CONTROL-THEORY AND ADVANCED TECHNOLOGY
Vol.4, No.4, pp.429-446, December, 1988

C87046R
©MITA PRESS

Contributed Paper

LOW-GAIN CONTROL OF DISTRIBUTED PARAMETER SYSTEMS WITH UNBOUNDED CONTROL AND OBSERVATION*

H. LOGEMANN,[1] J. BONTSEMA[2] AND D. H. OWENS[3]

Abstract. This paper deals with the problem of multivariable low-gain PI-control of stable distributed parameter systems with unbounded control and observation operators. We show that, under very mild assumptions, a fairly large class of distributed parameter systems can be stabilized and regulated by multivariable PI-controllers with sufficiently low gain. The controller design can be accomplished using plant step data, no exact knowledge of the plant being required.

Key Words—Infinite dimensional systems, unbounded control and observation, low-gain control, robust control, multivariable tuning regulators, PI-controllers.

1. Introduction

The problem of finite-dimensional control of infinite-dimensional systems by output feedback has received a considerable amount of attention in recent years, cf. e.g. Balas (1986), Curtain (1984), Curtain and Salamon (1986), Jacobson and Nett (1988), Kamen et al. (1985), Logemann (1986), Nett (1984) and Schumacher (1983). Unfortunately, the order of the controllers derived by the above authors may be quite high in certain cases. Moreover, if approximation techniques are used (cf. Balas, 1986; Jacobson and Nett 1988; Kamen et al., 1985; Logemann, 1986; Nett, 1984) the relationship between the particular approximation method and the order of the stabilizing controller is not yet understood. Intuitively, it is clear that restrictions on the plant, such as minimum-phase or stability should lead to simple low-order controllers. Logemann and Owens (1987) have shown that a large class of infinite-dimensional *minimum-phase* systems can be stabilized by PI-controllers with sufficiently *high gains*. Furthermore, it is proved in Logemann and Owens (1987) that the proposed PI-controllers have nice robustness properties and that they achieve almost decoupling and almost perfect tracking at high-gain.

In this paper, we shall study the "dual" situation, i.e., we investigate the problem of *low-gain* PI-control of a certain class of *stable* distributed parameter

* Received by the editors October 30, 1987 and in revised form September 19, 1988.
This work was supported by SERC under grant No. GR/D/45710 and the European Economic Community Contract No. EEC ST2J/0153-1-UK.

[1] Institut für Dynamische Systeme, Universität Bremen, Postfach 330 440, 2800 Bremen 33, West Germany.
[2] Mathematics Institute, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands.
[3] Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, Scotland.

systems. We do not assume that the plant is exactly known. However, we suppose that the designer has access to reliable plant step data.

In finite dimensions, this problem has been considered by Davison (1976). He proved that, under very mild assumptions, a lumped stable plant can be stabilized and regulated by a multivariable PI-controller of the form $(kK_I/s)+K_p$ in some interval $0<k<k^*$. Davisons result (Davison, 1976) was generalized to a class of distributed parameter systems by Pohjolainen (1982; 1985) and to certain time-delay systems by Koivo and Pohjolainen (1985) and Jussila and Koivo (1986). The papers (Davison, 1976; Jussila and Koivo, 1986; Koivo and Pohjolainen, 1985; Pohjolainen, 1982) and Pohjolainen (1985) are all based on state-space methods. Logemann and Owens presented in Logemann and Owens (to appear) a systematic input-output theory of Davisons multivariable tuning regulator for infinite-dimensional systems using the frequency-domain framework provided by Callier and Desoer (1978; 1980). In particular, they showed that Davisons result (Davison, 1976) extends to neutral systems, Volterra integrodifferential systems and Volterra integral systems.

In this paper, we prove the existence of multivariable tuning regulators for a certain class of distributed parameter systems with unbounded control and observation using the frequency-domain results of Logemann and Owens (to appear). The class of distributed parameter systems considered in this paper contains a large number of systems which are not covered by the theory developed in Pohjolainen (1982; 1985).

The organisation of the paper is as follows. Section 2 is devoted to preliminaries. In Sec. 3, we introduce the class of distributed systems we shall deal with. We show that the impulse response of a system belonging to this particular class is an element of $(L_-^1(\mathscr{R}_+))^{m\times m}$. The proof of existence of multivariable tuning regulators for the class of systems under consideration is broken up into two parts. In Sec. 4, we prove that *input-output* stability of the closed loop implies *internal* stability of the closed loop. The tracking property of the feedback system in the presence of certain disturbances is shown in Sec. 5. As an example, we consider the heat equation with homogeneous Dirichlet boundary conditions, point control and point observations in Sec. 6. An auxiliary result used in Sec. 4 is proved in the Appendix.

2. Preliminaries

Let $\mathscr{R}_+$ denote the interval $[0, \infty)$ and set $\mathscr{C}_+ \triangleq \{s\in\mathscr{C} \mid \mathrm{Re}(s)\geq 0\}$. Suppose f is a distribution form

$$f = \sum_{i=0}^{\infty} f_i\delta_{t_i} + f_a, \tag{2.1}$$

where $t_0 \triangleq 0$, $t_i>0$, $\forall i\geq 1$, δ_{t_i} denotes the Dirac distribution at t_i, $f_i\in\mathscr{C}$ and f_a is a $\mathscr{C}$-valued Lebesgue measurable function. The set A consists of all distributions f of the form (2.1) such that

$$\sum_{i=0}^{\infty} |f_i| + \int_0^{\infty} |f_a(t)|\,dt < \infty.$$

A is a convolution algebra. It is useful to define the following subalgebras of A:

$$A_- \triangleq \{f \in A \mid \exists \varepsilon > 0\colon f(\cdot)\exp(\varepsilon(\cdot)) \in A\},$$
$$L^1_-(\mathcal{R}_+) \triangleq \{f \in L^1(\mathcal{R}_+) \mid \exists \varepsilon > 0\colon f(\cdot)\exp(\varepsilon(\cdot)) \in L^1(\mathcal{R}_+)\},$$
$$L_e(\mathcal{R}_+) \triangleq \{f\colon \mathcal{R}_+ \to \mathcal{C} \mid \exists \varepsilon > 0,\ M > 0\colon |f(t)| \le M\exp(-\varepsilon t) \text{ a.e.}\}.$$

Moreover we define

$$\hat{A} \triangleq \{\hat{f} \mid f \in A\},$$

where $\hat{f}$ denotes the Laplace transform of f. It is now clear what is meant by $\hat{A}_-$. If $f \in \hat{A}$ let $\check{f}$ denote its inverse Laplace transform. Finally if M is a square matrix let $\sigma(M)$ denote the spectrum of M.

Remark 2.1: (i) Let $\theta(t) \triangleq \begin{cases} 1, & t>0 \\ 0, & t\le 0 \end{cases}$ denote the unit step function and let $f \in A_-$. Then, there exist $M>0$ and $\varepsilon>0$ such that

$$|(f*r\theta)(t) - \hat{f}(0)r| \le M\exp(-\varepsilon t)|r|,$$

$\forall t \ge 0$, $\forall r \in \mathcal{R}$ (for the proof, see Callier and Winkin, 1986).

