# Computations and Bounds for Surfaces in Weighted Projective Four–Spaces

submitted by

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## Summary

Many researchers have constructed examples of non general type surfaces in weighted projective spaces in various dimensions. Most of these constructions so far have been concentrated on complete intersections, and in the past three decades there has been a lot of success in this direction. Nowadays we have seen use of computer algebra systems to handle examples that are too cumbersome to do by hand.

All smooth projective surfaces can be embedded in $\mathbb{P}^5$, but only few of them in $\mathbb{P}^4$. The most amazing fact is that the numerical invariants of any smooth surface in $\mathbb{P}^4$ must satisfy the double point formula.

A natural question is whether there are any non general type surfaces in four dimensional weighted projective space, $\mathbb{P}^4(w)$, which are not complete intersections.

We believe that the answer is "yes, but they are not abundant".

This thesis shows the first part, and justifies the second part. That is, this thesis has two distinctive parts. First we prove that families of non general type surfaces in weighted projective four–space, $\mathbb{P}^4(w)$ are rare by showing that their corresponding covers in straight $\mathbb{P}^4$, which are usually general type surfaces, are rare.

In the second part we construct explicit examples of these rare objects in $\mathbb{P}^4$ using a technique involving sheaf cohomology and the Beilinson monad. We concentrate on the case of weights $w = (1, 1, 1, 1, 2)$ for our particular examples. We present three explicit examples, one of which is symmetric. The main computer algebra system used is *Macaulay2, Version 1.1* developed by D. Grayson and M. Stillman.

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Chapter 0

# Introduction

Ellingsrud and Peskine [24] proved that there exists an integer $d_0$ such that all smooth non general type surfaces in $\mathbb{P}^4$ have degree less than or equal to $d_0$. This motivated a search for such surfaces, partly by computational methods, and also an effort to find an effective bound on $d_0$, begun by Braun and Fløystad in [12]. As far as we know the smallest proven bound is 52 by Decker and Schreyer [19]. This upper bound whose existence is proved by [24] is generally believed to be fifteen. Examples in degree 15 were constructed by Popescu [41]. In fact examples are known in all degrees up to 15. The following table shows the chronological contributions in finding an effective bound for the degrees of smooth non general type surface in in $\mathbb{P}^4$.

Table 1: The race is still on!

| Authors | year | degree bound |
|---|---|---|
| Ellingsrud and Peskine [24] | 1989 | $\leq$ several thousand |
| Braun and Fløystad [12] | 1994 | $\leq 105$ |
| Cook [16] | 1995 | $\leq 80$ |
| Braun and Cook [11] | 1997 | $\leq 66$ |
| Cook [15] | 1997 | $\leq 46^*$ |
| Decker and Schreyer [19] | 2000 | $\leq 52$ *in response to errors they found above |

In recent years computer algebra systems such as Macaulay 2, Magma and Singular, have been used extensively in these computational methods. Many researchers have followed ideas developed by Decker, Ein and Schreyer in [18] to construct explicit examples. In [18] the authors give concrete constructions of non general type surfaces in $\mathbb{P}^4$.

Perhaps one can see some similarities among these construction methods, the most noteworthy being that they utilise a globalized form of the Hilbert-Burch theorem that allows one to realise any surface (more precisely, any codimension 2 locally Cohen-Macaulay subscheme) as the degeneracy locus of a map of vector bundles. In other words: for every codimension 2 subvariety $X$ in $\mathbb{P}^n$ there is a short exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0,$$

where $\mathcal{F}$ and $\mathcal{G}$ are vector bundles with $\operatorname{rk}\mathcal{G} = \operatorname{rk}\mathcal{F} + 1$ and $\psi$ is locally given by the maximal minors of $\varphi$ taken with alternating signs. (see [45])

There are two main parts to this thesis.

The first part is a proof that there exists a bound on the degree of quasismooth non general type surfaces in weighted projective four–space, $\mathbb{P}^4(w)$.

Some of the methods used to find non general type surfaces in $\mathbb{P}^4$ are also applicable to surfaces in weighted projective spaces $\mathbb{P}^4(w)$. It is therefore natural to ask whether a bound similar to the one whose existence is proved in [24] can be found for the degree of quasismooth non general type surfaces in a weighted projective space with given weights. In chapter three we show that such a readily computable bound (of course depending on the weights) does exist, and we compute it in some cases.

