INTRODUCTION TO ABELIAN VARIETIES

Let us begin with some general chat about what abelian varieties are and why they are interesting. Anything significant said before the start of section 1 will be repeated later.

I'm going to work over $\mathbb{C}$. This doesn't in the least mean that you can't do anything without complex analysis. On the contrary, abelian varieties, especially elliptic curves, over number fields are the main objects of study in large areas of number theory. But I am a complex geometer and the study of abelian varieties over this one very special field contains quite enough to be getting on with, as well as being beautiful. Before all the number theorists lose interest, I should point out that complex abelian varieties and number theory are also inextricably linked and no-one can study either without some knowledge of the other.

There are lots of books. The one that I have come to regard as the standard handbook is:

H. Lange & Ch. Birkenhake, *Complex Abelian Varieties* (Springer)

but this covers a lot of material and assumes rather more knowledge of algebraic geometry than most recent graduates have. A surprisingly accessible introduction can be found in the first 80 pages or so of

D. Mumford, *Abelian Varieties* (OUP, Bombay).

I should mention two other books by the same author, which explore related topics; the first one could serve as a text for some parts of the course, the second is just an object of beauty:

D. Mumford, *Curves and their Jacobians* (Ann Arbor, Mich)

D. Mumford, *Tata Lecture Notes on Theta I* (Birkhäuser)

Another fairly modern book on abelian varieties is

G.R. Kempf, *Complex Abelian Varieties and Theta Functions* (Springer)

which is not bad, though it is not error-free and the approach taken is not the one I propose to take. There are two older books:

H.P.F. Swinnerton–Dyer, *Analytic Theory of Abelian Varieties* (CUP)

and, inevitably

S. Lang, *Abelian Varieties*

of which the first can be recommended. One or another of these books will have the answer to most questions.

So what are abelian varieties and why are they interesting? The most basic example is a smooth cubic curve in $\mathbb{P}^2$, for instance

$$E = \{y^2z = 4x^3 - g_2xz^2 - g_3z^3\}$$

for general g_2, $g_3 \in \mathbb{C}$. (Any smooth cubic can be put in this form by a change of coordinate, as long as the characteristic of the ground field is not 2.) This is the simplest kind of non-rational variety you can have, so if we don't understand it we are not going to get very far. And indeed it stopped you in your tracks in your schooldays, when you thought mathematics meant doing more difficult integrals, because you couldn't do

$$\int \left(4x^3 - g_2x - g_3\right)^{-1/2} dx$$

or indeed $\int y^{-1}dx$ if y^2 was given by any polynomial in x of degree ≥ 3.

Another basic thing that we are going to have to understand if we are to make any progress at all with complex manifolds is $\mathbb{C}/\Lambda$, where $\Lambda \subseteq \mathbb{C}$ is a lattice of rank 2: say $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$. After all, this object has complex dimension 1, so it has real dimension 2, and we know what it is as a real manifold: it's a torus, next to the sphere the simplest kind of compact surface there is.

In fact these are the same objects. Given Λ we define the Weiersrtraß $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right]$$

so that $\wp'(z) = \sum_{\omega\in\Lambda} -2(z-\omega)^{-3}$. Among the good properties of $\wp$ is that it is a doubly periodic – that is, Λ-invariant – meromorphic function on $\mathbb{C}$ and that if

$$g_2 = 60 \sum_{\omega\in\Lambda\setminus\{0\}} \omega^{-4}, \qquad g_3 = 140 \sum_{\omega\in\Lambda\setminus\{0\}} \omega^{-6},$$

then $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$. So the map $u\colon \mathbb{C}/\Lambda \to \mathbb{P}^2$ given by $u(z+\Lambda) = \big(\wp(z) : \wp'(z) : 1\big)$ for $z \notin \Lambda$ and $u(0+\Lambda) = \big(0:1:0\big)$ actually maps $\mathbb{C}/\Lambda$ onto $E_\Lambda = \{y^2z = 4x^3 - g_2xz^2 - g_3z^3\} \subseteq \mathbb{P}^2$. With a certain amount of work (nothing too strenuous) you can show that u is a biholomorphic map; moreover, every smooth cubic curve in $\mathbb{P}^2$ is projectively equivalent to E_Λ for some Λ. So if we are only interested in complex analysis, plane cubic curves and 1-dimensional complex tori are the same things.

But $\mathbb{C}/\Lambda$ has more structure than that: it's an abelian group. That makes E_Λ into a group, too, by $P + Q = u\big(u^{-1}(P) + u^{-1}(Q)\big)$, and the identity element is $(0:1:0)$. We should like to have a geometric picture of the addition: that is, we should like $+\colon E_\Lambda \times E_\Lambda \to E_\Lambda$ to be a morphism of algebraic varieties, and one that we can describe in terms of projective geometry. The answer is well-known: $P+Q+R=0$ if P, Q and R are collinear. Of course you could just write that down and use it as the definition of addition, first choosing some inflexion point to be 0. If you do, you have a rather messy job proving that what you have defined is associative. Historically at least, it's better to do what we were doing and start with $\mathbb{C}/\Lambda$, and then we need to understand u^{-1}, so as to reconstruct Λ from E_Λ.

Consider $\eta = u^*(y^{-1}dx)$, a meromorphic differential on $\mathbb{C}/\Lambda$. Let $\pi\colon \mathbb{C} \to \mathbb{C}/\Lambda$ be the projection: then $\pi^*\eta = (u\pi)^*(y^{-1}dx) = (\wp')^{-1}d\wp = dz$, which is holomorphic. So $y^{-1}dx$ is actually a global holomorphic differential form on E_Λ. Moreover, elements of Λ are just the periods of this form: if γ is a closed path in $\mathbb{C}/\Lambda$ and $\tilde{\gamma}$ is a path in $\mathbb{C}$ which lifts γ then $\int_\gamma \eta = \int_{\tilde{\gamma}} dz = \gamma(1) - \gamma(0) \in \Lambda$, and obviously every element of Λ can be got in this way.

From this it follows that $u^{-1}(P) = \int_{(0:1:0)}^P y^{-1}dx + \Lambda \in \mathbb{C}/\Lambda$, and the statement that $P+Q+R=0$ if and only if they are collinear comes down to Abel's Theorem: if P, Q, $R \in E_\Lambda$ then

$$\int_{(0:1:0)}^P y^{-1}dx + \int_{(0:1:0)}^Q y^{-1}dx + \int_{(0:1:0)}^R y^{-1}dx \equiv 0 \bmod \Lambda$$

if and only if P, Q and R are collinear. This is an addition formula for elliptic integrals (and that is of course the form in which Abel proved it). It is quite easy now that we know all about complex analysis but it made Abel a Norwegian national hero. It is this connexion that gave rise to the name "abelian variety".

One other thing that we have learned is that E_Λ has a global holomorphic differential 1-form, which has no zeros either. This is pretty unusual and is something to celebrate: global forms are as common as mud but only a few privileged varieties are accorded nowhere vanishing ones. It's only got one global holomorphic form, though, up to a constant: otherwise, we could divide another form by this one and get a global nonconstant holomorphic function, which is against the rules. This is the differential geometer's way of saying that E_Λ has genus 1.

If we want to generalise we could try several things:

a) Curves of higher genus

b) Quartics in $\mathbb{P}^3$ and quintics in $\mathbb{P}^4$

c) $\mathbb{C}^g/\Lambda$ for $g > 1$.

All these things are sensible: we are going to do (c). Doing (a) leads you straight back to (c) anyway, as I will explain in a moment. Doing (b) leads you to K3 surfaces and Calabi-Yau manifolds, which are fascinating objects but not quite of such universal occurrence as abelian varieties. Mind you, if you believe some physicists there is a Calabi-Yau in the room you are in, or perhaps the room you are in is in a Calabi-Yau.

Why do curves lead you straight back to things like $\mathbb{C}^g/\Lambda$? Because if you have a curve of genus g then it has g differentials and you integrate each one of them against each of the $2g$ loops, getting $2g$ points in $\mathbb{C}^g$ which generate Λ. It turns out that the quotient $\mathbb{C}^g/\Lambda$, called the Jacobian, captures all information about the curve and is easier to study in some ways.

But actually $\wp$ is something of a miracle. If you just write down $2g$ elements of $\mathbb{C}^g$ generating a lattice Λ then there will probably be no meromorphic functions at all whose periods are exactly those $2g$ numbers, so if you consider $\mathbb{C}^g/\Lambda$ it won't have any meromorphic functions and in particular won't embed in any projective space. If it will embed in projective space it is called an abelian variety. The abelian varieties of dimension g form a family of dimension $g(g-1)/2$ and as this is bigger than the dimension of the family of curves of genus g, which is $3g-3$ for $g \geq 2$, most abelian varieties cannot be Jacobians. It is a hard question (called the Schottky problem) to determine which ones are Jacobians. But there are other ways as well in which abelian varieties (and even things of the form $\mathbb{C}^g/\Lambda$ that are not abelian varieties) arise in geometry, such as Albanese varieties and intermediate Jacobians, so that abelian varieties which are not Jacobians are still important.

One warning is useful. The word "torus" is used to mean three different things. It is used by topologists to mean a topological space that is a product of S^1s. As a topological space, $\mathbb{C}^g/\Lambda$ is a torus so it is often called a torus even when one is thinking about the complex structure. But the algebraic group $(\mathbb{C}^*)^n$ is also referred to as a torus. Ideally, $\mathbb{C}^g/\Lambda$ should always be referred to as a complex torus and $(\mathbb{C}^*)^n$ as an algebraic torus, to avoid confusion. Alas, this is not always done. Beware!

1. Complex tori and line bundles.

In giving a course on abelian varieties, it is best to say what an abelian variety is. There are several possible definitions, depending on one's point of view.

Definition: A **complex torus** is a quotient V/Λ of a complex vector space V by a cocompact lattice Λ of rank $2g$, where $g = \dim_{\mathbb{C}} V$ (so $\Lambda \otimes \mathbb{R} = V$).

Definition: A complex torus T is called an **abelian variety** if there exists a holomorphic embedding of T into $\mathbb{P}^N_{\mathbb{C}}$ for some positive integer N.

Not every complex torus has such an embedding. So we had better see how far we can get just thinking about complex tori and then try to decide which complex tori are in fact abelian varieties. It is possible to do all this without mentioning line bundles (Swinnerton-Dyer's book does), but I think it is worth the extra effort because modern books do use bundles and you will need them soon.

Warning. The word "torus" is used to mean three things: topological torus, algebraic torus and complex torus. In books on algebraic geometry the word "torus" tends to mean "algebraic torus", because complex tori are mostly only interesting if they are abelian varieties, and then we call them that.

Let $V \cong \mathbb{C}^g$ have basis $\mathbf{e}_1, \ldots, \mathbf{e}_g$ and suppose $\Lambda = \bigoplus_{i=1}^{2g} \lambda_i \mathbb{Z}$ (so $\lambda_i \in V$): write

$$\lambda_i = \sum_{j=1}^{g} \lambda_{ji} \mathbf{e}_j.$$

The matrix $\Pi = (\lambda_{ji}) \in M_{g\times 2g}(\mathbb{C})$ is called the **period matrix** of the complex torus $T = V/\Lambda$. Given a matrix $\Pi \in M_{g\times 2g}(\mathbb{C})$ we can easily check whether it is the period matrix of a complex torus or not.

Lemma 1.1. *$\Pi \in M_{g\times 2g}(\mathbb{C})$ is the period matrix of a complex torus if and only if $\begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix} \in M_{2g\times 2g}(\mathbb{C})$ is nonsingular.*

Proof: To say that Π is a period matrix is to say that its columns span a lattice Λ in $V = \mathbb{C}^g$. This means that $\Lambda \otimes \mathbb{R}$ should be the whole of V as a set, i.e. that the columns of Π should be linearly independent over $\mathbb{R}$. If they are not then $\Pi\mathbf{x} = 0$ for some non-zero $\mathbf{x} \in \mathbb{R}^{2g}$, so $\bar{\Pi}\mathbf{x} = \bar{\Pi}\bar{\mathbf{x}} = 0$, and thus $\begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix}\mathbf{x} = 0$ so $\begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix}$ is singular. Conversely, if $\begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix}$ is singular then for some $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{2g}$, not both zero, $\begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix}(\mathbf{x}+i\mathbf{y}) = 0$. So $\Pi\mathbf{x} + i\Pi\mathbf{y} = 0$ and $\bar{\Pi}(\mathbf{x} + i\mathbf{y}) = \Pi\mathbf{x} - i\Pi\mathbf{y} = 0$. So $\Pi\mathbf{x} = \Pi\mathbf{y} = 0$ and the columns of Π are linearly dependent over $\mathbb{R}$. ■

Having described our objects – complex tori – in terms of linear algebra, which is always a good thing to do, we should like to do the same for morphisms, i.e. for holomorphic maps between complex tori. Here the picture is very nice. It's just like affine space: an isometry of linear spaces is got by moving the origin to the right place and then using a linear map, and the following result is similar. First we need a definition.

Definition: If $y \in T$ the **translation** $t_y\colon T \to T$ by y is just $x \mapsto x + y$. If T' is another complex torus, a homomorphism $f\colon T \to T'$ is a holomorphic group homomorphism.

Proposition 1.2. *If $h\colon T \to T'$ is a holomorphic map then there is a unique homomorphism $f\colon T \to T'$ and a unique $y \in T'$ such that $h = t_y f$. Furthermore there is a unique $\mathbb{C}$-linear map $F\colon V \to V'$ with $F(\Lambda) \subseteq \Lambda'$, inducing f.*

Proof: Obviously we want to take $y = h(0)$ and $f = t_y^{-1}h = t_{-y}h$. Look at $f\,\mathrm{pr}\colon V \to T'$, where $\mathrm{pr}\colon V \to T$ is the quotient map. By the universal property of the map $\mathrm{pr}'\colon V' \to T'$ it lifts to a holomorphic map $F\colon V \to V'$. F is not unique but it is unique modulo the action of Λ', so if we specify that $F(0) = 0$ (we know that $F(0) \in \Lambda'$) then we fix F. But $F(\mathbf{v} + \lambda) \equiv F(\mathbf{v}) \bmod \Lambda'$ if $\lambda \in \Lambda$, so $\frac{\partial F}{\partial v_i}(\mathbf{v} + \lambda) = \frac{\partial F}{\partial v_i}(\mathbf{v})$ for all $\lambda \in \Lambda$. So by Liouville's theorem all partial derivatives of F are constant, so F is linear. So F is a homomorphism and therefore f is. ■

We also want to know about kernels and images.

Proposition 1.3. *If $f: T \to T'$ is a homomorphism then* $\operatorname{Im} f$ *is a subtorus of* T' *and* $\operatorname{Ker} f$ *is a closed subgroup of* T*: the connected component* $(\operatorname{Ker} f)^0$ *is a subtorus and is of finite index in* $\operatorname{Ker} f$*.*

Proof: With F as in the proof of (1.2), we have $\operatorname{Im} f = F(V)/\big(F(V) \cap \Lambda'\big)$. Since $F(\Lambda) \subseteq \Lambda'$, the discrete subgroup $F(V) \cap \Lambda'$ generates $F(V)$ as an $\mathbb{R}$-vector space, so $F(V) \cap \Lambda'$ is a lattice in $F(V)$, so $\operatorname{Im} f$ is a torus. The kernel, on the other hand, consists of the image in T of $\{\mathbf{v} \in V \mid F(\mathbf{v}) \in \Lambda'\} = F^{-1}(\Lambda')$. The component $F^{-1}(\Lambda')^0$ is a $\mathbb{C}$-vector space because F is linear, so $(\operatorname{Ker} f)^0 = F^{-1}(\Lambda')^0/\big(F^{-1}(\Lambda')^0 \cap \Lambda\big)$. But $F^{-1}(\Lambda')^0 \cap \Lambda$ is a discrete subgroup of $F^{-1}(\Lambda')^0$ and it must have maximal rank because $(\operatorname{Ker} f)^0$ is compact. Since $\operatorname{Ker} f$ is compact it can have only finitely many components, so $(\operatorname{Ker} f)^0$ is of finite index ■

A particularly interesting and important case is when $\operatorname{Im} f = T'$ and $(\operatorname{Ker} f)^0$ is trivial, i.e. $\# \operatorname{Ker} f < \infty$. Such an f is called an isogeny. You get isogenies by taking the quotient of T by a finite subgroup $\Gamma \subseteq T$: the only thing to be checked here is that T/Γ is a torus, but it is $V/\operatorname{pr}^{-1}(\Gamma)$ and $\operatorname{pr}^{-1}(\Gamma) \subseteq \Lambda$ is discrete and therefore a lattice.

What takes a bit of getting used to is that isogeny is an equivalence relation.

Proposition 1.4. *Suppose $f: T \to T'$ is an isogeny and $\# \operatorname{Ker} f = n$ (n is called the exponent of the isogeny). Then there is a unique isogeny $g: T' \to T$ such that $gf = n_T$ and $fg = n_{T'}$, where $n_T: T \to T$ is the map $x \mapsto nx$.*

Proof: $\operatorname{Ker} f \subseteq \operatorname{Ker} n_T$, because if $x \in \operatorname{Ker} f$ then $nx = 0$ as $\# \operatorname{Ker} f = n$. So there is a unique map $g: T' \to T$ such that $gf = n_T$. This is just group theory: you define g by its kernel, which is $\operatorname{Ker} n_T / \operatorname{Ker} f$. Obviously g is an isogeny: we have fixed it so as to have finite kernel and it must be surjective simply because $\dim T' = \dim T$. Suppose $y = x + \operatorname{Ker} f \in \operatorname{Ker} g$. Then $ny = nx + \operatorname{Ker} f = 0 + \operatorname{Ker} f \in T'$, so $y \in \operatorname{Ker} n_{T'}$. So by the same as before there is an isogeny $f': T \to T'$ such that $f'g = n_{T'}$. Now $f'n_T = f'gf = n_{T'}f$, but $n_{T'}f(x) = nf(x) = f(nx) = fn_T(x)$, so this shows that $f'n_T = fn_T$. Since n_T is surjective (we can divide by n in V and thus also in T), we must have $f = f'$. ■

So it makes sense to talk about two complex tori being isogenous, meaning there is an isogeny between them, and this is an equivalence relation. It's nearly isomorphism for some purposes. Number theorists usually find it just as good as isomorphism but it frequently wrecks geometric structures. This isn't all that surprising: we constructed it by essentially group-theoretic methods and we are still at the level of complex tori where there isn't really any geometry. But it's not too bad an equivalence relation even for geometers – a complex torus isogenous to an abelian variety is again an abelian variety, for instance.

We are now going to try to find an analogue of $\wp$, i.e. find some periodic functions whose periods are Λ. It doesn't work to write down hopeful-looking infinite sums: they all diverge. You have to do it, if at all, by getting at two functions on V which are not periodic but which do have some regular behaviour relative to Λ, and fix up periodic functions by taking the quotient of one by the other. These not-quite-periodic functions are examples of theta functions, though because we are still looking at complex tori one at a time we see them only as in a glass, darkly.

Another way to look at theta functions is to think of them as sections in some line bundle on T. This is how I want to introduce them, but to do that I'm going to have to introduce (holomorphic) line bundles. Some people may already be familiar with vector bundles (of which line bundles are a special case) from differential geometry, but I won't assume that. Let's have a digression.

Definition: Suppose X is a complex manifold. A holomorphic **line bundle** on X is a manifold $\mathcal{L}$ together with a surjective holomorphic map $\pi: \mathcal{L} \to X$ such that

i) $\pi^{-1}(x) \cong \mathbb{C}$ for any $x \in X$;

ii) there is an open cover $\big(U_\alpha\big)_{\alpha \in A}$ of X such that $\pi: \pi^{-1}(U_\alpha) \to U_\alpha$ is the projection of a product, that is, there is a biholomorphic map $\phi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$ such that $\operatorname{pr}_1 \phi_\alpha = \pi|_{\pi^{-1}(U_\alpha)}$

iii) the transition functions are well-behaved: if $U_\alpha \cap U_\beta \neq \emptyset$ then

$$\phi_{\alpha\beta} = \phi_\alpha \phi_\beta^{-1}: (U_\alpha \cap U_\beta) \times \mathbb{C} \to (U_\alpha \cap U_\beta) \times \mathbb{C}$$

is biholomorphic and if $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ then $\phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\gamma}$ where these make sense.