(ii) The subalgebra $L_e(\mathcal{R}_+) \subset A_-$ is an ideal of A_- (cf. Callier and Winkin, 1986).

In order to deal with unstable systems, it is useful to introduce the algebra,

$$\hat{B} \triangleq \hat{A}_-(\hat{A}^\infty_-)^{-1} = \left\{\frac{n}{d} \mid n \in \hat{A}_-,\ d \in \hat{A}^\infty_-\right\},$$

i.e., the quotient ring of $\hat{A}_-$ with respect to the multiplicative subset

$$\hat{A}^\infty_- \triangleq \{f \in \hat{A}_- \mid \exists R>0\colon \inf_{\substack{s \in \mathcal{C}_+ \\ |s| \ge R}} |f(s)| > 0\}$$

(cf. Callier and Desoer, 1978; 1980).

Definition 2.1. Let $G \in \hat{B}^{m\times q}$ and $K \in \hat{B}^{q\times m}$. The feedback system shown in Fig. 1 is called stable, if the matrix

$$H(G,K) \triangleq \begin{bmatrix} (I+GK)^{-1}K & -(I+KG)^{-1}KG \\ (I+GK)^{-1}GK & (I+GK)^{-1}G \end{bmatrix} \tag{2.2}$$

is in $\hat{A}_-^{(m\times q)\times(m\times q)}$. If the feedback scheme in Fig. 1 is stable we shall say that K stabilizes G.

Remark 2.2: (i) If K stabilizes G, then it follows in particular that $(I+GK)^{-1} \in \hat{A}_-^{m\times m}$ and $(I+KG)^{-1} \in \hat{A}_-^{q\times q}$.

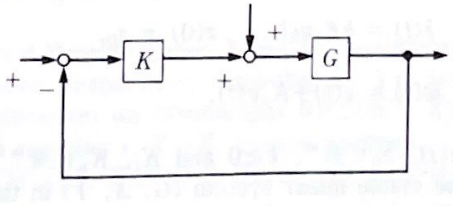


Fig. 1.

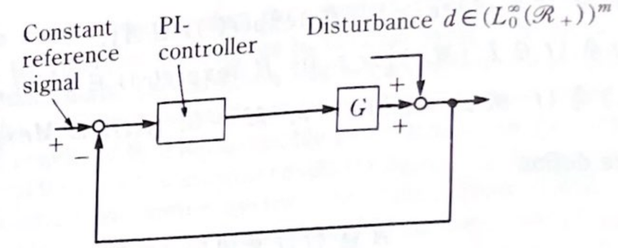


Fig. 2.

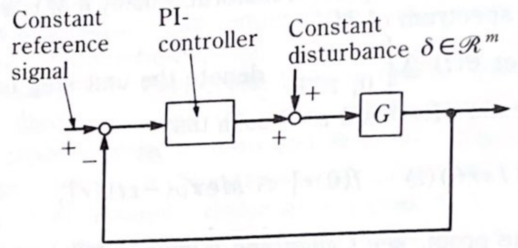


Fig. 3.

(ii) Let $G\in\hat{A}^{m\times q}$ and suppose that $\check{K}\in\hat{B}^{q\times q}$. Then, K stabilizes G, if and only if $(I+KG)^{-1}K\in\hat{A}_-^{q\times m}$.

In the sequel, we shall study the tracking properties of the feedback system in Fig. 2. We assume that the impulse response of the plant is in $A_-^{m\times m}$ and that the disturbance d is a function in the space $(L_0^\infty(\mathscr{R}_+))^m$, where

$$L_0^\infty(\mathscr{R}_+)\triangleq\{f\in L^\infty(\mathscr{R}_+)\,|\lim_{t\to\infty}f(t)\text{ exists}\}.$$

Realize that Fig. 3 (where $G\in\hat{A}_-^{m\times m}$ and $\delta\in\mathscr{R}^m$) is a special case of Fig. 2 with $d(t)\triangleq(\check{G}*\delta\theta)(t)$. Indeed, by Remark 2.1 (i), it is obvious that $\check{G}*\delta\theta\in(L_0^\infty(\mathscr{R}_+))^m$.

In order to allow for non-zero initial conditions in the plant the following definition is useful:

Definition 2.2. A stable linear system with m inputs and m outputs is a triple (G, X, T), where $G\in\hat{A}_-^{m\times m}$, X denotes a vector space and T is a linear operator mapping X into $(L_0^\infty(\mathscr{R}_+))^m$. The output y of the system (G, X, T) according to the input u and the initial condition $x_0\in X$ is given by $y=\check{G}*u+Tx_0$.

Consider the PI-controller

$$\dot{z}(t)=kK_Iv(t),\qquad z(0)=z_0,\tag{2.3a}$$

$$w(t)=z(t)+K_pv(t),\tag{2.3b}$$

where $v(t)$, $w(t)$, $z(t)$, $z_0\in\mathscr{R}^m$, $k\geq 0$ and K_I, $K_p\in\mathscr{R}^{m\times m}$. We apply the controller (2.3) to the stable linear system (G, X, T) in the presence of the initial condition $x_0\in X$ and the disturbance $d\in(L_0^\infty(\mathscr{R}_+))^m$,

$$y(t) = (\check{G}*w)(t) + (Tx_0)(t) + d(t), \tag{2.4}$$

$$v(t) = r\theta(t) - y(t), \qquad r \in \mathcal{R}^m. \tag{2.5}$$

Theorem 2.1. Let (G, X, T) be a given stable linear system with m inputs and m outputs. Suppose that $G(0)$ is non-singular. Let $(K_p, K_I) \in \mathcal{R}^{m\times m} \times \mathcal{R}^{m\times m}$ be a pair of matrices such that K_p stabilizes G and K_I satisfies the condition

$$\sigma((I+G(0)K_p)^{-1}G(0)K_I) \subset \mathring{\mathscr{C}}_+.$$

Then there exists $k^*>0$ such that the PI-controller $K_k(s)=(kK_I/s)+K_p$ stabilizes G, $\forall 0<k<k^*$. Moreover in the range $0<k<k^*$ the closed-loop system defined by (2.3)–(2.5) tracks constant reference signals (i.e., $\lim_{t\to\infty}y(t)=r$) in the presence of arbitrary initial conditions $(x_0, z_0) \in X\times\mathcal{R}^m$ and arbitrary disturbances $d \in (L_0{}^\infty(\mathcal{R}_+))^m$.

The proof of Theorem 2.5 can be found in Logemann and Owens (to appear).

Remark 2.3: There exist always pairs of matrices (K_p, K_I) which satisfy the conditions of the theorem. For example choose K_p such that the condition,

$$\bar{\sigma}(K_p) < \frac{1}{\sup\limits_{\omega\in\mathcal{R}} (\bar{\sigma}(G(i\omega))}, \tag{2.6}$$

is satisfied and set $K_I = G(0)^{-1}(I+G(0)K_p)$. In (2.6), $\bar{\sigma}(.)$ denotes the largest singular value of its argument.

