function g is not Lipschitz on $\mathbb{R}$.
(b) For $z \in (0, \varepsilon)$,

$$\frac{|g(z) - g(0)|}{|z - 0|} = \left|\frac{z\ln z}{z}\right| = |\ln z| \to \infty \quad \text{as} \quad z \downarrow 0\,.$$

It follows that g is not Lipschitz on $\mathbb{R}$.

Exercise 4.10
Let $z \in V$ and choose $\varepsilon > 0$ such that $U := \{w \in \mathbb{R}^Q : \|w - z\| \leq \varepsilon\} \subset V$. It follows from the continuity of the first order partial derivatives of g and compactness of U that

$$\gamma := \max_{1\leq i,\, j\leq N}\left(\sup_{w\in U} |\partial_i g_j(w)|\right) < \infty\,,$$

wherein $\partial_i g_j$ denotes the partial derivative of component j of g with respect to argument i.
Let $z_1, z_2 \in U$ and define $h_j : [0, 1] \to \mathbb{R}$ by

$$h_j(t) = g_j((1 - t)z_1 + tz_2)\,, \quad \forall\, t \in [0, 1]\,.$$

Note that

$$\dot{h}_j(t) = \langle(\nabla g_j)((1 - t)z_1 + tz_2), z_2 - z_1\rangle\,, \quad \forall\, t \in [0, 1]\,,$$

and so, by the Cauchy-Schwarz inequality,

$$|\dot{h}_j(t)| \leq \|(\nabla g_j)((1 - t)z_1 + tz_2)\|\|z_2 - z_1\| \leq \gamma\sqrt{Q}\|z_2 - z_1\|\,, \quad \forall t \in [0, 1]\,.$$

By the mean-value theorem of differentiation, there exists $\tau \in [0, 1]$ such that

$$|g_j(z_2) - g_j(z_1)| = |h_j(1) - h_j(0)| = |\dot{h}_j(\tau)|\,.$$

Hence, $|g_j(z_2) - g_j(z_1)| \leq \gamma\sqrt{Q}\|z_2 - z_1\|$, and thus,

$$\|g(z_2) - g(z_1)\| \leq \gamma\sqrt{MQ}\|z_2 - z_1\|\,.$$

This holds for all $z_1, z_2 \in U$ and the claim follows.

Exercise 4.11
We show that f is continuously differentiable. It follows then, from Proposition 4.14, that f is locally Lipschitz. Clearly,

$$f'(z) = \frac{1}{z}\cos z - \frac{1}{z^2}\sin z\,, \quad \text{if } z > 0 \qquad \text{and} \qquad f'(z) = 0\,, \quad \text{if } z < 0\,.$$

It remains to show that f is differentiable at 0 and that f' is continuous at 0. To this end, we use l'Hôpital's rule to obtain,

$$\lim_{z\downarrow 0}\frac{f(z) - f(0)}{z - 0} = \lim_{z\downarrow 0}\frac{\sin z - z}{z^2} \quad = \quad \lim_{z\downarrow 0}\frac{1}{2}\,\frac{\cos z - 1}{z} = \frac{1}{2}\cos' 0 = 0\,.$$

Moreover,

$$\lim_{z\uparrow 0}\frac{f(z) - f(0)}{z} = \lim_{z\uparrow 0}\frac{1 - 1}{z} = 0\,.$$

Therefore, $f'(0) = 0$. Clearly, f' is left-continuous at 0 and, since

$$\lim_{z\downarrow 0} f'(z) = \lim_{z\downarrow 0}\frac{z\cos z - \sin z}{z^2} = \lim_{z\downarrow 0}\frac{1}{2}\,\frac{\cos z - z\sin z - \cos z}{z} = -\lim_{z\downarrow 0}\frac{1}{2}\sin z = 0\,,$$

we conclude that f' is also right-continuous at 0. Hence, f' is continuous at 0, showing that f' is continuously differentiable.

Exercise 4.12
(a) Let $(t_0, z_0) \in J \times G$ be arbitrary. The hypotheses ensure that there exist neighbourhoods J_0 and $G_0 \subset G$ of t_0 and z_0, respectively, and a constant $L_2 \geq 0$ such that $J_0 \cap J$ and G_0 are compact, $C := \mathrm{cl}\{(f_1(t), f_2(z)) \colon (t, z) \in (J_0 \cap J) \times G_0\} \subset D$ and

$$\|f_2(x) - f_2(y)\| \leq L_2\|x - y\| \quad \forall\, x, y \in G_0.$$

By compactness of $J_0 \cap J$ and G_0, piecewise continuity of f_1 and continuity of f_2, there exists $K > 0$ such that $\|(f_1(t), f_2(z))\| \leq K$ for all $(t, z) \in (J_0 \cap J) \times G_0$. Therefore, the set C is compact. By Corollary 4.16, there exists $L_3 \geq 0$ such that

$$\|f_3(s, u) - f_3(s, v)\| \leq L_3\|u - v\| \quad \forall\, (s, u), (s, v) \in C.$$

Defining $L := L_3 L_2$, we have, for all $(t, x), (t, y) \in (J_0 \cap J) \times G_0$,

$$\begin{aligned}\|f(t, x) - f(t, y)\| &= \|f_3(f_1(t), f_2(x)) - f_3(f_1(t), f_2(y))\| \\ &\leq L_3\|f_2(x) - f_2(y)\| \leq L\|x - y\|\,,\end{aligned}$$

and so f is locally Lipschitz with respect to its second argument.
Finally, let $y \colon J \to G$ be continuous. By piecewise continuity of f_1 and continuity of f_2 and f_3, it immediately follows that the function $t \mapsto f(t, y(t)) = f_3(f_1(t), f_2(y(t)))$ is piecewise continuous. Therefore, f satisfies Assumption **A**.
(b) Let f be given by $f(t, z) := g(z) + k(t)h(z)$, where $g, h \colon \mathbb{R}^N \to \mathbb{R}^N$ are locally Lipschitz and $k \colon \mathbb{R} \to \mathbb{R}$ is piecewise continuous. Defining the piecewise continuous function $f_1 := k$, the locally Lipschitz function $f_2 := (g, h) \colon \mathbb{R}^N \to \mathbb{R}^{2N}$ and the continuous function $f_3 \colon \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}^N$ by $f_3(r, s) = f_3(r, (s_1, s_2)) := s_1 + rs_2$, we

see that f_3 is locally Lipschitz in its second argument $s = (s_1, s_2)$ and $f(t,z) = f_3(f_1(t), f_2(z))$. By Proposition 4.20, f satisfies Assumption **A**.

Exercise 4.13
Let $(\tau,\xi), (\sigma,\eta), (\rho,\theta) \in J \times G$ be arbitrary. Since $\psi(\tau,\tau,\xi) = \xi$, it follows that $(\tau,\xi) \sim (\tau,\xi)$ and so the relation $\sim$ is reflexive. Next, assume $(\tau,\xi) \sim (\sigma,\eta)$ and so $\psi(\sigma,\tau,\xi) = \eta$. By Theorem 4.26, we have $I(\tau,\xi) = I(\sigma,\eta)$ and $\psi(\tau,\sigma,\eta) = \psi(\tau,\sigma,\psi(\sigma,\tau,\xi)) = \psi(\tau,\tau,\xi) = \xi$. Therefore, $(\sigma,\eta) \sim (\tau,\xi)$ and so the relation $\sim$ is symmetric. Finally, assume $(\tau,\xi) \sim (\sigma,\eta)$ and $(\sigma,\eta) \sim (\rho,\theta)$. Then $\psi(\sigma,\tau,\xi) = \eta$ and $\psi(\rho,\sigma,\eta) = \theta$. Hence,

$$\psi(\rho,\tau,\xi) = \psi(\rho,\sigma,\psi(\sigma,\tau,\xi)) = \psi(\rho,\sigma,\eta) = \theta$$

and so $(\tau,\xi) \sim (\rho,\theta)$. Therefore, the reflexive and symmetric relation $\sim$ is also transitive and so is an equivalence relation.
Let $\mathcal{G}$ denote the graph of the maximal solution $\psi(\cdot,\tau,\xi)$, that is, $\mathcal{G} := \{(t,\psi(t,\tau,\xi)) : t \in I(\tau,\xi)\}$. Observe that

$$(\tau,\xi) \sim (\sigma,\eta) \Leftrightarrow \psi(\sigma,\tau,\xi) = \eta \Leftrightarrow (\sigma,\eta) \in \mathcal{G}$$

and so the equivalence class of (τ,ξ) coincides with $\mathcal{G}$.

Exercise 4.14
In Example 4.32, for the initial-value problem $\dot{x}(t) = A(t)x(t)$, $x(\tau) = \xi$, A p-periodic, the following equivalence was established:

$$\begin{aligned}&\exists\ np\text{-periodic solution}\\ &\qquad \Leftrightarrow\ 1 \text{ is an eigenvalue of } \Phi^n(p,0) \text{ and } \Phi(\tau,0)\xi \text{ is an associated eigenvector.}\end{aligned}$$

We use this equivalence to prove Proposition 2.20.

First, assume that there exists a np-periodic solution of $\dot{x}(t) = A(t)x(t)$. Then, by the above equivalence

$$1 \in \sigma\big(\Phi^n(p,0)\big) = \{\lambda^n : \lambda \in \sigma\big(\Phi(p,0)\}$$

and so $\Phi(p,0)$ has an eigenvalue λ with $\lambda^n = 1$.

Now, assume that $\Phi(p,0)$ has an eigenvalue λ with $\lambda^n = 1$. Let ξ be an associated eigenvector. Then $\Phi^n(p,0)\xi = \lambda^n\xi = \xi$ and so 1 is an eigenvalue of $\Phi^n(p,0)$ with associated eigenvector ξ. By the above equivalence, it follows that $t \mapsto \Phi(t,0)\xi$ is a np-periodic solution.

Exercise 4.15
Consider the differential equation $\dot{x} = x(1-x)$ with initial condition $x(0) = \xi$. Separation of variables gives

$$\int_\xi^x \frac{ds}{s(1-s)} = \int_0^t ds = t \quad \Rightarrow \quad \int_\xi^x \frac{ds}{s} + \int_\xi^x \frac{ds}{1-s} = t \quad \Rightarrow \quad \ln\frac{x}{\xi} - \ln\frac{1-x}{1-\xi} = t\,.$$

Consequently,

$$\frac{x}{\xi}\,\frac{1-\xi}{1-x} = e^t \quad \Rightarrow \quad x(1-\xi+e^t\xi) = \xi e^t \quad \Rightarrow \quad x(t) = \frac{\xi}{\xi + (1-\xi)e^{-t}}\,.$$

Thus $\varphi(t,\xi)$ is given by

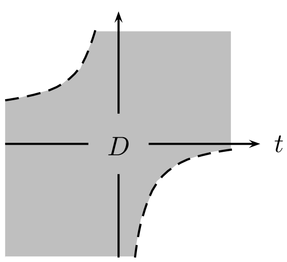
$$\varphi(t,\xi) = \frac{\xi}{\xi + (1-\xi)e^{-t}}\,.$$

Fr $\xi \in \mathbb{R}$, the maximal interval of existence I_ξ is

$$I_\xi = \begin{cases} \big(-\infty, \ln((\xi-1)/\xi)\big), & \xi < 0 \\ \mathbb{R}, & \xi \in [0,1] \\ \big(\ln((\xi-1)/\xi), \infty\big), & \xi > 1. \end{cases}$$

Consequently, the domain D of φ is given by $D = D_1 \cup D_2 \cup D_3$, where

$$D_1 := \bigcup_{\xi<0} \big(-\infty, \ln\frac{\xi-1}{\xi}\big) \times \{\xi\}, \quad D_2 := \mathbb{R}\times[0,1], \quad D_3 := \bigcup_{\xi>1} \big(\ln\frac{\xi-1}{\xi}, \infty\big) \times \{\xi\}.$$



Exercise 4.15: sketch of the domain D of the local flow φ

Exercise 4.16
For $(\xi_1,\xi_2) = \xi$, consider the initial-value problem $\dot{x} = f(x)$, $x(0) = \xi$. If $\xi = 0$, then it is clear that $\varphi(t,\xi) = \varphi(t,0) = 0$ for all $t \in \mathbb{R}$. Assume $\xi \neq 0$. A straightforward calculation gives the polar form of the initial-value problem

$$\dot{r} = r(1-r^2), \quad \dot{\theta} = -1, \quad (r(0), \theta(0)) = (\rho, \sigma),$$

where $\rho = \|\xi\|$, $\xi_1 = \rho\cos\sigma$, and $\xi_2 = \rho\sin\sigma$. Clearly, $\theta(t) = \sigma - t$.
Assume $\rho = 1$, then, from the first of the differential equations, it is clear that $r(t) = 1$ for all t and so the solution of the original initial-value problem is given componentwise by

$$\begin{aligned} x_1(t) &= \rho\cos(t-\sigma) = \rho\big(\cos\sigma\cos t + \sin\sigma\sin t\big) = \xi_1\cos t + \xi_2 \sin t \\ x_2(t) &= \rho\sin(\sigma - t) = \rho\big(\sin\sigma\cos t - \cos\sigma\sin t\big) = -\xi_1\sin t + \xi_2\cos t \end{aligned}$$

for all $t \in \mathbb{R}$. Writing $R(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$, it follows that

$$\varphi(t,\xi) = R(t)\xi \quad \forall\, t \in \mathbb{R}\,, \ \forall\, \xi \in \mathbb{R}^2\,, \ \|\xi\| = 1.$$

We proceed to resolve the cases of $\rho > 1$ and $\rho < 1$. Observe that, if $\rho > 1$, then $r(t) > 1$ for all t and, if $\rho < 1$, then $r(t) < 1$ for all t. Therefore, in each case $(1 - r(t))/(1-\rho) > 0$ for all t. Separating variables in the differential equation for r, we have

$$t = \int_0^t \mathrm{d}s = \int_\rho^{r(t)} \frac{ds}{s(1-s^2)} = \int_\rho^{r(t)} \left(\frac{1}{s} + \frac{1}{2(1-s)} - \frac{1}{2(1+s)}\right) \mathrm{d}s\,,$$

and so

$$t = \ln \frac{r(t)}{\rho} - \frac{1}{2} \ln \frac{1 - r(t)}{1 - \rho} - \frac{1}{2} \ln \frac{1 + r(t)}{1 + \rho} \quad \Longrightarrow \quad r(t) = \rho\big(\rho^2 + (1 - \rho^2)e^{-2t}\big)^{-1/2}.$$

Consequently,

$$\varphi(t, \xi) = \big(\|\xi\|^2 + (1 - \|\xi\|^2)e^{-2t}\big)^{-1/2} R(t)\xi \quad \forall\, t \in I_\xi.$$

If $\rho = \|\xi\| < 1$, then we may infer that $I_\xi = \mathbb{R}$. Furthermore, if $\rho = \|\xi\| > 1$, then the above expression for $\varphi(t, \xi)$ has a singularity: the maximal interval of existence of the solution is given by $I_\xi = (\alpha_\xi, \infty)$ with

$$\alpha_\xi := -\ln\big(\|\xi\|/\sqrt{\|\xi\|^2 - 1}\big).$$

Assembling the four cases (viz. $\xi = 0$, $0 < \|\xi\| < 1$, $\|\xi\| = 1$ and $\|\xi\| > 1$) treated above, we may infer that the local flow $\varphi\colon D \to \mathbb{R}^2$ has domain

$$D := \{(t.\xi) \in \mathbb{R} \times \mathbb{R}^2 \colon \|\xi\|^2 + (1 - \|\xi\|^2)e^{-2t} > 0\}$$

and is given by

$$\varphi(t, \xi) := \big(\|\xi\|^2 + (1 - \|\xi\|^2)\big)^{-1/2} R(t)\xi \quad \forall\,(t, \xi) \in D.$$

Exercise 4.17
We will show that $\mathcal{G} := \{\Phi_t \colon t \in \mathbb{R}\}$ satisfies the axioms of a commutative group.
Closure. $\Phi_s, \Phi_t \in \mathcal{G} \implies \Phi_s \circ \Phi_t = \Phi_{s+t} \in \mathcal{G}$.
Associativity. For all $\Phi_r, \Phi_s, \Phi_t \in \mathcal{G}$, we have

$$\big(\Phi_r \circ \Phi_s\big) \circ \Phi_t = \Phi_{r+s} \circ \Phi_t = \Phi_{r+s+t} = \Phi_r \circ \Phi_{s+t} = \Phi_r \circ \big(\Phi_s \circ \Phi_t\big).$$

Identity element. $I = \Phi_0 \in \mathcal{G}$, $\Phi_0 \circ \Phi_t = \Phi_t = \Phi_t \circ \Phi_0$ for all $\Phi_t \in \mathcal{G}$.
Inverse element. For each $\Phi_t \in \mathcal{G}$, $\Phi_{-t} \in \mathcal{G}$ and

$$\Phi_t \circ \Phi_{-t} = \Phi_0 = I = \Phi_{-t} \circ \Phi_t.$$

Commutativity. For all $\Phi_s \Phi_t \in \mathcal{G}$,

$$\Phi_s \circ \Phi_t = \Phi_{s+t} = \Phi_{t+s} = \Phi_t \circ \Phi_s.$$

Exercise 4.18
Let $\xi, \eta, \theta \in G$ be arbitrary. Since $\xi \in O(\xi)$, it follows that $\xi \sim \xi$ and so the relation $\sim$ is reflexive. Next, assume $\xi \sim \eta$ and so $\varphi(\tau, \xi) = \eta$ for some $\tau \in I_\xi$. Invoking Theorem 4.35, we have $-\tau \in I_\xi - \tau = I_\eta$ and $\varphi(-\tau, \eta) = \varphi(-\tau, \varphi(\tau, \xi)) = \xi$ and so $\eta \sim \xi$. Therefore, the relation $\sim$ is symmetric. Assume $\xi \sim \eta$ and $\eta \sim \theta$. Then $\varphi(\tau, \xi) = \eta$ for some $\tau \in I_\xi$ and $\varphi(\sigma, \eta) = \theta$ for some $\sigma \in I_\eta = I_\xi - \tau$. Then $\sigma + \tau \in I_\xi$ and $\varphi(\sigma + \tau, \xi) = \varphi(\sigma, \varphi(\tau, \xi)) = \varphi(\sigma, \eta) = \theta$ and so $\xi \sim \theta$. Therefore, the reflexive and symmetric relation $\sim$ is also transitive and so is an equivalence relation.
Finally, observe that

$$\xi \sim \eta \;\Leftrightarrow\; \big(\exists\, \tau \in I_\xi \colon \varphi(\tau, \xi) = \eta\big) \;\Leftrightarrow\; \eta \in O(\xi),$$

and so the equivalence class of ξ coincides with $O(\xi)$.