In particular, if $x \in U_\alpha \cap U_\beta$ then $\phi_{\alpha\beta}|_{\pi^{-1}(x)}: \mathbb{C} \to \mathbb{C}$ is an element of $\operatorname{GL}(\mathbb{C}) = \mathbb{C}^*$. So the idea is that $\mathcal{L}$ isn't necessarily trivial but is locally trivial.

A section in a line bundle is a map $\sigma\colon X \to \mathcal{L}$ such that $\pi\sigma = \mathrm{id}$. In other words, it's a twisted function. If $\mathcal{L}$ is, in fact, trivial, then σ really is a global holomorphic function. There is always one section, namely the zero section, but there need not be any more. The space of sections (it's obviously a $\mathbb{C}$-vector space) is denoted $\Gamma(\mathcal{L})$ or $H^0(\mathcal{L})$. In general it will be infinite-dimensional but in many important cases it isn't. In particular if X is compact then $\dim H^0(\mathcal{L}) < \infty$ for any line bundle $\mathcal{L}$.

If σ_0 and σ_1 are non-zero sections of $\mathcal{L}$ then σ_0/σ_1 is a meromorphic function. More generally, if $\sigma_0, \ldots, \sigma_N \in H^0(\mathcal{L})$ are linearly independent then we get a map $X \to \mathbb{P}^N$ by $x \mapsto \big(\sigma_0(x) : \ldots : \sigma_N(x)\big)$, as long as the σ_i don't all vanish at once. So if we want to embed X in some projective space a good place to start looking is at line bundles.

A line bundle on X is said to be trivial if it is biholomorphic to $\mathbb{C} \times X$. If $\psi\colon Y \to X$ is a holomorphic map of manifolds and $\mathcal{L}$ is a line bundle on X then there is a line bundle $\psi^*\mathcal{L}$ on Y, given by a cover $\tilde{U}_\alpha = \psi^{-1}(U_\alpha)$ of Y.

Proposition 1.5. *Every line bundle on $\mathbb{C}^g$ is trivial.*

Proof: (Optional: if you don't know what it means, ignore it for now.) The sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \stackrel{e^{2\pi i(\,)}}{\longrightarrow} \mathcal{O}^* \longrightarrow 0$$

gives a long exact sequence

$$\cdots \longrightarrow H^1(\mathbb{C}^g, \mathcal{O}) \longrightarrow H^1(\mathbb{C}^g, \mathcal{O}^*) \longrightarrow H^2(\mathbb{C}^g, \mathbb{Z}) \longrightarrow \cdots$$

and both $H^1(\mathbb{C}^g, \mathcal{O})$ and $H^2(\mathbb{C}^g, \mathbb{Z})$ are trivial. ∎

We can use this to describe holomorphic line bundles on $T = V/\Lambda$. If we have a line bundle $\mathcal{L}$ on T then $\mathrm{pr}^*\mathcal{L}$ is a line bundle on $V = \mathbb{C}^g$ and thus trivial. So Λ acts, not just on V, but on $V \times \mathbb{C} = \mathrm{pr}^*\mathcal{L}$, in such a way that $(V \times \mathbb{C})/\Lambda = \mathcal{L}$. The action is given by

$$\lambda\colon (\mathbf{v}, \alpha) \mapsto \big(\mathbf{v} + \lambda, \alpha f(\lambda, \mathbf{v})\big)$$

and the function $\mathbf{v} \mapsto f(\lambda, \mathbf{v})$ is a holomorphic nowhere vanishing function on V. The condition for this to define an action of Λ is

$$f(\lambda + \mu, \mathbf{v}) = f(\lambda, \mathbf{v} + \mu) f(\mu, \mathbf{v}) \qquad (*)$$

and a thing satisfying this relation is called a **1-cocycle** (for Λ, with coefficients in the nowhere vanishing functions on V) or, in this particular case only, a **factor of automorphy**. Thus every line bundle on T is determined by a factor of automorphy. However, different factors of automorphy may determine the same line bundle. The reason is that if we pick a different isomorphism $\mathrm{pr}^*\mathcal{L} \to V \times \mathbb{C}$ our factor of automorphy will be twisted by an automorphism of $V \times \mathbb{C}$, i.e. by a nonvanishing holomorphic function $h\colon V \longrightarrow \mathbb{C}^*$. In fact the change to $f(\lambda, \mathbf{v})$ is that it is multiplied by a coboundary, namely $h(\lambda + \mathbf{v})h(\mathbf{v})^{-1}$.

Again we want to get back to linear algebra. Since I do not want to teach you group cohomology either I shall produce a map out of thin air: we can write $f\colon \Lambda \times V \to \mathbb{C}^*$ as

$$f(\lambda, \mathbf{v}) = \exp\{2\pi i g(\lambda, \mathbf{v})\}$$

where $g\colon \lambda \times V \to \mathbb{C}$ is holomorphic in $\mathbf{v}$, and we put

$$\delta f(\lambda, \mu) = g(\mu, \mathbf{v} + \lambda) + g(\lambda + \mu, \mathbf{v}) - g(\lambda, \mathbf{v})$$

for λ, $\mu \in \Lambda$, $\mathbf{v} \in V$. This makes sense (that is, $\delta f(\lambda, \mu)$ does not depend on $\mathbf{v}$) and in fact $\delta f\colon \Lambda^2 \to \mathbb{Z}$, because $(*)$ gives

$$g(\lambda + \mu, \mathbf{v}) + g(\lambda, \mathbf{v}) - g(\mu, \mathbf{v} + \lambda) \equiv 0 \qquad \mathrm{mod}\ 1.$$

δf is an example of a 2-cocycle: a map $F\colon \Lambda^2 \to \mathbb{Z}$ is called a 2-cocycle if

$$\partial F(\lambda, \mu, \nu) = F(\mu, \nu) - F(\lambda + \mu, \nu) + F(\lambda, \mu + \nu) - F(\lambda, \nu) = 0$$

for all λ, μ, $\nu \in \Lambda$. If F is a cocycle we define $\alpha F(\lambda, \mu) = F(\lambda, \mu) - F(\mu, \lambda)$.

Proposition 1.6. *$\alpha F: \Lambda^2 \to \mathbb{Z}$ is an integer-valued alternating bilinear form.*

Proof:

$$\alpha F(\lambda+\mu,\nu) - \alpha F(\lambda,\nu) - \alpha F(\mu,\nu) = \partial F(\lambda,\mu,\nu) - \partial F(\nu,\lambda,\mu) - \partial F(\lambda,\mu,\nu)$$

which is zero. ■

In particular a factor of automorphy gives rise to an integral alternating bilinear form $E = \alpha\delta f$ on Λ, so a line bundle on T does likewise. This form is actually $c_1(\mathcal{L})$, or to be precise the image of $c_1(\mathcal{L})$ under an isomorphism $H^2(T,\mathbb{Z}) \xrightarrow{\sim} \mathrm{Alt}^2(\Lambda,\mathbb{Z})$. There are lots of things that we ought to do, such as check that different factors of automorphy for the same $\mathcal{L}$ do give the same value of $c_1(\mathcal{L})$.

Note that $E(\lambda,\mu) = \alpha\delta f(\lambda,\mu)$ is given by

$$E(\lambda,\mu) = g(\mu, \mathbf{v}+\lambda) + g(\lambda,\mathbf{v}) - g(\lambda,\mathbf{v}+\mu) - g(\mu,\mathbf{v}).$$

In fact this form, after being extended $\mathbb{R}$-linearly to V, satisfies $E(i\mathbf{x}, i\mathbf{y}) = E(\mathbf{x},\mathbf{y})$ (by a type argument which I won't do) and is thus the imaginary part of a Hermitian form H.

We summarise the above (which we haven't really proved) as follows.

Theorem 1.7. *Every line bundle on T is determined by a factor of automorphy f. There is a well-defined map c_1 from $\mathrm{Pic}(T)$ (the set of line bundles on T) to $\mathrm{Alt}^2(\Lambda,\mathbb{Z})$ given by*

$$c_1(\mathcal{L})(\lambda,\mu) = g(\mu,\lambda) + g(\lambda,0) - g(\lambda,\mu) - g(\mu,0)$$

where $g = \frac{1}{2\pi i}\log f_{\mathcal{L}}: \Lambda \times V \to \mathbb{C}$. The image of c_1, called the Néron-Severi group $\mathrm{NS}(T)$, is the set of imaginary parts of Hermitian forms whose imaginary part is integral on Λ.

$\mathrm{Pic}(T)$ is in fact a group, but we don't know that yet. But in fact it's easy to see: we just define the product $\mathcal{L}_1\mathcal{L}_2$ to be the bundle given by the factor of automorphy $f_{\mathcal{L}_1}f_{\mathcal{L}_2}$, so that $\mathcal{L}^{-1}$ corresponds to $f_{\mathcal{L}}^{-1}$. It is not hard to see that it is equivalent to take $\mathcal{L}_1\mathcal{L}_2 = \mathcal{L}_1 \otimes \mathcal{L}_2$ (which suggests how to make $\mathrm{Pic}(X)$ a group for general X, where the theory of factors of automorphy fails). The existence of $\mathcal{L}^{-1}$ is the reason why line bundles are sometimes called invertible sheaves. Since the group $\mathrm{Pic}(X)$ is abelian it is sometimes written additively, but usually not if one is actually thinking of its elements as being line bundles (we shall see another way of thinking of them later). Still, this does serve to remind us that $\mathcal{O}$, the trivial line bundle, corresponding to (untwisted) functions, is the identity element.

In order fully to describe line bundles on T in terms of linear algebra we need to understand the kernel of c_1, which is called $\mathrm{Pic}^0(T)$.

Definition: A semicharacter for $H \in \mathrm{NS}(T)$ (think of H as a Hermitian form) is a map $\chi: \Lambda \to U(1)$ ($U(1)$ is the circle group) such that

$$\chi(\lambda+\mu) = \chi(\lambda)\chi(\mu)\exp\{i\pi \,\mathrm{Im}\, H(\lambda,\mu)\}$$

so that if $H = 0$ then χ is a character.

Let $\mathcal{P}(\Lambda)$ be the set of all pairs (H,χ) with $H \in \mathrm{NS}(T)$ and χ a semicharacter for H. $\mathcal{P}(\Lambda)$ becomes a group if we define $(H_1,\chi_1)(H_2,\chi_2) = (H_1+H_2, \chi_1\chi_2)$, since $\chi_1\chi_2$ is a semicharacter for H_1+H_2.

The following theorem is one of the things that is called the Appel-Humbert Theorem (Mumford uses the term for a slightly different result).

Theorem 1.8. *There are maps L giving a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Hom}\big(\Lambda, U(1)\big) & \xrightarrow{\iota} & \mathcal{P}(\Lambda) & \xrightarrow{\mathrm{pr}} & \mathrm{NS}(T) & \longrightarrow & 0 \\ & & L\downarrow\wr & & L\downarrow\wr & & \| & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(T) & \longrightarrow & \mathrm{Pic}(T) & \xrightarrow{c_1} & \mathrm{NS}(T) & \longrightarrow & 0 \end{array}$$

Proof: The top row is exact by definition of ι and $\mathrm{pr}: (H,\chi) \mapsto H$. The bottom row is exact by the definitions of NS and Pic^0. We need to define $L: \mathcal{P}(\Lambda) \to \mathrm{Pic}(T)$, show that the diagram commutes and check that $L: \mathrm{Hom}\big(\Lambda, U(1)\big) \to \mathrm{Pic}^0(T)$ is iso.

If $D = (H, \chi) \in \mathcal{P}(\Lambda)$, define a factor of automorphy by

$$a_D(\lambda, \mathbf{v}) = \chi(\lambda) \exp\{\pi H(\mathbf{v}, \lambda) + \frac{\pi}{2} H(\lambda, \lambda)\}$$

so $a_D: \Lambda \times V \to \mathbb{C}^*$. Then a_D is a cocycle, since

$$\begin{aligned}
a_D(\lambda + \mu, \mathbf{v}) &= \chi(\lambda + \mu) \exp\big\{\pi H(\mathbf{v}, \lambda + \mu) + \frac{\pi}{2} H(\lambda + \mu, \lambda + \mu)\big\} \\
&= \chi(\lambda)\chi(\mu) \exp\big\{\pi H(\mathbf{v}, \lambda) + \pi H(\mathbf{v}, \mu) + \frac{\pi}{2} H(\lambda, \lambda) + \frac{\pi}{2} H(\mu, \mu) + \frac{\pi}{2} H(\lambda, \mu) + \frac{\pi}{2} H(\mu, \lambda)\big\} \\
&= \chi(\lambda)\chi(\mu) \exp\big\{\pi H(\mathbf{v}, \lambda) + \frac{\pi}{2} H(\lambda, \lambda) + \pi H(\mathbf{v}, \mu) + \frac{\pi}{2} H(\mu, \mu) + \pi \operatorname{Re} H(\lambda, \mu)\big\} \\
&= \chi(\lambda) \exp\big\{\pi H(\mathbf{v} + \mu, \lambda) + \frac{\pi}{2} H(\lambda, \lambda) - i\pi \operatorname{Im} H(\mu, \lambda)\big\} \chi(\mu) \exp\big\{\pi H(\mathbf{v}, \mu) + \frac{\pi}{2} H(\mu, \mu)\big\} \\
&= \chi(\lambda) \exp\big\{\pi H(\mathbf{v} + \mu, \lambda) + \frac{\pi}{2} H(\lambda, \lambda)\big\} \chi(\mu) \exp\big\{\pi H(\mathbf{v}, \mu) + \frac{\pi}{2} H(\mu, \mu)\big\} \\
&= a_D(\mathbf{v} + \mu, \lambda) a_D(\mathbf{v}, \mu).
\end{aligned}$$

From this we get a line bundle $\mathcal{L} = L(D) = \mathcal{L}(H, \chi)$ given by $(V \times \mathbb{C})/\Lambda$, where Λ acts by

$$\lambda: (\mathbf{v}, \alpha) \longmapsto \big(\mathbf{v} + \lambda, a_D(\mathbf{v}, \lambda)\alpha\big).$$

Obviously $D \mapsto a_D$ is a homomorphism.

The right-hand square commutes if $c_1\big(L(D)\big) = \operatorname{pr}(D)$, that is, if $c_1\big(\mathcal{L}(H, \chi)\big) = H$. To check this, put $\chi(\lambda) = \exp\big\{2\pi i \psi(\lambda)\big\}$, so that

$$a_D = \exp\big\{2\pi i g_D(\lambda, \mathbf{v})\big\}$$

where

$$g_D(\lambda, \mathbf{v}) = \psi(\lambda) - \frac{i}{2} H(\mathbf{v}, \lambda) - \frac{i}{4} H(\lambda, \lambda).$$

Then

$$\begin{aligned}
\operatorname{Im} c_1\big(L(D)\big) &= g_D(\mu, \lambda) + g_D(\lambda, 0) - g_D(\lambda, \mu) - g(\mu, 0) \\
&= \frac{1}{2i} \big[H(\lambda, \mu) - H(\mu, \lambda)\big] \\
&= \operatorname{Im} H
\end{aligned}$$

and since a Hermitian form is determined by its imaginary part it follows that $c_1\big(L(D)\big) = H$. This also implies that L maps $\operatorname{Hom}\big(\Lambda, U(1)\big)$ into $\operatorname{Pic}^0(T)$ and the left-hand square commutes automatically.

It remains to check that $L: \operatorname{Hom}\big(\Lambda, U(1)\big) \to \operatorname{Pic}^0(T)$ is an isomorphism. We need to recall something mentioned briefly earlier: two factors of automorphy define the same line bundle if they differ by coming from different trivialisations on $V \times \mathbb{C}$, i.e. by a nonvanishing function on V. More precisely, f_1 and f_2 define the same bundle if there is a holomorphic function $F: V \to \mathbb{C}^*$ such that $f_2(\lambda, \mathbf{v}) = f_1(\lambda, \mathbf{v}) F(\mathbf{v}) F(\mathbf{v} + \lambda)^{-1}$.

I want to show that $L: \operatorname{Hom}\big(\Lambda, U(1)\big) \to \operatorname{Pic}^0(T)$ is surjective, that is, that I can get any line bundle whose Chern class (c_1) is zero from a homomorphism $\Lambda \to U(1)$. Suppose $\mathcal{L} \in \operatorname{Pic}^0(T)$ and f is a factor of automorphy defining $\mathcal{L}$. Take $g = \frac{1}{2\pi i} \log f$ as usual. I claim that f might as well be independent of $\mathbf{v} \in V$, because I can find $f_0: V \to \mathbb{C}^*$ such that $f_1(\lambda, \mathbf{v}) f_0(\mathbf{v}) f_0(\mathbf{v} + \lambda)^{-1}$ is independent of $\mathbf{v}$. We have the cocycle condition

$$g(\lambda + \mu, \mathbf{v}) = g(\lambda, \mathbf{v} + \mu) + g(\mu, \mathbf{v})$$

and the condition that $c_1 = 0$

$$g(\mu, \lambda) - g(\mu, \mathbf{0}) - g(\lambda, \mu) + g(\lambda, \mathbf{0}) = 0$$

both holding for all λ, μ and $\mathbf{v}$. Take $h(\mathbf{v}) = -g(0,\mathbf{v})$. Then

$$\begin{aligned} g(\lambda,\mathbf{v}) - h(\lambda+\mathbf{v}) + h(\mathbf{v}) &= g(\lambda,\mathbf{v}) + g(0\lambda+\mathbf{v}) - g(0,\mathbf{v}) \\ &= g(\lambda,\mathbf{v}) - g(0,\mathbf{v}) \text{ as } g(0,\lambda+\mathbf{v}) = 0 \text{ by cocycle condition} \\ &= g(0,\lambda) - g(\lambda,0) \text{ by } c_1 = 0 \text{ condition} \end{aligned}$$

and this is independent of $\mathbf{v}$, so we can take $F(\mathbf{v}) = \exp\{2\pi i h(\mathbf{v})\}$.

If f is independent of $\mathbf{v}$ then the cocycle condition says $f\colon \Lambda \to \mathbb{C}^*$ is a homomorphism, so $\arg f\colon \Lambda \to U(1)$ is a character. Moreover, $\arg f$ and f define the same line bundle, because, since f is a homomorphism, $\log|f|\colon \Lambda \to \mathbb{R}$ is an additive homomorphism, i.e. an $\mathbb{R}$-linear map. So if we extend it to a function $\ell\colon V \to \mathbb{R}$ by $\mathbb{R}$-linearity, we can also define $\hat{\ell}\colon V \to \mathbb{C}$ by $\hat{\ell}(\mathbf{v}) = \ell(i\mathbf{v}) + i\ell(\mathbf{v})$ and then take $F = \exp\{i\hat{\ell}\}$, making f and $\arg f$ cohomologous. This proves that L is surjective.

Finally, we must show that L is injective on $\operatorname{Hom}(\Lambda, U(1))$. Suppose $\chi \in \operatorname{Hom}(\Lambda, U(1))$ and $\mathcal{L}(0,\chi)$ is trivial, i.e. $\mathcal{L}(0,\chi) = \mathcal{L}(0,1)$. Then there is an $F\colon V \to \mathbb{C}^*$ such that $\chi(\lambda) = F(\mathbf{v}+\lambda)F(\mathbf{v})^{-1}$ for all $\lambda \in \Lambda$, $\mathbf{v} \in V$. As $|\chi(\lambda)| = 1$ this implies that $|F(\mathbf{v}+\lambda)| = |F(\mathbf{v})|$ and hence that F is bounded. So F must be constant, and $\chi = 1$. ∎

Corollary 1.9. *Any line bundle $\mathcal{L} = \mathcal{L}(H,\chi)$ has a canonical factor of automorphy $a_{\mathcal{L}}$, which is the a_D occurring above.*

Summary. We have introduced the following general objects:

- Line bundles
- The Picard group $\operatorname{Pic}(X) = \{$ line bundles on $X\}$/isomorphism with multiplication given by $\otimes$.

and in the special case of complex tori we have also introduced

- The first Chern class $c_1(\mathcal{L})$ of a line bundle $\mathcal{L}$
- The Néron-Severi group $\operatorname{NS}(X) = \{c_1(\mathcal{L}) \mid \mathcal{L} \in \operatorname{Pic}(X)\}$
- $\operatorname{Pic}^0(X) = \operatorname{Ker} c_1$.

I have not said, and we do not need to know, what these are in general. But they do exist in general.

We have also introduced

- Factors of automorphy
- Semicharacters and Hermitian forms integral on Λ

as ways of describing $\operatorname{Pic}(X)$. If X isn't a complex torus then $\operatorname{Pic}(X)$ doesn't have such a nice description. Since our definitions of c_1, NS and Pic^0 used these descriptions we have defined them only for complex tori.