3. System description

Consider the following linear process

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)(Bu(\tau)+d_0)d\tau, \tag{3.1a}$$

$$y(t) = Cx(t) + d_1, \tag{3.1b}$$

where $S(t)$ is an exponentially stable C_0-semigroup with infinitesimal generator A on the real Banach space X, $u(t)$ and $y(t)$ are vectors in $\mathcal{R}^m$ and $x_0 \in X$. The disturbances d_0 and d_1 are assumed to be constant vectors in X and $\mathcal{R}^m$, respectively. Let $M_0>0$ and $\alpha_0>0$ be constants such that

$$\|S(t)\|_{L(X,X)} \le M_0\exp(-\alpha_0 t), \qquad \forall t \ge 0. \tag{3.2}$$

We need the following assumptions on the system (3.1) (cf. Curtain and Pritchard, 1978)

(A1) There exist real Banach spaces $\underline{X}$ and $\bar{X}$ (the output state-space and the input state-space, respectively) such that $\underline{X} \subset X$ is dense in X and $X \subset \bar{X}$ is dense in $\bar{X}$. Moreover we assume that $B \in L(\mathcal{R}^m, \bar{X})$ and $C \in L(\underline{X}, \mathcal{R}^m)$.

(A2) The canonical injection ι: $\underline{X} \to X$, $x \to x$ is bounded.

(A3) $S(t) \in L(\bar{X}, X) \cap L(X, \underline{X})$, $\forall t>0$. There exists $t_1>0$ such that

$$\|S(t)\|_{L(X,\underline{X})} \in L^p(0,\ t_1) \quad \text{and} \quad \|S(t)\|_{L(\bar{X},X)} \in L^q(0,\ t_1),$$
$$\frac{1}{p} + \frac{1}{q} = 1, \qquad p \geq 1.$$

It follows from (A2) and (A3) that $S(t) \in L(\underline{X},\ \underline{X})$, $\forall t>0$. Furthermore, we assume

(A4) $$\|S(t)\|_{L(\underline{X},\underline{X})} \in L^\infty(0,\ t_1).$$

The following technical lemma will be useful for later purposes:

Lemma 3.1. Suppose that (A1) and (A3) are satisfied. Then, we have $S(t) \in L(\bar{X},\ \underline{X})$, $\forall t>0$ and $\|S(t)\|_{L(\bar{X},\underline{X})} \in L^1_-(\mathscr{R}_+)$.

Proof. Let $x \in \bar{X}$ and $t>0$. Then, it follows from (A1) and (A3) that

$$S(t)x = S\left(\frac{t}{2}\right)S\left(\frac{t}{2}\right)x \in \underline{X}$$

and

$$\|S(t)x\|_{\underline{X}} \leq \left\|S\left(\frac{t}{2}\right)\right\|_{L(X,\underline{X})}\left\|S\left(\frac{t}{2}\right)\right\|_{L(\bar{X},X)}\|x\|_{\bar{X}}.$$

Hence, we have shown that $S(t) \in L(\bar{X},\ \underline{X})$. Moreover, we obtain for fixed $0<t_0\leq t_1$ and $0<\varepsilon<\alpha_0$.

$$\begin{aligned}
&\int_0^\infty \|S(t)\|_{L(\bar{X},\underline{X})}\exp(\varepsilon t)dt \\
&\quad = \sum_{i=0}^{\infty}\int_0^{t_0} \|S(t+it_0)\|_{L(\bar{X},\underline{X})}\exp(\varepsilon(t+it_0))dt \\
&\quad = \sum_{i=0}^{\infty}\int_0^{t_0} \left\|S\left(\frac{t}{2}+it_0+\frac{t}{2}\right)\right\|_{L(\bar{X},\underline{X})}\exp(\varepsilon(t+it_0))dt \\
&\quad \leq \sum_{i=0}^{\infty}\int_0^{t_0} \left\|S\left(\frac{t}{2}\right)\right\|_{L(X,\underline{X})}\|S(it_0)\|_{L(X,X)}\exp(\varepsilon it_0) \\
&\qquad \times \left\|S\left(\frac{t}{2}\right)\right\|_{L(\bar{X},X)}\exp(\varepsilon t)dt \\
&\quad \leq \sum_{i=0}^{\infty} M_0 N\exp((\varepsilon-\alpha_0)it_0) \\
&\quad = M_0 N\frac{1}{1-\exp((\varepsilon-\alpha_0)t_0)} < \infty,
\end{aligned}$$

where $N \triangleq \exp(\varepsilon t_0)\int_0^{t_0}\|S(t/2)\|_{L(X,\underline{X})}\|S(t/2)\|_{L(\bar{X},X)}dt<\infty$ by (A3), and Hölder's inequality.

Remark 3.1: The idea behind the proof of Lemma 3.1 is due to Przyluski (1980).

Corollary 3.1. Assume that (A1) and (A3) are satisfied. Under this condition the function $CS(t)B$ is an element in $(L^1_-(\mathscr{R}_+))^{m\times m} \subset (A_-)^{m\times m}$. Hence, the transfer matrix of (3.1) is in $(\hat{A}_-)^{m\times m}$.

Proof. Notice that

$$\|CS(t)B\|_{L(\mathscr{R}^m,\mathscr{R}^m)} \leq \|C\|_{L(\mathscr{X},\mathscr{R}^m)}\|S(t)\|_{L(\mathscr{X},\mathscr{X})}\|B\|_{L(\mathscr{R}^m,\mathscr{X})}$$

and apply Lemma 3.1.

4. Internal stabilization of system (3.1) by PI-control

Consider the PI-controller

$$\dot{z}(t) = K_I v(t), \qquad z(0) = z_0, \tag{4.1a}$$

$$w(t) = z(t) + K_p v(t), \tag{4.1b}$$

where $v(t)$, $w(t)$, $z(t)$, $z_0 \in \mathscr{R}^m$ and K_I, $K_p \in \mathscr{R}^{m\times m}$.