To show that a bound exists all we need is a fairly simple adaptation of the way in which the results of [24] (or [12]) are applied. For a computable bound we use the results of [12] together with some information about the contribution from the singularities of the surface in $\mathbb{P}^4(w)$.

Our procedure is to exploit the representation of $\mathbb{P}^4(w)$ as a quotient of $\mathbb{P}^4$ by a finite group action. Starting with a quasismooth non general type surface $X$ in weighted projective 4-space $\mathbb{P}^4(w)$, we take its cover in $\mathbb{P}^4$. This will (usually) be of general type, but it will have invariants bounded in terms of those of $X$, and the results of [12] still apply in this situation.

The second part consists of construction of new examples of general type surfaces in straight $\mathbb{P}^4$ which could possibly arise as double covers of non general type surfaces in weighted projective 4-space. We start by assuming existence of these non general type surfaces and end up by constructing general type surfaces: given enough information,

which we do not have at the moment, one could conceivably prove that they arise as double covers of non general type surfaces.

We present **three examples**, one of which is symmetric under an involution $\psi$ on $\mathbb{P}^4$, therefore indicating that maybe there exist non general type surfaces in $\mathbb{P}^4(w)$ which are not complete intersections. We reiterate that the difficulty here is that we are not yet able to determine whether the quotient of a general type surface is the one we were trying to obtain.

The main computer algebra system used is *Macaulay2, Version 1.1* developed by D. Grayson and M. Stillman [27]. We provide the programs that we used to construct the three examples, both in an Appendix and on DVD. The DVD also contains partial output of each program.

## 0.1. Brief overview of our construction method

In this section we give an outline of our construction method. A detailed account will come in chapter three along with the degree bound proof and chapter four via an example. Our construction technique follows the model of Decker, Ein and Schreyer in [18]. This model has been used over the last decade with a lot of success in constructing non general type surfaces in $\mathbb{P}^4$. It provides a very efficient way to construct new surfaces in $\mathbb{P}^4$.

We first guess the numerical invariants of our desired non general type surface $X$ in $\mathbb{P}^4(w)$. Then we consider $\widehat{X} \subset \mathbb{P}^4$, which is a cover of $X \subset \mathbb{P}^4(w)$. This $\widehat{X}$ is usually a general type surface with particular numerical invariants. Moreover $\widehat{X}$ is a smooth surface. We then construct $\widehat{X}$ as the degeneracy locus of a morphism between vector bundles.

At this point we recall the definition of the degeneracy locus $U_r$. (A glance at [3, page 83] or [26, page 2] might be useful.)

**Definition 1.** Given an $l \times m$ matrix $A = (i, j)$ of forms in variables $x_0, \ldots, x_N$, let $0 \leq r < \min(l, m)$. Consider the locus $U_r$ of points in $\mathbb{P}^N$ at which the rank of $A$ is at most $r$. This will be defined (cut out) by the minors of size $r + 1$ of $A$. This locus $U_r$ is called the degeneracy locus.

**Lemma 1.** *If $\mathcal{F}$ and $\mathcal{G}$ are vector bundles on $\mathbb{P}^4$ of rank $\operatorname{rk}\mathcal{F} = f$, and $\operatorname{rk}\mathcal{G} = f+1$ and if $\phi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ is a morphism, then $V(\phi) = \{p \in \mathbb{P}^4 | \operatorname{rk}\phi(p) < f\}$ has codimension $\leq 2$. If equality holds then $\widehat{X} = V(\phi)$ is a locally Cohen-Macaulay surface, and the*

*Eagon-Northcott complex*

$$0 \longleftarrow \mathcal{O}_{\widehat{X}}(j) \longleftarrow \bigwedge^{f} \mathcal{F}^* \otimes \bigwedge^{f+1} \mathcal{G} \longleftarrow \mathcal{G} \xleftarrow{\phi} \mathcal{F} \longleftarrow 0 \qquad (1)$$

*is exact and identifies* $\operatorname{coker}\phi$ *with the twisted ideal sheaf*

$$\operatorname{coker}\phi \cong \mathcal{J}_{\widehat{X}}(j)$$

*of* $\widehat{X}$*.*

That is, to construct $\widehat{X}$ with the particular numerical invariants, we find the appropriate $\mathcal{F}$ and $\mathcal{G}$ then we analyse the resulting Beilinson monad (see Definition 7) for the suitably twisted ideal sheaf $\mathcal{J}_{\widehat{X}}(j)$.