Twice I have asserted things without proof:

- All line bundles on $\mathbb{C}^g$ are trivial
- The alternating form E is the imaginary part of some Hermitian form H

Our original motivation for introducing line bundles was to get embeddings of abelian varieties, i.e. complex tori in projective space. So we want to get at sections of line bundles: the idea is that these will serve as coordinate functions on the complex torus T. There is another reason why line bundles are good: once you've got varieties you can go from line bundles to divisors (formal sums of codimension 1 subvarieties) and back, thus getting a much more geometric description of what is going on.

If $\mathcal{L}$ is a line bundle on some compact complex manifold X and $\sigma_0,\ldots,\sigma_N$ are a basis for $H^0(\mathcal{L})$ (which we assume to be finite dimensional – actually it always is) then we can define a map

$$\phi_{\mathcal{L}}\colon X \longrightarrow \mathbb{P}^N$$

by $\phi_{\mathcal{L}}(x) = (\sigma_0(x) : \ldots : \sigma_N(x))$, as long as the σ_i don't all vanish at once. We say that $\mathcal{L}$ is very ample if $\phi_{\mathcal{L}}$ is an embedding, that is $\phi_{\mathcal{L}}(X) \cong X$. We say that $\mathcal{L}$ is ample if $\mathcal{L}^{\otimes k}$ is very ample for some $k > 0$. You should think of a very ample line bundle as specifying what a hyperplane section will be.

We are going to identify the ample line bundles on T: in particular we are going to find out when there are any, i.e. when T is an abelian variety. In the process we shall find out that $H^0(\mathcal{L})$ is always finite-dimensional on a complex torus, though in fact this is true for any compact complex space. Recall that if $\mathcal{L}$ has a factor of automorphy $f_{\mathcal{L}}$ then $\mathcal{L}^{\otimes k}$ is given by the factor of automorphy $f_{\mathcal{L}}^k$: equivalently if $\mathcal{L} = \mathcal{L}(H,\chi)$ then $\mathcal{L}^{\otimes k} = \mathcal{L}(kH,\chi^k)$.

Definition: If f is a factor of automorphy, a **theta function** for f is a holomorphic function $\theta\colon V \to \mathbb{C}$ such that

$$\theta(\mathbf{v} + \lambda) = f(\lambda, \mathbf{v})\theta(\mathbf{v})$$

Clearly, if f defines $\mathcal{L}$ then θ gives a section of $\mathcal{L}$ and every section of $\mathcal{L}$ comes from a theta function. A canonical theta function for $\mathcal{L} = \mathcal{L}(H, \chi)$ is a theta function for the canonical factor of automorphy for $\mathcal{L}$,

$$f(\lambda, \mathbf{v}) = \chi(\lambda) \exp\left\{\pi H(\mathbf{v}, \lambda) + \frac{\pi}{2} H(\lambda, \lambda)\right\}.$$

Lemma 1.10. *Suppose H is degenerate. Then $\mathcal{L} = \mathcal{L}(H, \chi)$ is not ample.*

Proof: Put $N = \operatorname{Ker}_L H = \left\{\mathbf{v} \in V \mid H(\mathbf{v}, \mathbf{w}) = 0 \text{ for all } \mathbf{w} \in V\right\}$. If $E = \operatorname{Im} H$ then $H(\mathbf{v}, \mathbf{w}) = E(i\mathbf{v}, \mathbf{w}) + iE(\mathbf{v}, \mathbf{w})$ so $\mathbf{v} \in N$ if and only if $E(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w} \in V$. So N is a complex subspace of V and $N \cap \Lambda$ is a lattice in N, since E is integral on Λ. If θ is a canonical theta function then for any $\mathbf{v} \in V$

$$\theta(\mathbf{v} + \lambda) = \chi(\lambda)\theta(\mathbf{v}) \qquad\qquad \text{if } \lambda \in N \cap \Lambda.$$

Thus $|\theta(\mathbf{v} + \mathbf{w})|$ is a periodic function of $\mathbf{w} \in N$ and hence constant: that is to say, $\theta(\mathbf{v})$ depends only on the coset $\mathbf{v} + N$. (So $\theta(\mathbf{v} + \lambda) = \theta(\mathbf{v})$ if $\lambda \in N \cap \Lambda$, so $\chi(\lambda) = 1$ if $\lambda \in N \cap \Lambda$: this means that actually we might as well work with a nondegenerate H on V/N and $\Lambda/(N \cap \Lambda)$). In particular, $\mathcal{L}$ cannot be very ample as $\sigma_i(x) = \sigma_i(x + y)$ if $y \in x + N/(N \cap \Lambda)$, so the σ_i don't separate points. Since N is the same for $\mathcal{L}^{\otimes k}$ as for $\mathcal{L}$ it follows that $\mathcal{L}$ is not ample. ∎

Lemma 1.11. *Suppose $H(\mathbf{w}, \mathbf{v}) < 0$ for some $\mathbf{w}$. Then $h^0(\mathcal{L}) = 0$: in particular $\mathcal{L}$ is not very ample or even ample.*

Proof: We can write $\mathbf{w} = \mathbf{z} + \lambda$ for some λ with $\mathbf{z} \in K$, K compact. Then

$$\begin{aligned}
\left\|\theta(\mathbf{v} + \mathbf{w})\right\| &= \left\|\theta(\mathbf{v} + \mathbf{z} + \lambda)\right\| \\
&= \left\|\theta(\mathbf{v} + \mathbf{z})\right\| \left\|\chi(\lambda)\right\| \left\|\exp\left\{\pi H(\mathbf{v} + \mathbf{z}, \lambda) + \frac{\pi}{2} H(\lambda, \lambda)\right\}\right\| \\
&= \left\|\theta(\mathbf{v} + \mathbf{z})\right\| \exp\left\{\pi \operatorname{Re} H(\mathbf{v} + \mathbf{z}, \lambda) + \frac{\pi}{2} H(\lambda, \lambda)\right\}.
\end{aligned}$$

But

$$\begin{aligned}
\operatorname{Re} H(\mathbf{v} + \mathbf{z}, \lambda) + \frac{1}{2} H(\lambda, \lambda) &= \operatorname{Re} H(\mathbf{v} + \mathbf{z}, \mathbf{w} - \mathbf{z}) + \frac{1}{2} H(\mathbf{w} - \mathbf{z}, \mathbf{w} - \mathbf{z}) \\
&= \operatorname{Re} H(\mathbf{v} + \mathbf{z}, \mathbf{w}) - \operatorname{Re} H(\mathbf{v} + \mathbf{z}, \mathbf{z}) + \frac{1}{2} H(\mathbf{w}, \mathbf{w}) + \frac{1}{2} H(\mathbf{z}, \mathbf{z}) - \operatorname{Re} H(\mathbf{w}, \mathbf{z}) \\
&= \operatorname{Re} H(\mathbf{v}, \mathbf{w}) + \frac{1}{2} H(\mathbf{w}, \mathbf{w}) + \text{ a function of } \mathbf{z} \text{ and } \mathbf{v}
\end{aligned}$$

so for fixed $\mathbf{v}$ we have a linear term in $\mathbf{w}$ + a negative quadratic term in $\mathbf{w}$ + something bounded, and this tends to $-\infty$ as $\mathbf{w} \to \infty$. So $\left\|\theta(\mathbf{v} + \mathbf{w})\right\| \to 0$ as $\mathbf{w} \to \infty$, and so $\theta \equiv 0$. Thus $h^0(\mathcal{L}) = 0$. ∎

Corollary 1.12. *If $\mathcal{L} = \mathcal{L}(H, \chi)$ is ample then H is positive definite.* ∎

To get at the converse to this (and more) we need a supply of sections.

Theorem 1.13. *Suppose H is positive definite and write E as a matrix relative to a $\mathbb{Z}$-basis of Λ. Then*

$$\dim H^0\big(\mathcal{L}(H,\chi)\big) = \sqrt{\det E}.$$

Proof: The idea is to use a slightly different factor of automorphy and hence slightly different theta functions – classical theta functions – which are actually periodic with respect to about half of Λ. This enables us to write down Fourier expansions for the theta functions and then see how many coefficients we can choose before the behaviour with respect to the rest of Λ fixes everything else.

I can certainly choose a basis of Λ such that E has matrix $\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$. Let Λ_1 and Λ_2 be the $\mathbb{Z}$-spans of the first and second g elements and let V_1 and V_2 be the $\mathbb{R}$-spans. Thus $E|_{\Lambda_2\times\Lambda_2} = 0$ and $V_j \cap \Lambda = \Lambda_j$. Certainly $V_2 \cap iV_2 = 0$ because $H = 0$ there and H is nondegenerate, so $\Lambda_2 \otimes \mathbb{C} = V$. The restriction of H to V_2 is real symmetric (because $E = 0$ there), so there is a unique complex symmetric extension B of $H|_{V_2\times V_2}$ to the whole of V.

Put $\theta^*(\mathbf{v}) = \exp\{\frac{\pi}{2}B(\mathbf{v},\mathbf{v})\}\theta(\mathbf{v})$, so that

$$\begin{aligned}\theta^*(\mathbf{v}+\lambda) &= \chi(\lambda)\exp\Big\{\pi\big(H-B\big)(\mathbf{v},\lambda) + \frac{\pi}{2}\big(H-B\big)(\lambda,\lambda)\Big\}\theta^*(\mathbf{v}) \\ &= f^*(\lambda,\mathbf{v})\theta^*(\mathbf{v}).\end{aligned}$$

Since $f^*(\lambda,\mathbf{v}) = f(\lambda,\mathbf{v})\exp\big\{\frac{\pi}{2}B(\mathbf{v},\mathbf{v})\big\}\exp\big\{\frac{\pi}{2}B(\mathbf{v}+\lambda,\mathbf{v}+\lambda)\big\}^{-1}$, we see that f^* is also a factor of automorphy for $\mathcal{L}$ and θ^* is a theta function for it: these are the classical factor and theta functions. It isn't quite true that θ^* is periodic for Λ_2, but very nearly: the map $\chi\colon\Lambda_2 \to U(1)$ is a homomorphism so $\chi(\lambda) = \exp\{2\pi i l(\lambda)\}$ with $l\colon\Lambda_2 \to \mathbb{R}$ being $\mathbb{Z}$-linear. Extend l to a $\mathbb{C}$-linear map $l\colon V \to \mathbb{C}$ (recall that $\Lambda_2 \otimes \mathbb{C} = V$) and consider

$$\bar\theta(\mathbf{v}) = \exp\{-2\pi i l(\mathbf{v})\}\theta^*(\mathbf{v}).$$

Then $\bar\theta(\mathbf{v}+\lambda) = \bar\theta(\mathbf{v})$ for all $\lambda \in \Lambda_2$, because $\big(H-B\big)(\lambda,\lambda) = 0$ for $\lambda \in \Lambda_2$.

By Fourier analysis, with $\Lambda_2^* = \mathrm{Hom}(\Lambda_2,\mathbb{Z}) \subseteq \mathrm{Hom}(V,\mathbb{C})$

$$\bar\theta(\mathbf{v}) = \sum_{m\in\Lambda_2^*} a_m \exp\big\{2\pi i m(\mathbf{v})\big\}$$

so

$$\theta^*(\mathbf{v}) = \sum_{m\in\Lambda_2^*} a_m \exp\big\{2\pi i\big(m(\mathbf{v}) + l(\mathbf{v})\big)\big\}.$$

What conditions do the a_m satisfy? We need to look at $\theta^*(\mathbf{v}+\mu)$ for $\mu \in \Lambda$.

$$\begin{aligned}\theta^*(\mathbf{v}+\mu) &= \chi(\mu)\exp\Big\{\pi\big(H-B\big)(\mathbf{v},\mu) + \frac{\pi}{2}\big(H-B\big)(\mu,\mu)\Big\}\theta^*(\mathbf{v}) \\ &= \chi(\mu)\exp\big\{2\pi i\hat\mu(\mathbf{v}) + \pi i\hat\mu(\mu)\big\}\theta^*(\mathbf{V})\end{aligned}$$

where $\hat\mu(\lambda) = E(\lambda,\mu)$ if $\lambda \in \Lambda_2$ and $\hat\mu$ is the $\mathbb{C}$-linear extension of $E(\bullet,\mu)$ to $\Lambda_2 \otimes \mathbb{C} = V$. This is because $\big(H-B\big)(\lambda,\mu) = \overline{H(\mu,\lambda)} - B(\mu,\lambda) = -2i\,\mathrm{Im}\,H(\mu,\lambda) = 2iE(\lambda,\mu)$ if $\mu \in \Lambda$ and $\lambda \in \Lambda_2$.

Comparing coefficients in the Fourier series gives

$$a_m = \chi(\mu)\exp\big\{\pi i\hat\mu(\mu) - 2\pi i\big(m(\mu) - l(\mu)\big)\big\}a_{m-\hat\mu}.$$

So we only need to know a_m for one m in each coset of the image in Λ_2^* of Λ: call this image $\hat\Lambda$. There is a little well-definedness to be checked here, for instance that $\mathrm{Ker}(\mu \mapsto \hat\mu) \subseteq \Lambda_2$, so that if $\hat\mu_1 = \hat\mu_2$ we get the same equation for both $a_{m-\hat\mu_1}$ and $a_{m-\hat\mu_2}$, but subject to that we have proved that

$$h^0(\mathcal{L}) \le \|\Lambda_2^* : \hat\Lambda\|.$$

In fact $h^0(\mathcal{L}) = \|\Lambda_2^* : \hat\Lambda\|$. To show this is a matter of showing that the Fourier series converges if the a_m satisfy the right equation. It is enough to do so for $m \in \hat\Lambda + m_0$ for each m_0, as that splits the series into finitely many convergent bits. But $\|a_{m-\hat\mu}\| \sim \exp\big\{\,\mathrm{Im}\,\big(\hat\mu(\mu)\big)\big\}$ and if $\mu \in \Lambda_2$ (which it might as well be as we are only concerned with $\hat\mu$) then $\mathrm{Im}\,\big(\hat\mu(\mu)\big) = -H(\mu,\mu)$, so $\hat\mu \mapsto \mathrm{Im}\,\big(\hat\mu(\mu)\big)$ is a negative definite quadratic form on $\hat\Lambda$.

Finally, $\|\Lambda_2^* : \hat\Lambda\|$ is the index of the sublattice of Λ_2 spanned by the rows of D, which is $\det D$, and this is equal to the Pfaffian $\sqrt{\det E}$. ∎

Theorem 1.14. (Lefschetz) *Suppose H is positive definite. Then $\mathcal{L}(H,\chi)$ is ample: in fact $\mathcal{L}(H,\chi)^{\otimes 3}$ is very ample.*

Proof: We need to show that $\mathcal{L}^{\otimes 3}$ defines an embedding. That means three things:

i) It defines a map: for any $x \in T$ there is a $\sigma \in H^0(\mathcal{L}^{\otimes 3})$ such that $\sigma(x) \neq 0$.

ii) The map $\phi_{\mathcal{L}^{\otimes 3}}$ separates points: for all x, $y \in T$ we have $\phi_{\mathcal{L}^{\otimes 3}}(x) \neq \phi_{\mathcal{L}^{\otimes 3}}(y)$.

iii) The map $\phi_{\mathcal{L}^{\otimes 3}}$ separates tangent directions: $d\phi_{\mathcal{L}^{\otimes 3}}$ is injective at x.

It is (ii) that is difficult: the idea is that if $\phi_{\mathcal{L}^{\otimes 3}}$ fails to separate points then all the sections actually come from some quotient torus, but there aren't enough such sections.

Suppose θ is a canonical theta function for $\mathcal{L} = \mathcal{L}(H,\chi)$. If $\mathbf{a}$, $\mathbf{b} \in V$ then we can get a theta function for $\mathcal{L}(3H,\chi^3) = \mathcal{L}^{\otimes 3} = \mathcal{L}^3$ by considering

$$\hat{\theta}(\mathbf{v}) = \theta(\mathbf{v}-\mathbf{a})\theta(\mathbf{v}-\mathbf{b})\theta(\mathbf{v}+\mathbf{a}+\mathbf{b})$$

since

$$\begin{aligned}\hat{\theta}(\mathbf{v}+\lambda) &= \hat{\theta}(\mathbf{v})\chi(\lambda)^3 \exp\bigl\{\pi H(\mathbf{v}-\mathbf{a},\lambda) + \pi H(\mathbf{v}-\mathbf{b},\lambda) + \pi H(\mathbf{v}+\mathbf{a}+\mathbf{b}) + \frac{3\pi}{2}H(\lambda,\lambda)\bigr\} \\ &= \hat{\theta}(\mathbf{v})\chi(\lambda)^3 \exp\bigl\{3\pi H(\mathbf{v},\lambda) + \frac{3\pi}{2}H(\lambda,\lambda)\bigr\}.\end{aligned}$$

So if we choose a nontrivial theta function θ for $\mathcal{L}(H\chi)$, which we can do if $H > 0$, and a point $\mathbf{v}_0 \in V$, then we can certainly find $\mathbf{a}$, $\mathbf{b} \in V$ such that $\theta(\mathbf{v}_0-\mathbf{a})$, $\theta(\mathbf{v}_0-\mathbf{b})$ and $\theta(\mathbf{v}_0+\mathbf{a}+\mathbf{b})$ are all nonzero. Then $\hat{\theta}(\mathbf{v})$ is a theta function for $\mathcal{L}^3$ such that $\theta(\mathbf{v}_0) \neq 0$, and it gives a section $\sigma \in H^0(\mathcal{L}^3)$ with $\sigma(\Lambda+\mathbf{v}_0) \neq 0$. This proves (i).

Now for (ii). Suppose $\phi_{\mathcal{L}^3}\colon T \to \mathbb{P}^N$, given by $\phi_{\mathcal{L}^3}(x) = \bigl(\sigma_0(x) : \ldots : \sigma_N(x)\bigr)$ where $\sigma_0,\ldots,\sigma_N$ is a basis for $H^0(\mathcal{L}^3)$, is not injective. Then there exist $\mathbf{v}_1$, $\mathbf{v}_2 \in V$ such that $\mathbf{u} = \mathbf{v}_1 - \mathbf{v}_2 \notin \Lambda$ and there is a constant $\kappa \in \mathbb{C}^*$ such that $\psi(\mathbf{v}_2) = \kappa\psi(\mathbf{v}_1)$ for every theta function ψ for $\mathcal{L}^3$. In particular this means that if $\mathbf{a}$, $\mathbf{b} \in V$ and θ is a theta function for $\mathcal{L}$ then $\hat{\theta}(\mathbf{v}_2) = \kappa\hat{\theta}(\mathbf{v}_1)$, i.e.

$$\theta(\mathbf{v}_1-\mathbf{a})\theta(\mathbf{v}_1-\mathbf{v})\theta(\mathbf{v}_1+\mathbf{a}+\mathbf{b}) = \kappa\theta(\mathbf{v}_2-\mathbf{a})\theta(\mathbf{v}_2-\mathbf{v})\theta(\mathbf{v}_2+\mathbf{a}+\mathbf{b}).$$

So, taking logarithmic differentials

$$-\frac{\partial}{\partial\mathbf{a}}\log\theta(\mathbf{v}_1-\mathbf{a}) + \frac{\partial}{\partial\mathbf{a}}\log\theta(\mathbf{v}_1+\mathbf{a}+\mathbf{b}) = -\frac{\partial}{\partial\mathbf{a}}\log\theta(\mathbf{v}_2-\mathbf{a}) + \frac{\partial}{\partial\mathbf{a}}\log\theta(\mathbf{v}_1+\mathbf{a}+\mathbf{b})$$

and, writing ω for the meromorphic differential $d\theta/\theta$,

$$-\omega(\mathbf{v}_1-\mathbf{a}) + \omega(\mathbf{v}_1+\mathbf{a}+\mathbf{b}) = -\omega(\mathbf{v}_2-\mathbf{a}) + \omega(\mathbf{v}_1+\mathbf{a}+\mathbf{b})$$

so that $\eta(\mathbf{v}) = \omega(\mathbf{v}_2-\mathbf{v}) - \omega(\mathbf{v}_1-\mathbf{v})$ is independent of $\mathbf{v}$.