Exercise 4.19
Write $I_\xi \cap [0, \infty) = [0, \omega_\xi)$. First assume that $z \in \Omega(\xi)$. Then there exists a sequence

(t_n) in $[0, \omega_\xi)$ such that $t_n \to \omega_\xi$ and $\varphi(t_n, \xi) \to z$ as $n \to \infty$. For arbitrary $\tau \in [0, \omega_\xi)$ we have that

$$t_n \in (\tau, \omega_\xi)\,, \quad \text{for all suffciently large } n.$$

Consequently, since $I_{\varphi(\tau,\xi)} = I_\xi - \tau$ (by Proposition 3.2), we obtain that $\varphi(t_n - \tau, \varphi(\tau, \xi)) \in O^+(\varphi(\tau, \xi))$ for all sufficiently large n and, moreover,

$$\varphi(t_n - \tau, \varphi(\tau, \xi)) = \varphi(t_n, \xi) \to z \quad \text{as} \quad n \to \infty\,.$$

Therefore, $z \in \overline{O^+(\varphi(\tau, \xi))}$. This holds for every $\tau \in [0, \omega_\xi)$ and thus,

$$z \in \bigcap_{\tau\in[0,\omega_\xi)} \overline{O^+(\varphi(\tau, \xi))}\,.$$

Conversely, let us now assume that

$$z \in \bigcap_{\tau\in[0,\omega_\xi)} \overline{O^+(\varphi(\tau, \xi))}\,.$$

Let (τ_n) be a sequence in $[0, \omega_\xi)$ such that $\tau_n \to \omega_\xi$ as $n \to \infty$. Then, for each $n \in \mathbb{N}$, there exists $\sigma_n \in [0, \omega_\xi - \tau_n) = I_{\varphi(\tau_n,\xi)} \cap [0, \infty)$ such that

$$\|\varphi(\sigma_n, \varphi(\tau_n, \xi)) - z\| \le \frac{1}{n}\,, \quad \forall\, n \in \mathbb{N}\,.$$

Since $\varphi(\sigma_n + \tau_n, \xi) = \varphi(\sigma_n, \varphi(\tau_n, \xi))$, it follows that $\varphi(\sigma_n + \tau_n, \xi) \to z$ as $n \to \infty$, showing that $z \in \Omega(\xi)$.

The same argument applies *mutatis mutandis* to conclude that

$$A(\xi) = \bigcap_{\tau\in I_\xi\cap(-\infty,0]} \overline{O^-(\varphi(\tau, \xi))}.$$

Exercise 4.20

Let $\xi \in G$. Since the hypotheses of Theorem 4.38 hold, $\Omega(\xi)$ is non-empty, compact and is approached by $\varphi(t, \xi)$ as $t \to \infty$. Assume that $S \subset \mathbb{R}^N$ is non-empty and closed, and is approached by $\varphi(t, \xi)$ as $t \to \infty$. Seeking a contradiction, suppose that $\Omega(\xi) \not\subset S$. Then there exists $z \in \Omega(\xi)$ with $z \notin S$. Since S is closed, it follows that $\varepsilon := \mathrm{dist}(z, S) > 0$. Since $z \in \Omega(\xi)$, there exists (t_n), with $t_n \to \infty$ as $n \to \infty$, such that $\varphi(t_n, \xi) \to z$ as $n \to \infty$. By continuity of the map $u \mapsto \mathrm{dist}(u, S)$ (recall Exercise 4.7), we have $\mathrm{dist}(\varphi(t_n, \xi), S) \ge \varepsilon/2$ for all sufficiently large n. This contradicts the fact that $\mathrm{dist}(\varphi(t, \xi), S) \to 0$ as $t \to \infty$. Therefore, $\Omega(\xi) \subset S$.

Exercise 4.21

It is straightforward to verify that $f(z) = 0$ if, and only if, $z = 0$. Thus, the compact annulus $\mathcal{A} := \{z \in \mathbb{R}^2 \colon 1 \le \|z\| \le 3\}$ contains no equilibrium points. The circle $C_1 := \{z \in \mathbb{R}^2 \colon \|z\| = 1\}$ forms the inner boundary of $\mathcal{A}$ and

$$\langle z, f(z)\rangle = g((z)\|z\|^2 = (3 + 2z_1 - \|z\|^2)\|z\|^2 \ge 3 - 2|z_1| - \|z\|^2)\|z\|^2 \ge 0 \;\; \forall\, z \in C_1.$$

The circle $C_3 := \{z \in \mathbb{R}^2 \colon \|z\| = 3\}$ forma the outer boundary of $\mathcal{A}$ and

$$\langle z, f(z)\rangle = g((z)\|z\|^2 = (3 + 2z_1 - \|z\|^2)\|z\|^2 \le 3 + 2|z_1| - \|z\|^2)\|z\|^2 \le 0 \;\; \forall\, z \in C_3.$$

Therefore, the vector $f(z)$ is not directed outward at any point z of the boundary of $\mathcal{A}$. Thus, every $\xi \in \mathcal{A}$ has semiorbit $O^+(\xi)$ is the compact set $\mathcal{A}$ and so$\Omega(\xi)$ is a non-empty subset of $\mathcal{A}$; moreover, since $\mathcal{A}$ contains no equilibrium points, $\Omega(\xi)$ contains no equilibrium points. Therefore, by Theorem 4.46, for every $\xi \in \mathcal{A}$, $\Omega(\xi)$ is the orbit of a periodic point.

Exercise 4.22

(a) By the fundamental theorem of calculus, D is continuously differentiable. Moreover,

$$D(-u) = \int_0^{-u} d(v)\mathrm{d}v = -\int_0^{u} d(v)\mathrm{d}v = -D(u) \quad \forall\, u \in \mathbb{R}$$

and so D is an odd function. Since $D'(0) = d(0) < 0$, there exists $\varepsilon > 0$ such that $D(u) < 0$ for all $u \in (0,\varepsilon)$ and, since $D(u) \to \infty$ as $u \to \infty$, there exists $E > \varepsilon$ such that $D(u) > 0$ for all $u > E$. By continuity of D, the set $Z := \{u \in [\varepsilon, E] \colon D(u) = 0\}$ is non-empty, and the requisite properties hold for $a := \inf Z$ and $b := \sup Z$.

(b) By direct calculation we obtain $\dot{x}_1(t) = \dot{y}(t) = x_2(t) - D(y(t)) = x_2(t) - D(x_1(t))$ and $\dot{x}_2(t) = \ddot{y}(t) + d(y(t))\dot{y}(t) = -y(t) = -x_1(t)$.

(c) Note that, if $(z_1, z_2) = z \in \mathbb{R}^2$ is such that $\|z\| = a$, then $|z_1| \le a$ and so $z_1 D(z_1) \le 0$. Therefore,

$$\|z\| = a \implies \langle z, f(z)\rangle = z_1 z_2 - z_1 D(z_1) - z_1 z_2 = -z_1 D(z_1) \ge 0$$

and so the vector $f(z)$ does not point into the disc of radius a centred at 0 at any point z of its boundary. Therefore the exterior of the open disc of radius a centred at 0 is positively invariant under the (local) flow.

(d) For $(z_1, z_2) = (0, \gamma)$, we have $z_1^2 + 2z_2^2 = 2(b^2 + c^2 + 4m^2) = 2(c^2 + 4m^2 + 3b^2/2) - b^2 = 2r_1^2 - b^2$ and so $(0,\gamma) \in E_1 \subset \Gamma$. For $(z_1, z_2) = (0, -\gamma)$, we have $2z_1^2 + z_2^2 = b^2 + c^2 + 4m^2 = r_2^2 + b^2$ and so $(0, -\gamma) \in E_2 \subset \Gamma$.

(e) We first investigate the nature of f on $\Gamma = E_1 \cup C_1 \cup \cup L \cup C_2 \cup E_2$. Let $z = (z_1, z_2) \in E_1$. Then the vector $n = (z_1, 2z_2)$ is an outward pointing normal to Γ^*. Moreover, $\langle n, f(z)\rangle = z_1 z_2 - z_1 D(z_1) - 2z_1 z_2 = -z_1 z_2 - z_1 D(z_1) \le -|z_1|(|z_2| - m) \le 0$. Now, let $z = (z_1, z_2) \in C_1 \cup C_2$. Then z is an outward pointing normal to Γ^* and $\langle z, f(z)\rangle = -z_1 D(z_1) \le 0$. Next, let $z = (z_1, z_2) \in L$. Then the vector $n = (1, 0)$ is an outward pointing normal to Γ^* and $\langle n, f(z)\rangle = z_2 - D(z_1) = z_2 - D(c) \le 0$. Finally, let $z = (z_1, z_2) \in E_2$. Then the vector $n = (2z_1, z_2)$ is an outward pointing normal to Γ^* and $\langle n, f(z)\rangle = 2z_1 z_2 - 2z_1 D(z_1) - z_1 z_2 \le -|z_1|(|z_2| - m) \le 0$. By symmetry, the above analysis may be extended to the entire closed curve Γ^* to conclude that, at all points $z \in \Gamma^*$ the vector $f(z)$ is not outward pointing. This fact, in conjunction with the result in part (c), implies that the annular region $\mathcal{A}$ is positively invariant under the (local) flow. Moreover, $\mathcal{A}$ contains no equilibrium point. By the Poincaré-Bendixson theorem, we may infer the existence of a periodic solution $x = (x_1, x_2)$ of (4.38) with orbit in $\mathcal{A}$. Therefore, $y = x_1$ is a periodic solution of the Liénard equation (4.37).

(f) By part (e), the system (4.38) has a periodic solution (x_1, x_2) in $\mathcal{A}$. Since $\dot{x}_2 = -x_1$, there exists $\tau > 0$ such that $x_1(\tau) = 0$. Setting $v := x_2(\tau) \in [-\gamma, -a] \cup [a, \gamma]$, consider the solution y of the Liénard equation (4.37) satisfying $y(0) = 0$ and $\dot{y}(0) = v$. Then (y_1, y_2) given by

$$y_1 = y, \;\; y_2 = \dot{y} + D(y),$$

solves system (4.38) and satisfies $y_1(0) = 0 = x_1(\tau)$ and $y_2(0) = v = x_2(\tau)$. Consequently, $(y_1(t), y_2(t)) = (x_1(t+\tau), x_2(t+\tau))$ for all $t \in \mathbb{R}$ and so, the function $y = y_1$ is periodic.

Exercise 4.23

First observe that the system has precisely one critical point $0 \in G$. By part (b) of Exercise 1.1, we may deduce that

$$E \colon G \to \mathbb{R}_+, \quad z = (z_1, z_2) \mapsto (1 - \cos z_1 + z_2^2/2$$

is a first integral. Clearly $E(\xi) > E(0) = 0$ for all $\xi \in G\backslash\{0\}$. Let $\cos^{-1}\colon [-1,1] \to [0,\pi]$ denote the inverse of the function $\cos|_{[0,\pi]}$. Let $\alpha \in (0,2)$ be arbitrary. Set $a := \cos^{-1}(1-\alpha)$. Define $\gamma_0\colon [0,1] \to [-a,a]$ by

$$\gamma_0(t) := \begin{cases} a(4t-1), & 0 \le t \le 1/2 \\ a(3-4t), & 1/2 < t \le 1. \end{cases}$$

Now define $\gamma_1\colon [-a,a] \to [0,2), \quad t \mapsto \sqrt{2(\alpha - 1 + \cos t)}$ and finally define $\gamma\colon [0,1] \to G$ by

$$\gamma(t) := \begin{cases} \big(\gamma_0(t), \gamma_1(t)\big), & 0 \le t \le 1/2 \\ \big(\gamma_0(t), -\gamma_1(t)\big), & 1/2 < t \le 1. \end{cases}$$

Then the level set $E^{-1}(\alpha)$ is non-empty and is given by

$$E^{-1}(\alpha) = \{z \in G\colon E(z) = \alpha\} = \{\gamma(t)\colon t \in [0,1]\}$$

and is evidently a closed Jordan curve. Moreover, $\cup_{\alpha\in(0,2)} E^{-1}(\alpha) = G\backslash\{0\}$. Let $\xi \in G\backslash\{0\}$ be arbitrary and set $\alpha = E(\xi)$. Then, by Proposition 4.54, $O(\xi) = E^{-1}(\alpha)$ is a periodic orbit.

Chapter 5

Exercise 5.1
In this case, $G := (-1,\infty)\times(-1,\infty)$ and $f\colon G\to\mathbb{R}^2$ is given by

$$f(z)=f(z_1,z_2):=\big((z_1+1)z_2\,,\,-z_1(z_2+1)\big).$$

Set $U:=\{(z_1,z_2)\in\mathbb{R}^2 : z_1^2+z_2^2<1\}$ and define $V:U\to\mathbb{R}$ by

$$V(z)=V(z_1,z_2):=z_1+z_2-\ln(z_1+1)-\ln(z_2+1)\,.$$

Clearly, $V(0)=0$. Moreover, since $\ln(s+1)<s$ for all $s\in(-1,1)\setminus\{0\}$, we have $V(z)>0$ for all $z\in U\setminus\{0\}$. Furthermore,

$$(\nabla V)(z)=(\nabla V)(z_1,z_2)=\left(\frac{z_1}{z_1+1},\frac{z_2}{z_2+1}\right)\,,$$

and so $V_f(z)=V_f(z_1,z_2)=z_1z_2-z_1z_2=0$ for all $z\in U$. It now follows from Theorem 5.2 that the equilibrium 0 is stable.

Exercise 5.2
By hypothesis, there exists $\varepsilon>0$ such that

$$\frac{g(w)}{w}<1\,,\quad \forall\, w\in(-\varepsilon,\varepsilon)\setminus\{0\}\,.$$

Define $U:=\mathbb{R}\times(-\varepsilon,\varepsilon)$ and consider

$$V:U\to\mathbb{R}\,,\quad z=(z_1,z_2)\mapsto z_1^2+z_2^2\,.$$

Then $V(0)=0$ and $V(z)>0$ for all $z\in U\setminus\{0\}$. Moreover, setting $f(z)=f(z_1,z_2):=(z_2,-z_1-z_2+g(z_2))$, it follows that

$$\langle(\nabla V)(z),f(z)\rangle=2z_1z_2+2z_2(-z_1-z_2+g(z_2))=2z_2^2\left(\frac{g(z_2)}{z_2}-1\right)\le 0\,,\quad \forall z\in U\,.$$

By Theorem 5.2, the equilibrium 0 is stable.

Exercise 5.3
Define $f\colon\mathbb{R}^2\to\mathbb{R}^2$ by $f(z)=f(z_1,z_2):=(z_2,b\sin z_1)$. Set $U:=(-\pi,\pi)\times\mathbb{R}$ and define $V\colon U\to\mathbb{R}$ by $V(z)=V(z_1,z_2):=z_1z_2$. Let $z=(z_1,z_2)\in U$ be such that $V(z)=z_1z_2>0$. Then, $z_1\ne 0$, $z_2\ne 0$ and so $V_f(z)=z_2^2+bz_1\sin z_1>0$. Therefore, hypothesis (1) of Theorem 5.7 holds. Let $\delta>0$ be arbitrary and set $\theta:=\min\{\delta,\pi\}/2$. For $\xi:=(\theta,\theta)$, we have $\xi\in U$, $\|\xi\|<\delta$ and $V(\xi)=\theta^2>0$. Therefore, hypothesis (2) of Theorem 5.7 also holds and so $(0,0)$ is an unstable equilibrium.