We carry out this analysis in steps: first we (carefully) choose a Beilinson cohomology table. We apply the Beilinson theorems and the Riemann-Roch formula for surfaces to determine some of the dimensions $h^i(\mathcal{J}_{\widehat{X}}(j))$; $0 \leq i, j \leq 4$. In all the examples discussed we have applied the Beilinson theorems to the twisted ideal sheaf $\mathcal{J}_{\widehat{X}}(4)$.

**Remark 1.** *Decker et al. [18, Proposition 1.7] say that if* $X \subset \mathbb{P}^4$ *is a smooth surface not of general type then* $h^3(\mathcal{J}_{\widehat{X}}(j)) = h^2(\mathcal{O}_{\mathbb{P}^4}(j)) = h^0\omega_X(-j) = 0$ *for* $j \geq 1$*. We do not know if a similar vanishing result holds for general type surfaces in* $\mathbb{P}^4$*.*

**Proposition 1.** *All our Beilinson cohomology tables will be of the kind depicted in Table 2.*

*In there, the blanks mean we have reason to put zeros and the rest can be filled according to the Euler characteristic* $\chi(\mathcal{J}_{\widehat{X}}(j)) = \sum_i (-1)^i h^i(\mathcal{J}_{\widehat{X}}(j))$*; and for each* $j$ *this sum is known to us by the Riemann-Roch formula.*

We carry out the justification of this choice of cohomology table in two lemmas.

**Theorem 1.** $h^2(\mathcal{J}_{\widehat{X}}) = q(\widehat{X})$ *and* $h^3(\mathcal{J}_{\widehat{X}}) = p_g(\widehat{X})$*.*

*Proof.* From the short exact sequence

$$0 \longrightarrow \mathcal{J}_{\widehat{X}} \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_{\widehat{X}} \longrightarrow 0 \qquad (2)$$

we obtain a long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(\mathcal{J}_{\widehat{X}}) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^4}) \longrightarrow H^0(\mathcal{O}_{\widehat{X}}) \longrightarrow \\ H^1(\mathcal{J}_{\widehat{X}}) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}^4}) \longrightarrow H^1(\mathcal{O}_{\widehat{X}}) \longrightarrow \\ H^2(\mathcal{J}_{\widehat{X}}) \longrightarrow H^2(\mathcal{O}_{\mathbb{P}^4}) \longrightarrow \dots \end{aligned} \qquad (3)$$

Table 2: Our General Beilinson Cohomology Table

| | | | | | |
|---|---|---|---|---|---|
| $i=4$ | | | | | |
| $i=3$ | $p_g(\widehat{X})$ | $h^3(\mathcal{J}_{\widehat{X}}(1))$ | $h^3(\mathcal{J}_{\widehat{X}}(2))$ | $h^3(\mathcal{J}_{\widehat{X}}(3))$ | $h^3(\mathcal{J}_{\widehat{X}}(4))$ |
| $i=2$ | $q(\widehat{X})$ | $h^2(\mathcal{J}_{\widehat{X}}(1))$ | $h^2(\mathcal{J}_{\widehat{X}}(2))$ | $h^2(\mathcal{J}_{\widehat{X}}(3))$ | $h^2(\mathcal{J}_{\widehat{X}}(4))$ |
| $i=1$ | | | $h^1(\mathcal{J}_{\widehat{X}}(2))$ | $h^1(\mathcal{J}_{\widehat{X}}(3))$ | $h^1(\mathcal{J}_{\widehat{X}}(4))$ |
| $i=0$ | | | | | $h^0(\mathcal{J}_{\widehat{X}}(4))$ |
| | $j=0$ | $j=1$ | $j=2$ | $j=3$ | $j=4$ |

By Theorem 10 we know that $H^1(\mathcal{O}_{\mathbb{P}^4}) = H^2(\mathcal{O}_{\mathbb{P}^4}) = H^3(\mathcal{O}_{\mathbb{P}^4}) = 0$ so

$$H^1(\mathcal{O}_{\widehat{X}}) \cong H^2(\mathcal{J}_{\widehat{X}}),\ H^2(\mathcal{O}_{\widehat{X}}) \cong H^3(\mathcal{J}_{\widehat{X}})$$

so $h^2(\mathcal{J}_{\widehat{X}}) = q(\widehat{X})$ and $h^3(\mathcal{J}_{\widehat{X}}) = p_g(\widehat{X})$. □

In the examples we shall have $q(\widehat{X}) = 0$.