Therefore $\eta = d\ell(\mathbf{v})$, where $\ell\colon V \to \mathbb{C}$ is linear. But

$$\eta = d\log\frac{\theta(\mathbf{v}_2+\mathbf{v})}{\theta(\mathbf{v}_1+\mathbf{v})}$$

so $\theta(\mathbf{v}_2+\mathbf{v}) = \kappa' e^{\ell(\mathbf{v})}\theta(\mathbf{v}_1+\mathbf{v})$, and so $\theta(\mathbf{u}+\mathbf{v}) = \kappa'' e^{\ell(\mathbf{v})}\theta(\mathbf{v})$. Using the fundamental equation for θ we obtain

$$e^{\pi H(\mathbf{u},\lambda)} = e^{\ell(\lambda)} \qquad \text{for all } \lambda \in \Lambda.$$

So $\pi H(\mathbf{u},\lambda) - \ell(\lambda) \in 2\pi i\mathbb{Z}$ and in particular it is imaginary. Therefore $\pi H(\lambda,\mathbf{u}) - \ell(\lambda)$ is imaginary (as $\pi h(\lambda,\mathbf{u}) - \pi H(\mathbf{u}\lambda) \in \mathbb{R}$) for all $\lambda \in \Lambda$. I claim that in fact $\pi H(\lambda,\mathbf{u}) - \ell(\lambda) = 0$ for any $\lambda \in \Lambda$. Suppose not. Then $\lambda \neq 0$ and we can find $\lambda' \in \Lambda$ such that $\lambda' = k\lambda$ for some $k \notin \mathbb{R}$. Then

$$\begin{aligned}\pi H(\lambda',\mathbf{u}) - \ell(\lambda') &= \pi H(k\lambda,\mathbf{u}) - \ell(k\lambda) \\ &= k\bigl(\pi H(\lambda,\mathbf{u}) - \ell(\Lambda)\bigr) \notin i\mathbb{R}.\end{aligned}$$

If $\pi H(\lambda, \mathbf{u}) - \ell(\lambda) = 0$ for all $\lambda \in \Lambda$ then

$$\begin{aligned} 2\pi i\mathbb{Z} \ni \pi H(\mathbf{u}, \lambda) - \ell(\lambda) \\ &= \pi H(\lambda, \mathbf{u}) - \ell(\lambda) + \pi H(\mathbf{u}, \lambda) - \pi H(\lambda, \mathbf{u}) \\ &= 0 + 2\pi i \operatorname{Im} H(\mathbf{u}, \lambda) \\ &= 2\pi i E(\mathbf{u}, \lambda) \end{aligned}$$

so $E(\mathbf{u}, \lambda) \in \mathbb{Z}$ for all $\lambda \in \Lambda$. Consider $\Lambda^\perp = \{\mathbf{v} \in V \mid E(\mathbf{v}, \lambda) \in \mathbb{Z}\ \forall \lambda \in \Lambda\}$. It is a discrete subgroup of V and it contains Λ (necessarily as a subgroup of finite index), so it is a lattice in V. Put $\Lambda' = \Lambda + \mathbb{Z}\mathbf{u} \subseteq \Lambda^\perp$: clearly Λ' is also a lattice, and Λ' strictly contains Λ. However

$$\begin{aligned} \theta(\mathbf{u} + \mathbf{v}) &= \kappa'' e^{\ell(\mathbf{v})} \theta(\mathbf{v}) \\ &= \kappa''' e^{\pi H(\mathbf{v}, \mathbf{u}) + \frac{\pi}{2} H(\mathbf{u}, \mathbf{u})} \theta(\mathbf{v}) \end{aligned}$$

where $\kappa''' = \kappa'' e^{\frac{-\pi}{2}} H(\mathbf{u}, \mathbf{u})$, since if $\pi H(\lambda, \mathbf{u}) = \ell(\lambda)$ then $\pi H(\mathbf{v}, \mathbf{u}) = \ell(\mathbf{v})$, by $\mathbb{R}$-linearity. Now if we put $\chi'(\mathbf{u}) = \kappa'''$ then $\chi' \in \operatorname{Hom}\big(\Lambda', U(1)\big)$, and we have shown that θ is actually a theta function for $\mathcal{L}(H, \chi')$ on the torus $T' = V/\Lambda'$. But the dimension of the space of such theta functions is $\det_{\Lambda'} E$, which is strictly less that $\det_\Lambda E$ which is the dimension of the space of all theta functions: so this cannot be true for all theta functions, contradicting our assumption.

Finally, for (iii), suppose $\mathbf{v}_0 \in V$ and that there is a non-trivial tangent vector

$$\sum_{i=1}^g \alpha_i \left.\frac{\partial}{\partial z_i}\right|_{\mathbf{v}_0} \in T_{V, \mathbf{v}_0} = T_{T, \bar{\mathbf{v}}_0}$$

that is mapped to zero by $\phi_{\mathcal{L}}$. Then there is an $\alpha_0 \in \mathbb{C}$ such that for all theta functions ψ for $\mathcal{L}(3H, \chi^3) = \mathcal{L}^{\otimes 3}$

$$\alpha_0 \psi(\mathbf{v}_0) = \sum_{i=1}^g \alpha_i \frac{\partial \psi}{\partial z_i}(\mathbf{v}_0),$$

that is,

$$\Big(\sum_{i=1}^g \alpha_i \frac{\partial}{\partial z_i}\Big)(\log \psi)(\mathbf{v}_0) = 0$$

(remember $\log \psi \colon T \to \mathcal{L}^{\otimes 3}$). Take $\mathbf{a}, \mathbf{b} \in V$ and θ a theta function for $\mathcal{L}$: put $\psi = \hat{\theta}$ and $t(\mathbf{v}) = \sum_{i=1}^g \alpha_i \frac{\partial}{\partial z_i} (\log \hat{\theta})(\mathbf{v})$. Then

$$t(\mathbf{v}_0 - \mathbf{a}) + t(\mathbf{v}_0 - \mathbf{b}) + t(\mathbf{v}_0 + \mathbf{a} + \mathbf{b}) = \alpha_0$$

so t is linear in $\mathbf{v}$. Thus

$$\theta(\mathbf{v} + c\mathbf{u}) = e^{c'\mathbf{u}^2 + ct(\mathbf{u})} \theta(\mathbf{v})$$

for all $c \in \mathbb{C}$ and some $\mathbf{u} \in V$, $c' \in \mathbb{C}$. So $c\mathbf{u} \in \Lambda^\perp$ for all $c \in \mathbb{C}$, but this is impossible because $\Lambda^\perp$ is discrete. ■

Let us take another look at the view. We started out with complex tori and we have got as far as determining which ones are in fact abelian varieties: we were able to embed $T = V/\Lambda$ in $\mathbb{P}^N$ if we could find a positive definite Hermitian form H on V such that the imaginary part E takes integer values on Λ. This is an arithmetic condition, and a highly nontrivial one: most lattices will not satisfy it.

We get the embedding by taking a line bundle constructed out of H and some extra data χ and looking at sections. We describe line bundles by means of factors of automorphy, i.e. by specifying an action of Λ on $V \times \mathbb{C}$, and we describe sections by means of theta functions, i.e. Λ-invariant functions on V.

In two places I have asserted things without proof:

- $V \times \mathbb{C}$ is the only line bundle on $V = \mathbb{C}^g$, so we haven't missed anything;
- the form $E = c_1(\mathcal{L})$ that you get from a line bundle $\mathcal{L}$ via a factor of automorphy in in fact $\operatorname{Im} H$.

Actually, I haven't really used the first of these yet. All the constructions – factor of automorphy, c_1, theta functions, ampleness – have been made for bundles coming from $V \times \mathbb{C}$, and it is conceivable that there are more bundles on V and hence on T that I haven't told you about. But in fact that is not the case. Moreover – and this I haven't said, though it's not hard – there aren't any other ways of embedding T in $\mathbb{P}^N$ apart from using a line bundle: given any smooth compact complex manifold $X \subseteq \mathbb{P}^N$ I can find a line bundle called $\mathcal{O}_X(1)$ which determines the embedding. So the only tori that embed in $\mathbb{P}^N$ are the ones for which a positive definite H is available.

This fact is a special case of something much more general which I'm going to want anyway: the correspondence between line bundles and divisors, mentioned in passing earlier. It provides an interpretation of line bundles (not just very ample ones) in geometric terms.

A divisor D is a sum of codimension 1 subvarieties with multiplicity. We can get a divisor D from a line bundle $\mathcal{L}$ by taking σ to be a meromorphic section of $\mathcal{L}$ and then taking D to be (σ) =(zeros of σ)−(poles of σ). Suppose I have two different meromorphic sections of $\mathcal{L}$, σ_1 and σ_2: then $f = \sigma_1/\sigma_2$ is a global meromorphic function so $(f) = (\sigma_1) - (\sigma_2)$. We say the two divisors (σ_1) and (σ_2) are linearly equivalent if this happens.

To go from D back to $\mathcal{L}$, define D locally as being given by $(f_\alpha = 0)$ on an open set U_α and take as transition functions f_α/f_β on $U_\alpha \cap U_\beta$. In particular if $D = (f)$ then $\mathcal{L}$ is trivial, as then $f_\alpha \equiv f_\beta \equiv f$. Call the bundle constructed in this way $\mathcal{O}(D)$. If $D > 0$ then f_α is holomorphic.

If X is a curve and D is a divisor on X then $D = \sum a_i P_i$, where $P_i \in X$ are points and the a_1 are the multiplicities. The degree $\deg D$ is defined to be $\sum a_i$: note that $\deg D = 0$ is not at all the same as saying that D is trivial. For instance the divisor $P - Q$, where P and Q are distinct points on an elliptic curve, has degree zero but is not trivial as then f would give a one-to-one map from a torus to the sphere. The collection of all degree zero divisors is called $\mathrm{Pic}^0(X)$: it turns out to be an abelian variety called the Jacobian $\mathrm{Jac}(X)$.

2 Curves and Jacobians

From now on we are going to be using abelian varieties and algebraic varieties in general, and the first thing we do is give, rather more precisely than before, the correspondence between line bundles and divisors.

Let X be a smooth (this is important) projective variety. There is a general principle, known as GAGA ("géométrie algébrique et géométrie analytique") to the effect that on projective varieties over $\mathbb{C}$ holomorphic=algebraic and meromorphic=rational, and I intend to be careless about the distinctions.

Definition: A **divisor** on X is a finite formal sum $\sum a_i D_i$ of irreducible codimension 1 subvarieties with multiplicities $a_i \in \mathbb{Z}$.

The group $\mathrm{Div}(X)$ of all divisors is just the free abelian subgroup on the set of irreducible codimension 1 subvarieties. A divisor D is said to be effective if $a_i \geq 0$ for all i. Because X is smooth a prime divisor D_0 – that is, an irreducible subvariety of codimension 1 – is necessarily given locally by the vanishing of some function, so if D is a divisor there are an open cover $\{U_\alpha\}$ of X and rational functions f_α on U_α such that $\mathrm{ord}_{D_i} f_\alpha = a_i$: thus $D|_{U_\alpha} = (f_\alpha)$. The line bundle corresponding to D is $\mathcal{O}(D)$ and is given by the transition functions $\phi_{\alpha\beta} = f_\alpha/f_\beta$. Conversely if $\mathcal{L}$ is a line bundle with a rational section σ (and at least if X is projective any $\mathcal{L}$ has a rational section), then $\mathcal{L} \mapsto (\sigma)$ inverts this.

Definition: Two divisors D_1 and D_2 are **linearly equivalent** (denoted $D_1 \sim D_2$) if $D_1 - D_2 = (f)$ for some rational function f on X.

Lemma 2.1. *There is a one-to-one correspondence between linear equivalence classes of divisors and line bundles, on smooth projective varieties.*

Proof: Two linearly equivalent divisors give the same bundle since $f_\alpha f/f_\beta f = f_\alpha/f_\beta$. If σ_1, σ_2 are rational sections then $\sigma_1/\sigma_2 = f$ is a rational function so $(\sigma_1) - (\sigma_2) = (f)$. ■

Lemma 2.2. $\text{Div}(X)/\sim$ *is an additive group and* $\text{Pic}(X) \to \text{Div}(X)/\sim$ *is an isomorphism.*

Proof: If D_1, $D_2 \sim 0$ then $D_1 - D_2 \sim 0$ as it is the divisor of f_1/f_2, so $[D_1 + D_2]$ and $[-D_1]$ are well-defined and $\text{Div}(X)/\sim$ is a group.

If $\mathcal{L}_1$, $\mathcal{L}_2 \in \text{Pic}(X)$ have transition functions $\phi^1_{\alpha\beta}$, $\phi^2_{\alpha\beta}$, then the bundle with transition functions $\phi^1_{\alpha_\beta}(\phi^2_{\alpha\beta})^{-1}$ is $\mathcal{L}_1\mathcal{L}_2^{-1}$, so $\text{Div}(X) \to \text{Pic}(X)$ is a group homomorphism. Conversely, if σ_i are rational sections of $\mathcal{L}_i$, then $\sigma_1\sigma_2^{-1}$ is a rational section of $\mathcal{L}_1\mathcal{L}_2^{-1}$, so $\text{Pic}(X) \to \text{Div}(X)$ is also a group homomorphism. ■

Clearly D is effective if and only if the f that defines it is actually a section, not just a rational section, in $\mathcal{L}$. Two elements σ_1 and σ_2 of $H^0\big(\mathcal{O}(D)\big)$ define the same divisor if and only if $\sigma_1 = k\sigma_2$ for some constant k. Hence if we denote by $|D|$ the set of effective divisors linearly equivalent to D, we have $|D| = \mathbb{P}H^0\big(\mathcal{O}(D)\big)$, so $\dim |D| = h^0\big(\mathcal{O}(d)\big) - 1$.

Now suppose that $X = C$ is a curve, so that a prime divisor is just a point. We define the degree of a divisor D by

$$\deg \sum a_i D_i = \sum a_i$$

so $\deg D \in \mathbb{Z}$. Since a rational function has as many zeros as poles, the degree is actually defined on $\text{Pic}(C)$. We can introduce $\text{Pic}^0(X) = \text{Ker} \deg = \{\mathcal{L} \mid \deg \mathcal{L} = 0\}$. This is of interest two us for two reasons, both surprising. It's an abelian variety, and it contains all the information about the curve C.

Let C be a curve. There are various ways of thinking of the genus $g(C)$. You can think of it as being the number of handles that C has, or the number of independent differential forms. For now, I'm going to assume that these are the same. So we have $2g$ paths $\gamma_1, \dots, \gamma_{2g}$ starting from some base point P_0 and returning there, which generate the fundamental group of C, and g 1-forms $\omega_1, \dots, \omega_g$. We put

$$\lambda_{ji} = \int_{\gamma_i} \omega_j$$

and look at the corresponding matrix $\Pi = (\lambda_{ji})$. Note that Stokes' Theorem tells us that $\int_{\gamma_i'} \omega_j = \int_{\gamma_i} \omega_j$ if γ_i and γ_i' determine the same homotopy class. Please believe, for the moment, that $\Lambda = \sum \lambda_i \mathbb{Z}$, the integer span of the columns of Π, is indeed a lattice.

Definition: The quotient $\mathbb{C}^g/\Lambda$ is called the **Jacobian**, $J(C)$ or $\text{Jac}(C)$.

In fact $J(C)$ is an abelian variety and has a natural polarisation.

Now let me beg a few questions. When talking about abelian varieties I feel a duty (not always performed) to justify my assertions, but when talking about curves I am willing to impose a certain amount of dogma.

Let C be an algebraic curve of genus $g \geq 1$. There is a "very basic but nonelementary" (to quote a standard book on curves, the one by Arbarello, Cornalba, Griffiths and Harris) fact, that the number of 1-forms (that is, $H^0(K_C)$, where K_C is the cotangent bundle) is equal to the topological genus g.

I also need to be able to use De Rham cohomology. All I need of it is H^1_{DR}, though the fact above may be interpreted as De Rham's theorem for H^2. We define

$$H^1_{\text{DR}}(X) = \{\text{Closed differential 1-forms}\}/\{\text{Exact forms}\}$$

By a differential 1-form we mean something which is locally of the form

$$\eta = \sum (f_i dx_i + g_i dy_i)$$

with f_i and g_i complex-valued C^∞ functions. If I prefer, I can write it as

$$\eta = \sum (\phi_i dz_i + \psi_i d\bar{z}_i)$$

instead. The De Rham theorem says that $H^1_{\text{DR}}(X) \cong H^1(X; \mathbb{C})$ or, to be more precise, that $H^1_{\text{DR}}(X; \mathbb{R}) \cong H^1(X; \mathbb{R})$. A similar statement holds for differential q-forms and H^q for any q, but to prove the case $q = 1$ you need only the Poincaré Lemma (every closed form on $\mathbb{R}^n$ is exact) and a belief in Čech cohomology.

The Hodge decomposition says (for curves) that $H^1_{\text{DR}}(C) = H^0(K_C) \oplus \overline{H^0(K_C)}$; that is, that I can always choose ϕ and ψ in $\eta = \phi dz = \psi d\bar{z}$ to be holomorphic and antiholomorphic respectively, without changing the cohomology class of η. This is a very special case of something far more general.

One other thing you will have to believe is that wedge product of forms agrees with intersection: I will explain this when I need it.

Theorem 2.3. *The matrix* $\Pi \in M_{g\times 2g}(\mathbb{C})$ *given by*

$$\Pi = \begin{pmatrix} \int_{\gamma_1}\omega_1 & \dots & \int_{\gamma_{2g}}\omega_1 \\ \vdots & \ddots & \vdots \\ \int_{\gamma_1}\omega_g & \dots & \int_{\gamma_{2g}}\omega_g \end{pmatrix}$$

is the period matrix of a complex torus.

Proof: Note first of all that $\int_\gamma \omega$ is well-defined for $\gamma \in H_1(C;\mathbb{Z})$ by Stokes' Theorem, so the assertion makes sense. We need to show that the matrix $\begin{pmatrix}\Pi \\ \bar{\Pi}\end{pmatrix}$ is nonsingular. Suppose that $\mathbf{x}\begin{pmatrix}\Pi \\ \bar{\Pi}\end{pmatrix} = 0$: then

$$\sum_{j=1}^{g}\Big(\int_{\gamma_i}(x_j\omega_j + y_j\bar{\omega}_j)\Big) = 0$$

(where $\mathbf{x} = (x_1,\dots,x_g,y_1,\dots,y_g) \in \mathbb{C}^{2g}$), and therefore

$$\int_{\gamma_i}\Big(\sum_{j=1}^{g}(x_j\omega_j + y_j\bar{\omega}_j)\Big) = 0$$

for all i. The isomorphism

$$H^1_{\mathrm{DR}}(X) \longrightarrow H^1(X;\mathbb{C}) = \big(H_1(X;\mathbb{Z})\otimes\mathbb{C}\big)^*$$

is given by

$$\eta \longmapsto \Big(\sum c_i \otimes \gamma_i \longmapsto \sum c_i \int_{\gamma_i}\eta\Big).$$

It is clear that this is at least plausible in that if η is exact it returns zero, so we have given a well-defined map from H^1_{DR} to H^1. Moreover, if we believe De Rham's theorem, if $\int_{\gamma_i}\big(\sum_{j=1}^{g}(x_j\omega_j + y_j\bar{\omega}_j)\big) = 0$ then $\sum_{j=1}^{g}(x_j\omega_j + y_j\bar{\omega}_j) = 0$ also. But $\{\omega_j\}$ and $\{\bar{\omega}_j\}$ between them span $H^0(K_C)\oplus\overline{H^0(K_C)} \cong H^1_{\mathrm{DR}}(C)$, so this implies $\mathbf{x} = 0$. ■

Now I want to check that $J(C)$ is in fact an abelian variety, i.e. that there exists a positive definite Hermitian form H on $V = \mathbb{C}^g$ taking integer values on Λ.

Let us now decide which basis of $H_1(C;\mathbb{Z})$ we are talking about. We want one such that the intersection number $\gamma_i\gamma_j$ (strictly speaking, the dual of the cup product of the Poincaré duals) is given by the matrix $\begin{pmatrix}0 & -\mathbf{1} \\ \mathbf{1} & 0\end{pmatrix}$.

Define an alternating $\mathbb{R}$-bilinear form E on $H^0(K_C)^*$ by choosing as $\mathbb{R}$-basis for $V = H^0(K_C)^*$ the set $\{\lambda_i = (\omega \mapsto \int_{\gamma_i}\omega)\}$ and declaring E to have matrix $\begin{pmatrix}0 & -\mathbf{1} \\ \mathbf{1} & 0\end{pmatrix}$ with respect to this basis. Then define H on $H^0(K_C)^*$ by $H(\mathbf{u},\mathbf{v}) = E(i\mathbf{u},\mathbf{v}) + iE(\mathbf{u},\mathbf{v})$. Clearly this determines a (not obviously Hermitian) form that takes integer values on Λ, because $(\int_{\gamma_i}\omega_1,\dots,\int_{\gamma_i}\omega_g)$ is just λ_i expressed in terms of the basis $\omega_1,\dots\omega_g$ for $H^0(K_C)$.