We shall study the feedback interconnection of (3.1) and (4.1), i.e.,

$$v(t) = r(t) - y(t), \tag{4.2}$$

$$u(t) = w(t), \tag{4.3}$$

where the external signal r is assumed to be locally integrable. Define

$$S_e(t) \triangleq \begin{bmatrix} S(t) & 0 \\ 0 & I_m \end{bmatrix}, \qquad B_e \triangleq \begin{bmatrix} B & 0 \\ 0 & K_I \end{bmatrix},$$

$$F_e \triangleq \begin{bmatrix} -K_pC & I \\ -C & 0 \end{bmatrix}, \qquad x_e(t) \triangleq \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \qquad x_{e0} \triangleq x_e(0).$$

Using these notations the dynamics of the closed-loop system given by (3.1), (4.1), (4.2) and (4.3) are determined by

$$x_e(t) = S_e(t)x_{e0} + \int_0^t S_e(t-\tau)B_e u_e(\tau)d\tau + \int_0^t S_e(t-\tau)B_e\begin{bmatrix} K_p(r(\tau)-d_1) \\ r(\tau)-d_1 \end{bmatrix}d\tau + \int_0^t S_e(t-\tau)\begin{bmatrix} d_0 \\ 0 \end{bmatrix}d\tau, \tag{4.4}$$

$$u_e(t) = F_e x_e(t). \tag{4.5}$$

Let us assume for a moment that $d_0 = 0$, $d_1 = 0$ and $r(t) \equiv 0$. Then (4.4) and (4.5) become

$$x_e(t) = S_e(t)x_{e0} + \int_0^t S_e(t-\tau)B_e u_e(\tau)d\tau, \tag{4.6}$$

$$u_e(t) = F_e x_e(t). \tag{4.7}$$

It follows from (A1) and (A3) that $S_e(t)$, B_e and F_e satisfy the assumptions of

Theorem 8.9 in Curtain and Pritchard (1978). Hence, Eqs. (4.6) and (4.7) have a unique solution,

$$x_e(t) = V(t)x_{e0},$$

where $V(t)$ is a C_0-semigroup on $X_e \triangleq X \times \mathcal{R}^m$ which is the unique solution of

$$V(t)x_{e0} = S_e(t)x_{e0} + \int_0^t S_e(t-\tau)B_eF_eV(\tau)x_{e0}d\tau. \tag{4.8}$$

Moreover, it follows from (A1) and (A3) that the function

$$x_e(t) \triangleq V(t)x_{e0} + \int_0^t V(t-\tau)\left\{B_e\begin{bmatrix} K_p(r(\tau)-d_1) \\ r(\tau)-d_1 \end{bmatrix} + \begin{bmatrix} d_0 \\ 0 \end{bmatrix}\right\}d\tau \tag{4.9}$$

is well defined and solves Eqs. (4.4) and (4.5). This can be shown using the methods of Curtain and Pritchard (1978, p.223) (cf. also Curtain, 1984). We omit the proof because we shall make no further use of the variation of constants formula (4.9).

Let $G(s)$ and $K(s)$ denote the transfer matrix of (3.1) and (4.1), respectively, i.e.,

$$G(s) \triangleq \int_0^\infty CS(t)B\exp(-st)dt,$$

$$K(s) \triangleq \frac{1}{s}K_I + K_p.$$

The next result shows that under certain conditions i/o-stability of the closed-loop system implies internal stability of the closed-loop system.

Theorem 4.1. Suppose that (A1)–(A4) are satisfied and that $\det(K_I) \neq 0$. If the feedback system given by (3.1), (4.1), (4.2) and (4.3) is i/o-stable, i.e., $H(G, K) \in \hat{A}_-^{2m\times 2m}$, then the feedback semigroup $V(t)$ given by (4.8) is exponentially stable.

Proof. Let $d_0 = 0$, $d_1 = 0$ and $r(t) \equiv 0$ (cf. (3.1) and (4.2)). Then an easy computation yields

$$\hat{u} = (I+KG)^{-1}\hat{z}_0 - (I+KG)^{-1}K\hat{y}_0, \tag{4.10}$$

where $z_0(t) \triangleq z_0\theta(t)$, $y_0(t) \triangleq CS(t)x_0$.

Using the fact that $\det(K_I) \neq 0$ and the i/o-stability of the closed-loop system it is not difficult to show that $G(0)$ is non-singular (cf. Logemann and Owens, to appear). Therefore we have $(I+KG)^{-1}(0) = 0$ and by Remark 2.1 (i) there exist M_1, $\alpha_1 > 0$ such that

$$|((\delta_0 I + \check{K} * \check{G})^{-1} * z_0\theta)(t)| < M_1\exp(-\alpha_1 t)|z_0|, \quad \forall t \geq 0. \tag{4.11}$$

Define

$$F \triangleq (I+KG)^{-1}K,$$

$$y_1(t) \triangleq \begin{cases} y_0(t), & 0 \le t \le t_0, \\ 0, & t > t_0, \end{cases}$$

$$y_2(t) \triangleq \begin{cases} 0, & 0 \le t \le t_0, \\ y_0(t), & t > t_0, \end{cases}$$

where $t_0 \in (0, t_1]$ is fixed.

Let us suppose for a while that $x_0 \in \underline{X}$. Then

$$|y_2(t)| \le \|C\|_{L(\underline{X},\mathcal{R}^m)} \|\,\|M_0\| S(t_0)\|_{L(X,\underline{X})} \times \exp(\alpha_0 t_0)\exp(-\alpha_0 t)\|x_0\|_{\underline{X}}, \quad \forall t \ge 0, \tag{4.12}$$

where we have used the equality

$$S(t)x_0 = S(t_0)S(t-t_0)\iota x_0,$$

(A2), (A3) and (3.2). Moreover, we obtain

$$|y_1(t)| \le \|C\|_{L(\underline{X},\mathcal{R}^m)} \|S(t)\|_{L(\underline{X},\underline{X})} \|x_0\|_{\underline{X}}, \quad \forall t > 0,$$

and hence by (A4)

$$|y_1(t)| \le \|C\|_{L(\underline{X},\mathcal{R}^m)} \operatorname*{ess\,sup}_{0\le t\le t_0} \|S(t)\|_{L(\underline{X},\underline{X})} \times \exp(t_0)\exp(-t)\|x_0\|_{\underline{X}} \quad \text{a.e. on } \mathcal{R}_+. \tag{4.13}$$

It follows from (4.12) and (4.13) via Remark 2.1 (ii) that there exist M_2, $\alpha_2 > 0$ such that

$$|(\check{F} * y_0)(t)| \le M_2 \exp(-\alpha_2 t)\|x_0\|_{\underline{X}} \quad \text{a.e. on } \mathcal{R}_+. \tag{4.14}$$

By (4.10), (4.11) and (4.14), there exist constants M_3, $\alpha_3 > 0$ such that

$$|u(t)| \le M_3 \exp(-\alpha_3 t)(\|x_0\|_{\underline{X}} + |z_0|) \quad \text{a.e. on } \mathcal{R}_+. \tag{4.15}$$

Using (3.1a) we obtain

$$\|x(t)\|_{\underline{X}} \le \|S(t_0)\|_{L(X,\underline{X})} \|\iota\| M_0 \exp(\alpha_0 t_0)\exp(-\alpha_0 t)\|x_0\|_{\underline{X}} + M_4 \exp(-\alpha_4 t)(\|x_0\|_{\underline{X}} + |z_0|), \quad \forall t \ge t_0, \tag{4.16}$$

where the positive constants M_4 and α_4 exist by Lemma 3.1, (4.15) and Remark 2.1 (ii). With appropriately chosen constants M_5, $\alpha_5 > 0$, we can rewrite (4.16) as follows:

$$\|x(t)\|_{\underline{X}} \le M_5 \exp(-\alpha_5 t)(\|x_0\|_{\underline{X}} + |z_0|), \quad \forall t \ge t_0. \tag{4.17}$$