Exercise 5.4
(a) Define $h\colon\mathbb{R}_+\to\mathbb{R}$ by

$$h(t)=\begin{cases}2\cos t^2-\dfrac{\sin t^2}{t^2}, & t>0\,,\\ 1, & t=0\,.\end{cases}$$

Then, h is continuous on $\mathbb{R}_+$ and, clearly, $h(t)$ does not converge to 0 as $t\to\infty$. Moreover,

$$\int_0^t h(s)\mathrm{d}s=\left.\frac{\sin s^2}{s}\right|_0^t=\frac{1}{t}\sin t^2\,,$$

showing that $\int_0^t h(s)ds \to 0$ as $t \to \infty$.

Note that, by Lemma 5.9, the function h cannot be uniformly continuous. To show this directly (that this, without appealing to Barbălat's lemma), define

$$s_n := \sqrt{2n\pi} \quad \text{and} \quad t_n := \sqrt{(2n+1/2)\pi}; \quad \forall\, n \in \mathbb{N}\,.$$

Then

$$\lim_{n\to\infty}(s_n - t_n) = 0 \quad \text{and} \quad \lim_{n\to\infty}(h(s_n) - h(t_n)) = 2\,,$$

showing that h is not uniformly continuous.

Exercise 5.5
Let $\xi = (\xi_1, \xi_2) \in G = \mathbb{R}^2$ and write $\varphi(t,\xi) := (x(t), y(t))$ for all $t \in [0, \omega_\xi) := I_\xi \cap \mathbb{R}_+$. Then

$$x\dot{x} = x^3 \tanh(x)(1-y) = \dot{y}(1-y) = \dot{y} - y\dot{y}\,.$$

Integration yields $x^2(t) - \xi_1^2 = 2y(t) - 2\xi_2 - y^2(t) + \xi_2^2$ for all $t \in [0, \omega_\xi)$. Rearranging, we have

$$0 \le x^2(t) = \|\xi\|^2 - 2\xi_2 + 2y(t) - y^2(t)\ \forall\, t \in [0, \omega_\xi)\,,$$

whence boundedness of y and x. Therefore, by Theorem 4.11, $\omega_\xi = \infty$ and $O^+(\xi)$ is bounded. Moreover, since $\dot{y}(t) = x^3(t)\tanh(x(t)) \ge 0$ for all $t \in [0, \omega_\xi)$, y is non-decreasing. Combining this with the fact that y is bounded shows that $\lim_{t\to\infty} y(t) =: \lambda$ exists and is finite. Consequently,

$$\lim_{t\to\infty} \int_0^t x^3(s)\tanh(x(s))\mathrm{d}s = \lim_{t\to\infty}(y(t) - \xi_2) = \lambda - \xi_2\,.$$

By the integral-invariance principle (Theorem 5.10) with $U = \mathbb{R}^2$ and g given by $g(z) = g(z_1, z_2) = z_1^3 \tanh(z_1)$ for all $z \in \mathbb{R}^2$, it follows that $\lim_{t\to\infty} x(t) = 0$. Note that any point of the form $(0, z_2)$ is an equilibrium point and thus, $g^{-1}(0) = \{(0, z_2) : z_2 \in \mathbb{R}\}$ is an invariant set.

Exercise 5.6
If the hypothesis "$V_f(z) \le 0$ for all $z \in U$" is replaced by "$V_f(z) \ge 0$ for all $z \in U$", then inspection of the proof of Theorem 5.12 reveals that the same argument applies with only one modification, namely, the phrase "$V \circ x$ is non-increasing" should be replaced by "$V \circ x$ is non-decreasing. There is no anomaly: note that there is no requirement that V be sign definite; the crucial ingredient is that $\lim_{t\to\infty} V(x(t))$ should exist and be finite.

Exercise 5.7
As in Example 5.3, introducing the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(z) = f(z_1, z_2) := (z_2\,, -b\sin z_1 - az_2)$, the system may be expressed in the form $\dot{x} = f(x)$. Let φ denote the local flow generated by f. Define the vertical strip $S := (-\pi, \pi) \times \mathbb{R}$. By Example 5.3, the function $V\colon S \to \mathbb{R}$ given by

$$V(z) = V(z_1, z_2) := z_2^2 + 2b(1 - \cos z_1).$$

is a Lyapunov function with $V_f(z_1, z_2) = -2az_2^2 \le 0$ for all $(z_1, z_2) \in S$, and so, the equilibrium 0 is stable. Consequently, there exists a neighbourhood $U \subset S$ of 0 such that, for every $\xi \in U$, the the closure of the semi-orbit $O^+(\xi)$ is contained in S. By Theorem 5.12, it follows that, for every $\xi \in U$, $\mathbb{R}_+ \subset I_\xi$, and moreover, as $t \to \infty$, $\varphi(t,\xi)$ approaches the largest invariant set M in $V_f^{-1}(0) = \{z = (z_1, z_2) \in S\colon z_2 = 0\}$. Let $z = (z_1, 0)$ be an arbitrary point of M and write $(x_1(t), x_2(t)) = \varphi(t, z)$ for all $t \in$

I_z. Obviously, $x_2(t) = 0$ for all $t \in \mathbb{R}$. Therefore, $0 = \dot{x}_2(t) = -ax_2(t) - b\sin x_1(t) = -b\sin x_1(t)$ for all $t \in \mathbb{R}$. Since $x_1(t) \in (-\pi, \pi)$ for all $t \in \mathbb{R}$, it follows that $x_1(t) = 0$ for all $t \in \mathbb{R}$. In particular, $0 = x_1(0) = z_1$ and so, $z = 0$. Therefore $M = \{0\}$ and thus, $\varphi(t, \xi) \to 0$ as $t \to \infty$.

Exercise 5.8
With $V\colon \mathbb{R}^2 \to \mathbb{R}$ and $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$V(z) = V(z_1, z_2) := z_1^2 + z_2^2 = \|z\|^2$$

and

$$f(z) = f(z_1, z_2) := \big(z_2 - z_1^3(a_1 + b_1 z_1^2)\,,\, -z_1 - z_2^3(a_2 + b_2 z_2^2)\big),$$

respectively, we have that

$$V_f(z) = \langle(\nabla V)(z), f(z)\rangle = -2z_1^4(a_1 + b_1 z_1^2) - 2z_2^4(a_2 + b_2 z_2^2) \le 0 \quad \forall\, z \in \mathbb{R}^2.$$

Therefore,

$$\frac{d}{dt}V(\varphi(t,\xi)) = V_f(\varphi(t,\xi)) \le 0\,, \quad \forall t \in [0, \omega_\xi)\,,$$

where $\omega_\xi = \sup I_\xi$. Consequently,

$$\|\varphi(t,\xi)\| \le \|\varphi(0,\xi)\| = \|\xi\|\,, \quad \forall t \in [0, \omega_\xi).$$

Therefore, $O^+(\xi)$ is compact. By Theorem 5.12 (with $U = \mathbb{R}^2$), we may infer that $\omega_\xi = \infty$ and, since $V_f^{-1}(0) = \{0\}$, we have $\varphi(t,\xi) \to 0$ as $t \to \infty$.

Exercise 5.9
Set $x(\cdot) := \varphi(\cdot, \xi)$. By continuity of V and compactness of $\mathrm{cl}(O^+(\xi))$, V is bounded on $O^+(\xi)$ and so the function $V \circ x$ is bounded. Since $(\mathrm{d}/\mathrm{d}t)(V \circ x))(t) = V_f(x(t)) \le 0$ for all $t \in \mathbb{R}_+$, $V \circ x$ is non-increasing. We conclude that the limit $\lim_{t\to\infty} V(x(t)) =: \lambda$ exists and is finite. Let $z \in \Omega(\xi)$ be arbitrary. Then there exists a sequence (t_n) in $\mathbb{R}_+$ such that $t_n \to \infty$ and $x(t_n) \to z$ as $n \to \infty$. By continuity of V, it follows that $V(z) = \lambda$. Consequently,

$$V(z) = \lambda \;\; \forall z \in \Omega(\xi). \tag{$*$}$$

By invariance of $\Omega(\xi)$, if $z \in \Omega(\xi)$, then $\varphi(t,z) \in \Omega(\xi)$ for all $t \in \mathbb{R}$ and so $V(\varphi(t,z)) = \lambda$ for all $t \in \mathbb{R}$. Therefore, $V_f(\varphi(t,z)) = 0$ for all $t \in \mathbb{R}$. Since $\varphi(0,z) = z$ and z is an arbitrary point of $\Omega(\xi)$, it follows that

$$V_f(z) = 0 \;\; \forall\, z \in \Omega(\xi), \tag{$**$}$$

and so $\Omega(\xi) \subset V_f^{-1}(0)$. The claim now follows because, by Theorem 4.38, $\Omega(\xi)$ is invariant and $x(t)$ approaches $\Omega(\xi)$ as $t \to \infty$.

Comment. It might be tempting to conclude from $(*)$ that $(\nabla V)(z) = 0$ for all $z \in \Omega(\xi)$, which then immediately would yield $(**)$. However, this conclusion is not correct: the set $\Omega(\xi)$ is not open and therefore $(*)$ does not imply that $(\nabla V)(z) = 0$ for all $z \in \Omega(\xi)$. (The invalidity of the conclusion is illustrated by the following simple example: if $V(z) = \|z\|^2$ and $\Omega(\xi) = \{z \in \mathbb{R}^N : \|z\| = 1\}$, then $V(z) = 1$ for all $z \in \Omega(\xi)$, but $(\nabla V)(z) = 2z \ne 0$ for all $z \in \Omega(\xi)$.)

Exercise 5.10
(a) For $r^0, \theta^0 \in (0,\infty) \times [0, 2\pi)$, let $r(\cdot\,; r^0)$ and $\theta(\cdot\,; \theta^0)$ denote the unique maximal solutions of the initial-value problems

$$\dot{r} = r(1-r)\,,\; r(0) = r^0 \qquad \text{and} \qquad \dot{\theta} = \sin^2(\theta/2)\,,\; \theta(0) = \theta^0\,,$$

respectively. Invoking separation of variables, a routine calculation shows that

$$r(t;r^0) = \frac{r^0}{r^0 + (1-r^0)e^{-t}} \quad \forall\, t \geq 0,$$

and hence, $\lim_{t\to\infty} r(t;r^0) = 1$.
If $\theta^0 = 0$, then $\theta(t;\theta^0) = \theta(t;0) = 0$ for all $t \in \mathbb{R}$ and the claim in part (i) follows. Assume now that $\theta^0 \in (0, 2\pi)$. Then, $\theta(t;\theta^0) < 2\pi$ for all $t \geq 0$, because otherwise there would exist $\tau > 0$ such that $\theta(\tau;\theta^0) = 2\pi$, in which case the initial-value problem

$$\dot\theta(t) = \sin^2(\theta(t)/2)\,, \quad \theta(t_0) = 2\pi$$

would have two solutions on $\mathbb{R}$, namely, $\theta(\cdot\,;\theta^0)$ and $\theta(\cdot) = 2\pi$, contradicting uniqueness. Since $\theta(\cdot\,;\theta^0)$ is strictly increasing and $\theta(t;\theta) \in [\theta^0, 2\pi)$ for all $t \geq 0$, it follows that $\theta^* := \lim_{t\to\infty} \theta(t;\theta^0)$ exists and is contained in $(\theta^0, 2\pi]$. Suppose $\theta^* < 2\pi$. Then, $c := \sin^2(\theta^*/2) > 0$ and, for all $t > 0$ sufficiently large, $(\mathrm{d}/\mathrm{d}t)\theta(t;\theta^0) \geq c/2 > 0$ which contradicts the fact that $\theta(t;\theta^0) \in [\theta^0, 2\pi)$ for all $t \geq 0$. Therefore, $\lim_{t\to\infty} \theta(t;\theta^0) = 2\pi$. Since

$$\psi(t;(r^0,\theta^0)) = (r(t;r^0), \theta(t;\theta^0)) \quad \forall\, t \geq 0,$$

the claim in part (ii) now follows.

(b) Writing $x = r\cos\theta$ and $y = r\sin\theta$, a straightforward calculation gives the system

$$\dot x = g(x,y)x - h(x,y)y, \quad \dot y = g(x,y)y + h(x,y)x$$

on $\mathbb{R}^2\backslash\{(0,0)\}$. The point $(1,0)$ is an equilibrium of this system. Denoting the corresponding local flow by ψ_c, it follows from (a) that
- $\lim_{t\to\infty} \psi_c(t,(x^0,y^0)) = (1,0)$ for all $(x^0,y^0) \in \mathbb{R}^2 \setminus \{(0,0)\}$;
- $\|\psi_c(t,(\cos\theta^0, \sin\theta^0))\| = 1$ for all $t \geq 0$ and all $\theta^0 \in [0, 2\pi)$;
- for each $n \in \mathbb{N}$, there exists $t_n > 0$ such that $\psi_c(t_n, (\cos(1/n), \sin(1/n))) = (-1, 0)$.

(c) Applying the coordinate transformation $x \mapsto x+1$ to the system in (b) yields the equivalent system

$$\dot x = g(x+1,y)(x+1) - h(x+1,y)y, \quad \dot y = g(x+1,y)y + h(x+1,y)(x+1)$$

on $G := \mathbb{R}^2\backslash\{(-1,0)\}$, with equilibrium $(0,0)$. Let φ denote the local flow generated by this system. Then, for all $(x^0,y^0) \in G$, $\varphi(t,(x^0,y^0)) = \psi_c(t,(x^0+1,y^0)) - (1,0)$ and $\lim_{t\to\infty} \varphi(t,(x^0,y^0)) = 0$. Therefore, the equilibrium is globally attractive. To see that the equilibrium is not stable, define $\xi_n := \big(\cos(1/n) - 1, \sin(1/n)\big)$. Then there exists $\delta > 0$ such that, for all $\xi \in G$ with $\|\xi\| \leq \delta$, $\|\varphi(t,\xi)\| \leq 1$ for all $t \geq 0$. For $n \in \mathbb{N}$, define $\xi_n := \big(\cos(1/n) - 1, \sin(1/n)\big)$ and observe that, by the result in the third bullet item in (b),

$$\|\varphi(t_n,\xi_n)\| = \|\psi_c(t_n,(\cos(1/n),\sin(1/n))) - (1,0)\| = \|(-2,0)\| = 2.$$

Since $\xi_n \to (0,0)$ as $n \to \infty$, it follows that the equilibrium $(0,0)$ is not stable.

Exercise 5.11
The planar system in Example 5.4 is encompassed by Example 5.16 with $g\colon \mathbb{R}^2 \to \mathbb{R}$ given by $g(z) = g(z_1,z_2) := z_1^3 + z_2 - z_2|z_2|$. Define $U := (-1,1) \times (-1,1)$ and observe that $z_1 g(z_1,0) = z_1^4 > 0$ for all $z_1 \neq 0$ and $\partial_2 g(z_1,z_2) = 1 - |z_2| > 0$ for all $(z_1,z_2) \in U$. Therefore, by Example 5.16, we may deduce that the equilibrium $0 \in \mathbb{R}^2$ is asymptotically stable.

Exercise 5.12

(a) The Liénard system is of the form (5.8) with $g\colon \mathbb{R}^2 \to \mathbb{R}$ given by $g(z) = g(z_1, z_2) = k(z_1) + d(z_1)z_2$. By assumption, there exists $\varepsilon > 0$ such that $z_1 k(z_1) > 0$ and $d(z_1) > 0$ for all $z_1 \in (-\varepsilon, \varepsilon) \setminus \{0\}$. Define $U := (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$. Then $z_1 g(z_1, 0) = z_1 k(z_1) > 0$ for all $z_1 \in (-\varepsilon, \varepsilon) \setminus \{0\}$ and $\partial_2 g(z_1, z_2) = d(z_1) > 0$ for all $(z_1, z_2) \in U$ with $z_1 z_2 \neq 0$. Therefore, by the result in Example 5.16, $0 \in \mathbb{R}^2$ is an asymptotically stable equilibrium.

(b) Define $K\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$ by $K(z_1) := \int_0^{z_1} k(s)\mathrm{d}s$. Observe that, by hypothesis (a), $k(s) \geq 0$ for all $s \in [0, \varepsilon)$ and $k(s) \leq 0$ for all $s \in (-\varepsilon, 0)$ which, together with continuity of k and hypothesis (b), ensures that $K(z_1) > 0$ for all $z_1 \in (-\varepsilon, \varepsilon) \setminus \{0\}$. Set $U := (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$ and define $V\colon U \to \mathbb{R}$ by $V(z) = V(z_1, z_2) := K(z_1) + z_2^2/2$. Define $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $f(z) = f(z_1, z_2) := \big(z_2, -g(z)\big) = \big(z_2, -k(z_2) - d(z_1)z_2\big)$, in which case, the Liénard system may be expressed in the form $\dot{x} = f(x)$. We may now infer that $V(0) = 0$, $V(z) > 0$ for all $z \in U \setminus \{0\}$ and

$$V_f(z) = V_f(z_2, z_2) = k(z_1)z_2 + z_2\big(-k(z_1) - d(z_1)z_2\big) = -z_2^2 d(z_1) \leq 0 \quad \forall\, z \in U.$$

By Theorem 5.2 (with $G = \mathbb{R}^2$), it follows that 0 is a stable equilibrium.
Finally, set $d = 0$ and let k be the identity map. In this case, the Liénard system reduces to the harmonic oscillator $\ddot{y} + y = 0$. Hypotheses (i) and (ii) clearly hold and so the equilibrium 0 is stable but is not asymptotically stable since (maximal) solutions of the harmonic oscillator have the property that $\|(y(t), \dot{y}(t))\| = \|(y(0), \dot{y}(0))\|$ for all $t \in \mathbb{R}$.