We need to check that H is Hermitian and positive definite.

Theorem 2.4. *Suppose* $\Pi \in M_{g\times 2g}(\mathbb{C})$ *is a period matrix for some complex torus* X*. Then* X *is an abelian variety if and only if the Riemann relations*

$$\Pi A^{-1\top}\Pi = 0, \qquad i\Pi A^{-1\top}\bar{\Pi} > 0$$

are satisfied for some nondegenerate integral skew-symmetric matrix A*.*

This follows at once from the two lemmas below. Take the basis $\lambda_1,\dots\lambda_{2g}$ for Λ obtained from Π (that is, think of Λ as being spanned by the columns of Π) and let E be the alternating form whose matrix with respect to $\{\lambda_i\}$ is A. Put $H(\mathbf{u},\mathbf{v}) = E(i\mathbf{u},\mathbf{v}) + iE(\mathbf{u},\mathbf{v})$.

Lemma 2.5. *H is Hermitian if and only if* $\Pi A^{-1\top}\Pi = 0$.

Proof: H is Hermitian if and only if $E(i\mathbf{u}, i\mathbf{v}) = E(\mathbf{u}, \mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$. Put $P = \begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix}$ and $S = \begin{pmatrix} i\mathbf{1} & 0 \\ 0 & -i\mathbf{1} \end{pmatrix}$, and let $I = P^{-1}SP$. Thus $i\Pi = \Pi I$ and $-i\bar{\Pi} = \bar{\Pi}I$. The statement that the matrix of E with respect to $\{\lambda_i\}$ is A means that

$$E(\Pi\mathbf{x}, \Pi\mathbf{y}) = {}^{\top}\mathbf{x}A\mathbf{y}$$

for all $\mathbf{x}, \mathbf{y} \in V$, so if H is Hermitian exactly when

$$\begin{aligned} {}^{\top}\mathbf{x}A\mathbf{y} &= E(\Pi\mathbf{x}, \Pi\mathbf{y}) \\ &= E(i\Pi\mathbf{x}, i\Pi\mathbf{y}) \\ &= E(\Pi I\mathbf{x}, \Pi I\mathbf{y}) \\ &= {}^{\top}\mathbf{x}{}^{\top}IAI\mathbf{y}, \end{aligned}$$

that is, when $A = {}^{\top}\mathbf{x}{}^{\top}IAI\mathbf{y}$. Hence

$$A = {}^{\top}PS^{\top}(P^{-1})AP^{-1}SP$$

which simplifies to

$$(PA^{-1\top}P)^{-1} = S(PA^{-1\top}P)^{-1}S.$$

This says

$$\begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix} A^{-1\top} \begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix} = \begin{pmatrix} i\mathbf{1} & 0 \\ 0 & -i\mathbf{1} \end{pmatrix} \begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix} A^{-1\top} \begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix} \begin{pmatrix} i\mathbf{1} & 0 \\ 0 & -i\mathbf{1} \end{pmatrix}$$

and hence

$$\Pi A^{-1\top}\Pi = -\Pi A^{-1\top}\Pi$$

as required. ■

Lemma 2.6. *H is positive definite if and only if $i\Pi A^{-1\top}\bar{\Pi}$ is positive definite.*

Proof: In fact the matrix of H is $2i\bar{\Pi}A^{-1\top}\Pi$. To see this, put $\mathbf{u} = \Pi\mathbf{x}$, $\mathbf{v} = \Pi\mathbf{y}$ and calculate $E(i\mathbf{u}, \mathbf{v})$ and $E(\mathbf{u}, \mathbf{v})$, thus:

$$\begin{aligned} E(i\mathbf{u}, \mathbf{v}) &= E(i\Pi\mathbf{x}, \Pi\mathbf{y}) \\ &= E(\Pi I\mathbf{x}, \Pi\mathbf{y}) \\ &= {}^{\top}\mathbf{x}{}^{\top}IA\mathbf{y} \\ &= {}^{\top}\begin{pmatrix} \mathbf{u} \\ \bar{\mathbf{u}} \end{pmatrix} {}^{\top}P^{-1\top}IAP^{-1} \begin{pmatrix} \mathbf{v} \\ \bar{\mathbf{v}} \end{pmatrix} \\ &= {}^{\top}\begin{pmatrix} \mathbf{u} \\ \bar{\mathbf{u}} \end{pmatrix} {}^{\top}P^{-1\top}P^{\top}S^{\top}P^{-1}AP^{-1} \begin{pmatrix} \mathbf{v} \\ \bar{\mathbf{v}} \end{pmatrix} \\ &= {}^{\top}\begin{pmatrix} \mathbf{u} \\ \bar{\mathbf{u}} \end{pmatrix} S(PA^{-1\top}P)^{-1} \begin{pmatrix} \mathbf{v} \\ \bar{\mathbf{v}} \end{pmatrix} \\ &= {}^{\top}\begin{pmatrix} \mathbf{u} \\ \bar{\mathbf{u}} \end{pmatrix} \begin{pmatrix} 0 & i(\bar{\Pi}A^{-1\top}\Pi)^{-1} \\ -i(\Pi A^{-1\top}\bar{\Pi}) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \bar{\mathbf{v}} \end{pmatrix} \\ &= {}^{\top}\mathbf{u}i(\bar{\Pi}A^{-1}\Pi)^{-1}\bar{\mathbf{v}} - {}^{\top}\bar{\mathbf{u}}i(\bar{\Pi}A^{-1\top}\bar{\Pi})\mathbf{v} \end{aligned}$$

since $\Pi A^{-1\top}\Pi = 0$; and similarly for $E(\mathbf{u}, \mathbf{v})$. ■

Now we want to apply the Riemann relations to the Jacobian, in order to show that the Jacobian is indeed an abelian variety.

Theorem 2.7. Jac(C) *is an abelian variety with a principal polarisation defined by* E.

Proof: We need
$$\Pi \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} {}^{\top}\Pi = 0$$
and
$$i\Pi \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \bar{\Pi} > 0.$$

The first of these is straightforward:
$$\begin{aligned}\sum_{j,k=1}^{2g} \Pi_{ij} E_{jk} \Pi_{lk} &= \sum_{j,k=1}^{2g} \left(\int_{\gamma_j} \omega_i E_{jk} \int_{\gamma_k} \omega_l \right) \\ &= \sum_{j=1}^{g} \left(\int_{\gamma_j} \omega_i \int_{\gamma_{j+g}} \omega_l \right) + \sum_{j=g+1}^{2g} \left(- \int_{\gamma_j} \omega_i \int_{\gamma_{j-g}} \omega_l \right) \\ &= 0.\end{aligned}$$

The other needs a fact. As before
$$\sqrt{-1}\Pi_{ij} E_{jk} \bar{\Pi}_{lk} = \sqrt{-1} \sum_{j,k=1}^{2g} \left(\int_{\gamma_j} \omega_i \right) \left(\int_{\gamma_k} \bar{\omega}_l \right) E_{jk}.$$

Let $\eta_1, \ldots, \eta_{2g}$ be the basis of $H^1_{\mathrm{DR}}(C)$ dual to $\left\{ \int_{\gamma_j} \right\}$: that is, $\int_{\gamma_j} \eta_i = \delta_{ij}$. Then $\omega_i = \sum_{j=1}^{2g} \left(\int_{\gamma_j} \omega_i \right) \eta_j$ (just calculating the coordinates). Because cup products in $H^1_{\mathrm{DR}}(C)$ are given by $\wedge$ and agree with intersection numbers
$$\int_C \eta_i \wedge \eta_j = \gamma_i \cdot \gamma_j = E_{ij},$$
so
$$\begin{aligned}\sqrt{-1}\Pi_{ij} E_{jk} \bar{\Pi}_{lk} &= \sqrt{-1} \sum_{j,k=1}^{2g} \left(\int_{\gamma_j} \omega_i \right) \left(\int_{\gamma_k} \bar{\omega}_l \right) \int_C \eta_j \wedge \eta_k \\ &= \sqrt{-1} \int_C \omega_i \wedge \bar{\omega}_l\end{aligned}$$
and in particular $\omega\sqrt{-1}\Pi E {}^{\top}\bar{\Pi}\bar{\omega} = i \int_C \omega \wedge \bar{\omega}$, which is positive as it is the volume of C with respect to the positive real 2-form $i\omega \wedge \bar{\omega}$. ■

Now we come to something interesting and important: the Abel-Jacobi map. This is one of the most fundamental tools in the theory of curves (and it has important generalisations to higher-dimensional varieties as well).

Suppose D is a divisor of degree 0 on a curve C (we write $D \in \mathrm{Div}^0(C)$): this means that $D = P_1 + \cdots + P_k - Q_1 - \cdots - Q_k$, where P_i and Q_k are (not necessarily distinct) points of C. Define the Abel-Jacobi map
$$\alpha\colon \mathrm{Div}^0(C) \longrightarrow \mathrm{Jac}(C)$$
by
$$\alpha\colon D \longrightarrow \left(\sum_{i=1}^{k} \int_{P_i}^{Q_i} \omega_1, \ldots, \sum_{i=1}^{k} \int_{P_i}^{Q_i} \omega_g \right).$$

Lemma 2.8. *The map* α *is well-defined: that is, it does not depend on the representation of* D.

Proof: The representation of D is non-unique in two ways: we could add and subtract a point P (thus $0 = P - P$) and we could re-order the P_i and Q_j. Also, $\int_{P_i}^{Q_i} \omega$ is not well-defined because we have to specify a path from P_i to Q_i. Let us deal with the last difficulty first: if $gamma_i$ and γ_i' are two paths from P_i to Q_i then

$$\sum_i \int_{\gamma_i} \omega_j - \sum_i \int_{\gamma_i'} \omega_j = \sum_i \int_{\gamma_i - \gamma_i'} \omega_j \in \Lambda$$

so the two integrals define the same point of $\mathrm{Jac}(C)$. Similarly, any path γ from P to P simply gives an extra term $\int_\gamma \omega_j$ which is in Λ, so adding and subtracting a point P makes no difference either. Finally

$$\begin{aligned}\int_{P_1}^{Q_1} \omega_j + \int_{P_2}^{Q_2} \omega_j - \int_{P_1}^{Q_2} \omega_j - \int_{P_2}^{Q_1} \omega_j &= \int_{P_1}^{Q_1} \omega_j + \int_{Q_1}^{P_2} \omega_j + \int_{P_2}^{Q_2} \omega_j + \int_{Q_2}^{P_1} \omega_j \\ &= \int_{P_1}^{P_1} \omega_j \in \Lambda\end{aligned}$$

and we are done. ■

So that was easy. However, much more is true. Abel's theorem states that the kernel of α is precisely the set of linearly trivial divisors, in other words, that α induces a map $\alpha\colon \mathrm{Pic}^0(C) \to \mathrm{Jac}(C)$, which is injective. And the Jacobi inversion theorem says that this α is also surjective.

Before proving either of these statements I'd like to think about what they mean. One way of looking at it is to say that we have classified all line bundles of degree zero, and hence all line bundles, on C. Note that there is also a map $\alpha^{(d)}\colon \mathrm{Pic}^d(C) = \{\text{line bundles of degree } d\} \to \mathrm{Jac}(C)$, which is also an isomorphism, though not so natural a one as it depends on the choice of one divisor of degree d, say $D_0 = dP$ for some point $P \in C$. It is given by

$$\alpha^{(d)}(D) = \alpha(D - D_0).$$

Another useful thing to look at is the symmetric product $S^dC = \{P_1 + \cdots + P_d \mid P_i \in C\}$. This is a complex manifold of dimension d, and there is a map $\psi_d\colon S^dC \to \mathrm{Jac}(C)$ given by $\psi_d(P_1 + \cdots + P_d) = \alpha^{(d)}(P_1 + \cdots + P_d)$. ψ_d is well-defined up to translation by an element of $\mathrm{Jac}(C)$: we had to choose an element $D_0 \in \mathrm{Pic}^d(C)$ to start with and if we choose D_0' instead we move ψ_d by $D_0 - D_0' \in \mathrm{Pic}^0(C) = \mathrm{Jac}(C)$. The fibre $\psi_d^{-1}(D)$, if $D \in \mathrm{Im}\,\psi_d$, is the linear system $|D - D_0|$ and this turns out to be a good way of thinking about linear systems. For example, if $\psi_d\colon S^dC \to W_d = \mathrm{Im}\,\psi_d$ is an isomorphism then every degree d linear system is trivial, but if some fibre has dimension at least 1 then there is a d-to-1 map $C \to \mathbb{P}^1$. In particular, $\psi_1 = \alpha\colon C \to \mathrm{Jac}(C)$ is an embedding.

Theorem 2.9. (Abel's Theorem) *If $D \in \mathrm{Div}^0(C)$ then $\alpha(D) = 0$ if and only if D is linearly equivalent to zero.*

Proof: First we show that $\alpha\colon \mathrm{Pic}^0(C) \to \mathrm{Jac}(C)$ is well defined, i.e. that if $D \sim 0$ then $\alpha(D) = 0$. Suppose, then, that $D = (f)$ for some rational function f on C. Define

$$\mu\colon \mathbb{P}^1 \longrightarrow \mathrm{Jac}(C)$$

by $\mu\colon (x_0 : x_1) \mapsto \alpha\big((x_0 f - x_1)\big)$ (here we are thinking of f as a map from C to $\mathbb{P}^1$ and x_0 and x_1 as homogeneous coordinates on $\mathbb{P}^1$). Then μ must be constant. There are various ways to see this. One argument is topological: if μ is nonconstant it must be open and therefore an injective map from a 2-sphere to a torus, which is impossible. A better argument, from our point of view, is that $\mu^* dz_i$ must be identically zero as it is a global 1-form on $\mathbb{P}^1$, but then $d\mu = 0$ so μ is constant. Since μ is constant, we have $\alpha(D) = \mu(1:0) = \mu(0:1) = \alpha\big((-1)\big) = 0$.

The converse is much harder. We start by translating the problem into one about differential forms with poles. Suppose that $D = \sum(P_i - Q_i) = (f)$. We can express this by saying that the differential

$$\eta = \frac{1}{2\pi i}\frac{df}{f} = \frac{1}{2\pi i} d\log f$$

has simple poles at P_i and Q_i and (assuming for the moment that the P_i and Q_i are all distinct) it has residue 1 at each P_i and -1 at each Q_i. If the P_i and Q_i are not distinct we simply write $D = \sum a_i P_i + \sum b_j Q_j$, with the P_i and Q_j all distinct: then η has simple poles at P_i and Q_j with residues a_i at P_i and b_j at Q_j. Moreover we have fixed things so that

$$\int_\gamma \eta \in \mathbb{Z}$$

for any loop $\gamma \subseteq C \setminus \{P_i, Q_j\}$.

Suppose we have an η with all these properties. Then choose a base point $O \in C \setminus \{P_i, Q_j\}$ and put

$$f(P) = \exp\{2\pi i \textstyle\int_O^P \eta\}.$$

Then f is a well-defined meromorphic function and $(f) = D$. So if we start with some divisor D and assume $\deg D = 0$, that is, $\sum a_i + \sum b_j = 0$, and produce a differential form η with simple poles with the right residues and such that $\int_\gamma \eta \in \mathbb{Z}$ for loops γ missing P_i and Q_j, then we can produce a function f such that $(f) = D$ and we shall have proved Abel's Theorem.

We first try to produce a differential with the specified poles and residues, without worrying about $\int_\gamma \eta \in \mathbb{Z}$. If you know sheaf cohomology this can be done in two lines: the short exact sequence

$$0 \longrightarrow \Omega^1_C \longrightarrow \Omega^1_C(\textstyle\sum P_i + \sum Q_j) \longrightarrow \displaystyle\bigoplus_{P_i} \mathbb{C}_{P_i} \oplus \bigoplus_{Q_j} \mathbb{C}_{Q_j} \longrightarrow 0$$

induces

$$\cdots \longrightarrow H^0\big(\Omega^1_C(\textstyle\sum P_i + \sum Q_j)\big) \xrightarrow{\delta} \mathbb{C}^n \longrightarrow H^1(\Omega^1_C) \longrightarrow \cdots$$

and $h^{1,1}(C) = 1$ so $\dim \operatorname{coker} \delta \leq 1$; but $\operatorname{Im} \delta \subseteq \{\sum a_i + \sum b_j = 0\}$. This, however, uses quite heavy machinery: I intend to give, essentially, this proof, but in an elementary way.

Observe first that if η is a 1-form with (perhaps) poles at P_i and Q_j and residues a_i, b_j there, then

$$\begin{aligned} 2\pi i \textstyle\sum a_i + 2\pi i \sum b_j &= \displaystyle\sum \int_{\text{loops around } P_i} \eta + \sum \int_{\text{loops around } Q_j} \eta \\ &= -\int_{\text{curve with holes}} d\eta \\ &= 0. \end{aligned}$$

This is just Stokes' Theorem. We calculate the residues at each important point by taking a small disc centred there and integrating η around the boundary of that disc, but we can equally consider the boundaries of the discs as being the boundary of what is left of the curve after we remove those discs. What we want to know is that this is the only condition on the a_i and b_j.

Choose, as above, a small disc Δ_i around each P_i and similarly Δ'_j for each Q_j. Take a 1-form η_i on Δ_i with just a simple pole at P_i, having the right residue: if z_i is a local coordinate at P_i we can use $\eta_i \frac{a_i dz_i}{z_i}$: do the same for Q_j. In other words, find local solutions to the problem. Use the Δ_i and Δ'_j as part of an open cover $\{U_\nu\}$ of C with a 1-form η_ν on each U_ν, holomorphic except for the singularities we have just described.

Now take a C^∞ bump function β_i which is equal to 1 near P_i and is zero outside Δ_i (and similarly for Q_j, β'_j, Δ'_j). Let $\psi = 0$ outside the Δ_i and Δ'_j and on Δ_i put

$$\psi = \frac{\partial}{\partial \bar{z}} \beta_i \eta_i \wedge d\bar{z}$$

(and similarly on Δ'_j). If there is a global C^∞ $(1,0)$-form ϕ, that is, something which is everywhere locally of the form $\phi = g dz$ with g a local C^∞ function, such that $\psi = \bar{\partial}\phi$, then $\eta = \sum \beta_i \eta_i + \sum \beta'_j \eta'_j - \phi$ has the right poles and it also has $\bar{\partial}\eta = 0$, so it is holomorphic. (Recall that $\bar{\partial}\phi = \frac{\partial g}{\partial \bar{z}}(dz \wedge d\bar{z})$, and note that $d = \partial + \bar{\partial}$ so $\bar{\partial}\phi = d\phi$.) So we are all right as long as we can find an appropriate ϕ.

All C^∞ $(1,1)$-forms are d-closed (since there are no nontrivial 3-forms on the 2-manifold C), so the statement that $\psi = \bar{\partial}\phi = d\phi$ amounts to the statement that ψ is cohomologically trivial: to be precise, that $[\psi] = 0$ in $H^2_{\mathrm{DR}}(C) = \{\text{closed 2-forms}\}/\{\text{exact 2-forms}\}$. But $H^2_{\mathrm{DR}}(C) \cong H^2(C;\mathbb{C}) = \big(H_2(C;\mathbb{Z}) \otimes \mathbb{C}\big)^*$ by

$$\xi \longmapsto \Big(k \longmapsto k \sum_{P \in C} \mathrm{res}_P(\xi)\Big)$$

so $\psi \mapsto \big(k \mapsto k(\sum a_i + \sum b_i)\big)$ which is zero. Consequently (assuming we believe De Rham's Theorem, as usual) such a ϕ does exist.

Next, we need to adjust the η we have found, without changing the poles, so as to arrange for its periods to be integral, that is, for $\int_{\gamma_i} \eta \in \mathbb{Z}$. We can certainly arrange this for the first g loops: in fact, by adding on an appropriate holomorphic 1-form (a sum of ω_i's) we can arrange for $\int_{\gamma_i} \eta = 0$ if $1 \le i \le g$. Suppose we have done this. We need to be able to tell what the other $\int_{\gamma_i} \eta$ are so that we can adjust them. For now I will simply say what the answer is and prove it later as a separate, not especially hard, lemma.