**Theorem 2.** *Some more cohomologies vanish.*

1. $h^4(\mathcal{J}_{\widehat{X}}(j)) = h^4\mathcal{O}_{\mathbb{P}^4}(j) = 0$ *for* $j \geq -4$*;*
2. $h^1(\mathcal{J}_{\widehat{X}}(1)) = 0$ *because* $\widehat{X}$ *is not a Veronese surface.*
3. *By Kodaira vanishing* $h^2(\mathcal{J}_{\widehat{X}}(j)) = h^1(\mathcal{O}_{\widehat{X}}(j)) = h^1(\omega_{\widehat{X}}(-j)) = 0$ *for* $j \leq -1$
4. *We will also assume that* $\widehat{X}$ *is non-degenerate (does not lie in any hyperplane) so* $h^0(\mathcal{J}_{\widehat{X}}(1)) = 0$
5. *Since all smooth surfaces that lie in a quadric or in a cubic have been fully classified, (see [5, 36]) we assume* $\widehat{X}$ *neither lies in a quadric nor in a cubic, hence we can also set* $h^0(\mathcal{J}_{\widehat{X}}(2)) = 0$, $h^0(\mathcal{J}_{\widehat{X}}(3)) = 0$.

## 0.2. Structure of the thesis

Chapter 0 is the introduction. Here we explain what this thesis studies as briefly as possible.

Chapter 1 is the background material. In this chapter we include material that we feel helps the thesis to be self-contained.

Chapter 2 is continuation of background information with particular emphasis on algebraic surfaces. The main aim is to give the briefest outline and point the reader to references that give detailed treatments.

Chapter 3 provides the general construction method that we use throughout the remaining chapters. In this chapter we present the **first main result** of this thesis:

**Theorem.** *There exists $d_w \in \mathbb{N}$ depending only on the weights $w_i$ such that any quasi-smooth normal surface $X \in \mathbb{P}^4(w)$ of degree $d > d_w$ is of general type.*

Put another way, we prove that there exists a bound on the degrees of quasismooth non general type surfaces in weighted projective four–space and this bound depends on the weights. We compute explicit bounds in a few interesting cases and point out that this is a mild generalisation of the result of Ellingsrud and Peskine in [24].

Chapter 4 provides **the second main result** of this thesis. We elaborate more on the construction method using particular weights $w = (1, 1, 1, 1, 2)$. We present an explicit example and extract some code from the program used to help realise this result.

**Proposition.** *There exists a smooth general type surface $\widehat{X}$ in $\mathbb{P}^4$, of degree $\widehat{d} = 14$, sectional genus $\widehat{\pi} = 18$, topological Euler characteristic $c_2(\widehat{X}) = 64$, first Chern number $c_1^2(\widehat{X}) = 20$, Euler characteristic of the structure sheaf $\chi\left(\mathcal{O}_{\widehat{X}}\right) = 7$, irregularity $q = 0$ and geometric genus $p_g = 6$.*

Moreover, $\mathbb{P}^4$ has an involution under which $\widehat{X}$ is invariant, giving a quotient $X \in \mathbb{P}^4(w)$.

Chapter 5 contains **the third main result** of this thesis:

**Proposition.** *There exists a smooth general type surface $\widehat{X}$ in $\mathbb{P}^4$, of degree $\widehat{d} = 9$, sectional genus $\widehat{\pi} = 10$, topological Euler characteristic $c_2(\widehat{X}) = 63$, first Chern number $c_1^2(\widehat{X}) = 9$, Euler characteristic of the structure sheaf $\chi\left(\mathcal{O}_{\widehat{X}}\right) = 6$, irregularity $q = 0$ and geometric genus $p_g = 5$.*

We also give some evidence for

**Conjecture.** *There exists a smooth general type surface* $\widehat{X}$ *in* $\mathbb{P}^4$*, of degree* $\widehat{d} = 5$*, sectional genus* $\widehat{\pi} = 6$*, topological Euler characteristic* $c_2(\widehat{X}) = 55$*, first Chern number* $c_1^2(\widehat{X}) = 5$*, Euler characteristic of the structure sheaf* $\chi\left(\mathcal{O}_{\widehat{X}}\right) = 5$*, irregularity* $q = 0$ *and geometric genus* $p_g = 4$*.*

Chapter 6 gives a few concluding remarks.