Exercise 5.13

Let U and V be as in Corollary 5.17. Stability of the equilibrium 0 is an immediate consequence of Theorem 5.2. The remaining issue is to establish attractivity. Let $\varepsilon > 0$ be such that $\overline{\mathbb{B}}(0, \varepsilon) \subset U$. By stability, there exists $\delta > 0$ such that, if $\xi \in \overline{\mathbb{B}}(0, \delta)$, then $x(t) \in \overline{\mathbb{B}}(0, \varepsilon)$ for all $t \in \mathbb{R}_+$ and for every maximal solution x with $x(0) = \xi$. Let x be any such solution. By boundedness of x and continuity of f, we may infer boundedness of $\dot{x}$ and so x is uniformly continuous. Since $V_f(x(t)) \leq 0$ for all $t \in \mathbb{R}_+$, it follows that $V \circ x$ is bounded ($0 \leq V(x(t)) \leq V(\xi)$ for all $t \in \mathbb{R}_+$) and non-increasing. Hence, $V \circ x$ converges, in particular, there exists $c \in [0, V(\xi)]$ such that $V(x(t)) \to c$ as $t \to \infty$. Therefore,

$$\lim_{t\to\infty} \int_0^t V_f(x(s))\mathrm{d}s = \lim_{t\to\infty} V(x(t)) - V(\xi) = c - V(\xi).$$

Furthermore, by continuity of V_f, together with uniform continuity and boundedness of x, $V_f \circ x$ is uniformly continuous. By Barbălat's lemma (Lemma 5.9), we may conclude that $V_f(x(t)) \to 0$ as $t \to \infty$. Seeking a contradiction, suppose that $x(t) \not\to 0$ as $t \to \infty$. Then there exist $\theta \in (0, \varepsilon)$ and a sequence (t_n) in $\mathbb{R}_+$ with $t_n \to \infty$ as $n \to \infty$ and $\|x(t_n)\| \geq \theta$. By continuity and negativity of V_f on the annulus $A := \{z \in U\colon \theta \leq \|z\| \leq \varepsilon\}$, there exists $\mu > 0$ such that $V_f(z) \leq -\mu$ for all $z \in A$. Therefore, $V_f(x(t_n)) \leq -\mu$ for all $n \in \mathbb{N}$, which contradicts the fact that $V_f(x(t)) \to 0$ as $t \to \infty$.

Exercise 5.14

By attractivity of the equilibrium, there exists $\varepsilon > 0$ such that

$$\lim_{t\to\infty} \varphi(t, \zeta) = 0 \quad \forall\, \zeta \in \overline{\mathbb{B}}(0, 2\varepsilon). \tag{$*$}$$

Let $\xi \in \mathcal{A}$ be arbitrary. It suffices to show that ξ has a neighbourhood U contained in $\mathcal{A}$. Since $\xi \in \mathcal{A}$, there exists $T \geq 0$ such that $(T, \eta) \in \mathrm{dom}(\varphi)$ and $\|\varphi(T, \xi)\| \leq \varepsilon$.

By openness of dom(φ) and continuity of φ (see Theorem 4.34), there exists $\delta > 0$ such that

$$\|\varphi(T,\eta) - \varphi(T,\xi)\| \le \varepsilon \quad \forall\, \eta \in U := \overline{\mathbb{B}}(\xi,\delta).$$

Therefore,

$$\|\varphi(T,\eta)\| \le \|\varphi(T,\eta) - \varphi(T,\xi)\| + \|\varphi(T,\xi)\| \le 2\varepsilon \quad \forall\, \eta \in U.$$

By $(*)$, it follows that

$$\lim_{t\to\infty} \varphi(t+T,\eta) = \lim_{t\to\infty} \varphi(t,\varphi(T,\eta)) = 0 \quad \forall\, \eta \in U.$$

Therefore, the neighbourhood U of ξ is contained in $\mathcal{A}$. Since $\xi \in \mathcal{A}$ is arbitrary, it follows that $\mathcal{A}$ is an open set.

Exercise 5.15
First, assume that $V\colon \mathbb{R}^N \to \mathbb{R}_+$ is radially unbounded. Let $c \in \mathbb{R}_+$ be arbitrary and set $\Sigma_c := \{z \in \mathbb{R}^N : V(z) \le c\}$. Clearly, Σ_c is a closed set. The set Σ_c is also bounded as, otherwise, there must exist a sequence (z_n) in Σ_c with $\|z_n\| \to \infty$ as $n \to \infty$ and so, by radial unboundedness, $V(z_n) \to \infty$ as $n \to \infty$, which is impossible since $V(z_n) \le c$ for all $n \in \mathbb{N}$. Therefore, Σ_c is closed and bounded, and so is compact.

Now, assume that Σ_c is compact for all $c \in \mathbb{R}_+$. Suppose that V is not radially unbounded. Then there exists $c \in \mathbb{R}_+$ and a sequence (z_n) in $\mathbb{R}^N$ with $\|z_n\| \to \infty$ as $n \to \infty$ and $V(z_n) \le c$ for all $n \in \mathbb{N}$. Therefore, (z_n) is an unbounded sequence in Σ_c, which contradicts compactness of $\Sigma + c$. Therefore, V is radially unbounded.

Exercise 5.16
The Lorenz system is of the form $\dot{x} = f(x)$, with continuously differentiable $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(z) = f(z_1,z_2,z_3) := \big(\sigma(z_2 - z_1), rz_1 - z_2 - z_1z_3, z_1z_2 - bz_3\big)$$

with $\sigma > 0$, $b > 0$ and $0 < r < 1$. Consider the function $V\colon \mathbb{R}^3 \to \mathbb{R}$ given by $V(z) = V(z_1,z_2,z_3) := rz_1^2 + \sigma z_2^2 + \sigma z_3^2$. Clearly, $V(0) = 0$, $V(z) > 0$ for all $z \in \mathbb{R}^3 \backslash \{0\}$ and V is radially unbounded. Moreover,

$$\begin{aligned} V_f(z_1,z_2,z_3) &= 2r\sigma z_1(z_2 - z_1) + 2\sigma z_2(rz_1 - z_2 - z_1z_3) + 2\sigma z_3(z_1z_2 - bz_3) \\ &= -2\sigma(rz_1^2 - 2rz_1z_2 + z_2^2) - 2b\sigma z_3^2 \quad \forall (z_1,z_2,z_3) \in \mathbb{R}^3, \end{aligned}$$

Since $0 < r < 1$, we may choose ρ such that $0 < r < \rho < 1$. Write $\mu := \min\{r(1-\rho), (1 - r/\rho)\}$. Then, $\mu > 0$ and, since $2z_1z_2 \le \rho z_1^2 + z_2^2/\rho$, we have

$$rz_1^2 - 2rz_1z_2 + z_2^2 \ge r(1-\rho)z_1^2 + (1 - r/\rho)z_2^2 \ge \mu\big(z_1^2 + z_2^2\big).$$

Therefore,

$$V_f(z) = V_f(z_1,z_2,z_3) \le -2\sigma\big(\mu z_1^2 + \mu z_2^2 + bz_3^2\big) \le 0 \;\forall\, z \in \mathbb{R}^3$$

Moreover, $V_f^{-1}(0) = \{0\}$. Hence, by Theorem 5.22, the equilibrium 0 is globally asymptotically stable.

Exercise 5.17
(a) A routine calculation gives $(\nabla V)(z_1,z_2) = 2\big(z_1\,,\, z_2(1+z_2^2)^{-2}\big)$ for all $(z_1,z_2) \in \mathbb{R}^2$.
If $z_1^2z_2^2 \ge 1$, then

$$V_f(z_1,z_2) = 2\big(-z_1^2 + z_2^2(1+z_2^2)^{-2}\big) = 2(1+z_2^2)^{-2}\big((1 - z_1^2z_2^2)z_2^2 - z_1^2 - 2z_1^2z_2^2\big) < 0.$$

If $z_1^2 z_2^2 < 1$ and $(z_1, z_2) \neq 0$, then

$$\begin{aligned} V_f(z_1, z_2) &= 2\big(-z_1^2 + (2z_1^2 z_2^4 - z_2^2)(1+z_2^2)^{-2}\big) \\ &= 2(1+z_2^2)^{-2}\big((z_1^2 z_2^2 - 1)z_2^2 - z_1^2 - 2z_1^2 z_2^2\big) < 0. \end{aligned}$$

Clearly, $V(0) = 0$ and $V(z) > 0$ for all $z \in \mathbb{R}^2 \setminus \{0\}$. By Corollary 5.17, it follows that the equilibrium 0 is asymptotically stable.

(b) Let $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ be such that $\xi_1^2 \xi_2^2 \geq 1$. Then $x \colon \mathbb{R} \to \mathbb{R}^2$ given by

$$x(t) = (x_1(t), x_2(t)) = (e^{-t}\xi_1, e^t \xi_2)$$

solves the initial-value problem $\dot{x} = f(x)$, $x(0) = \xi$. Indeed, $x(0) = \xi$. Also, $\dot{x}_1 = -x_1$, $\dot{x}_2 = x_2$ and $x_1^2(t) x_2^2(t) = \xi_1^2 \xi_2^2 \geq 1$, showing that $\dot{x} = f(x)$. Since $|x_2(t)| \to \infty$ as $t \to \infty$, we may conclude that 0 is not globally asymptotically stable.

(c) Setting $z_n = (0, n)$, it follows that $\|z_n\| = n \to \infty$ and $V(z_n) = n^2/(1+n^2) \to 1$ as $n \to \infty$. Hence, V is not radially unbounded.

Exercise 5.18
Write $M = \big(M_{ij}\big)$, where M_{ij} denotes the entry in row i and column j of M. Then, for $k = 1, \ldots, N$,

$$q(z) = \sum_{i=1}^N \sum_{j=1}^N M_{ij} z_i z_j = \sum_{i \neq k} \sum_{j \neq k} M_{ij} z_i z_j + \sum_{j \neq k} M_{kj} z_k z_j + \sum_{i \neq k} M_{ik} z_i z_k + M_{kk} z_k^2$$

and so

$$\begin{aligned} (\partial_k q)(z) &= \sum_{j \neq k} M_{kj} z_j + \sum_{i \neq k} M_{ik} z_i + 2M_{kk} z_k = \sum_{j=1}^N M_{kj} z_j + \sum_{i=1}^N M_{ik} z_i \\ &= k\text{-th component of } (M + M^*)z. \end{aligned}$$

Therefore, $(\nabla q)(z) = (M + M^*)z$ for all $z \in \mathbb{R}^N$.

Exercise 5.19
The system is of the form $\dot{x} = Ax$ with $A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.

(a) Setting $Q = I$, we seek $P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$ such that $PA + A^*P + I = 0$. Direct calculation gives $0 = -2p_2 + 1 = p_1 - p_2 - p_3 = 2(p_2 - p_3) + 1$, whence

$$P = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1 \end{pmatrix}.$$

Defining $V \colon \mathbb{R}^2 \to \mathbb{R}_+$ by $V(z) := \langle z, Pz \rangle$, we have

$$\langle \nabla V(z), Az \rangle = 2\langle Pz, Az \rangle = \langle (PA + A^*P)z, z \rangle = -\langle z, z \rangle = -\|z\|^2 \quad \forall\, z \in \mathbb{R}^2.$$

Therefore, the derivative of V along non-zero solutions is negative.
(b) With $V \colon \mathbb{R}^2 \to \mathbb{R}_+$ given by $V(z) = \|z\|^2$, we find

$$\langle \nabla V(z), Az \rangle = 2\langle z, Az \rangle = -2z_2 \quad \forall\, z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^2.$$

Therefore, V qualifies as a Lyapunov function. However, in contrast with part (a), the derivative of V along non-zero solutions is only non-positive.

Exercise 5.20
The system is of the form $\dot{x} = Ax + h(x)$ with

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad h\colon \mathbb{R}^3 \to \mathbb{R}^3,\ z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} z_1^2|z_3| \\ z_1 \sin z_3 \\ z_1 z_2 - z_2 z_3 \end{pmatrix}.$$

Clearly, $\sigma(A) = \{-1, -2\}$ and so A is Hurwitz. Let $\varepsilon > 0$ be arbitrary. Choose $\delta > 0$ sufficiently small so that $\delta\sqrt{\delta^2 + 5} < \varepsilon$. Then, for all $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{R}^3$, we have

$$\begin{aligned} \|z\| < \delta \implies \frac{\|h(z)\|}{\|z\|} &= \sqrt{\frac{z_1^4 z_3^2 + z_1^2 \sin^2 z_3 + (z_1 z_2 - z_2 z_3)^2}{\delta^2}} \\ &< \sqrt{\frac{\delta^6 + \delta^4 + 4\delta^4}{\delta^2}} = \delta\sqrt{\delta^2 + 5} < \varepsilon. \end{aligned}$$

Therefore, $\lim_{z\to 0} h(z)/\|z\| = 0$ and so, by Theorem 5.27, 0 is an asymptotically stable equilibrium.

Exercise 5.21
Let $\psi\colon \mathbb{R} \to \mathbb{R}$ be continuously differentiable with $\psi(0) = 0$ and $\psi'(0) \in (\alpha, \beta)$. The feedback system is given by $\dot{x} = f(x)$, where $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f(z) := Az - b\psi(c^* z) \quad \forall\, z \in \mathbb{R}^2.$$

Clearly, f is continuously differentiable and $f(0) = 0$. Moreover,

$$\tilde{A} := (Df)(0) = A - \psi'(0)bc^* = A - kbc^*$$

and, since $k \in (\alpha, \beta)$, it follows that $\tilde{A}$ is Hurwitx. By Corollary 5.29, we may infer that 0 is an asymptotically stable equilibrium.

Exercise 5.22
Define $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\begin{aligned} f(z) &= \big(f_1(z_1, z_2, z_3), f_2(z_1, z_2, z_3), f_3(z_1, z_2, z_3)\big) \\ &= \big(-2z_1 + z_1^2|z_3| + z_2\,,\ z_1 \sin z_3 - z_2 + 4z_3\,,\ z_1 z_2 - z_2 z_3 - z_3\big). \end{aligned}$$

A straightforward calculation reveals that all (nine) first partial derivatives $\partial_i f_j(0)$ exist. However, f is not differentiable at points $(z_1, z_2, 0)$ with $z_1 \neq 0$ and so the hypotheses of Corollary 5.29 fail to hold.

Exercise 5.23
The Lorenz system is of the form $\dot{x} = f(x)$, with continuously differentiable $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(z) = f(z_1, z_2, z_3) := \big(\sigma(z_2 - z_1), rz_1 - z_2 - z_1 z_3, z_1 z_2 - bz_3\big).$$

Therefore,

$$A := (Df)(0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

with characteristic polynomial given by $(\lambda+b)(\lambda^2+(\sigma+1)\lambda+\sigma(1-r))$. Given that $\sigma>0$ and $r>1$, it immediately follows that A has a positive eigenvalue. Therefore, by Theorem 5.31, 0 is an unstable equilibrium of the Lorenz system.

Exercise 5.24
As in Exercise 5.16, define $V\colon \mathbb{R}^3 \to \mathbb{R}$ by $V(0z) = V(z_1,z_2,z_3) := rz_1^2+\sigma z_2^2+\sigma z_3^2$. Writing $a_1 := \min\{r,\sigma\} > 0$ and $a_2 := \max\{r,\sigma\}$, we have $a_1\|z\|^2 \le V(z) \le a_2\|z\|^2$ for all $z\in\mathbb{R}^3$. Furthermore, by the calculation in the solution to Exercise 5.16, we have $V_f(z) \le -a_3\|z\|^2$ for all $z\in\mathbb{R}^3$, where $a_3 := 2\sigma\min\{\mu,b\} > 0$. Therefore, by Theorem 5.35, 0 is an exponentially stable equilibrium.

Exercise 5.25
Let $u\colon \mathbb{R}_+ \to \mathbb{R}^M$ be piecewise continuous with $u(t)\to u^\infty \in \mathbb{R}$ as $t\to\infty$. Let $\xi\in\mathbb{R}^N$ be arbitrary and let $x\colon\mathbb{R}_+\to\mathbb{R}^N$ be the solution of the initial-value problem $\dot{x} = Ax+bu$, $x(0)=\xi$. Define $w\colon\mathbb{R}_+\to\mathbb{R}^N$ by $w(t) := x(t)+A^{-1}Bu^\infty$. Then,

$$\dot{w}(t)\dot{x}(t) = Ax(t)+Bu(t) = Aw(t)+B(u(t)-u^\infty) \quad \forall\, t\in\mathbb{R}_+\backslash E$$

where E is the set of points at which u fails to be differentiable. Thus, writing $\theta := \xi + A^{-1}Bu^\infty$ and $v(\cdot) := u(\cdot)-u^\infty$, we see that w solves the initial-value problem $\dot{w} = Aw+Bv$, $w(0)=\theta$. Since A is Hurwitz and $v(t)\to 0$ as $t\to\infty$, the 0-CICS property holds and so $Ax(t)+A^{-1}Bu^\infty = w(t)\to 0$ as $t\to\infty$.