Now realize that, by (4.3),

$$u(t) = w(t) = z(t) - K_p C x(t). \tag{4.3'}$$

Hence, by (4.15) and (4.17), there exist positive constants M_6 and α_6 such that

$$\|z(t)| \leq M_6 \exp(-\alpha_6 t)(\|x_0\|_X + |z_0|), \qquad \forall t \geq t_0.$$

Therefore

$$\|V(t)\|_{L(\underline{X}_e, \underline{X}_e)} \leq M_7 \exp(-\alpha_7 t), \qquad \forall t \geq t_0, \tag{4.18}$$

where $X_e \triangleq X \times \mathscr{R}^m$ and M_7 and α_7 are suitable positive constants. Define ι_e: $\underline{X}_e \to X_e$, $x_e \to x_e$. For $x_e \in \underline{X}_e$, we have

$$\|V(t)x_e\|_{X_e} \leq \|\iota_e\| \, \|V(t-t_0)\|_{L(X_e, \underline{X}_e)} \|V(t_0)\|_{L(X_e, \underline{X}_e)} \|x_e\|_{X_e}. \tag{4.19}$$

Without restriction of generality, we can assume that $V(t_0) \in L(X_e, \underline{X}_e)$. This is justified by Appendix. Combining (4.18) and (4.19), it follows for all $x_e \in \underline{X}_e$

$$\|V(t)x_e\|_{X_e} \leq \|\iota_e\| \, \|V(t_0)\|_{L(X_e, \underline{X}_e)} M_7 \exp(\alpha_7 t_0) \exp(-\alpha_7 t) \|x_e\|_{X_e}, \qquad \forall t \geq 2t_0. \tag{4.20}$$

Now $\underline{X}_e$ is dense in X_e and hence (4.20) holds $\forall x_e \in X_e$. Therefore, letting

$$M_8 \triangleq \|\iota_e\| \, \|V(t_0)\|_{L(X_e, \underline{X}_e)} M_7 \exp(\alpha_7 t_0)$$

and

$$M_9 \triangleq \sup_{0 \leq t \leq 2t_0} \|V(t)\|_{L(X_e, X_e)},$$

we obtain finally

$$\|V(t)\|_{L(X_e, X_e)} \leq \max(M_8, M_9) \exp(2\alpha_7 t_0) \exp(-\alpha_7 t), \qquad \forall t \geq 0.$$

5. Regulation of system (3.1) by PI-control

We shall study the feedback interconnection of (3.1) and (2.3), i.e.,

$$v(t) = r\theta(t) - y(t), \tag{5.1}$$

$$u(t) = w(t), \tag{5.2}$$

where $r\theta(t)$, $r \in \mathscr{R}^m$, is a constant reference signal.

Theorem 5.1. Suppose that the system (3.1) satisfies (A1)–(A4) and that a $\det(G(0)) \neq 0$. Choose matrices K_p, $K_I \in \mathscr{R}^{m \times m}$ such that K_p stabilizes G and K_I satisfies the condition

$$\sigma((I + G(0)K_p)^{-1} G(0) K_I) \subset \mathring{\mathscr{C}}_+.$$

Then, there exists a number $k^*>0$ such that $\forall 0<k<k^*$ the feedback semigroup $V_k(t)$ on X_e given by

$$V_k(t)x_e = S_e(t)x_e + \int_0^t S_e(t-\tau)\begin{bmatrix} I_{\hat{X}} & 0 \\ 0 & kI_m \end{bmatrix} B_e F_e V_k(\tau)d\tau, \qquad x_e \in X_e,$$

is exponentially stable.

Furthermore, in the range $0<k<k^*$, the closed-loop system given by (2.3), (3.1), (5.1) and (5.2) tracks constant reference signals (i.e., $\lim_{t\to\infty} y(t)=r$) in the presence of arbitrary initial conditions $(x_0, z_0)\in \underline{X}\times\mathscr{R}^m$ and arbitrary disturbances $d_0 \in X$ and $d_1 \in \mathscr{R}^m$.

Proof. The exponential stability follows from Theorem 2.1 and Theorem 4.1. Moreover, define

$$(Tx_0)(t) \triangleq CS(t)x_0, \tag{5.3}$$

$$d(t) \triangleq C\int_0^t S(\tau)d_0 d\tau + d_1. \tag{5.4}$$

If we show that T maps $\underline{X}$ into $(L_0^\infty(\mathscr{R}_+))^m$ and that $d\in (L_0^\infty(\mathscr{R}_+))^m$, then application of Theorem 2.1 to the system $(G, \underline{X}, T)$ and disturbances d of the form (5.4) proves the remaining part of the theorem. Let $x_0 \in \underline{X}$, then $(Tx_0)|_{[0,t_1]}$ is in $(L^\infty(0, t_1))^m$ by (A4). For $t\geq t_1$, we have

$$\begin{aligned} |(Tx_0)(t)| &= |CS(t_1)S(t-t_1)x_0| \\ &\leq \|C\|_{L(\underline{X},\mathscr{R}^m)} \|S(t_1)\|_{L(X,\underline{X})} \|S(t-t_1)\|_{L(X,X)} \|x_0\|_X. \end{aligned}$$

Therefore $(Tx_0)|_{[t_1,\infty)} \in (L^\infty(t_1, \infty))^m$ and $\lim_{t\to\infty} |(Tx_0)(t)|=0$, because $S(t)$ is exponentially stable on X.

Moreover, realize that the spectrum of the infinitesimal generator A of the semigroup $S(t)$ does not contain 0 ($S(t)$ is exponentially stable), hence $A^{-1}\in L(X, X)$. Recall from semigroup theory that

$$S(t)x - x = \int_0^t S(\tau)Axd\tau, \qquad \forall x \in D(A)$$

$$\Rightarrow \quad S(t)A^{-1}d_0 - A^{-1}d_0 = \int_0^t S(\tau)d_0 d\tau, \qquad \forall d_0 \in X$$

$$\Rightarrow \quad \int_0^\infty S(\tau)d_0 d\tau = -A^{-1}d_0, \qquad \forall d_0 \in X. \tag{5.5}$$

The Bochner integral on the LHS of (5.5) has to be understood in the space X. However, it follows from the assumptions that it exists also with respect to $\underline{X}$ and that both integrals coincide. It follows that the function d defined by (5.4) is in $(L_0^\infty(\mathscr{R}_+))^m$.