Fact: If we choose a base point O and a form η with $\int_{\gamma_i} \eta = 0$ for $i \le g$ (and γ_i as usual) and with residues $1/2\pi i$ at a point P and $-1/2\pi i$ at a point Q, then

$$\int_{\gamma_{i+g}} \eta = \int_O^P \omega_i - \int_O^Q \omega_i = \int_Q^P \omega_i$$

where $\omega_1, \ldots, \omega_g$ is a basis for the space of 1-forms on C such that $\int_{\gamma_i} \omega_j = \delta_{ij}$ (we can arrange this as we know the corresponding quadratic form is positive definite), and the integrals $\int_O^P$ and $\int_O^Q$ are taken along some paths not depending on i. So if we write our divisor D as $P_1 - Q_1 + P_2 - Q_2 + \cdots + P_d - Q_d$ we can assign an η_k to each $P_k - Q_k$ and then take $\eta = \sum \eta_k$. With this notation (so the points P_k and Q_k are not necessarily distinct, but we do not have to think about multiplicity)

$$\int_{\gamma_{i+g}} \eta = \sum_k \int_{Q_k}^{P^k} \omega_i.$$

In fact I might as well assume from now on that $D = P - Q$, since I can add Ds by adding ηs or multiplying fs.

By hypothesis

$$\alpha(D) = \Big(\int_P^Q \omega_1, \ldots, \int_P^Q \omega_g\Big) \in \Lambda$$

so

$$\begin{aligned}
\alpha(D) &= \Big(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g\Big) \\
&= \Big(\sum_{j=1}^{2g} m_j \int_{\gamma_j} \omega_1, \ldots, \sum_{j=1}^{2g} m_j \int_{\gamma_j} \omega_g\Big)
\end{aligned}$$

where $\gamma = \sum_{j=1}^{2g} m_j \gamma_j$. Now take $\eta' = \eta - \sum_{j=1}^{g} m_{j+g} \omega_j$. Then for $i \le g$

$$\begin{aligned}
\int_{\gamma_i} \eta' &= \int_{\gamma_i} \eta - \sum_{j=1}^{g} m_{j+g} \int_{\gamma_i} \omega_j \\
&= m_{i+g} \in \mathbb{Z}
\end{aligned}$$

since $\int_{\gamma_i} \eta = 0$ and $\int_{\gamma_i} \omega_j = \delta_{ij}$. On the other hand

$$\begin{aligned}
\int_{\gamma_{i+g}} \eta' &= \int_{\gamma_{i=g}} \eta - \sum_{j=1}^{g} m_{j+g} \int_{\gamma_{i+g}} \omega_j \\
&= \int_P^Q \omega_i - \sum_{j=1}^{g} m_{j+g} \int_{\gamma_{i+g}} \omega_j \\
&= \sum_{j=1}^{2g} m_j \int_{\gamma_j} \omega_i - \sum_{j=1}^{g} m_{j+g} \int_{\gamma_{i+g}} \omega_j \\
&= m_i + \sum_{j=1}^{g} m_{j+g} \int_{\gamma_{j+g}} \omega_i - \sum_{j=1}^{g} m_{j+g} \int_{\gamma_{i+g}} \omega_j \\
&= m_i \in \mathbb{Z}
\end{aligned}$$

using the fact that $\int_{\gamma_j} \omega_i = \delta_{ij}$ and, from the Riemann relations, $\int_{\gamma_{j+g}} \omega_i = \int_{\gamma_{i+g}} \omega_j$. ∎

We still have to find out about residues. We do this by cutting the curve C open and integrating. It won't make any difference to the periods of η if we assume that all the loops γ_i start from a common base point $S \in C$.

Lemma 2.10. *If η is a 1-form having simple poles only at points S_k (not lying on any of the γ_is) then for any holomorphic 1-form ω*

$$\sum_{i=1}^{g} \Big(\int_{\gamma_i} \omega \int_{\gamma_{i+g}} \eta - \int_{\gamma_{i+g}} \omega \int_{\gamma_i} \eta \Big) = 2\pi i \sum_k \operatorname{res}_{S_k}(\eta) \Big(\int_S^{S_k} \omega \Big),$$

where the path of the integral $\int_S^{S_k} \omega$ does not cross any of the γ_is.

Proof: Cut C open along all the γ_is and call the resulting closed $4g$-gon Δ. Then $\partial\Delta = \sum_i \gamma_i + \gamma_{i+g} + \gamma_i^{-1} + \gamma_{i+g}^{-1}$, where γ^{-1} denotes γ with the opposite orientation: we simply go round the edge of Δ identifying alternate edges if we want to recover C. On Δ we can integrate ω and define a function

$$h(P) = \int_S^P \omega$$

as Δ is simply-connected. Obviously if P and P' are points of Δ that are identified in C then $h(P)$ and $h(P')$ differ by a period of ω. In fact it is very easy to see that if $P \in \gamma_i$ and $P' \in \gamma_i^{-1}$ then $h(P) - h(P') = -\int_{\gamma_{i+g}} \omega$ for $i \leq g$ and $\int_{\gamma_{i-g}} \omega$ for $i > g$.

Now we integrate $h\eta$ around the edge of Δ:

$$\begin{aligned}
\int_{\partial\Delta} h\eta &= 2\pi i \sum_k \operatorname{res}_{S_k}(h\eta) \\
&= 2\pi i \sum_k \operatorname{res}_{S_k}(\eta) h(S_k) \\
&= 2\pi i \sum_k \operatorname{res}_{S_k}(\eta) \int_S^{S^k} \omega.
\end{aligned}$$

But

$$\begin{aligned}\int_{\partial\Delta} h\eta &= \sum_{i=1}^{g}\Big(\int_{\gamma_i+\gamma_i^{-1}} + \int_{\gamma_{i+g}+\gamma_{i+g}^{-1}}\Big)\\ &= \sum_{i=1}^{g}\Big(\int P\in\gamma_i\big(h(P)-h(P')\big)\eta(P) + \int P\in\gamma_{i+g}\big(h(P)-h(P')\big)\eta(P)\Big)\\ &= \sum_{i=1}^{g}\Big(-\int_{\gamma_{i+g}}\omega\int_{\gamma_i}\eta + \int_{\gamma_i}\omega\int_{\gamma_{i+g}}\eta\Big)\end{aligned}$$

which is what is claimed. ■

When we used this we were in the special case $\omega=\omega_j$ and $\int_{\gamma_i}\eta=0$, and we solved for $\int_{\gamma_{i+g}}\eta$.

Much of this account follows Griffiths and Harris.

Now we come to the converse result. We are going to see that the injective map $\alpha\colon \mathrm{Pic}^0(C)\to \mathrm{Jac}(C)$ is in fact an isomorphism. In fact we can prove rather more than that.

Theorem 2.11. (Jacobi Inversion Theorem) *Suppose $Q\in C$ and $\omega_1,\dots\omega_g$ form a basis for $H^0(K_C)$. Then for any point $\mathbf{a}\in\mathrm{Jac}(C)$ there exist points $P_1,\dots,P_g\in C$, not necessarily distinct, such that*

$$\alpha\Big(\sum_{i=1}^{g}(P_i-Q)\Big)=\mathbf{a}.$$

In particular $\alpha\colon\mathrm{Pic}^0(C)\to\mathrm{Jac}(C)$ is an isomorphism.

If we were interested only in proving the surjectivity it would be enough to show the existence of $P_1,\dots,P_k$ for $k\gg 0$ having this property but we can get this rather handy bound without any extra effort. *Proof:* Consider the gth symmetric power S^gC. I mentioned this at the start of the section. It is the set of unordered g-tuples of not necessarily distinct points in C, and an element of S^gC is normally written as $P_1+\cdots+P_g$. Since we don't care what order the P_i mentioned in the theorem come in it is clear that S^gC rather than the Cartesian product C^g is what we should be looking at. It is also clear that α and Q jointly induce a map

$$\alpha^{(g)}\colon S^gC\longrightarrow \mathrm{Jac}(C)$$

given by

$$\alpha^{(g)}\colon P_1+\cdots+P_g\longmapsto\Big(\sum_{i=1}^{g}\int_Q^{P_i}\omega_1,\dots\sum_{i=1}^{g}\int_Q^{P_i}\omega_g\Big).$$

The theorem asserts that $\alpha^{(g)}$ is surjective. This is actually not very hard – not nearly as hard as Abel's Theorem, anyway. The first thing to do is to notice that S^gC is a compact complex manifold. Actually we don't even need that much. S^gC is the quotient of C^g by a finite group (the symmetric group on g elements) so it is certainly compact. Near a point $(P_1,\dots,P_g)\in C^g$ with all the P_i distinct – that is, on a dense open set – the quotient map is an isomorphism, so S^gC is smooth there. That is enough, but with very little more work one can see that S^gC really is smooth everywhere, though we shan't need it. If there are coincidences among the P_i then there is a nontrivial local isotropy group, which is a product of smaller symmetric groups. These are generated by transpositions, which act as reflections, so by a theorem of Chevalley the quotient is still smooth. You can see this directly by writing down charts, using elementary symmetric polynomials in the local coordinates in C^g to get local coordinates on S^gC, or (what comes to the same thing) thinking about the tangent space to S^gC.

Let $D=P_1+\cdots+P_g$ be a point of S^gC with the P_i distinct, and take local coordinates z_i near P_i on C, so that the z_i can also be thought of as local coordinates on S^gC. A point near D is thus $D'=z_1+\cdots+z_g$ and

$$\begin{aligned}\frac{\partial}{\partial z_i}\alpha^{(g)}(D') &= \Big(\int_Q^{z_i}\omega_j\Big)\\ &= \frac{\omega_j}{dz_i}\end{aligned}.$$

(Here we are dividing one holomorphic 1-form by another so as to get a function locally: i.e. $\omega_j = \sum_i h_{ij} dz_i$ in a neighbourhood of D, because every 1-form looks like that, and $\omega_j/dz_i = h_{ij}$ by definition.) We can consider the Jacobian matrix – the other kind of Jacobian, but the same Jacobi – $\left(\frac{\partial \alpha^{(g)}}{\partial z_i}\right)$, which is (ω_j/dz_i) near D. I claim that it is generically non-singular, that is, that for D in an open dense set it is of maximal rank g. Choose D such that ω_1 does not vanish at P_1 (a nontrivial but harmless condition). Since we are on a curve, $\omega_i(P_j)$ is just a number (the cotangent bundle is a line bundle) and for $i > 1$ we can replace ω_i by $\omega_1(P_1)\omega_i - \omega_i(P_1)\omega_1$. By doing this, we can assume that $\omega_i(P_1) = 0$ for $i > 1$. Next we assume that $\omega_2(P_2) \neq 0$ and repeat the process, ending up with an upper-triangular matrix with $\omega_i(P_i)/dz_i$ along the main diagonal: this is still the Jacobian matrix, though expressed in different coordinates. It clearly has maximal rank, so the Jacobian matrix has maximal rank generically.

But this implies that $\alpha^{(g)}$ is surjective, because $S^g C$ and $\mathrm{Jac}(C)$ have the same dimension and $\alpha^{(g)}$ is proper (in the context of holomorphic maps, that means "compact fibres"). So by the Proper Mapping Theorem $\alpha^{(g)}(S^g C)$ is an analytic subvariety, and it contains an open set since $\alpha^{(g)}$ is an isomorphism at least somewhere, so it must be the whole of $\mathrm{Jac}(C)$. ■

This is, admittedly, a little unsatisfactory, since the Proper Mapping Theorem, though obvious, is rather hard (it's a little easier if you know, as in this case, that the varieties involved are smooth). An alternative way of finishing is to say this. Let ξ be a volume form on $\mathrm{Jac}(C)$, so $\int_{\mathrm{Jac}(C)} \xi > 0$. Then $\int_{S^g C} \alpha^{(g)*}\xi > 0$, because $\alpha^{(g)}$ is surjective and locally injective almost everywhere. But we can find a real C^∞ $(2g-1)$-form ζ on $\mathrm{Jac}(C) \setminus \{\mathbf{x}\}$, for any point $\mathbf{x} \in \mathrm{Jac}(C)$, such that $\xi = d\zeta$. We can do this because $H^{2g}_{\mathrm{DR}}\big(\mathrm{Jac}(C) \setminus \{\mathbf{x}\}\big) \cong H^{2g}(\text{punctured torus}; \mathbb{R}) = 0$. If we could do this for an $\mathbf{x} \notin \alpha^{(g)}(S^g C)$ then we should find

$$0 < \int_{S^g C} \alpha^{(g)*}\xi = \int_{\partial S^g C} d(\alpha^{(g)*}\zeta) = 0$$

which is absurd.

Corollary 2.12. *$\alpha^{(g)}$ is generically 1-to-1.*

This means that $\alpha^{(g)}$ is birational.

Proof: By Abel's Theorem, ${\alpha^{(g)}}^{-1}(\mathbf{a}) = |\mathbf{a} + gQ| = |D| = \mathbb{P}H^0\big(\mathcal{O}(D)\big)$. But since $S^g C$ and $\mathrm{Jac}(C)$ have the same dimension, this fibre is of dimension zero in general, and a zero-dimensional projective space is a point. ■

Corollary 2.13. *Every divisor D on a curve C of genus g such that $\deg D \geq g$ is linearly equivalent to some effective divisor. If $\deg D = g$ then for almost all D the effective divisor is unique.*

Corollary 2.14. *If C is of genus 1 then $C \cong \mathrm{Jac}(C)$. In particular, every curve of genus 1 is $\mathbb{C}/\Lambda$ for some lattice Λ (and therefore has the structure of an abelian group once a base point is given).*

Just to establish that something can really be done with this I will use Jacobians to prove Riemann-Roch for curves, and I will say a lot more about what else can be done.

Theorem 2.15. (Riemann-Roch) *Let C be a smooth curve of genus $g \geq 1$. then for any line bundle $\mathcal{O}(D)$ on C*

$$h^0\big(\mathcal{O}(D)\big) - h^0\big(\mathcal{O}(K-D)\big) = \deg(D) - g + 1.$$

Proof: It will be enough to show R-R (as Riemann-Roch is frequently abbreviated) for the case $|D| \neq \emptyset$ because then we can argue as follows: it must be true for $D = K$ because $K > 0$ (there are global 1-forms, indeed g of them), so $\deg K = 2g-2$. So either $\deg D \geq g$, or $\deg(K-D) \geq g$, or $\deg(K-D) = \deg D = g-1$. If $\deg(K-D) = \deg D = g-1$ and neither D nor $K-D$ is equivalent to an effective divisor, so $|D| = |K-D| = \emptyset$, then $h^0\big(\mathcal{O}(D)\big) = h^0\big(\mathcal{O}(K-D)\big) = 0 = \deg(D) - g + 1$ anyway. Otherwise one of $|D|$ and $|K-D|$ is nonempty, by assumption if $\deg D = g-1$ and by Corollary 2.13 otherwise. Without loss of generality we may assume it is D.

So suppose $|D| \neq \emptyset$. Then $h^0\big(\mathcal{O}(D)\big) = \dim|D| + 1 = r(D) + 1$ say. We may as well assume that $D = P_1 + \cdots + P_d$ actually is effective (but the P_i may not be distinct). Take local coordinates $t_1, \ldots, t_r$ in $|D| = \mathbb{P}H^0\big(\mathcal{O}(D)\big) \cong \mathbb{P}^r$ near D. Thus $D = D_{\mathbf{0}} = P_1 + \cdots + P_d = P_1(\mathbf{0}) + \cdots + P_d(\mathbf{0})$ and a nearby divisor is $D_{\mathbf{t}} = P_1(\mathbf{t}) + \cdots + P_d(\mathbf{t})$. Let z_i be a local coordinate at P_i on C, so that $P_i(\mathbf{t})$ has coordinate

$z_i\big(P_i(\mathbf{t})\big) = z_i(\mathbf{t})$ (and $z_i(\mathbf{0}) = 0$). (Think of $\mathbf{t}$ as time: $z_i(\mathbf{t})$ is the amount that P_i, and hence that bit of D, strays in time $\mathbf{t}$.) We can also write any form ω as $\omega = h_i(z_i)dz_i$ near P_i, with h_i holomorphic.

Consider the matrix $\left(\frac{\partial z_i}{\partial t_j}\right)$. It must have rank r at any $\mathbf{t}$ because for a suitable choice of $\delta\mathbf{t} = (\delta t_1, \ldots, \delta t_r)$ we have

$$\left(\frac{\partial z_i}{\partial t_j}\right) \delta\mathbf{t} = \delta\mathbf{z} = \delta D_t$$

and this moves in an r-dimensional space (a time $\delta\mathbf{t}$ later D could have moved in any of the r directions in $|D|$).

By Abel's theorem

$$\sum_i \int_Q^{P_i(\mathbf{t})} \omega = \text{ constant} \qquad \bmod \Lambda$$

so

$$\sum_i \int_{P_i}^{P_i(\mathbf{t})} \omega = \sum_i \int_0^{z_i(\mathbf{t})} h_i(z_i)dz_i = \text{ constant} \qquad \bmod \Lambda$$

and if we take ∇ we get

$$\sum_i h_i\big(z_i(\mathbf{t})\big) \frac{\partial z_i}{\partial t_j}(\mathbf{t}) = 0.$$

We can simply put $\mathbf{t} = 0$ in this equation, as everything is continuous, so

$$\Big(h_i\big(z_i(\mathbf{0})\big)\Big)_i = \big(\omega(P_i)\big)_i \in \operatorname{Ker}\left(\frac{\partial z_i}{\partial t_j}\right).$$

But $\dim \operatorname{Ker}\left(\frac{\partial z_i}{\partial t_j}\right) = d - r$ (we calculated that the rank was r a little while ago), so the dimension of the space of vectors $\big\{\omega(P_i)\big\}$ is at most $d - r$. But this is precisely the space of all ωs modulo the ones that vanish at P_i, which is $H^0(K)/H^0(K - D)$. So

$$\dim \Big(H^0(K)/H^0(K - D)\Big) \leq d - r$$

and since $h^0(K) = g$ this implies $h^0(K - D) \geq g - d + r = g - d + h^0(D) - 1$. So

$$h^0(D) - h^0(K - D) \leq \deg(D) - g + 1.$$

For $D = K$ this says $\deg K \geq 2g - 2$. We need to know that in fact $\deg K = 2g - 2$. You can think of this as Gauß-Bonnet if you like. If we accept this we can get the equality for all divisors. Looking at $\alpha^{(d)} \colon S^d C \to \operatorname{Jac}(C)$ we see that

$$\begin{aligned} h^0(D) - 1 = \dim |D| &= \dim {\alpha^{(d)}}^{-1}(D - dQ) \\ &= \dim S^d C - \dim \operatorname{Jac}(C) \\ &= d - g \end{aligned}$$

(trivially if $d < g$), so if $h^0(K - D) = 0$ we have $h^0(D) = d - g + 1$. If $h^0(K - D) \neq 0$ we can use the above inequality for $K - D$ to show that

$$\begin{aligned} h^0(K - D) - h^0(D) &\leq \deg(K - D) - g + 1 \\ &= 2g - 2 - \deg(D) - g + 1 \\ &= g - 1 - \deg(D) \end{aligned}$$

whence the result. ■

Time for another breather. I want to have a look at what we've done, discuss vaguely what we are going to do, and mention one or two things that don't fit in elsewhere.

We saw some examples of real-life abelian varieties, namely Jacobians. The first step was to go back and forth between divisors and line bundles: this is a basic procedure and the fact that it is possible is one of the reasons why line bundles are easier to understand than other vector bundles and why divisors are better behaved than other algebraic cycles. We used this to get an isomorphism between an entirely algebraic object, $\mathrm{Pic}^0(C)$, and a transcendental object, $\mathrm{Jac}(C)$. This in itself is obviously nontrivial. To do it, we had to spend a lot of time integrating forms with or without poles, and here I assumed two things: the De Rham theorem

$$H^i_{\mathrm{DR}}(X)\Big(= \text{ closed } i\text{-forms / exact } i\text{-forms } \Big) \cong H^i(X;\mathbb{R})$$

and that there are g 1-forms on a curve of genus g. I also used the fact that wedge product of forms agrees with intersection or cup product, that is, that the De Rham isomorphism is a ring isomorphism. But this we used in only one place, when we showed that the Jacobian is actually an abelian variety. However, note the way we did this: we wrote down an explicit and natural H, thereby equipping the Jacobian with a special ample line bundle (and even a special divisor, Θ, the divisor of zeros of the theta function, which we didn't need for what we did but is important). One thing we must do is think about this situation, of polarised abelian varieties, more generally.