Exercise 5.26
The claim follows from a straightforward application of Proposition 4.20.

Exercise 5.27
With $u=0$, the system is given by $\dot{x} = -x|x|$. The function $V\colon\mathbb{R}\to\mathbb{R}_+$, $z\mapsto z^2$ is a radially-unbounded Lyapunov function with $(V'(z))(-z|z|) = -2|z|^3 < 0$ for all $z\neq 0$. Therefore, the equilibrium 0 is globally asymptotically stable.

Exercise 5.28
This is a straightforward consequence of the facts that, for $a,b\in\mathbb{R}_+$, $(a+b)\le \max\{2a,2b\}$ and $\max\{a,b\}\le a+b$.

Exercise 5.29
(a) & (b) Since $\psi(\cdot,0)=0$ and $g(s)s\le 0$ for all $s\in\mathbb{R}_+$, it follows that $0\le\psi(t,\xi)\le\xi$ for all $\xi\in\mathbb{R}_+$ and all $t\in I_\xi\cap\mathbb{R}_+$, where I_ξ denotes the maximal interval of existence of the solution of the initial-value problem $\dot{x}=g(x)$, $x(0)=\xi$. Therefore, $\mathbb{R}_+\subset I_\xi$ for all $\xi\in\mathbb{R}_+$ (by Theorem 4.11) and so $\mathbb{R}_+\times\mathbb{R}_+\subset\operatorname{dom}(\psi)$.

(c) Since $g(s)s<0$ for all $s>0$, we may infer that, for every, $\xi>0$, $\psi(\cdot,\xi)$ is decreasing and $\psi(t,\xi)\to 0$ as $t\to\infty$. Moreover, if $0\le\xi_1<\xi_2$, then $\psi(t,\xi_1)<\psi(t,\xi_2)$ for all $t\in\mathbb{R}_+$ (by Corollary 4.36). Define $\theta\colon\mathbb{R}_+\times\mathbb{R}_= \to\mathbb{R}_+$ by $\theta(r,t) := \psi(t,r)$. Then, for each $r>0$, $\theta(r,\cdot)$ is decreasing and $\theta(r,t)\to 0$ as $t\to\infty$. Moreover, for each $t\in\mathbb{R}_+$, $\theta(0,t)=0$ and, if $0\le r_1<r_2$, then $\theta(r_1,t)<\theta(r_2,t)$. Furthermore, by continuity of ψ, $\theta(\cdot,t)$ is continuous for each $t\in\mathbb{R}_+$. Therefore, for each $t\in\mathbb{R}_+$, $\theta(\cdot,t)$ is a $\mathcal{K}$ function. We may now conclude that θ is of class $\mathcal{KL}$.

Exercise 5.30
Define $c\colon\mathbb{R}_+\to\mathbb{R}_+$ by

$$c(s) := \sup\{|V_f(z,w)|\colon \|z\|\le b_1(s),\ \|w\|\le s\} \quad \forall\, s\in\mathbb{R}_+$$

and observe that c is non-decreasing, with $c(0)=0$. Moreover, $b_3(s) = c(s)+b_2(b_1(s))$ for all $s\in\mathbb{R}_+$. Since $b_1,b_2\in\mathcal{K}_\infty$, it follows that $b_2\circ b_1$ is in $\mathcal{K}_\infty$. Therefore, to conclude that b_3 is in $\mathcal{K}_\infty$ it suffices to show that the function c is continuous. Continuity at $s = 0$ is clear. Let $s > 0$ be arbitrary. We will show that c is continuous at s. Let (s_n) be a sequence in $\mathbb{R}_+$ with $s_n \to s$ as $n \to \infty$. We may assume that $s_n > 0$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define $\rho_n := \min\{s_n, s\}$ and $\sigma_n := \max\{s_n, s\} \geq \rho_n > 0$. Then $|c(s) - c(s_n)| = c(\sigma_n) - c(\rho_n)$ for all $n \in \mathbb{N}$ and so, to conclude that c is continuous at s, it is sufficient to show that $\lim_{n\to\infty} \big(c(\sigma_n) - c(\rho_n)\big) = 0$. For each $n \in \mathbb{N}$, the set

$$K_n := \{(z, w) \colon \ \|z\| \leq b_1(\sigma_n), \ \|w\| \leq \sigma_n\}$$

is compact which, together with continuity of $(z, w) \mapsto |V_f(z, w)|$, ensures the existence of $(y_n, v_n) \in K_n$ such that $c(\sigma_n) = |V_f(y_n, v_n)|$. Define sequences (z_n) and (w_n) by

$$z_n := \frac{b_1(\rho_n)}{b_1(\sigma_n)}\, y_n, \quad w_n := \frac{\rho_n}{\sigma_n}\, v_n$$

and observe that

$$\|z_n\| \leq b_1(\rho_n), \quad \|w_n\| \leq \rho_n \ \ \forall\, n \in \mathbb{N}.$$

Therefore,

$$|V_f(z_n, w_n)| \leq c(\rho_n) \leq c(\sigma_n) = |V_f(y_n, v_n)| \ \ \forall\, n \in \mathbb{N}. \tag{$*$}$$

By boundedness of the sequence (σ_n), there exists $\sigma > 0$ such that $\sigma_n \leq \sigma$ for all $n \in \mathbb{N}$. Define $K := \{(z, w) \colon \ \|z\| \leq b_1(\sigma), \ \|w\| \leq \sigma\}$. Then K is compact and is such that $(y_n, v_n), (z_n, w_n) \in K$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ be arbitrary. Since V_f is uniformly continuous on K, there exists $\delta > 0$ such that, for all $(z, w), (y, v) \in K$,

$$\|z - y\| + \|w - v\| \leq \delta \implies \big||V_f(z, w)| - |V_f(y, v)|\big| \leq \varepsilon. \tag{$**$}$$

Since $\lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \sigma_n = s > 0$, we may infer that, as $n \to \infty$,

$$\|z_n - y_n\| = \left(1 - \frac{b_1(\rho_n)}{b_1(\sigma_n)}\right) \|y_n\| \to 0 \ \text{ and } \ \|w_n - v_n\| = \left(1 - \frac{\rho_n}{\sigma_n}\right) \|v_n\| \to 0,$$

and so there exists $N \in \mathbb{N}$ such that $\|z_n - y_n\| + \|w_n - v_n\| \leq \delta$ for all $n \geq N$. The conjunction of $(*)$ and $(**)$ now gives $0 \leq c(\sigma_n) - c(\rho_n) \leq \varepsilon$ for all $n \geq \mathbb{N}$ and so $\lim_{n\to\infty}(c(\sigma_n) - c(\rho_n) = 0$, completing the proof.

Exercise 5.31
It is clear that $a_1(0) = 0 = a_2(0)$ and that the functions a_1 and a_2 are non-decreasing and are continuous at 0. Let $s > 0$ be arbitrary. Let (s_n) be a sequence in $\mathbb{R}_+$ with $s_n \to s$ as $n \to \infty$. Since $s > 0$, we may assume that $s_n > 0$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ be arbitrary. We will establish continuity at s of both a_1 and a_2 by showing that, for $i = 1, 2$, there exists $N \in \mathbb{N}$ such that

$$|a_i(s) - a_i(s_n)| \leq \varepsilon \ \ \forall\, n \geq N. \tag{$\dagger$}$$

For each $n \in \mathbb{N}$, define $\rho_n := \min\{s, s_n\} > 0$ and $\sigma_n := \max\{s, s_n\} \geq \rho_n$. Clearly, $\lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \sigma_n = s > 0$ and so there exist $\rho > 0$ and $\sigma > 0$ such that $\rho \leq \rho_n \leq \sigma_n \leq \sigma$ for all $n \in \mathbb{N}$. Observe that $\sigma_n/\rho_n \leq \sigma/\rho$ for all $n \in \mathbb{N}$ and $\sigma_n/\rho_n \to 1$ as $n \to \infty$. Since W is radially unbounded, there exists $r \geq \rho$ such that $W(y) > a_1(\sigma)$ for all y with $\|y\| > r$. Write $R := r\sigma/\rho \geq \sigma$ and set $K := \overline{\mathbb{B}}(0, R)$. Since W is uniformly continuous on K, there exists $\delta > 0$ such that, for all $y, z \in K$,

$$\|y - z\| \leq \delta \implies |W(y) - W(z)| \leq \varepsilon. \tag{$\dagger\dagger$}$$

(a) First, we prove continuity of a_2 at s. By continuity of W, for each $n \in \mathbb{N}$, there exists y_n, with $\|y_n\| \leq \sigma_n$, such that $a_2(\sigma_n) = \sup\{W(y) \colon \ \|y\| \leq \sigma_n\} = W(y_n)$. For each $n \in \mathbb{N}$, set $z_n := (\rho_n/\sigma_n)y_n$. Then $\|z_n\| \leq \rho_n$ and

$$0 \leq a_2(\sigma_n) - a_2(\rho_n) \leq W(y_n) - W(z_n) \ \ \forall\, n \in \mathbb{N}. \tag{$*$}$$

Observe that the sequences (y_n) and (z_n) are in K and, since $\rho_n/\sigma_n \to 1$ as $n \to \infty$, we have

$$\|y_n - z_n\| = \left(1 - \frac{\rho_n}{\sigma_n}\right)\|y_n\| \to 0 \quad \text{as } n \to \infty.$$

In particular, there exists $N \in \mathbb{N}$ so that $\|y_n - z_n\| \leq \delta$ for all $n \geq N$ which, in conjunction with $(*)$ and $(\dagger\dagger)$, gives

$$|a_2(s) - a_2(s_n)| = a_2(\sigma_n) - a_2(\rho_n) \leq \varepsilon \quad \forall\, n \geq N.$$

Therefore, $(\dagger)$ holds for $i = 2$ and so a_2 is continuous at s.
(b) Next, we prove that a_1 is continuous at s. Recall that, for all y with $\|y\| > r$, we have $W(y) > a_1(\sigma) \geq a_1(\rho_n)$ for all $n \in \mathbb{N}$. Therefore,

$$a_1(\rho_n) = \inf\{W(y)\colon\ \rho_n \leq \|y\|\} = \inf\{W(y)\colon\ \rho_n \leq \|y\| \leq r\} \quad \forall\, n \in \mathbb{N}$$

and so there exists a sequence (y_n), such that $\rho_n \leq \|y_n\| \leq r$ and $a_1(\rho_n) = W(y_n)$ for all $n \in \mathbb{N}$. Define the sequence (z_n) by $z_n := (\sigma_n/\rho_n)y_n$. Then

$$\sigma_n \leq \frac{\sigma_n}{\rho_n}\|y_n\| = \|z_n\| \leq \frac{r\sigma}{\rho} = R \quad \forall\, n \in \mathbb{N}.$$

Therefore,

$$0 \leq a_1(\sigma_n) - a_1(\rho_n) \leq W(z_n) - W(y_n) \quad \forall\, n \in \mathbb{N}. \tag{$**$}$$

Observe that the sequences (y_n) and (z_n) are in K and, since $\sigma_n/\rho_n \to 1$ as $n \to \infty$, we have

$$\|y_n - z_n\| = \left(\frac{\sigma_n}{\rho_n} - 1\right)\|y_n\| \to 0 \quad \text{as } n \to \infty.$$

In particular, there exists $N \in \mathbb{N}$ so that $\|y_n - z_n\| \leq \delta$ for all $n \geq N$ which, in conjunction with $(**)$ and $(\dagger\dagger)$, gives

$$|a_1(s) - a_1(s_n)| = a_1(\sigma_n) - a_1(\rho_n) \leq \varepsilon \quad \forall\, n \geq N.$$

Therefore, $(\dagger)$ holds for $i = 1$ and so a_1 is continuous at s.

Exercise 5.32
(a) Define $V\colon \mathbb{R} \to \mathbb{R}$ by $V(z) = z^2/2$. Then

$$V_f(z,v) = -z^2(1+2z^2) + z(1+z^2)v^2 = -z^4 + (1+z^2)(zv^2 - z^2)\ \forall\,(z,v) \in \mathbb{R}\times\mathbb{R}.$$

Therefore, for $|z| \geq v^2$, we have $V_f(z,v) \leq -z^4$ and so, an application of Corollary 5.44 (with b_1 and b_2 given by $b_1(s) = s^2$ and $b_2(s) = s^4$) shows that the system is ISS.

(b) Define $V\colon \mathbb{R}^2 \to \mathbb{R}$ by $V(z) = V(z_1,z_2) := z_1^2/2 + z_2^4/4$. By Lemma 5.46, there exist $a_1, a_2 \in \mathcal{K}_\infty$ such that

$$a_1(\|z\|) \leq V(z) \leq a_2(\|z\|)\ \forall\, z \in \mathbb{R}^2.$$

Moreover,

$$V_f(z,v) = -z_1^2 - z_2^4 + z_1 z_2^2 + z_2^3 v \leq -z_1^2/2 - z_2^4/2 + z_2^3 v\ \forall\,(z,v) \in \mathbb{R}^2\times\mathbb{R}.$$

Let $\mu > 0$. By Young's inequality[1]

$$z_2^3 v = (\mu z_2^3)(v/\mu) \leq (\mu z_2^3)^{4/3}/(4/3) + (v/\mu)^4/4 \quad \forall\,(z_2,v) \in \mathbb{R}\times\mathbb{R},$$

[1] William Henry Young (1863-1942), English. Young's inequality says that if $a, b \geq 0$ and $p, q > 0$ are such that $1/p + 1/q = 1$, then $ab \leq a^p/p + b^q/q$.

and, setting $\mu = 3^{-3/4}$, we have $z_2^3 v \le z_2^4/4 + 27v^4/4$. Therefore,

$$V_f(z,v) \le -V(z) + 27v^4/4 \quad \forall\, (z,v) \in \mathbb{R}^2 \times \mathbb{R}.$$

An application of Theorem 5.41 (with $a_3 = a_1$ and a_4 given by $a_4(s) = 27s^4/4$) shows that the system is ISS.

(c) Define $V\colon \mathbb{R}^2 \to \mathbb{R}$ by $V(z) = V(z_1,z_2) := \|z\|^2/2$. Then

$$V_f(z,v) = V_f(z_1,z_2,v_1,v_2) = -z_1^2 - z_2^4 + z_1v_1 + z_2v_2 \quad \forall\, (z,v) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

For all $(z_1,v_1),(z_2,v_2) \in \mathbb{R}\times\mathbb{R}$, $z_1v_1 \le (z_1^2+v_1^2)/2$ and $z_2v_2 \le z_2^4/4 + 3v_2^{4/3}/4$ (by Young's inequality). Therefore, defining $W_1, W_2\colon \mathbb{R}^2 \to \mathbb{R}_+$ by $W_1(z) := z_1^2/2 + 3z_2^4/4$ and $W_2(v) = W_2(v_1,v_2) := v_1^2/2 + 3v_2^{4/3}/4$, we have

$$V_f(z,v) \le -W_1(z) + W_2(v) \quad \forall\, (z,v) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

By Lemma 5.46, there exist $a_3, a_4 \in \mathcal{K}_\infty$ such that

$$a_3(\|z\|) \le W_1(z) \quad \forall\, z \in \mathbb{R}^2 \quad \text{and} \quad W_2(v) \le a_4(\|v\|) \quad \forall\, v \in \mathbb{R}^2.$$

Therefore,

$$V_f(z,v) \le -a_3(\|z\|) + a_4(\|v\|) \quad \forall\, (z,v) \in \mathbb{R}^2 \times \mathbb{R}^2,$$

and so, by Theorem 5.41, it follows that the system is ISS.

Chapter 6

Exercise 6.1
By direct calculation

$$\mathcal{C}(A_c, b_c) = (b_c, A_c b_c, \ldots, A_c^{n-1} b_c) = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & * \\ \vdots & \vdots & .\cdot\cdot & \vdots & \vdots \\ 0 & 1 & \ldots & * & * \\ 1 & * & \ldots & * & * \end{pmatrix},$$

and so $\operatorname{rk} \mathcal{C}(A_c, b_c) = N$. Hence, (A_c, b_c) is controllable.

Exercise 6.2
(a) Let $s \in \mathcal{S}$ be arbitrary. Then, by property (i), $\Gamma(s) \sim s$ and so $\Gamma(s) \in [s]$. Therefore, every $s \in \mathcal{S}$ has at least one representative in $\Gamma(\mathcal{S})$. Assume s_1, s_2 are representatives of $s \in \mathcal{S}$. Then $s_1 \sim s_2$ and so, by property (ii), $\Gamma(s_1) = \Gamma(s_2)$. Therefore, every $s \in \mathcal{S}$ has precisely one representative in $\Gamma(\mathcal{S})$.