6. Example

Consider the problem of heating a bar of length 1 with both endpoints at temperature zero and with heat injection of magnitude $u_i(t)$ at the point ξ_i, $i=1$, 2. The measurements are taken at the points η_1 and η_2. The system to be controlled can then be formulated as

$$z_t(x,\ t) = z_{xx}(x,\ t) + \delta(x-\xi_1)u_1(t) + \delta(x-\xi_2)u_2(t), \tag{6.1a}$$

$$y_1(t) = z(\eta_1,\ t), \tag{6.1b}$$

$$y_2(t) = z(\eta_2,\ t), \tag{6.1c}$$

$$(0 \le x \le 1, \quad t > 0, \quad 0 < \xi_i < 1, \quad 0 < \eta_i < 1, \quad i = 1,\ 2)$$

$$z(0,\ t) = z(1,\ t) = 0, \quad \forall t \ge 0, \tag{6.1d}$$

$$z(x,\ 0) = z_0(x). \tag{6.1e}$$

It follows from Curtain and Pritchard (1978) that the system (6.1) can be written in the form (3.1) satisfying the assumption (A1)–(A4). In particular the semigroup $S(t)$ and the spaces X, $\underline{X}$ and $\bar{X}$ are given by

$$(S(t)z)(x) = \sum_{n=1}^{\infty} 2\exp(-n^2\pi^2 t)\sin(n\pi x)\int_0^1 \sin(n\pi\lambda)z(\lambda)d\lambda,$$

$$X = L^2(0,\ 1),$$
$$\underline{X} = H^{\frac{1}{2}+\varepsilon}(0,\ 1),$$
$$\bar{X} = (\underline{X})^* = H^{-(\frac{1}{2}+\varepsilon)}(0,\ 1),$$

where $H^{\varrho}(0,\ 1)$ denotes the Sobolev space on $[0,\ 1]$ of order ϱ ($\varrho \in \mathscr{R}$) and ε is some sufficiently small positive number.

Let us consider the following situation:

$$\eta_i \ge \xi_i \quad (i = 1,\ 2), \tag{6.2a}$$

$$\eta_2 \ge \xi_1, \tag{6.2b}$$

$$\eta_1 \le \xi_2. \tag{6.2c}$$

A direct calculation of the transfer matrix $G(s) = (g_{ij}(s))$ by taking Laplace transforms in (6.1a), (6.1b) and (6.1c) yields

$$\left.\begin{aligned} g_{ij}(s) &= \frac{1}{\sqrt{s}}\,\frac{\sinh((1-\eta_i)\sqrt{s}\,)\sinh(\xi_j\sqrt{s}\,)}{\sinh(\sqrt{s}\,)} \\ \text{if} \quad i &= j \text{ and } (i,\ j) = (2,\ 1) \end{aligned}\right\} \tag{6.3}$$

and

$$g_{12}(s) = \frac{1}{\sqrt{s}}\,\frac{\sinh((1-\xi_2)\sqrt{s}\,)\sinh(\eta_1\sqrt{s}\,)}{\sinh(\sqrt{s}\,)}. \tag{6.4}$$

Using (6.3) and (6.4) it is easy to show that $g_{ij}(0) = (1-\eta_i)\xi_j$ for $i = j$ and $(i,\ j) = (2,\ 1)$ and $g_{12}(0) = (1-\xi_2)\eta_1$. As a consequence, we have

$$\det(G(0)) = \xi_1(1-\eta_2)(\xi_2-\eta_1)$$

and, since $\xi_1 \neq 0$ and $\eta_2 \neq 1$, it follows that $\det(G(0)) \neq 0$, if and only if $\xi_2 \neq \eta_1$. Hence, Theorem 5.1 can be applied to the system given by (6.1a)–(6.1e), if and only if $\xi_2 \neq \eta_1$.

Remark 6.1: Similar considerations can be made, if the conditions (6.2a)–(6.2c) are not satisfied. However, if $\max(\eta_1, \eta_2) \leq \min(\xi_1, \xi_2)$ or $\max(\xi_1, \xi_2) \leq \min(\eta_1, \eta_2)$ then it is possible to show that $\det(G(0)) = 0$ and, therefore, Theorem 5.1 cannot be applied in this case.

For the special case that

$$\xi_1 = 0.2, \quad \xi_2 = 0.6, \quad \eta_1 = 0.4, \quad \eta_2 = 0.8,$$
$$z_0(x) = \sin(\pi x)$$

and

$$r = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

simulation results are shown in Fig. 4–Fig. 6, where

– in Fig. 4: $K_p = 0, \quad K_I = (G(0))^{-1} = \begin{bmatrix} 15 & -20 \\ -5 & 15 \end{bmatrix},$

$$k = 0.1, \quad d_0 = 0 \quad \text{and} \quad d_1 = 0.$$

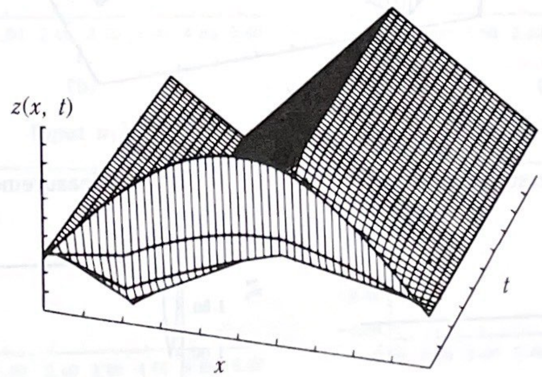
– in Fig. 5: $K_p = 0, \quad K_I = (G(0))^{-1},$

$$k = 0.5, \quad d_0 = 0 \quad \text{and} \quad d_1 = 0.$$

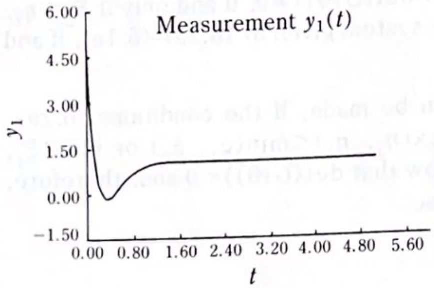
– in Fig. 6: $K_p = \begin{bmatrix} 0.5 & 0.25 \\ 0.25 & 0.5 \end{bmatrix},$

$$K_I = \begin{bmatrix} 15.5 & -19.75 \\ -4.75 & 15.5 \end{bmatrix},$$

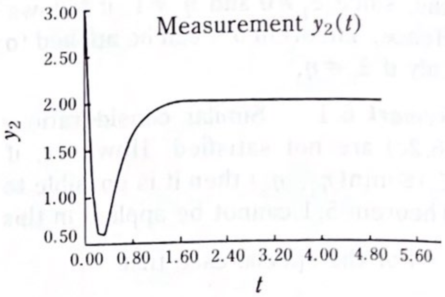
$$k = 1.0, \quad d_0(x) = \cos(\pi x) \quad \text{and} \quad d_1 = 0.$$



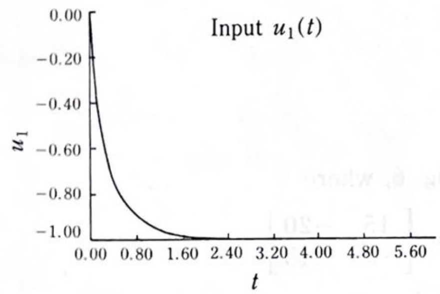
(a)