A polarised abelian variety is an abelian variety equipped with a member of the Néron-Severi group, that is, with an H. H is determined by E and with a suitable choice of basis for Λ, E has matrix $\begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}$, where $T = \mathrm{diag}(t_1, t_2, \ldots, t_n)$. The t_i are positive integers, determined by H, and $t_i | t_{i+1}$. The type of a polarisation is the n-tuple $(t_1, \ldots, t_n)$: the most important case, not least because it is what naturally happens in the case of Jacobians, is $t_i = 1$ for all i. This is called a principal polarisation, frequently abbreviated to p.p.; but other polarisations do arise in nature. Not every abelian variety has a principal polarisation but every abelian variety is isogenous to one that does.

It turns out that in practice one has to work almost all the time with polarised abelian varieties. In particular, it is possible to write down a sensible parameter space for polarised abelian varieties but you really need the polarisation to achieve this. For instance, an elliptic curve can always be thought of as a plane cubic (and this embedding corresponds to a polarisation – in dimension 1 we don't need to worry about type) with equation (in characteristic not 2) $Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$. The only parameter we need then is the famous j-invariant

$$j = \frac{1728 g_2^3}{g_2^3 - 27g_3^2}$$

which tells you exactly which curve you've got.

It is important to be aware that the canonical divisor of an abelian variety (indeed, the canonical bundle of any complex torus) is trivial. This just means that there is a global non-vanishing n-form, namely $dz_1 \wedge \ldots \wedge dz_n$, where the z_i are coordinates in $\mathbb{C}^n$ – clearly this is Λ-invariant and therefore descends to X. This is quite unlike projective space (where K is negative in the sense that $\mathcal{O}(-mK)$ has lots of sections if m is big) or most other things (in general you expect $\mathcal{O}(mK)$, not $\mathcal{O}(-mK)$, to have lots of sections – Mori theory is about trying to arrange for K to be ample). There are other varieties with K trivial, called Calabi-Yau varieties (or $K3$ surfaces, for obscure reasons, if they are of dimension 2), and they also hold endless fascination for geometers.

Another way to associate an abelian variety with a given variety is to look at the Albanese torus $\mathrm{Alb}(X)$. This is a torus with a map $\alpha\colon X \to \mathrm{Alb}(X)$ having the property that every map from X to a torus factors through α. We shall not discuss this here but it is another useful tool, not perhaps quite as fundamental in its importance as the Jacobian but nevertheless essential.

The theta functions associated with a polarisation actually have a second dimension, literally. Consider for a moment the case of plane cubic curves E and their j-invariants. Pretend that you could make a surface by gluing all the curves together, so you had a surface S and a map $j\colon S \to \mathbb{C}$ such that $j^{-1}(t)$ is the elliptic curve E_t whose j-invariant is t. Actually you can't quite do this satisfactorily, but you so nearly can that it doesn't really matter. The theta function on the fibre E_t is then just the restriction of a much better theta function which really is a function on S, in other words a function of two variables. This is what makes theta functions really valuable. We shall discuss this in more detail in the next section.

3 Moduli and theta functions.

We begin with the case of elliptic curves, that is, curves which are abelian varieties. By definition we have $X = \mathbb{C}/\Lambda$ for some lattice Λ. Let $P \in X$ be the origin. The we put, for $z \in \mathbb{C}$

$$\wp(z) = z^{-2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \Big[(z-\lambda)^{-2} - \lambda^{-2}\Big]$$

so that

$$\wp'(z) = \sum_{\lambda\in\Lambda} -2(z-\lambda)^{-3}.$$

1, $\wp$ and $\wp'$ are all periodic and hence give meromorphic functions on X. Moreover, they are all sections of $\mathcal{O}(3P)$, that is, they have at most triple poles at the origin and no others. On the other hand, $\mathcal{O}(P)$ corresponds to $E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and therefore $h^0\big(\mathcal{O}(3P)\big) = \begin{vmatrix} 0 & 3 \\ -3 & 0 \end{vmatrix}^{1/2} = 3$, by 1.13. So $H^0\big(\mathcal{O}(3P)\big) = \langle 1, \wp, \wp' \rangle$. By 1.14, $3P$ is very ample, and that proves the following.

Proposition 3.1. *Every elliptic curve can be embedded in $\mathbb{P}^2$ in such a way that $P = (0:1:0)$ is an inflexion point.* ■

In fact we can do better than that, and give an equation.

Proposition 3.2. *The Weierstraß $\wp$-function satisfies*

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

where $g_2 = 60\sum_{\lambda\in\Lambda\setminus\{0\}} \lambda^{-4}$ *and* $g_3 = 140\sum_{\lambda\in\Lambda\setminus\{0\}} \lambda^{-6}$.

Proof: $\wp(z) - z^{-2}$ is an even function, holomorphic near P and vanishing there. So by Taylor's theorem

$$\begin{aligned}\wp(z) &= z^{-2} + az^2 + bz^4 + O(z^6)\\ \wp'(z) &= -2z^{-3} + 2az + 4bz^3 + O(z^5)\end{aligned}$$

so we may consider

$$\begin{aligned} q(z) &= \wp'(z)^2 - 4\wp(z)^3 + 20a\wp(z) + 28b \\ &= 4z^{-6} - 8az^{-2} - 16b - 4z^{-6} - 12b + 20az^{-2} + 28b + O(z) \\ &= O(z) \end{aligned}$$

which is a holomorphic function near $z = 0$ and vanishes at $z = 0$. By periodicity $q(\lambda) = 0$ for all $\lambda \in \Lambda$ and is a bounded holomorphic function, so $q(z) \equiv 0$. We can recover g_2 and g_3 by noting that $2a$ and $24b$ are the second and fourth derivatives at $z = 0$ of $\sum_{\lambda\in\Lambda\setminus\{0\}} \big((z-\lambda)^{-2} - \lambda^{-2}\big)$, and this sum converges absolutely and uniformly so we can also calculate the derivatives by differentiating term by term. ■

Corollary 3.3. *Every elliptic curve over $\mathbb{C}$ is isomorphic to the plane curve*

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$$

for some g_2, g_3. ■

On the other hand, every smooth plane cubic curve has genus 1. You can either prove this directly by making a projective change of coordinates that transforms a general plane cubic into this special-looking one or use the adjunction formula to calculate the degree of K. Another argument is to observe that all the smooth plane cubics form one continuous family (they can all be deformed into one another) and so the genus must be the same for all of them. The upshot is that if we want to describe all elliptic curves we may as well describe all smooth plane cubics of this form.

Theorem 3.4. *$\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda'$ if and only if $j(\mathbb{C}/\Lambda) = j(\mathbb{C}/\Lambda')$, where*

$$j = \frac{1728 g_2^3}{g_2^3 - 27 g_3^2}.$$

Proof: Suppose first that $\phi\colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ is an isomorphism. Then $\tilde{\phi}\colon \mathbb{C} \to \mathbb{C}/\Lambda'$ is a holomorphic function which is periodic with respect to Λ. So $\frac{d\tilde{\phi}}{dz}$ is a periodic holomorphic function from $\mathbb{C}$ to $\mathbb{C}$ and thus constant: say $\frac{d\tilde{\phi}}{dz} = a$. Then $\phi(z) \equiv az \bmod \Lambda'$. In particular $a\lambda \in \Lambda'$ if $\lambda \in \Lambda$, that is, $a\Lambda \subseteq \Lambda'$. Similarly $a^{-1}\Lambda' \subseteq \Lambda$, so $a\Lambda = \Lambda'$. But then $g_2' = a^4 g_2$ and $g_3' = a^6 g_3$, so $j' = j$.

Conversely, if $j = j'$, then $(g_2^3 : g_3^2) = ({g_2'}^3 : {g_3'}^2)$ so there exists $b \in \mathbb{C}$ such that $b^{-12} g_2^3 = {g_2'}^3$ and $b^{-12} g_3^2 = {g_3'}^2$. Put $X' = bX$, $Y' = Y$ and $Z' = b^3 Z$. Then

$$b^{-3} {Y'}^2 Z' = 4 b^{-3} X^3 - g_2 b^{-7} X' {Z'}^2 - g_3 b^{-9} {Z'}^3$$

so

$$\begin{aligned} {Y'}^2 Z' &= 4{X'}^3 - g_2 b^{-4} X' {Z'}^2 - g_3 b^{-6} {Z'}^3 \\ &= 4{X'}^3 - g_2' X' {Z'}^2 - g_3' {Z'}^3 \end{aligned}$$

so the two curves are projectively equivalent. ■

The significance of the expression $g_2^3 - 27g_3^2$ is that it is what is non-zero if the curve is smooth.

What we have found is a parameter space, or moduli space, for the set of all pairs

$$\{\text{elliptic curve } E,\ \text{point } 0 \in E\}.$$

(Strictly speaking one ought to reserve the term "elliptic curve" for such pairs and refer to a curve of genus 1 as a curve of genus 1. People who work over $\mathbb{C}$ tend to be careless about this, but number theorists, who work over fields that are not algebraically closed, can't afford to be because a curve of genus 1 might not have any points at all over the field in question.) What about abelian varieties of higher dimension? It won't be possible to work in the same way because a good projective description won't be so easy to find. What we can do, though, is to give some kind of moduli space a priori, without thinking about specific projective embeddings, essentially by looking at the period matrix. The idea is to choose a basis for Λ in such a way that E has a good simple form and then write the period matrix in terms of that basis. Specifically, we can always choose a basis $\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g$ of Λ such that E has matrix $\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$, where D is a diagonal matrix. If $D = I$ we say that the abelian variety is principally polarised. There is no guarantee that we can arrange for a given abelian variety to be principally polarised, but I will accept the loss of generality. Observe, in any case, that if instead $D = \operatorname{diag}(t_1, \dots, t_g)$ we can take Λ' to be the lattice generated by the λ_i and $\frac{1}{t_i}\mu_i$ and then $\mathbb{C}^g/\Lambda'$ is isogenous to $\mathbb{C}^g/\Lambda$ and does have a principal polarisation.

From now on we shall work with principally polarised abelian varieties.

Lemma 3.5. *With respect to the bases $\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g$ for Λ over $\mathbb{Z}$ and $\mu_1, \dots, \mu_g$ for $\mathbb{C}^g = V$, the period matrix is*

$$\Pi = (Z, I)$$

for some $Z \in M_{g\times g}(\mathbb{C})$.

Proof: Z is just the matrix whose j-th column consists of the coordinates of λ_j with respect to $\{\mu_i\}$. ■

Lemma 3.6. *$Z = {}^{\top}Z$ and $\operatorname{Im} Z$ is positive definite.*

Proof: These are just the Riemann relations. Note that H has matrix $2i\left(\bar{\Pi}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}^{-1} {}^{\top}\Pi\right)^{-1} = \big(\operatorname{Im} Z\big)^{-1}$. ■

The Siegel upper half-plane of degree g is defined to be

$$\mathbb{H}_g = \left\{ Z \in M_{g\times g}(\mathbb{C}) \mid Z = {}^{\top}Z,\ \operatorname{Im} Z > 0 \right\}.$$

It is sometimes written $\mathfrak{H}$ or $\mathfrak{S}$. It is a subset of $M_{g\times g}(\mathbb{C})$ but we can also think of it as being an open (in the usual topology) subset of $\mathbb{C}^{\frac{1}{2}g(g+1)}$.

Proposition 3.7. *Points of* $\mathbb{H}_g$ *are in* 1*-to-*1 *correspondence with the set of abelian varieties* X *of dimension* g *with a principal polarisation and a symplectic basis for* $\Lambda = \Lambda_X$*.*

By a symplectic basis we mean a basis $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ with respect to which E has matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

Proof: We have already shown how to produce a point of $\mathbb{H}_g$ from such an X. Going the other way is just as easy: you let Λ be the lattice generated by the columns of (Z, I) and let H have matrix $(\operatorname{Im} Z)^{-1}$ with respect to the standard basis of $\mathbb{C}^g = V$ (which is $\mu_1, \ldots, \mu_g$). Then H is a positive definite Hermitian form. We want to show that $\operatorname{Im} H$ has matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ with respect to some basis for Λ, so as to justify our assertion that the polarisation is principal. But with respect to the basis given by the columns of (Z, I), the matrix of $\operatorname{Im} H$ is

$$\begin{aligned}
\operatorname{Im}\left({}^{\top}(Z,I)(\operatorname{Im} Z)^{-1}(\bar{Z}, I)\right) &= \operatorname{Im}\begin{pmatrix} Z \\ I \end{pmatrix}(\operatorname{Im} Z)^{-1}(\bar{Z}, I) \qquad\qquad (\text{as } Z = {}^{\top}Z) \\
&= \operatorname{Im}\begin{pmatrix} \operatorname{Re} Z + i \operatorname{Im} Z \\ I \end{pmatrix}(\operatorname{Im} Z)^{-1}(\operatorname{Re} Z - i \operatorname{Im} Z, I) \\
&= \begin{pmatrix} -\operatorname{Re} Z + \operatorname{Re} Z & (\operatorname{Im} Z)(\operatorname{Im} Z^{-1}) \\ -(\operatorname{Im} Z)(\operatorname{Im} Z^{-1}) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}
\end{aligned}$$

as required. ■

What we want to do is get rid of the choice of symplectic basis. Once it's put like that, it becomes clear that we are going to have an action of $\operatorname{Sp}(2g, \mathbb{Z})$ on $\mathbb{H}_g$ and the moduli space of principally polarised abelian varieties will be $\mathcal{A} = \mathbb{H}_g / \operatorname{Sp}(2g, \mathbb{Z})$.

To fix notation, we make the definition that

$$\operatorname{Sp}(2g, \mathbb{Z}) = \left\{ R \in M_{2g \times 2g}(\mathbb{Z}) \mid R \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} {}^{\top}R = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right\}.$$

This is not the only convention in use, unfortunately: sometimes $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is replaced with another standard alternating form of determinant 1 such as $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, and sometimes what I have called $\operatorname{Sp}(2g, \mathbb{Z})$ is referred to as $\operatorname{Sp}(g, \mathbb{Z})$ (the notation for dihedral groups is afflicted by the same ambiguity). Be careful! For us, $\operatorname{Sp}(2g, \mathbb{Z})$ is a subgroup of $\operatorname{SL}(2g, \mathbb{Z})$ and in particular $\operatorname{Sp}(2, \mathbb{Z}) = \operatorname{SL}(2, \mathbb{Z})$.

Theorem 3.8. $\operatorname{Sp}(2g, \mathbb{Z})$ *acts on* $\mathbb{H}_g$ *by*

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \longrightarrow R(Z) = (AZ + B)(CZ + D)^{-1}.$$

Proof: In fact we can even take $R \in \operatorname{Sp}(2g, \mathbb{R})$. Notice that if $R \in \operatorname{Sp}(2g, \mathbb{R})$ then so is ${}^{\top}R$, since

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} {}^{\top}R \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = R^{-1}.$$

Also ${}^{\top}AC$ and ${}^{\top}BD$ are symmetric and ${}^{\top}AC - {}^{\top}CB = I$: this follows straight from the definition and in fact these conditions are also sufficient for R to be symplectic. Now I claim that $CZ + D$ is invertible, which is one of the things we have to prove.

Consider ${}^\top(\overline{CZ+D})(AZ+B) - {}^\top(\overline{AZ+B})(CZ+D)$. Since A, B, C and D are real we have

$$\begin{aligned}{}^\top(\overline{CZ+D})(AZ+B) - {}^\top(\overline{AZ+B})(CZ+D) &= \\ &= {}^\top\bar{Z}({}^\top CA - {}^\top AC)Z + {}^\top\bar{Z}({}^\top CB - {}^\top AD) + ({}^\top DA - {}^\top BC) + {}^\top DB - {}^\top BD \\ &= Z - \bar{Z} \\ &= 2i \operatorname{Im} Z.\end{aligned}$$

If $(CZ+D)\mathbf{v} = 0$ for some $\mathbf{v} \in V$ then this gives

$$0 = 2i{}^\top\bar{\mathbf{v}} \operatorname{Im} Z\mathbf{v} = 2i{}^\top(\operatorname{Re}\mathbf{v}) \operatorname{Im} Z \operatorname{Re}\mathbf{v} + 2i(\operatorname{Im}\mathbf{v}) \operatorname{Im} Z \operatorname{Im}\mathbf{v}$$

so $\mathbf{v} = 0$, because $\operatorname{Im} Z > 0$.

Next, $R(Z) = {}^\top R(Z)$, because

$$\begin{aligned}{}^\top(CZ+D)\big(R(Z) - {}^\top R(Z)\big)(CZ+D) &= {}^\top(CZ+D)(AZ+B) - {}^\top(AZ+B)(CZ+D) \\ &= {}^\top Z({}^\top CA - {}^\top AC)Z + ({}^\top DA - {}^\top BC)Z + {}^\top Z({}^\top CB - {}^\top AD) \\ &\quad + {}^\top DB - {}^\top BD \\ &= Z - {}^\top Z \\ &= 0.\end{aligned}$$

Finally, we must check that $\operatorname{Im} R(Z)$ is positive definite. But

$$\begin{aligned}2i{}^\top(\overline{CZ+D}) \operatorname{Im} R(Z)(CZ+D) &= {}^\top(\overline{CZ+D})\big(R(Z) - \overline{R(Z)}\big)(CZ+D) \\ &= {}^\top(\overline{CZ+D})\big(R(Z) - {}^\top\overline{R(Z)}\big)(CZ+D) \\ &= (\overline{CZ+D})(AZ+B) - (\overline{AZ+B})(CZ+D) \\ &= \operatorname{Im} Z\end{aligned}$$

so $R(Z) \in \mathbb{H}_g$. It is clear that the map given describes a group action, that is, that $R_1\big(R_2(Z)\big) = R_1R_2(Z)$. ■

Obviously these are generalisations of Möbius transformations. We are going to work with $\operatorname{Sp}(2g, \mathbb{Z})$ but we could instead work with any sensible discrete subgroup of $\operatorname{Sp}(2g, \mathbb{R})$. In the case $g = 1$ this amounts to looking at the Poincaré sphere but looking at other discrete subgroups of $\operatorname{SL}(2, \mathbb{Q})$ gives other modular curves and these are beautiful and important objects.

Theorem 3.9. *If $Z, Z' \in \mathbb{H}_g$ then the principally polarised abelian varieties (X_Z, H_Z) and $(X_{Z'}, H_{Z'})$ are isomorphic if and only if Z and Z' are equivalent under the action of* $\operatorname{Sp}(2g, \mathbb{Z})$.

Proof: Suppose first $(X_Z, H_Z) \cong (X_{Z'}, H_{Z'})$. That means that there is a map $f\colon X_{Z'} \to X_Z$ which is an isomorphism of complex tori and satisfies $f^*H_Z = H_{Z'}$ (notice which way the maps go). We have long known how to express f by an isomorphism $F\colon V \to V$ such that $F(\Lambda') = \Lambda$. Let $T \in M_{g\times g}$ be the matrix of F with respect to the basis $\mu_1, \ldots, \mu_g$ of V and let $R \in M_{2g\times 2g}(\mathbb{Z})$ be the matrix of F with respect to the bases $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ for Λ and $\lambda'_1, \ldots, \lambda'_g, \mu_1, \ldots, \mu_g$ for Λ' (so λ_i is the i-th column of Z, etc.). T and R are called the matrices of the analytic and rational representations of f respectively. Since $F(\Lambda') \subseteq \Lambda$ we have

$$T(Z', I) = (Z, I)R. \tag{‡}$$

You just have to think about this: it is one of those elementary but confusing things (well, it confuses me). The left-hand side is $\big(F(\lambda'_1), \ldots, F(\lambda'_g), F(\mu_1), \ldots, F(\mu_g)\big)$ expressed in terms of $\mu_1, \ldots, \mu_g$. The right-hand side is the same thing expressed in terms of $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$.