(b) It is straightforward to verify that the requisite properties hold for $\sim$, namely, reflexivity (for all $(A, b) \in \mathcal{S}$, $(A, b) \sim (A, b)$), symmetry (for all $(A_1, b_1), (A_2, b_2) \in \mathcal{S}$, $(A_1, b_1) \sim (A_2, b_2)$ implies $(A_2, b_2) \sim (A_1, b_1)$), and transitivity (for all (A_1, b_1), (A_2, b_2), $(A_3, b_3) \in \mathcal{S}$, if $(A_1, b_1) \sim (A_2, b_2)$ and $(A_2, b_2) \sim (A_3, b_3)$, then $(A_1, b_1) \sim (A_3, b_3)$). Therefore, $\sim$ is an equivalence relation.To see that Γ is a canonical form, we show that the requisite properties (i) and (ii) hold. First note that, by Lemma 6.1, for all $(A, b) \in \mathcal{S}$ $\Gamma(A, b) = (A_c, b_c) \sim (A, b)$ and so property (i) holds. Let $(A_1, b_1), (A_2, b_2) \in \mathcal{S}$ be such that $(A_1, b_1) \sim (A_2, b_2)$. Then, A_1 and A_2 have the same characteristic polynomial and so have the same controller form, that is, $\Gamma(A_1, b_1) = \Gamma(A_2, b_2)$. Thus, property (ii) also holds. Therefore, Γ is a canonical form.

Exercise 6.3
For $n \in \mathbb{N}$, $N \geq 2$, let $\mathcal{P}(N)$ be the statement

$$\mathcal{P}(N): \quad P_M(s) = s^N + m_{N-1} s^{N-1} + \cdots + m_1 s + m_0$$

The matrix $m = \begin{pmatrix} 0 & 1 \\ -m_0 & -m_1 \end{pmatrix}$ has characteristic polynomial

$$P_M(s) = \begin{vmatrix} s & -1 \\ m_0 & s + m_1 \end{vmatrix} = s^2 + m_1 s + m_0$$

and so $\mathcal{P}(2)$ is a true statement. Assume $N \in \mathbb{N}$, $N \geq 2$, and $\mathcal{P}(N)$ true. The $(N+1) \times (N+1)$ matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -m_0 & -m_1 & -m_2 & \cdots & -m_{N-1} & -m_N \end{pmatrix}$$

has characteristic polynomial

$$P_M(s) = \begin{vmatrix} s & -1 & 0 & \cdots & 0 & 0 \\ 0 & s & -1 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & s & -1 \\ m_0 & m_1 & m_2 & \cdots & m_{N-1} & s + m_N \end{vmatrix}$$

which, by expansion on row 1 and invoking the truth of $\mathcal{P}(N)$, gives

$$P_M(s) = s \begin{vmatrix} s & -1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & s & -1 \\ m_1 & m_2 & \cdots & m_{N-1} & s+m_N \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & -1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & s & -1 \\ m_0 & m_2 & \cdots & m_{N-1} & s+m_N \end{vmatrix}$$

$$= s(s^N + m_N s^{N-1} + \cdots + m_2 s + m_1) + m_0 = s^{N+1} + m_n S^N + \cdots m_1 s + m_0$$

and so $\mathcal{P}(N+1)$ is a true statement. The result follows by induction.

Exercise 6.4

We first show that, for all $k \in \mathbb{N}$,

$$\hat{A}^k = A^k + \sum_{i=0}^{k-1} A^{k-1-i} b f^* \hat{A}^i \,. \qquad (*)$$

For $k = 1$, formula $(*)$ reduces to $\hat{A} = A + bf^*$, which is trivially true (by the definition of $\hat{A}$). Assume now that formula $(*)$ is true for $k = m$. Then,

$$\begin{aligned} \hat{A}^{m+1} = \hat{A}\hat{A}^m &= A(A^m + \sum_{i=0}^{m-1} A^{m-1-i} b f^* \hat{A}^i) + bf^*\hat{A}^m \\ &= A^{m+1} + \sum_{i=0}^{m-1} A^{m-i} b f^* \hat{A}^i + bf^*\hat{A}^m \\ &= A^{m+1} + \sum_{i=0}^{m} A^{m-i} b f^* \hat{A}^i \,, \end{aligned}$$

which is $(*)$ for $k = m+1$. We conclude that formula $(*)$ is true for all $k \in \mathbb{N}$.
Write $P(z) = \sum_{n=0}^{N} a_n z^n$, with $a_n \in \mathbb{R}$, $n = 1, \ldots, N$ and $a_N = 1$. Using $(*)$, we obtain

$$a_n \hat{A}^n = a_n A^n + a_n \sum_{i=0}^{n-1} A^{n-1-i} b f^* \hat{A}^i, \; n = 1, \ldots, N.$$

Therefore, there exist $g_n \in \mathbb{R}^N$, $n = 0, \ldots, N-1$, such that

$$P(\hat{A}) = \sum_{n=0}^{N} a_n \hat{A}^n = P(A) + bg_0^* + Abg_1^* + \cdots + A^{N-1} b f^*$$

By the Cayley-Hamilton theorem, $P(\hat{A}) = 0$, and thus,

$$P(A) = -(bg_0^* + Abg_1^* + \cdots + A^{n-2} b g_{n-2}^* + A^{N-1} b f^*) \,.$$

Writing $G := \big(g_0, g_1, \cdots, g_{N-1}, f\big) \in \mathbb{R}^{N \times N}$, the above formula for $P(A)$ can be written in the form $P(A) = -\mathcal{C}(A,b)G^*$ and so $G^* = -\mathcal{C}(A,b)^{-1}P(A)$, where $\mathcal{C}(A,b)^{-1}$ exists by controllability. Since the last row of G^* coincides with f^*, it follows that

$$f^* = -(0, \ldots, 0, 1)\mathcal{C}(A,b)^{-1}P(A).$$

Exercise 6.5
(a) The matrix $\mathcal{C}(A,b)$ is given by

$$\mathcal{C}(A,b) = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 1 & 0 & -4 & 0 \end{pmatrix}.$$

To calculate f, the last row of $\mathcal{C}^{-1}(A,b)$ is needed:

$$cC^{-1}(A,b) = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & \frac{1}{6} & -\frac{1}{3} & 0 \end{pmatrix}.$$

The polynomial p to be assigned is given by

$$p(\lambda) = (\lambda+1)^2(\lambda+2)^2 = \lambda^4 + 6\lambda^3 + 13\lambda^2 + 12\lambda + 4\,.$$

Now

$$A^2 = \begin{pmatrix} * & * & * & * \\ 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ * & * & * & * \end{pmatrix},\ A^3 = \begin{pmatrix} * & * & * & * \\ -3 & 0 & 0 & -2 \\ -6 & 0 & 0 & -4 \\ * & * & * & * \end{pmatrix},\ A^4 = \begin{pmatrix} * & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ * & * & * & * \end{pmatrix},$$

and thus,

$$\begin{aligned} p(A) &= \begin{pmatrix} * & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ * & * & * & * \end{pmatrix} + \begin{pmatrix} * & * & * & * \\ -18 & 0 & 0 & -12 \\ -36 & 0 & 0 & -24 \\ * & * & * & * \end{pmatrix} + \begin{pmatrix} * & * & * & * \\ 0 & -13 & 0 & 0 \\ 0 & -26 & 0 & 0 \\ * & * & * & * \end{pmatrix} + \\ &\quad \begin{pmatrix} * & * & * & * \\ 36 & 0 & 0 & 24 \\ 0 & 0 & 0 & 12 \\ * & * & * & * \end{pmatrix} + \begin{pmatrix} * & * & * & * \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ * & * & * & * \end{pmatrix} \\ &= \begin{pmatrix} * & * & * & * \\ 18 & -8 & 0 & 12 \\ -36 & -24 & 4 & -12 \\ * & * & * & * \end{pmatrix}. \end{aligned}$$

By Ackermann's formula,

$$\begin{aligned} f^* &= -(0,0,\ldots,0,1)\mathcal{C}^{-1}(A,b)p(A) \\ &= \left(0, -\frac{1}{6}, \frac{1}{3}, 0\right) \begin{pmatrix} * & * & * & * \\ 18 & -8 & 0 & 12 \\ -36 & -24 & 4 & -12 \\ * & * & * & * \end{pmatrix} \\ &= \left(-15, -\frac{20}{3}, \frac{4}{3}, -6\right). \end{aligned}$$

(b) Simply define $F \in \mathbb{R}^{2\times 4}$ by $F := \begin{pmatrix} 0 \\ f^* \end{pmatrix}$, in which case, $A + BF = A + bf^*$ and so $\sigma(A+BF) = \{-1,-2\}$, and each eigenvalue has multiplicity 2.

Exercise 6.6
(a) Let $S \subset \mathbb{R}^L$ be a proper algebraic set. Then there exists a real polynomial Γ in L variables, not equal to the zero polynomial, such that $S = \{z \in \mathbb{R}^L :\ \Gamma(z) = 0\}$. Set $S^c := \mathbb{R}^L \backslash S$. If $w \in S^c$, then $\Gamma(w) \neq 0$ and by continuity of Γ there exists a neighbourhood $W \subset \mathbb{R}^L$ of w such that $W \subset S^c$. Consequently, S^c is open. Next we show that S^c is dense in $\mathbb{R}^L$. Seeking a contradiction, suppose that S^c is not dense in $\mathbb{R}^L$. Then there exists $z \in S$ and an open neighbourhood $Z \subset \mathbb{R}^L$ of z such that $Z \subset S$. The polynomial Γ_0 defined by $\Gamma_0(s) := \Gamma(s + z)$ for all $s \in \mathbb{R}^L$ has the property that $\Gamma_0(s) = 0$ for all $s \in Z_0$, where $Z_0 := \{s - z : s \in Z\}$. Obviously, Z_0 is an open neighbourhood of 0 and it follows from repeated partial differentiation that all coefficients of Γ_0 are zero. Thus, Γ_0 is the zero polynomial and so is Γ, yielding the desired contradiction.

(b) We prove the claim by induction over L. Trivially, the claim is true for $L = 1$. Let S be a proper algebraic set in $\mathbb{R}^{L+1}$. Then there exists a non-zero polynomial Γ in $L + 1$ variables such that $S = \{z \in \mathbb{R}^{L+1} : \Gamma(z) = 0\}$. Write Γ in the form

$$\Gamma(s_1, \ldots, s_{L+1}) = \sum_{i=0}^{k} \Delta_i(s_1, \ldots, s_L) s_{L+1}^i, \qquad (*)$$

where the Δ_i, $0 \leq i \leq k$, are polynomials in L variables. Set

$$Z := \bigcap_{i=1}^{k} Z_i, \quad \text{where } Z_i := \{z \in \mathbb{R}^L : \Delta_i(z) = 0\}, \quad 1 \leq i \leq k,$$

and let λ_L denote Lebesgue measure in $\mathbb{R}^L$. Since Γ is not the zero polynomial, there exists $j \in \{1, \ldots, k\}$ such that Δ_j is not the zero polynomial, and so, Z_j is a proper algebraic set in $\mathbb{R}^L$. By induction hypothesis, $\lambda_L(Z_j) = 0$, and consequently, $\lambda_L(Z) = 0$. Let $\sigma : \mathbb{R}^{L+1} \to \{0, 1\}$ be the characteristic function of S. Defining $\rho : \mathbb{R}^L \to \mathbb{R}$ by

$$\rho(s_1, \ldots, s_L) := \int_{-\infty}^{\infty} \sigma(s_1, \ldots, s_L, s_{L+1}) \mathrm{d}s_{L+1},$$

it follows from Fubini's theorem[2] that

$$\lambda_{L+1}(S) = \int_{\mathbb{R}^{L+1}} \sigma(s_1, \ldots, s_{L+1}) \mathrm{d}s_1 \ldots \mathrm{d}s_{L+1} = \int_{\mathbb{R}^L} \rho(s_1, \ldots, s_L) \mathrm{d}s_1 \ldots \mathrm{d}s_L. \quad (**)$$

Note that if $(s_1, \ldots, s_L) \in \mathbb{R}^L \backslash Z$, then, invoking $(*)$, we conclude that there are at most finitely many (not more than k) numbers $z \in \mathbb{R}$ such that $(s_1, \ldots, s_L, z) \in S$. Therefore, $\rho(s_1, \ldots, s_L) = 0$ for all $(s_1, \ldots, s_L) \in \mathbb{R}^L \backslash Z$ and, since $\lambda_L(Z) = 0$, it now follows from $(**)$ that $\lambda_{L+1}(S) = 0$.

Exercise 6.7
The monic polynomial P is given by $P(s) = (s+1)(s+2)(s+5) = s^3 + 8s^2 + 17s + 10$. Set

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b = Bv = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

[2] Guido Fubini (1897-1943), Italian.

in which case we have

$$A + BE = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad \mathcal{C}(A + BE, b) = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

The matrix $\mathcal{C}(A + BE, b)$ has full rank and so $(A + BE, b)$ is controllable. Moreover,

$$\mathcal{C}(A + BE, b)^{-1} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 1 & -1/2 \end{pmatrix}, \quad P(A + BE) = \begin{pmatrix} 46 & 0 & 60 \\ 41 & 84 & 22 \\ 60 & 0 & 76 \end{pmatrix}.$$

Therefore,

$$f^* = -(0, 0, 1)\mathcal{C}(A + BE, b)^{-1}P(A + BE) = (-11, -84, 16)$$

and

$$F = E + vf^* = \begin{pmatrix} -11 & -84 & 16 \\ 1 & 0 & 0 \end{pmatrix}.$$

Exercise 6.8
Let $A = -I$ and $B = 0$. Then $-I = A + BF$ is Hurwitz for all $F \in \mathbb{R}^{M \times N}$ but (A, B) is evidently not controllable.

Exercise 6.9
Note initially that

$$\mathcal{C}(A, b) = (b, Ab, A^2b) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

has determinant 0, and so the system fails to be controllable for all $\alpha \in \mathbb{R}$. On the other hand, we will show that the system is stabilizable for all $\alpha \in \mathbb{R}$. This we do by an application of the Hautus criterion for stabilizability. We have

$$\big(sI - A, b\big) = \begin{pmatrix} s-1 & -\alpha & 0 & 1 \\ 0 & s+1 & 0 & 0 \\ -1 & -1 & s-1 & 1 \end{pmatrix}.$$

Consider columns 1,2 and 4, in which case we have

$$\begin{vmatrix} s-1 & -\alpha & 1 \\ 0 & s+1 & 0 \\ -1 & -1 & 1 \end{vmatrix} = s(s+1)$$

which is non-zero for all $s \in \overline{\mathbb{C}}_+ \backslash \{0\}$ and all $\alpha \in \mathbb{R}$. By the Hautus criterion for stabilizability, we may conclude that the system is stabilizable for all $\alpha \in \mathbb{R}$ if we can show that, for $s = 0$, $\operatorname{rk}(sI - A, b) = 3$. Considering columns 2,3 and 4 of $(-A, b)$, we have

$$\begin{vmatrix} -\alpha & 0 & 1 \\ 1 & 0 & 0 \\ -1 & -1 & 1 \end{vmatrix} = -1.$$

Therefore, we have shown that, for all $\alpha \in \mathbb{R}$, $\operatorname{rk}\big(sI - A, b\big) = 3$ for all $s \in \overline{\mathbb{C}}_+$ and so, by the Hautus criterion, the system is stabilizable for all $\alpha \in \mathbb{R}$.

Exercise 6.10
Let $\lambda \in \sigma(A)$. If λ is uncontrollable, then an argument identical to that used in the proof of the necessity part of the eigenvalue-assignment theorem (Theorem 6.3) shows that $\lambda \in \sigma(A+BF)$ for all $F \in \mathbb{R}^{M\times N}$.
Conversely, assume that $\lambda \in \sigma(A+BF)$ for all $F \in \mathbb{R}^{M\times N}$. Then any monic real polynomial P of degree N such that $P(\lambda) \neq 0$ cannot be assigned to (A,B) and therefore, by the eigenvalue-assignment theorem, (A,B) is not controllable. If $B=0$, then, trivially, $\mathrm{rk}\,(\lambda I - A, 0) = \mathrm{rk}\,(\lambda I - A) < N$, showing that λ is uncontrollable. Let $B \neq 0$. Then, without loss of generality, we may assume that A and B take the form (Kalman controllability decomposition, Lemma 3.10):

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

where the pair (A_1, B_1) is controllable. For every $F = (F_1, F_2) \in \mathbb{R}^{M\times N}$, we have

$$A + BF = \begin{pmatrix} A_1 + B_1F_1 & A_2 + B_1F_2 \\ 0 & A_3 \end{pmatrix}, \quad \sigma(A+BF) = \sigma(A_1 + B_1F_1) \cup \sigma(A_3),$$

where the second identity follows form Theorem A.7. Since (A_1, B_1) is controllable, Theorem 6.3 ensures that we can choose F_1 such that $\lambda \notin \sigma(A_1 + B_1F_1)$. Consequently, $\lambda \in \sigma(A_3)$ and thus, $\mathrm{rk}\,(\lambda I - A, B) < N$, showing that λ is uncontrollable.