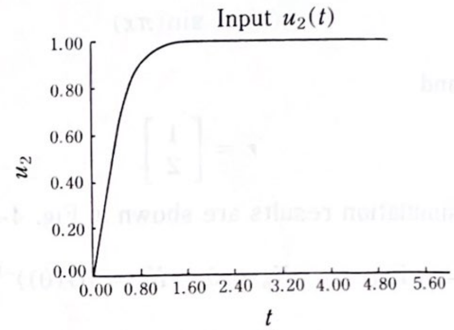
(b)



(c)

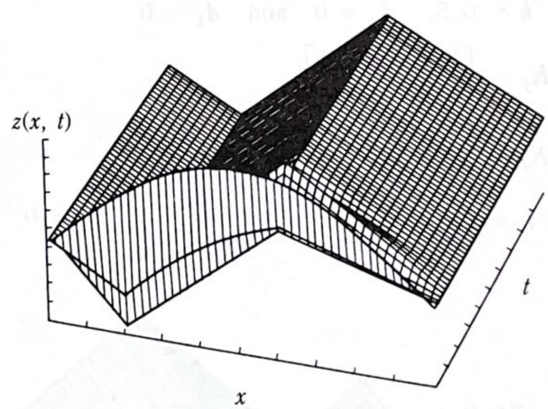


(d)

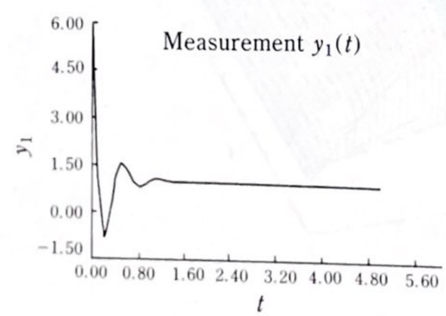


(e)

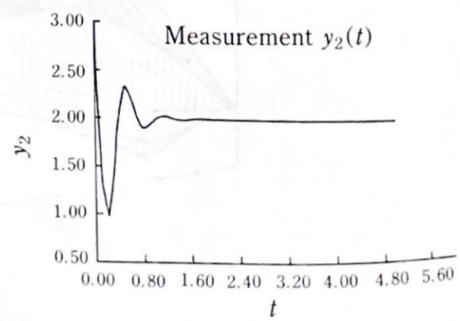
Fig. 4



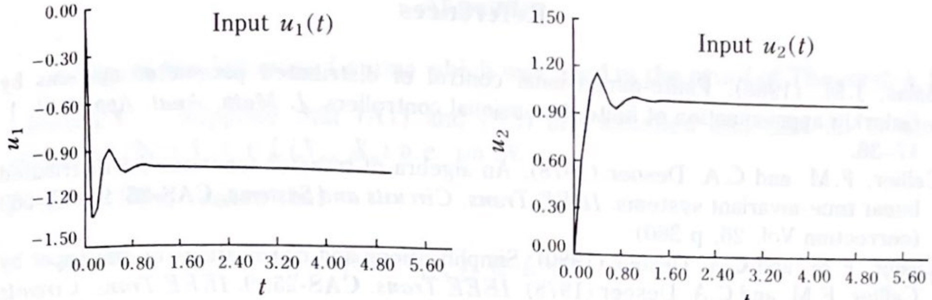
(a)



(b)

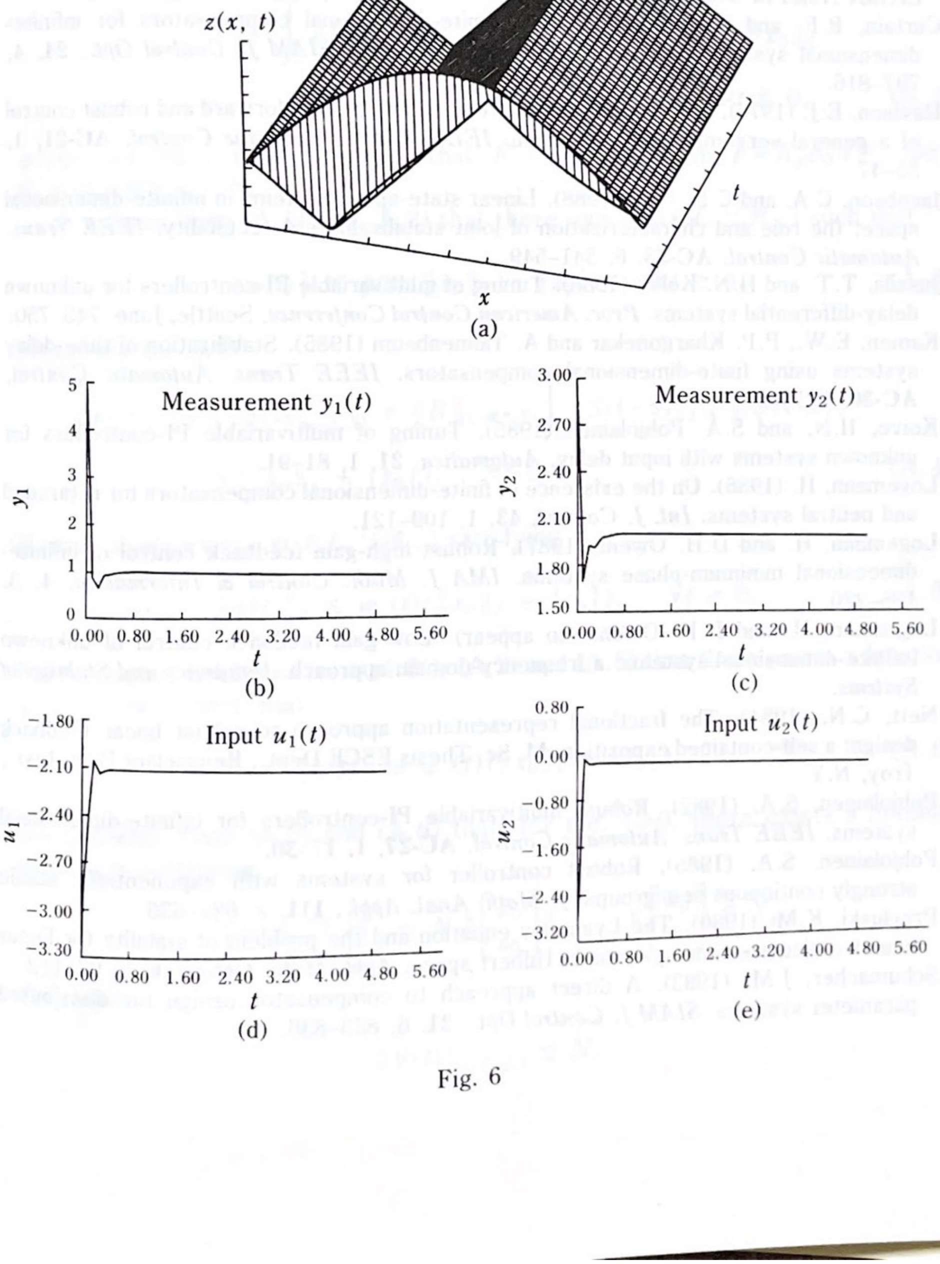


(c)



(d) (e)

Fig. 5



(a) (b) (c) (d) (e)

Fig. 6

References

Balas, J.M. (1986). Finite-dimensional control of distributed parameter systems by Galerkin approximation of finite-dimensional controllers. *J. Math. Anal. Appl.* **114**, 1, 17–36.