The appendix is reserved for our *Macaulay2* programs for the three examples. We attach a DVD with the programs and some partial output from each. Of course, if the reader so wishes they can remove all the semicolons at the end of each line and run the program to get all the output, but then they might need to have a lot of patience.

Chapter 1

# Background

In this chapter we give a brief summary of concepts relevant to our research. In particular we refer to the text books [6, 9, 23, 29, 38] and [44].

The first section is a glimpse at the Hilbert series. We refer to [2] for more details.

The second section discusses the Koszul complex on a projective space.

Section three deals with the Beilinson monad, the major tool that we use throughout our construction of examples.

Section four gives a brief overview of weighted projective spaces.

Section five tackles the interpretation of Betti tables produced by *Macaulay2*. The final section is dedicated to summarising the construction method of Decker, Ein and Schreyer [18] on which our own construction method is modelled.

## 1.1. Hilbert series

**Definition 2.** Let $R = \oplus_{n\geq 0} R_n$ be a finitely generated graded $k$-algebra. The Hilbert function of $R$ is defined to be

$$\mathcal{H}(R, n) = \dim_k(R_n)$$

where $\dim_k R_n$ is the dimension of the vector space $R_n$ over $k$.

If $I$ is a homogeneous ideal of $R$ we define the Hilbert function of $I$ as

$$\mathcal{H}(I, n) = \dim I_n.$$

The Hilbert series of $R$ is the generating function of the sequence given by the Hilbert function.

**Definition 3.** Let $R = \oplus_{n\geq 0} R_n$ be a finitely generated $k$-algebra. The Hilbert series of $R$ is

$$\mathcal{F}(R,t) = \sum_{n=0}^{\infty} \mathcal{H}(R,n)t^n.$$

Similarly, if $I$ is a homogeneous ideal of $R$ then the Hilbert series of $I$ is

$$\mathcal{F}(I,t) = \sum_{n=0}^{\infty} \mathcal{H}(I,n)t^n$$

The *Macaulay2* command `hilbertSeries I,` which might be expected to return the series for $I$, actually returns the series for $R/I$. This is because most of the time this is the interesting object one would be looking for anyway. In other words *Macaulay2* has been designed with the knowledge that, most of the time, one is more interested in how the ideal sits inside the ambient ring. However sometimes one really does want to find the series of $I$ without taking a quotient. In that case the following theorem from [17, Chapter 6] can be used.

**Theorem 3.** *Let $R = \oplus_{n\geq o} R_n$ be a graded $k$-algebra and $I = \oplus_{n\geq o} I_n$ be a graded ideal. Then*

$$\mathcal{F}(R/I,t) = \mathcal{F}(R,t) - \mathcal{F}(I,t)$$

We also use the following definition of the Hilbert series.

**Theorem 4** (Hilbert, Serre)**.** *Given a graded polynomial ring $R = k[x_0, x_1, \ldots, x_n]$ and graded ideal $I$ in $R$, then the Hilbert series of $R/I$ is expressed as*

$$\mathcal{F}(R/I,t) = \frac{p(t)}{(1-t)^{n+1}}$$

*where $p(t)$ is a polynomial in $t$ with integer coefficients.*

The polynomial $p(t)$ is called Poincaré polynomial. We end this section with an example from Chapter 6, where `IX` denotes the twisted ideal sheaf $\mathcal{J}_{\widehat{X}}(4)$.

```
ii85 :  hilbertSeries IX
             5     6
       1 - 6T  + 5T
oo85 = -------------
                 5
          (1 - T)

oo85 : Expression of class Divide
```

## 1.2. Koszul resolution

Let $V$ be a five-dimensional vector space with basis $\{e_0, \ldots, e_4\}$.