Put ${}^\top R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D \in M_{g\times g}(\mathbb{Z})$. Then (‡) says

$$TZ' = Z{}^\top A + {}^\top B \qquad \text{and} \qquad T = Z{}^\top C + {}^\top D.$$

Moreover, since Z is symmetric, ${}^{\top}T = CZ + D$ which is invertible because f is an isomorphism, so

$$Z' = {}^{\top}Z' = (AZ+B){}^{\top}T^{-1} = (AZ+B)(CZ+D)^{-1} = R(Z).$$

We need to check also that $R \in \mathrm{Sp}(2g,\mathbb{Z})$, but this is true simply because R preserves H, that is, ${}^{\top}R\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} R = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

Conversely, if $Z' = R(Z)$ for some $R \in \mathrm{Sp}(2g,\mathbb{Z})$ then R determines $F\colon V \to V$ and hence $f\colon X_Z \to X_{Z'}$, preserving H because R is symplectic, and F is an isomorphism because R is invertible. ■

Corollary 3.10. *There is a 1-to-1 correspondence between the set $\mathcal{A}_g$ of isomorphism classes of principally polarised abelian varieties and points of the orbit space $\mathbb{H}_g/\mathrm{Sp}(2g,\mathbb{Z})$.* ■

There is a difficulty with this, though. If it is going to be any use to us we need $\mathcal{A}_g$ to be something we can handle, such as a complex manifold. Actually it isn't a complex manifold. The reason why not is that $\mathrm{Sp}(2g,\mathbb{Z})$ has torsion and the torsion elements necessarily have fixed points (by the Brauer fixed-point theorem, for instance): that is to say, it can happen that $Z = R(Z)$ for some $R \neq I$. This will, in general, cause $\mathcal{A}_g$ to have some singularities, but they are pretty harmless ones. They correspond to abelian varieties having extra automorphisms, so that they can be looked at in more than one way. (I'm cheating slightly, because in fact this always happens: $-I \in \mathrm{Sp}(2g,\mathbb{Z})$ acts trivially on $\mathbb{H}_g$ and this corresponds to the automorphism -1 of (X,H). In other words, $\mathrm{Sp}(2g,\mathbb{Z})$ acts through the quotient $\mathrm{PSp}(2g,\mathbb{Z}) = \mathrm{Sp}(2g,\mathbb{Z})/\pm I$. This doesn't really change anything, but it is what prevents there being a universal family of elliptic curves. You can get round it by choosing a 3-torsion point, because that won't be preserved by -1.)

In actual fact $\mathcal{A}_g$ is a quasi-projective variety. All I will prove here is that it is Hausdorff (and I shan't even do all the details of that), by showing that the action of $\mathrm{Sp}(2g,\mathbb{Z})$ is properly discontinuous. Since it acts on $\mathbb{H}_g$ by biholomorphic maps this makes $\mathcal{A}_g$ into a complex analytic space, which is a big step in the right direction.

Theorem 3.11. *$\mathcal{A}_g$ is Hausdorff.*

Proof: We need to show that if $K_1, K_2 \subseteq \mathbb{H}_g$ are compact then $R(K_1) \cap K_2 = \emptyset$ except for finitely many $R \in \mathrm{Sp}(2g,\mathbb{Z})$: if we can do this then we can separate $x_1, x_2 \in \mathcal{A}_g$ by taking K_i to be a compact neighbourhood of some preimage $\tilde{x}_i \in \mathbb{H}_g$ and then using $K_1 \setminus \bigcup\limits_R R(K_2)$ and $K_2 \setminus \bigcup\limits_R R(K_1)$.

Consider the map $h\colon \mathrm{Sp}(2g,\mathbb{R}) \to \mathbb{H}_g$ given by $h(R) = R(iI)$, which is continuous. The fibre $h^{-1}(iI)$ is

$$\begin{aligned}
\mathrm{Stab}(iI) &= \left\{ R = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid (iA+B)(iC+D)^{-1} = iI, R \in \mathrm{Sp}(2g,\mathbb{R}) \right\} \\
&= \left\{ R \in \mathrm{Sp}(2g,\mathbb{R}) \mid R = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \right\} \\
&= \mathrm{Sp}(2g,\mathbb{R}) \cap \mathrm{O}(2g,\mathbb{R})
\end{aligned}$$

since

$$\begin{aligned}
R^{\top}R &= \begin{pmatrix} A^{\top}A + B^{\top}B & A^{\top}B - B^{\top}A \\ -B^{\top}A + A^{\top}B & A^{\top}A + B^{\top}B \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}
\end{aligned}$$

by the symplecticity conditions. As $\mathrm{O}(2g,\mathbb{R})$ is compact this fibre is compact. Furthermore, $\mathrm{Sp}(2g,\mathbb{R})$ acts transitively on $\mathbb{H}_g$ because if $X + iY \in \mathbb{H}_g$ then $Y = A^{\top}A$ for some A and $R = \begin{pmatrix} A & X^{\top}A^{-1} \\ 0 & {}^{\top}A^{-1} \end{pmatrix}$ satisfies $R(iI) = X + iY$. So all the fibres are conjugate and hence compact, and h is surjective. With a bit more similar work one can show that it is proper.

Now if $R(K_1) \cap K_2 \neq \emptyset$ then $R\big(h^{-1}(K_2)\big) \subseteq h^{-1}(K_2) \subseteq \mathrm{Sp}(2g,\mathbb{R})$, so $R \in h^{-1}(K_2)\big[h^{-1}(K_1)\big]^{-1}$. Since $\mathrm{Sp}(2g,\mathbb{Z})$ is discrete, a compact subset of $\mathrm{Sp}(2g,\mathbb{R})$ contains only finitely many elements of $\mathrm{Sp}(2g,\mathbb{Z})$. But

$H^{-1}(K_i)$ are compact and $h^{-1}(K_2)\big[h^{-1}(K_1)\big]^{-1} \subseteq \mathrm{Sp}(2g,\mathbb{R})$ is the image of the compact set $h^{-1}(K_1) \times h^{-1}(K_2) \subseteq \mathrm{Sp}(2g,\mathbb{R})^2$ under the continuous map $(R_1, R_2) \mapsto R_1 R_2^{-1}$. ■

All this works for any sensible subgroup of $\mathrm{Sp}(2g,\mathbb{Q})$ or even $\mathrm{Sp}(2g,\overline{\mathbb{Q}}\cap\mathbb{R})$. By "sensible" in this context I mean that one should replace $\mathrm{Sp}(2g,\mathbb{Z})$ by an arithmetic group Γ: an arithmetic group is one for which $\Gamma \cap \mathrm{Sp}(2g,\mathbb{Z})$ has finite index in both Γ and $\mathrm{Sp}(2g,\mathbb{Z})$. Such a Γ will arise from looking at more complicated structures associated with abelian varieties, for instance the choice of some l-torsion points for some integer l.

Since we are dealing with principal polarisations there is a unique (up to a constant) section of the line bundle corresponding to the polarisation (well, there are many such line bundles, but pick one). So for each point of $\mathcal{A}_g$ there is a canonical canonical theta function and a canonical classical theta function. Let us return to the case $g = 1$, so $\mathcal{A}_g = \mathbb{C}$, to see how these theta functions fit together.

The (Riemann) theta function is a function

$$\vartheta\colon \mathbb{C} \times \mathbb{H} \longrightarrow \mathbb{C}$$

given by the series

$$\vartheta(z,\tau) = \sum_{n\in\mathbb{Z}} \exp\{\pi i n^2 \tau + 2\pi i n z\}$$

(which converges, very fast).

Proposition 3.12. *The Riemann theta function satisfies*

$$\begin{aligned} &\vartheta(z+1,\tau) = \vartheta(z,\tau) \\ &\vartheta(z+\tau,\tau) = \exp\{-\pi i\tau - 2\pi i z\}\vartheta(z\tau) \end{aligned}$$

Proof: The first part is obvious. And

$$\begin{aligned} \vartheta(z+\tau,\tau) &= \sum_{n\in\mathbb{Z}} \exp\left\{(\pi i n^2 + 2\pi i n)\tau + 2\pi i n z\right\} \\ &= \sum_{n\in\mathbb{Z}} \exp\left\{\pi i (n+1)^2\tau - \pi i\tau + 2\pi i(n+1)z - 2\pi i z\right\} \\ &= \exp\left\{-\pi i\tau - 2\pi i z\right\} \vartheta(z,\tau) \end{aligned}$$

as stated. ■

If we think of τ as a constant we can use this to determine a factor of automorphy. In fact this is exactly what we had when we looked at classical theta functions: recall that we had $\Lambda = \Lambda_1 \oplus \Lambda_2$ and a function θ_1 which was Λ_1-periodic. If we put $g(\tau,z) = -\frac{1}{2}(\tau+2z)$ and $g(1,z) = 1$ we can recover $E = \mathrm{Im}\, H$ using the formula

$$\mathrm{Im}\, H(\lambda,\mu) = g(\mu,\lambda) + g(\lambda,\mathbf{0}) - g(\lambda,\mu) - g(\mu,\mathbf{0});$$

thus $\mathrm{Im}\, H(1,1) = 0$, $\mathrm{Im}\, H(1,\tau) = -\frac{1}{2}(\tau+2) + 1 - 1 + \frac{1}{2}\tau = -1$, $\mathrm{Im}\, H(\tau,1) = 1$ by a similar calculation and $\mathrm{Im}\, H(\tau,\tau) = 0$. So $E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so ϑ does indeed give a section – essentially the only section – of the line bundle $L(1,H)$ corresponding to the principal polarisation H and the trivial character on $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$. In particular ϑ is the only holomorphic function satisfying the relations above.

I want to describe two more properties of ϑ. One of them relates to the action of Sp, or in this case $\mathrm{SL}(2,\mathbb{Z})$ since $g = 1$. We want to have some functional equation relating the values of ϑ for given τ to those for $\frac{a\tau+b}{c\tau+d}$, which after all corresponds to the same elliptic curve. We can't actually do this for every element of $\mathrm{SL}(2,\mathbb{Z})$ and in any case I shall not give all the details of the proof.

Theorem 3.13. *Suppose* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ *and that* ab *and* cd *are even. Then*

$$\vartheta\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \zeta \cdot (c\tau+d)^{\frac{1}{2}} \exp\left\{\frac{i\pi c z^2}{c\tau+d}\right\} \vartheta(z,\tau)$$

where ζ *is an eighth root of unity.*

Proof: (Sketch) If we look at $\vartheta\big((c\tau+d)z,\tau\big)$ we get a function which is nearly periodic with respect to $z \mapsto z+1$. We can get real periodicity by inserting a fudge factor. Set

$$\Theta(z,\tau) = \exp\left\{i\pi c(c\tau+d)z^2\right\} \vartheta\big((c\tau+d)z,\tau\big).$$

Then $\Theta(z+1,\tau) = \Theta(z,\tau)$ by a simple but messy calculation (it matters that $2|cd$ because you get a factor of $e^{i\pi cd}$) and

$$\Theta\big(z + \frac{a\tau+b}{c\tau+d}, \tau\big) = \exp\left\{-i\pi\frac{a\tau+b}{c\tau+d} - 2i\pi z\right\} \Theta(z\tau),$$

by another messy calculation using $2|ab$ and $ad - bc = 1$. The details are on page 29 of Tata Lecture Notes on Theta I, where what I have called Θ is called Ψ. But this implies that

$$\Theta(z,\tau)\phi(\tau)\vartheta\left(z, \frac{a\tau+b}{c\tau+d}\right)$$

because of the uniqueness of ϑ which we proved above. The statement of the theorem is now that $\phi(\tau) = \zeta \cdot (c\tau+d)^{\frac{1}{2}}$. We have fixed the zeroth term in the Fourier series for ϑ to be 1, so $\int_0^1 \vartheta(z,\tau)dz = 1$. Hence

$$\begin{aligned}
\phi(\tau) &= \int_0^1 \Theta(z,\tau)dz \\
&= \int_0^1 \exp\left\{i\pi c(c\tau+d)y^2\right\} \vartheta\big((c\tau+d)z\tau\big)dz \\
&= \sum_{n\in\mathbb{Z}} \exp\{-i\pi n^2 d/c\} \int_0^1 \exp\left\{i\pi(cz+n)^2(\tau+d/c)\right\} dz \\
&= \sum_{n=1}^{c} \exp\{-i\pi n^2 d/c\} \int_{-\infty}^{\infty} \exp\left\{i\pi c^2 z^2(\tau+d/c)\right\} dz
\end{aligned}$$

because $\exp\{-i\pi d(n+c)^2/c\} = \exp\{-i\pi n^d/c\}$, since $2|cd$. But we know the value of $\int_{-\infty}^{\infty} e^{-t^2} dt$ and so this simplifies to

$$\phi(\tau) = \sum_{n=1}^{c} \exp\{-i\pi n^2 d/c\} c^{-1} \big[(\tau+d/c)/i\big]^{\frac{1}{2}}.$$

The mysterious factor of $\zeta c^{\frac{1}{2}}$ which makes everything work comes from the Gauss sum $\sum_{n=1}^{c} \exp\{-i\pi n^2 d/c\}$, and we aren't going to use its actual value so for the present we can just believe that it is what it is. ■

Actually we aren't going to use anything else now. What I will do is explain where the funny-looking condition that ab and cd should be even comes from. The trouble is that if $z = \frac{1}{2}(\tau+1)$ then

$$\begin{aligned}
\vartheta(z,\tau) &= \sum_{n\in\mathbb{Z}} \exp\left\{\pi i n^2\tau + \pi i n\tau + \pi i n\right\} \\
&= \sum_{n \text{even}} \exp\left\{\pi i n^2\tau + \pi i n\tau\right\} - \exp\left\{\pi i(n-1)^2\tau + \pi i(n-1)\tau\right\} \\
&= \sum_{n \text{even}} \exp\left\{\pi i n^2\tau + \pi i n\tau\right\} - \exp\left\{\pi i n^2\tau - \pi i n\tau\right\} \\
&= \sum_{n \text{even}} \exp\{\pi i n^2\tau\}\big[\exp\{\pi i n\tau\} - \exp\{-\pi i n\tau\}\big] \\
&= 0
\end{aligned}$$

as the n term cancels with the $-n$ term, leaving only the $n = 0$ term which vanishes.

Now in general $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ does not send $\frac{1}{2}(\tau+1)$ to $\frac{1}{2}(\frac{a\tau+b}{c\tau+d}+1)$ modulo $\Lambda_{\tau'} = \mathbb{Z} + \mathbb{Z}\frac{a\tau+b}{c\tau+d}$, but to some other 2-torsion point of $\mathbb{C}/\Lambda_{\tau'}$. There are three nontrivial 2-torsion points, $\frac{1}{2}\tau$, $\frac{1}{2}$ and $\frac{1}{2}\tau + \frac{1}{2}$, and $\mathrm{SL}(2,\mathbb{Z})$ permutes them. We are interested in the stabiliser of $\frac{1}{2}\tau + \frac{1}{2}$. In fact $\mathrm{SL}(2,\mathbb{Z})$ acts on the set of 2-torsion points via the quotients induced by reduction mod 2

$$\mathrm{SL}(2,\mathbb{Z}) \longrightarrow \mathrm{SL}(2,\mathbb{Z}/2) \cong S_3.$$

This is clear, because there is a subset of $\mathrm{SL}(2,\mathbb{Z})$ which is just $\mathrm{SL}(2,\mathbb{Z}/2)$, namely

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right\}$$

and these elements do the right things to $\frac{1}{2}\tau$, $\frac{1}{2}$ and $\frac{1}{2}\tau+\frac{1}{2}$. So one interesting subgroup is the kernel of reduction mod 2, called the principal congruence subgroup of level 2; another, and the one we need, is the preimage of $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$. This is called $\Gamma_{1,2}$ and it is precisely given by $ab \equiv cd \equiv 0 \pmod 2$. Of course it's not normal (a reflection doesn't generate a normal subgroup of the symmetry group of a triangle – this is the first example of a non-normal subgroup). The conjugates are the preimages of $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}$, given by $c \equiv 0 \pmod 2$, and similarly $b \equiv 0 \pmod 2$.

Incidentally, we have almost shown that ϑ is a modular form for $\Gamma_{1,2}$. This is because if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{1,2}$

$$\vartheta\Big(0, \frac{a\tau+b}{c\tau+d}\Big) = \zeta \cdot (c\tau+d)^{\frac{1}{2}}\vartheta(0,\tau)$$

which, but for the ζ, says that ϑ is a modular form of weight $\frac{1}{2}$. Of course we can get rid of this by taking ϑ^4 instead: it is a modular form for $\Gamma_{1,2}$ of weight 2.

The principal congruence subgroup $\Gamma(N)$ of level N in $\mathrm{SL}(2,\mathbb{Z})$ is the kernel of reduction mod N. A modular form of weight k and level N is a holomorphic function $f(\tau)$ on $\mathbb{H}$ such that for all $\tau \in \mathbb{H}$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$$

and f is bounded near the cusps in some sense. There is an analogous definition for $\mathrm{Sp}(2g,\mathbb{Z})$ for $g > 1$, and in that case the boundedness condition can be dropped as it is automatically satisfied.

Note that this definition only makes sense because if for $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we put $e_R(\tau) = (c\tau+d)^k$ then

$$e_{R_1R_2}(\tau) = e_{R_1}(R_2\tau)e_{R_2}(\tau),$$

in other words that e is a 1-cocycle. So modular forms of weight k and level N are precisely the sections of some line bundle on $\mathcal{A}_g(N)$. It turns out that even for level 1 this bundle is ample, and that is why $\mathcal{A}_g$ is a projective variety.

Here, to round things off, are two more objects in mathematics that relate to abelian varieties. Not everything does, and I have really just been showing some – quite hard – geometry in action. But many surprising things do.

Let us have a last look at ϑ and think about what happens if we take real parameters, replacing $z \in \mathbb{C}$ by $x \in \mathbb{R}$ and $\tau \in \mathbb{H}$ by it for $t \in \mathbb{R}_+$. Then

$$\vartheta(x+1, it) = \vartheta(x, it)$$

and

$$\begin{aligned}\vartheta(x, it) &= \sum_{n\in\mathbb{Z}} \exp(-\pi n^t) \exp(2\pi i n x) \\ &= 1 + 2\sum_{n\in\mathbb{N}} \exp(-\pi n^2 t) \cos(2\pi n x),\end{aligned}$$

which is real. Furthermore

$$\frac{\partial}{\partial t}\vartheta(x, it) = 2\sum_{n\in\mathbb{N}} -\pi n^2 \exp(\pi n^2 t) \cos(2\pi n x)$$

and

$$\frac{\partial^2}{\partial x^2}\vartheta(x, it) = 2\sum_{n\in\mathbb{N}} -4\pi^2 n^2 \exp(\pi n^2 t) \cos(2\pi n x)$$

so ϑ satisfies the PDE

$$\frac{\partial}{\partial t}\vartheta(x, it) = \frac{1}{4\pi}\frac{\partial^2}{\partial x^2}\vartheta(x, it).$$

This equation is well known, though possibly not to the average geometer: it is the heat equation in one variable, with certain boundary conditions. To explain what the boundary conditions are we need to take $\lim_{t\to 0}\vartheta(x, it)$, which doesn't exist. But as a distribution it does exist: that is, $\lim_{t\to 0}\int_0^1 f(x)\vartheta(x, it)dx$ exists if f is measurable. If we take f to be a function on the circle we can write $f(x) = \sum_m a_m \exp(2\pi i m x)$, and then

$$\begin{aligned}\int_0^1 f(x)\vartheta(x, it)dx &= \int_0^1 \sum_{n,m} a_m \exp(-\pi n^2 t) \exp\left\{2\pi i(n+m)x\right\}dx \\ &= \sum_{n,m} a_m \exp(-\pi n^2 t) \int_0^1 \exp\left\{2\pi i(n+m)x\right\}dx \\ &= \sum_n a_{-n} \exp(-\pi n^2 t)\end{aligned}$$

so

$$\begin{aligned}\lim_{t\to 0}\int_0^1 f(x)\vartheta(x, it)dx &= \sum_n a_n \\ &= f(0) \\ &= \int_0^1 f(x)\delta(x)dx\end{aligned} .$$

So if I take a circular piece of wire of length 1 and at time $t = 0$ apply a lighter to it at the origin, the temperature at time t at the point x will be $\vartheta(x, it)$.

Finally: what do higher-dimensional abelian varieties look like as projective varieties? An elliptic curve is a plane cubic, but what about surfaces? We can certainly get some embeddings, by taking, say, the third power of a principal polarisation, but that is very wasteful, embedding X in $\mathbb{P}^8$. Maybe we can do better by taking a polarisation but not using all the sections (i.e. not using a complete linear system to embed X) or by using a non-principal polarisation (this turns out to be more useful). How much better? We can't embed an abelian surface in $\mathbb{P}^3$ because a smooth hypersurface in $\mathbb{P}^3$ has to be simply-connected, so what about $\mathbb{P}^4$? There are indeed abelian surfaces embedded in $\mathbb{P}^4$. They were first discovered by Commesatti in 1915 when, of course, nobody was paying any attention, and then forgotten for fifty-seven years. But there is an amazing rank 2 vector bundle on $\mathbb{P}^4$, called the Horrocks-Mumford bundle, and it has sections (a four-dimensional family of them) whose zeros are, in general, an abelian surface.