Exercise 6.11
Since, for all $z \in \mathbb{R}^N$, $(\nabla V)(z) = Pz$ and $\langle Pz, Az\rangle = \langle PAz, z\rangle = \langle A^*Pz, z\rangle$, we have

$$\langle (\nabla V)(z), Az\rangle = \langle Pz, Az\rangle = \langle (PA + A^*P)z, z\rangle/2. \qquad (*)$$

It is now immediate that, if $PA + A^*P = 0$, then $\langle (\nabla V)(z), Az\rangle = 0$ for all $z \in \mathbb{R}^N$.
Conversely, assume that $\langle (\nabla V)(z), Az\rangle = 0$ for all $z \in \mathbb{R}^N$. Then, by $(*)$, the matrix $Q := PA + A^*P$ satisfies $\langle Qz, z\rangle = 0$ for all $z \in \mathbb{R}^N$. Let $y, z \in \mathbb{R}^N$ be arbitrary. Exploiting the symmetry of Q, we have $\langle Qy, z\rangle = \langle Qz, y\rangle$. Therefore

$$0 = \langle Q(y+z), y+z\rangle = \langle Qy, y\rangle + \langle Qz, z\rangle + 2\langle Qy, z\rangle = 2\langle Qy, z\rangle.$$

and, since y and z are arbitrary, it follows that $Q = 0$.

Exercise 6.12
Note that

$$\mathrm{span}\{Az, Bz, z\} = \mathrm{span}\left\{ \begin{pmatrix} z_2 \\ -z_1 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\} = \mathbb{R}^2 \quad \forall\, z \in \mathbb{R}^2\backslash\{0\}.$$

Since $\mathrm{ad}^1(A,B) = I$, it follows that

$$\mathrm{span}\{Az, Bz, \mathrm{ad}^1(A,B)z, \mathrm{ad}^2(A,B)z, \ldots\} = \mathrm{span}\{Az, Bz, z\} = \mathbb{R}^2 \quad \forall\, z \in \mathbb{R}^2\backslash\{0\}.$$

Noting that $A + A^* = 0$, it follows from Corollary 6.15 that the feedback law $u(t) = -\langle x(t), Bx(t)\rangle = -x_1(t)x_2(t)$ is globally asymptotically stabilizing.

Exercise 6.13
(a) Let $N = 1$, $A = 1$ and $S = \{1\} \subset \mathbb{R}$. Then S is A-invariant, but S is not positively $\exp(At)$-invariant because $\exp(At) = e^t \neq 1$ for all $t > 0$.
Let $N = 1$, $A = -1$ and $S = (0, \infty) \subset \mathbb{R}$. For each $\xi \in S$, we have $\exp(At)\xi = e^{-t}\xi \in S$ for all $t \in \mathbb{R}$ and so S is $\exp(At)$-invariant. However, S is not A-invariant because,

for each $\xi \in S$, $A\xi = -\xi \notin S$.
(b) Let $S \subset \mathbb{R}^N$ be a subspace; since S is finite dimensional, it is closed. Assume that S is A-invariant. Set $E_n(t) := \sum_{k=0}^{n}(1/k!)(At)^k$ for all $n \in \mathbb{N}$ and let $\xi \in S$. Since S is an A-invariant subspace, we have $E_n(t)\xi \in S$ for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}_+$. By closedness of S, it follows that $\lim_{n\to\infty} E_n(t)\xi = \exp(At)\xi$ is in S for all $t \in \mathbb{R}_+$.
Now assume that the subspace S is positively $\exp(At)$-invariant. Let $\xi \in S$ be arbitrary. Then, for each $n \in \mathbb{N}$, $\zeta_n := n(\exp(An^{-1}) - I)\xi$ is in S and so, by closedness of S, $A\xi = \lim_{n\to\infty} \zeta_n \in S$. Therefore, S is A-invariant.
(c) Let $N = 1$, $A = 1$ and $S = [1, \infty) \subset \mathbb{R}$. For each $\xi \in S$, we have $\exp(At)\xi = e^t\xi \in S$ for all $t \in \mathbb{R}_+$ and so S is positively $\exp(At)$-invariant. However, S is not $\exp(At)$-invariant because, for each $\xi \in S$, $\exp(At)\xi = e^t\xi \to 0$ as $t \to -\infty$.
(d) Let $S \subset \mathbb{R}^N$ be a subspace. As a finite-dimensional subspace S is closed. By part (b), if S is positively $\exp(At)$-invariant, then S is A-invariant, and thus, by the closedness and subspace property of S, we conclude that $\exp(At)\xi = \sum_{k=0}^{\infty}(1/k!)(At)^k\xi$ is in S for all $\xi \in S$ and all $t \in \mathbb{R}$.

Exercise 6.14
(a) Writing $x_1(t) = y(t)$, $x_2(t) = \dot{y}(t)$ and $x_3(t) = z(t)$, we have $\dot{x}(t) = Ax(t) + u(t)Bx(t)$ with $A, B \in \mathbb{R}^{3\times 3}$ as given.
(b) By direct calculation, we have

$$\begin{aligned}
\mathrm{ad}^1(A,B) = [A,B] = AB - BA &= \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\mathrm{ad}^2(A,B) = [A, \mathrm{ad}^1(A,B)] &= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

By induction, we find that, for all $k \in \mathbb{N}$,

$$\mathrm{ad}^{2k-1}(A,B) = (-4)^{k-1}\mathrm{ad}^1(A,B), \quad \mathrm{ad}^{2k}(A,B) = (-4)^{k-1}\mathrm{ad}^2(A,B).$$

Therefore,

$$\begin{aligned}
\mathrm{span}\{Az, Bz, \mathrm{ad}^1(A,B)z, \ldots\} &= \mathrm{span}\{Az, Bz, \mathrm{ad}^1(A,B)z, \mathrm{ad}^2(A,B)z\} \\
&= \mathrm{span}\left\{ \begin{pmatrix} z_2 \\ -z_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -z_2 \\ z_3 \end{pmatrix}, \begin{pmatrix} -z_2 \\ -z_1 \\ 0 \end{pmatrix}, \begin{pmatrix} -z_1 \\ z_2 \\ 0 \end{pmatrix} \right\}
\end{aligned}$$

which is not equal to $\mathbb{R}^3$ for all $z \in \mathbb{R}^3$ of the form $z = (z_1, z_2, 0)$. Therefore, the hypotheses of Corollary 6.15 fail to hold.
(c) Set $\Omega := \{(z_1, z_2, z_3) = z \in \mathbb{R}^3 \colon z_3(z_1^2 + z_2^2) \neq 0\}$. Observe that $A + A^* = 0$ and

$$\mathrm{span}\{Az, Bz, \mathrm{ad}^1(A,B)z, \mathrm{ad}^2(A,B)z\} = \mathbb{R}^3 \quad \forall\, z \in \Omega.$$

Setting $\Gamma := \{z \in \mathbb{R}^3 \colon \langle z, Bz\rangle = 0\} = \{(z_1, z_2, z_3) \in \mathbb{R}^3 \colon z_2^2 - z_3^2 = 0\}$, we see that $(\mathbb{R}^3\backslash\Omega) \cap \Gamma = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_2 = 0 = z_3\}$ and the only positively $\exp(At)$-invariant subset thereof is $\{0\}$. Therefore, by Theorem 6.15, we may conclude that the feedback $u(t) = -\langle x(t), Bx(t)\rangle = x_2^2(t) - x_3^2(t) = \dot{y}^2(t) - z^2(t)$ is globally asymptotically stabilizing.

Exercise 6.15
Recall that $O \in \mathbb{R}^{N\times N}$ is said to be orthogonal if its columns form an orthonormal

basis of $\mathbb{R}^N$; equivalently, O is orthogonal if $O^*O = I$ (and so $O^{-1} = O^*$). We first show that, for every $N \in \mathbb{N}$ and every symmetric $P \in \mathbb{R}^{N\times N}$, there exists an orthogonal matrix $O \in \mathbb{R}^{N\times N}$ such that $\Lambda := O^*PO$ is diagonal (of course, the diagonal entries of Λ are the eigenvalues of P, and so each eigenvalue is real and recurs up to its algebraic multiplicity). This we prove by induction on N. For each $N \in \mathbb{N}$, let $\mathcal{P}(N)$ denote the statement:

$\mathcal{P}(N)$: "For each symmetric $P \in \mathbb{R}^{n\times N}$ there exists orthogonal $O \in \mathbb{R}^{N\times N}$ such that $\Lambda := O^TPO$ is diagonal."

Clearly, $\mathcal{P}(1)$ is a true statement. Assume that $N \in \mathbb{N}$ and $\mathcal{P}(N)$ is true. Let $P \in \mathbb{R}^{(N+1)\times(N+1)}$ be symmetric. Let $\lambda \in \mathbb{R}$ be an eigenvalue of P and let $v_1 \in \mathbb{R}^{N+1}$ be an associated eigenvector with $\|v_1\| = 1$. Let $v_2, \ldots, v_{N+1} \in \mathbb{R}^{N+1}$ be such that $O_1 := \big(v_1, v_2, \ldots, v_{N+1}\big)$ is an orthogonal matrix. Write $P_1 = O_1^*PO_1 = O_1^{-1}PO_1$ and so $O_1P_1 = PO_1$. Now, the first column of PO_1 is $Pv_1 = \lambda v_1$ and so the first column of O_1P_1 is also λv_1. Therefore, the first column of P_1 is $(\lambda\,,\, 0\,,\, \ldots\,,\, 0)^*$ and, since P_1 is symmetric, we may infer that the first row of P_1 is $(\lambda\,,\, 0\,,\, \ldots\,,\, 0)$. Therefore, $P_1 = \mathrm{diag}\big(\lambda\,,\, P_2\big)$, where $P_2 \in \mathbb{R}^{N\times N}$ is symmetric. By the induction hypothesis, there exists orthogonal $O_2 \in \mathbb{R}^{N\times N}$ such that $\Lambda_2 := O_2^*P_2O_2$ is diagonal. Writing $O_0 := \mathrm{diag}\big(1\,,\, O_2\big) \in \mathbb{R}^{(N+1)\times(N+1)}$, then O_0 is orthogonal and we have $O_0^*P_1O_0 = \mathrm{diag}\big(\lambda\,,\, \Lambda_2\big) =: \Lambda$. Finally, writing $O = O_1O_0$, then $O^*PO = O_0^*O_i^*PO_1O_0 = O_0^*P_1O_0 = \Lambda$. Therefore, $\mathcal{P}(N+1)$ is a true statement. By induction, it follows that $\mathcal{P}(N)$ is true for all $n \in \mathbb{N}$.

Now, let $P \in \mathbb{R}^{N\times N}$ be symmetric and positive definite. Then each eigenvalue of P is real and positive. Let $O \in \mathbb{R}^{N\times N}$ be orthogonal and such that $O^*PO = \Lambda$, where is diagonal. Define $\mu := \min\{\lambda\colon \lambda \in \sigma(P)\} > 0$ and $nu := \max\{\lambda\colon \lambda \in \sigma(P)\}$, and so μ (respectively, ν) is the smallest (respectively, largest) of the positive entries on the diagonal o f Λ. Then,

$$\langle z, Pz\rangle = \langle (O^*z), O^*PO(O^*z)\rangle = \langle (O^*z), \Lambda(O^*z)\rangle$$

and noting that, since O is orthogonal, $\|z\| = 1$ implies $\|O^*z\| = 1$, we have

$$\min_{\|z\|=1} \langle z, Pz\rangle = \min_{\|w\|=1} \rangle w, \Lambda w\rangle = \mu$$

and

$$\|P\| = \max_{\|z\|=1} \langle a, Pz\rangle = \max_{\|w\|=1} \langle w, \Lambda w\rangle = \nu.$$

Moreover, since $\sigma(P^{-1}) = \{1/\lambda\colon \lambda \in \sigma(P)\}$, we have $\|P^{-1}\| = 1/\mu$ and so $\mu = 1/\|P^{-1}\|$.

Exercise 6.16
Noting that, for all $s \in \mathbb{C}$ and all $\alpha \in \mathbb{R}$, $\mathrm{rk}\,(sI - A, b) = \mathrm{rk}\,(sI - (A - \alpha bc^*), b)$ and $\mathrm{rk}\,(sI - A^*, c) = \mathrm{rk}\,(sI - (A - \alpha bc^*)^*, c)$, the requisite results follow by the Hautus criteria for controllability and observability (Theorems 3.11 and 3.21).

Exercise 6.17
The transfer function $\hat{G}$ for the system is given by

$$\hat{G}(s) = c^*(sI - A)^{-1}b = \frac{1}{s^2 + s - 2}$$

and, with $\alpha = 2$ and $\beta = 3$, the rational function R is given by

$$R(s) = \big(1 + \beta\hat{G}(s)\big)\big(1 + \alpha\hat{G}(s)\big)^{-1} = \frac{s^2 + s + 1}{s^2 + s}$$

Clearly, R does not have any poles in the open right half plane C_+. The pole at $s = 0$ is semisimple and $\lim_{s\to 0} sR(s) = 1 > 0$. Also, $\operatorname{Re} R(i\omega) = \frac{\omega^2}{1+\omega^2} \geq 0$ for all $\omega \in \mathbb{R}$. Therefore, by Lemma 6.16, we may infer that R is positive real. The requisite results now follow by Theorem 6.17.

Exercise 6.18
Setting $A_\alpha := A - \alpha bc^*$, then, as in the proof of Theorem 6.17, (A_α, b, c^*) is a minimal realization of the strictly-proper rational function $\hat{G}_\alpha := \hat{G}/(1+\alpha\hat{G})$. By assumption $\hat{G}_\alpha$ is positive real and so, by the positive real lemma (Lemma 6.18), there exist a symmetric positive-definite matrix $P \in \mathbb{R}^{N\times N}$ and a vector $l \in \mathbb{R}^N$ such that $PA_\alpha + A_\alpha^* P = -ll^*$ and $Pb = c$. Let k_α, f, φ and V be as in the proof of Theorem 6.17. Then

$$\begin{aligned} V_f(z) = \langle (\nabla V)(z), A_\alpha z - bk_\alpha(c^*z)\rangle &= -(l^*z)^2 - 2(c^*z)k_\alpha(c^*z) \\ &\leq -2(c^*z)k_\alpha(c^*z) \quad \forall\, z \in \mathbb{R}^N. \end{aligned}$$

(a) Assume $k \in S[\alpha, \infty)$. Then $wk_\alpha(w) = wk(w) - \alpha w^2 \geq 0$ for all $w \in \mathbb{R}$ and so $V_f(z) \leq 0$ for all $z \in \mathbb{R}^N$. By the same argument as that used in the proof of Theorem 6.17, it follows that (6.39) holds.
(b) Now assume that $k \in S(\alpha, \infty)$. Then $wk_\alpha(w) > 0$ for all $w \in \mathbb{R}\backslash\{0\}$. Therefore, $V_f(z) < 0$ for all $z \notin \ker c^*$ and so $V_f^{-1}(0) \subset \ker c^*$. The same argument (based on LaSalle's invariance principle) as that used in the proof of Theorem 6.17 now applies to conclude that the equilibrium is globally asymptotically stable.

Exercise 6.19
In this case,

$$A = \begin{pmatrix} 0 & 1 \\ -\mu & 0 \end{pmatrix}, \quad b = c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and so $\hat{G}$ is given by $\hat{G}(s) = s/(s^2+\mu)$, which has simple poles at $\pm i\sqrt{\mu}$, each with residue $1/2$. Moreover, $\operatorname{Re}\hat{G}(i\omega) = 0$ for all $w \neq \pm\sqrt{\mu}$. By Lemma 6.16, $\hat{G}$ is positive real and the requisite results follow from Theorem 6.19 (with $\alpha = 0$).