Callier, F.M. and C.A. Desoer (1978). An algebra of transfer functions for distributed linear time-invariant systems. *IEEE Trans. Circuits and Systems*, **CAS-25**, 9, 651–663 (correction Vol. 26, p.360).

Callier, F.M. and C.A. Desoer (1980). Simplifications and clarifications on the paper by Callier, F.M. and C.A. Desoer (1978), *IEEE Trans.* **CAS-25**, 9. *IEEE Trans. Circuits and Systems*, **CAS-27**, 4, 320–323.

Callier, F.M. and J. Winkin (1986). Distributed system transfer functions of exponential order. *Int. J. Control*, **43**, 5, 1353–1373.

Curtain, R.F. (1984). Finite-dimensional compensators for parabolic distributed systems with unbounded control and observation. *SIAM J. Control Opt.*, **22**, 2, 255–276.

Curtain, R.F. and A.J. Pritchard (1978). Infinite-dimensional linear system theory. *Lecture Notes in Control and Information Sciences*, 8, Springer-Verlag, Berlin.

Curtain, R.F. and D. Salamon (1986). Finite-dimensional compensators for infinite-dimensional systems with unbounded input operator. *SIAM J. Control Opt.*, **24**, 4, 797–816.

Davison, E.J. (1976). Multivariable tuning regulators: the feedforward and robust control of a general servomechanism problem. *IEEE Trans. Automatic Control*, **AC-21**, 1, 35–47.

Jacobson, C.A. and C.N. Nett (1988). Linear state-space systems in infinite-dimensional space: the role and characterization of joint stabilizability/detectability. *IEEE Trans. Automatic Control*, **AC-33**, 6, 541–549.

Jussila, T.T. and H.N. Koivo (1986). Tuning of multivariable PI-controllers for unknown delay-differential systems. *Proc. American Control Conference*, Seattle, June, 745–750.

Kamen, E.W., P.P. Khargonekar and A. Tannenbaum (1985). Stabilization of time-delay systems using finite-dimensional compensators. *IEEE Trans. Automatic Control*, **AC-30**, 1, 75–78.

Koivo, H.N. and S.A. Pohjolainen (1985). Tuning of multivariable PI-controllers for unknown systems with input delay. *Automatica*, **21**, 1, 81–91.

Logemann, H. (1986). On the existence of finite-dimensional compensators for retaraded and neutral systems. *Int. J. Control*, **43**, 1, 109–121.

Logemann, H. and D.H. Owens (1987). Robust high-gain feedback control of infinite-dimensional minimum-phase systems. *IMA J. Math. Control & Information*, **4**, 3, 195–220.

Logemann, H. and D.H. Owens (to appear). Low-gain feedback control of unknown infinite-dimensional systems: a frequency-domain approach. *Dynamics and Stability of Systems*.

Nett, C.N. (1984). The fractional representation approach to robust linear feedback design: a self-contained exposition. M. Sc. Thesis ESCE Dept., Rensselaer Poly. Inst., Troy, N.Y.

Pohjolainen, S.A. (1982). Robust multivariable PI-controllers for infinite-dimensional systems. *IEEE Trans. Automatic Control*, **AC-27**, 1, 17–30.

Pohjolainen, S.A. (1985). Robust controller for systems with exponentially stable strongly continuous semigroups. *J. Math. Anal. Appl.*, **111**, 2, 622–636.

Przyluski, K.M. (1980). The Lyapunov equation and the problem of stability for linear bounded discrete-time systems in Hilbert space. *Appl. Math. Optim.*, **6**, 2, 97–112.

Schumacher, J.M. (1983). A direct approach to compensator design for distributed parameter systems. *SIAM J. Control Opt.*, **21**, 6, 823–836.

Appendix

We prove the following Lemma which was used in the proof of Theorem 4.1.

Lemma 1. Suppose that (A1) and (A3) are satisfied and that K_I is non-singular. Then $V(t) \in L(X_e, \underline{X}_e)$ a.e. on $\mathscr{R}_+$.

Proof. First realize that

$$\|S(t)\|_{L(X,\underline{X})} \in L^1_-(\mathscr{R}_+). \tag{A.1}$$

This can be shown using the same methods as in the proof of Lemma 3.1. We obtain from (4.10) and (4.11)

$$\begin{aligned} |u(t)| \leq{}& M_1 \exp(-\alpha_1 t)|z_0| \\ &+ \|K_p\| \, \|C\|_{L(\underline{X},\mathscr{R}^m)} \|S(t)\|_{L(X,\underline{X})} \|x_0\|_X \\ &+ \int_0^t \|F_1(t-\tau)\| \, \|C\|_{L(\underline{X},\mathscr{R}^m)} \|S(\tau)\|_{L(X,\underline{X})} d\tau \|x_0\|_X, \\ & \qquad \forall t \geqq 0, \end{aligned} \tag{A.2}$$

where we have used the fact that $\check{F}$ is of the form $\check{F} = K_p \delta_0 + F_1$ with $F_1 \in (L_-^1(\mathscr{R}_+))^{m \times m}$.

It follows from (A.1) and (A.2) that there exists $\phi \in L_-^1(\mathscr{R}_+)$ such that

$$|u(t)| \leq \phi(t)(\|x_0\|_X + |z_0|), \qquad \forall t \geq 0. \tag{A.3}$$

Moreover by (3.1)

$$\begin{aligned} \|x(t)\|_{\underline{X}} \leq{}& \{\|S(t)\|_{L(X,\underline{X})} + \|B\|_{L(\mathscr{R}^m,\tilde{X})} \int_0^t \|S(t-\tau)\|_{L(\tilde{X},\underline{X})} \phi(\tau) d\tau\} \\ & \times (\|x_0\|_X + |z_0|), \qquad \forall t \geq 0. \end{aligned} \tag{A.4}$$

Hence, there exists $\psi_1 \in L_-^1(\mathscr{R}_+)$ satisfying

$$\|x(t)\|_{\underline{X}} \leq \psi_1(t)(\|x_0\|_X + |z_0|), \qquad \forall t \geq 0. \tag{A.5}$$

Furthermore, it follows from (4.3′), (A.3) and (A.5) that there exists a function $\psi_2 \in L_-^1(\mathscr{R}_+)$ such that

$$|z(t)| \leq \psi_2(t)(\|x_0\|_X + |z_0|). \tag{A.6}$$

We conclude from (A.5) and (A.6) that, for a.e. $t > 0$, there exists a number $0 < N_t < \infty$ satisfying

$$\left\| V(t) \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} \right\|_{\underline{X}_e} \leq N_t \left\| \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} \right\|_{X_e}, \qquad \forall \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} \in X_e.$$

Therefore

$$\|V(t)\|_{L(X_e,\underline{X}_e)} \leq N_t.$$