Exercise 6.20
(a) Let $\tilde{A}$, $\tilde{b}$ and $\tilde{c}$ be as in the proof of Theorem 6.21. Furthermore, let $f\colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ be the locally Lipschitz function given by

$$f(z) := \tilde{A}z - \tilde{b}k(\tilde{c}^*z) + \tilde{d}, \quad \text{where } \tilde{d} := \begin{pmatrix} 0 \\ \gamma\rho \end{pmatrix} \in \mathbb{R}^{N+1}.$$

Then the initial-value problem (6.49) may be expressed in the form

$$\dot{\eta}(t) = f(\eta(t)), \quad \eta(0) = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}, \quad \text{where } \eta(t) := \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}.$$

By the global Lipschitz property of k, there exists $\lambda > 0$ such that $|k(\tilde{c}^*z) - k(0)| \leq \lambda|\tilde{c}^*z| \leq \lambda\|\tilde{c}\|\|z\|$ for all $z \in \mathbb{R}^{N+1}$. Therefore,

$$\|f(z)\| \leq \|\tilde{A}\|\|z\| + \lambda\|\tilde{b}\|\|\tilde{c}\|\|z\| + \|\tilde{b}\||k(0)| + \|\tilde{d}\| \quad \forall\, z \in \mathbb{R}^{N+1}.$$

Writing $L := \max\{\|\tilde{A}\| + \lambda\|\tilde{b}\|\|\tilde{c}\|, \|\tilde{b}\||k(0)| + \|\tilde{d}\|\}$, we have

$$\|f(z)\| \leq L\big(1 + \|z\|\big) \quad \forall\, z \in \mathbb{R}^{N+1}.$$

By Proposition 4.12, it now follows that the maximal solution of the initial-value problem (6.49) has interval of existence $\mathbb{R}$.
(b) For all $t \in \mathbb{R}_+$, we have

$$\begin{aligned}\dot{z}(t) = \dot{x}(t) = Ax(t) + bk(u(t)) &= Az(t) + b\big(k(u(t)) - k(u^\rho)\big)\\ = Az(t) + b\big(k(v(t) + u^\rho) - k(u^\rho)\big) &= Az(t) + b\tilde{k}(v(t))\end{aligned}$$

and

$$\begin{aligned}\dot{v}(t) = \dot{u}(t) = \gamma\big(\rho - c^*x(t)\big) &= \gamma\big(\rho - c^*z(t) + c^*A^{-1}bk(u^\rho)\big)\\ = \gamma\big(\rho - c^*z(t) - \hat{G}(0)k(u^\rho)\big) &= \gamma\big(\rho - c^*z(t) - \rho\big) = -\gamma c^* z(t).\end{aligned}$$

(c) Let $s \in \mathbb{C}$ and $z \in \mathbb{C}^N$ be arbitrary and assume that $z^*(sI - \tilde{A}, \tilde{b}) = 0$, where $\tilde{A}$ and $\tilde{b}$ are given by (6.53). By the Hautus criterion for controllability, it is sufficient to show that $z = 0$. Writing $z^* = (w^*, \bar{v})$, where $w \in \mathbb{C}^N$ and $v \in \mathbb{C}$, we obtain

$$z^*(sI - \tilde{A},\, \tilde{b}) = (w^*(sI - A) + \bar{v}\gamma c^*,\, s\bar{v},\, -w^*b) = 0.$$

Assume that $s \neq 0$. Then $v = 0$, and thus $w^*(sI - A, b) = 0$. Since (A, b) is controllable, the Hautus criterion for controllability implies that $w = 0$, and hence, $z = 0$. Now assume that $s = 0$. Then

$$-w^*A + \bar{v}\gamma c^* = 0, \quad w^*b = 0, \tag{$*$}$$

and consequently,

$$\bar{v}\gamma\hat{G}(0) = -\bar{v}\gamma c^*A^{-1}b = 0.$$

Since $\gamma\hat{G}(0) > 0$, we now conclude that $v = 0$. By $(*)$, $w^*(-A, b) = 0$, and so controllability of (A, b) together with the Hautus criterion yields that $w = 0$, and hence, $z = 0$.
(d) Observability of $(\tilde{c}^*, \tilde{A})$ follows from an argument similar to that employed in the solution of part (c).

Appendix

Exercise A.1
Let $x_1, \ldots, x_K \in \mathbb{F}^P$ be a basis of $\big(\ker M\big)^\perp$. Choose $x_{K+1}, \ldots, x_N \in \mathbb{F}^P$ arbitrarily and set $X := \big(x_1, \ldots, x_N\big) \in \mathbb{F}^{P\times N}$. Define $y_1 := Mx_i \in \mathbb{F}^N$, $i = 1, \ldots K$. First, we show that $y_1, \ldots, y_K$ are linearly independent. Assume $\alpha_1 \in \mathbb{F}$, $i = 1, \ldots, K$, and $\sum_{i=1}^K \alpha_1 y_i = 0$. Then $M\left(\sum_{i=1}^k \alpha_i x_i\right) = 0$ and so $\sum_{i=1}^k \alpha_i x_i \in \big(\ker M\big) \cap \big(\ker M\big)^\perp$. Therefore, $\sum_{i=1}^K \alpha_i x_i = 0$ and, by linear independence of $x_1, \ldots, x_k$, we may infer that $\alpha_i = 0$, $i = 1, \ldots, K$, whence linear independence of $y_1, \ldots, y_K$. Now choose vectors $y_{K+1}, \ldots, y_N \in \mathbb{F}^N$ such that $y_1, \ldots, y_N$ is a basis of $\mathbb{F}^N$. Set $Y := \big(y_1, \ldots, y_N\big) \in \mathbb{F}^{N\times N}$ and note that Y is invertible. Define $M^\sharp := XY^{-1} \in \mathbb{F}^{P\times N}$. Then

$$M^\sharp M x_1 = M^\sharp y_i = XY^{-1} y_i = x_i, \quad i = 1, \ldots, K,$$

and, since $x_1, \ldots, x_K$ is a basis of $\big(\ker M\big)^\perp$, we have $M^\sharp M x = x$ for all $x \in \big(\ker M\big)^\perp$. Moreover, if M has full rank, then $M^\sharp M = I$ and so M has a left inverse.

Exercise A.2
(a) The result holds vacuously if $S = \emptyset$ (the empty set is both open and closed). Thus, we restrict to the case wherein $S \neq \emptyset$.
First assume that S is closed. Let (x_n) be a convergent sequence in S with limit $x \in X$. Suppose that $x \notin S$. Then x is a point of the open set $X\backslash S$. Therefore, x has an open neighbourhood U with $U \subset X\backslash S$. Since $x_n \to x$ as $n \to \infty$, it follows that $x_n \in U \subset X\backslash S$ for all n sufficiently large. This contradicts the fact that (x_n) is a sequence in S. Therefore, $x \in S$.
Now assume that every convergent sequence in S has its limit in S. Suppose that S is not closed. Then $X\backslash S$ is non-empty and is not open, and so there exists $x \in X\backslash S$ such that $\mathbb{B}(x, \varepsilon) \cap S \neq \emptyset$ for all $\varepsilon > 0$. Thus, for each $n \in \mathbb{N}$, there exists $x_n \in \mathbb{B}(x, 1/n) \cap S$. The sequence (x_n) so constructed is a sequence in S with limit $x \in X\backslash S$. This contradicts the hypothesis that every convergent sequence in S has its limit in S. Therefore, our supposition is false and so S is closed.

(b) First assume that $x \in \mathrm{cl}(S)$. Suppose, for contradiction, that there exists $\varepsilon > 0$ such that $\mathbb{B}(x, \varepsilon) \cap S = \emptyset$. Then $X\backslash\mathbb{B}(x, \varepsilon)$ contains S and so, since $X\backslash\mathbb{B}(x, \varepsilon)$ is a closed set, it must also contain $\mathrm{cl}(S)$. Therefore, $\mathrm{cl}(S) \cap \mathbb{B}(x, \varepsilon) = \emptyset$, whcih contradicts the hypothesis that $x \in \mathrm{cl}(S)$. We have now shown that, if $x \in \mathrm{cl}(s)$, then $\mathbb{B}(x, \varepsilon) \cap S \neq \emptyset$ for all ε.
We prove the converse by contraposition. Assume that $x \in X$ is such that $x \notin \mathrm{cl}(S)$. Then, since $X\backslash\mathrm{cl}(S)$ is an open set, there exists $\varepsilon > 0$ such that $\mathbb{B}(x, \varepsilon) \subset X\backslash\mathrm{cl}(S)$. In particular, $\emptyset = \mathbb{B}(x, \varepsilon) \cap \mathrm{cl}(S) \supset \mathbb{B}(x, \varepsilon) \cap S$.

Exercise A.3
Let (x_n) be a sequence in a Banach space X and, for aech $n \in \mathbb{N}$, write $s_n = \sum_{k=1}^n x_k$. Assume that the series $\sum_{k=1}^\infty x_k$ is absolutely convergent, that is, assume that $\sum_{k=1}^\infty \|x_k\| < \infty$. Let $\varepsilon > 0$ be arbitrary. By absolute convergence of the series, there exists $N \in \mathbb{N}$ such that $\sum_{k=n}^\infty \|x_k\| < \varepsilon$ for all $n > N$. Now, forall $n, m \in \mathbb{N}$ with $n \geq m > N$, we have

$$\|s_n - s_m\| = \left\|\sum_{k=1}^n x_k - \sum_{k=1}^m x_k\right\| = \left\|\sum_{k=m}^n x_k\right\| \leq \sum_{k=m}^n \|x_k\| < \varepsilon$$

and so (s_n) is a Cauchy sequence in the Banach space X and so, by completeness, (s_n) converges in X.

Exercise A.4
Let $\mathbb{F}$ denote the underlying scalar field. For convenience, write $B = C_b(S, Y)$. It is straightforward to verify that B ia s vector space. By boundedness of the elements of B, we have $\|f\|_\infty < \infty$ for all $f \in B$. Moreover, (i) $\|f\|_\infty = 0$ if, and only if, $f = 0$; (ii) $\|\lambda f\|_\infty = |\lambda|\|f\|_\infty$ for all $(\lambda, f) \in \mathbb{F} \times B$; (iii) for all $f, g \in B$,

$$\|f+g\|_\infty = \sup_{x\in S}\|f(x)+g(x)\| \le \sup_{x\in S}\|f(x) + \sup_{x\in S}\|g(x)\| = \|f\|_\infty + \|g\|_\infty.$$

Therefore, $\|\cdot\|_\infty$ is a norm on B. We proceed to prove completeness of this normed space. Let (f_n) be a Cauchy sequence in B. Then, for every $(f_n(x))$ is a Cauchy sequence in the Banach space Y and so converges to a limit $f(x) \in Y$. To complete the proof, it suffices to show that the function $f\colon S \to Y$ is in B and $\|f - f_n\|_\infty \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ be arbitrary. Since (f_n) is a Cauchy sequence in B, there exists $N \in \mathbb{N}$ such that $\|f_n(x) - f_m(x)\| \le \varepsilon$ for all $x \in S$ and all $n, m \in \mathbb{N}$ with $n, m > N$. Passing to the limit $n \to \infty$ gives $\|f(x) - f_m(x)\| \le \varepsilon$ for all $x \in S$ and all $m > N$. Therefore, $\|f - f_m\|_\infty \le \varepsilon$ for all $m > N$ and so we may infer that $\|f - f_n\|_\infty \to 0$ as $n \to \infty$. By boundedness of the functions f_n, we see that f is bounded. It remains only to show that f is continuous. Let $s \in S$ and $\varepsilon > 0$ be arbitrary. Fix $n \in \mathbb{N}$ sufficiently large so that $\|f - f_n\|_\infty \le \varepsilon/3$. By continuity of f_n, there exists $\delta > 0$ such that $\|f_n(x) - f_n(s)\| \le \varepsilon/3$ for all $x \in S$ with $\|x - s\| \le \delta$. Therefore, for all $x \in S$ with $\|x - s\| \le \delta$,

$$\|f(x) - f(s)\| \le \|f(x) - f_n(x)\| + \|f_n(x) - f_n(s)\| + \|f_n(s) - f(s)\| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

whence continuity of f at s. Since $s \in S$ is arbitrary, f is continuous. Thus, we have shown that $f \in B$.

Exercise A.5
Noting that

$$\{\|Mz\|_p/\|z\|_p \colon z \in \mathbb{F}^Q\backslash\{0\}\} = \{\|Mz\|_p \colon z \in \mathbb{F}^Q,\ \|z\|_p = 1\},$$

it immediately follows that $\|M\| = \sup_{\|z\|_p=1}\|Mz\|_p$. Clearly, $\sup_{\|z\|_p=1}\|Mz\|_p \le \sup_{\|z\|_p\le 1}\|Mz\|_p =: \mu$. We will show that $\mu = \|M\|$. By continuity of $z \mapsto \|Mz\|_p$ and compactness of $\{z \in \mathbb{F}^Q \colon \|z\|_p \le 1\}$, there exists $\hat z \in \mathbb{F}^Q$, with $\|\hat z\|_p \le 1$, such that $\|M\hat z\|_p = \mu$. If $\hat z = 0$, then $\mu = 0 = \|M\|$. If $\hat z \ne 0$, then

$$\begin{aligned}\|M\| = \sup_{\|z\|_p=1}\|Mz\|_p \le \sup_{\|z\|_p\le 1}\|Mz\|_p = \mu = \|M\hat z\|_p = \|\hat z\|_p\|\hat z\|_p^{-1}\|M\hat z\|_p \\ \le \|\hat z\|_p^{-1}\|M\hat z\|_p \le \sup_{\|z\|_p=1}\|Mz\|_p = \|M\|.\end{aligned}$$

Therefore, $\mu = \|M\|$. This completes the proof of (A.11).

To conclude that (A.12) also holds, simply note that

$$\begin{aligned}&\inf\{\gamma \ge 0 \colon \|Mz\|_p \le \gamma\|z\|_p\ \forall\, z \in \mathbb{F}^Q\} \\ &\qquad = \inf\{\gamma \ge 0 \colon \|Mz\|_p \le \gamma\ \forall\, z \in \mathbb{F}^Q,\ \|z\|_p = 1\} \\ &\qquad = \inf\big\{\gamma \ge 0 \colon \sup_{\|z\|_p=1}\|Mz\|_p = \|M\| \le \gamma\big\} = \|M\|.\end{aligned}$$

Exercise A.6
First, assume that the improper integral converges to $F \in \mathbb{M}_\mathbb{F}$. Let $\varepsilon > 0$ be arbitrary.

Then there exists $r \geq a$ such that $\| \int_a^T f(t)\mathrm{d}t - F\| \leq \varepsilon/2$ for all $T \in [r, \infty)$. Therefore,

$$\begin{aligned}\left\| \int_\sigma^\tau f(t)\mathrm{d}t \right\| &= \left\| \int_a^\tau f(t)\mathrm{d}t - \int_a^\sigma f(t)\mathrm{d}t \right\| \\ &\leq \left\| \int_a^\tau f(t)\mathrm{d}t - F \right\| + \left\| \int_a^\sigma f(t)\mathrm{d}t - F \right\| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ \ \forall\, \sigma, \tau \in [r, \infty), \ \sigma \leq \tau.\end{aligned}$$

Now, assume that, for every $\varepsilon > 0$, there exists $r \geq a$ such that $\| \int_\sigma^\tau f(t)\mathrm{d}t\| \leq \varepsilon$ for all $\sigma, \tau \in [r, \infty)$ with $\sigma \leq \tau$. For each $n \in \mathbb{N}$ with $n \geq a$, define $F_n := \int_a^n f(t)\mathrm{d}t$. Let $\varepsilon > 0$ be arbitrary. By the hypothesis, there exists $r \geq a$ such that

$$\|F_n - F_m\| = \left\| \int_m^n f(t)\mathrm{d}t \right\| \leq \varepsilon \ \ \forall\, m, n \in \mathbb{N}, \ r \leq m \leq n.$$

Therefore, (F_n) is a Cauchy sequence in the Banach space $\mathbb{M}_\mathbb{F}$ and so converges. Denote its limit by F. We will show that the improper integral converges to F. Let $\varepsilon > 0$ be arbitrary. By the hypothesis in conjunction with convergencs of (F_n) to f, there exists $r \geq a$ such that

$$\|F_n - F\| \leq \frac{\varepsilon}{2} \ \ \forall\, n \in \mathbb{N}, \ n \geq r \ \ \text{and} \ \ \left\| \int_\sigma^\tau f(t)\mathrm{d}t \right\| \leq \frac{\varepsilon}{2} \ \ \forall\, \sigma, \tau \in [r, \infty), \ \sigma \leq \tau.$$

Let $\tau \geq r + 1$ be arbitrary and denote its integer part by $n = \lfloor \tau \rfloor \geq r$. Then

$$\begin{aligned}\left\| \int_a^\tau f(t)\mathrm{d}t - F \right\| &= \left\| \int_a^n f(t)\mathrm{d}t + \int_n^\tau f(t)\mathrm{d}t - F \right\| \\ &\leq \|F_n - F\| + \left\| \int_n^\tau f(t)\mathrm{d}t \right\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

and so the improper integral converges to F.

Exercise A.7
If $\gamma < 0$, then F is bounded and so is of class $\mathcal{E}_0$. If $\gamma \geq 0$, then, for all $\delta > \gamma$, there exists $c \in \mathbb{R}_+$ such that $\|f(t)\| \leq ce^{\delta t}$ and so $\|F(t\| \leq \int_0^t \|f(\tau)\|\mathrm{d}\tau \leq (c/\delta)e^{\delta t}$. Therefore, if $\gamma \geq 0$, then F is of class $\mathcal{E}_\gamma$. Defining $\beta := \max\{\gamma, 0\}$, it follows that F is of class $\mathcal{E}_\beta$. Since $\beta \geq \gamma$ and f is of class $\mathcal{E}_\gamma$, f is *a fortiori* of class $\mathcal{E}_\beta$. Therefore, $f, F \in \mathcal{E}_\beta$. Define $F^\nabla := f$. Let E denote the set of points in $\mathbb{R}_+$ at which f fails to be continuous. Then, $f(t) = F'(t) = F^\nabla(t)$ for all $t \in \mathbb{R}_+ \backslash E$. Moreover, $F, F^\nabla \in \mathcal{E}_\beta$ and, by the result in part (1) of Theorem A.37, we have

$$\mathcal{L}\{f\}(s) = \mathcal{L}\{F^\nabla\}(s) = s\mathcal{L}\{F\}(s) - F(0) = s\mathcal{L}\{F\}(s) \ \ \forall\, s \in \mathbb{C}_\beta$$

and so $\mathcal{L}\{F\}(s) = (1/s)\mathcal{L}\{f\}(s)$ for all $s \in \mathbb{C}_\beta$.