**Definition 4.** The Koszul complex on $\mathbb{P}^4 = \mathbb{P}^4(V)$ is an exact sequence

$$\begin{aligned} 0 \xleftarrow{k_1} V^* \otimes \mathcal{O}(-1) \xleftarrow{k_2} \bigwedge^2 V^* \otimes \mathcal{O}(-2) \xleftarrow{k_3} \bigwedge^3 V^* \otimes \mathcal{O}(-3) \\ \xleftarrow{k_4} \bigwedge^4 V^* \otimes \mathcal{O}(-4) \xleftarrow{k_5} \bigwedge^5 V^* \otimes \mathcal{O}(-5) \longleftarrow 0 \end{aligned} \tag{1.1}$$

with $\ker(k_i) \cong \Omega^i,\ 0 \leq i \leq 4$, where $\Omega^i = \bigwedge^i T^*_{\mathbb{P}^4(V)}$.

**Remark 2.** *Notice that one can use the Koszul complex to prove that*

$$\mathrm{Hom}(\Omega^i(i), \Omega^j(j)) \cong \bigwedge^{i-j} V$$

*the isomorphism being defined by contraction.*

As a simple illustration we write down the Koszul complexes of some vector bundles.

**Example 1.** *(a)*

$$0 \longleftarrow \Omega^4(4) \longleftarrow \mathcal{O}(-1) \longleftarrow 0$$

*(b)*

$$0 \longleftarrow \Omega^3(3) \longleftarrow 5\mathcal{O}(-1) \longleftarrow \mathcal{O}(-2) \longleftarrow 0$$

*(c)*

$$0 \longleftarrow \Omega^2(2) \longleftarrow 10\mathcal{O}(-1) \longleftarrow 5\mathcal{O}(-2) \longleftarrow \mathcal{O}(-3) \longleftarrow 0$$

*(d)*

$$0 \longleftarrow \Omega^1(1) \longleftarrow 10\mathcal{O}(-1) \longleftarrow 10\mathcal{O}(-2) \longleftarrow 5\mathcal{O}(-3) \longleftarrow \mathcal{O}(-4) \longleftarrow 0$$

**Definition 5** (Linear determinantal varieties)**.** These are defined as follows: let $\Omega$ be an $m \times n$ matrix of homogeneous linear forms $\Omega = (L_{i,j})$ on a projective space $\mathbb{P}^l$, not all vanishing simultaneously. Then the variety

$$\sum_k(\Omega) = \{[Z_0, \ldots, Z_l] : \mathrm{rk}(\Omega(Z)) \leq k\}$$

is called a linear determinantal variety.

Thus, we can construct examples which are just the common zero locus of the $(k+1) \times (k+1)$ minors, which are homogeneous polynomials of degree $k+1$. For small enough values of $k$ we can do this explicitly in *Macaulay2*, but for large values of $k$ we run out of memory.

## 1.3. Monads and spectral sequences

The technique of monads provides powerful tools in construction and classification of coherent sheaves with prescribed invariants. The basic idea behind monads is to represent arbitrary coherent sheaves in terms of simpler sheaves such as line bundles or bundles of differentials, and in terms of homomorphisms between these simpler sheaves [23]. In particular, we have relied heavily on the Beilinson monad in our construction method. This is just a monad for a sheaf $\mathscr{F}$ that involves direct sums of twisted bundles of differentials, and thus homogeneous matrices over the exterior algebra on a vector space of finite dimension $n+1$ over a field $R$.

Below follows a formal definition of a monad. We start with the basic version from [40].

In this version, a monad over a compact complex manifold $X$ is a complex

$$0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0 \tag{1.2}$$

of holomorphic vector bundles over $X$ which is exact at $A$ and at $C$, such that $\mathrm{Im}(a)$ is subbundle of $B$. The holomorphic vector bundle

$$E = \frac{\mathrm{Ker}\, b}{\mathrm{Im}\, a}$$

is the homology of the monad. We also have the definition below [23, page 230].

**Definition 6.** A monad on $X$ is a bounded complex

$$\ldots \longrightarrow \mathcal{K}^{-1} \longrightarrow \mathcal{K}^0 \longrightarrow \mathcal{K}^1 \longrightarrow \ldots \tag{1.3}$$

of coherent sheaves on $X$ which is exact except at $\mathcal{K}^0$. The homology $\mathcal{F}$ at $\mathcal{K}^0$ is called the homology of the monad, and the monad is said to be a monad for $\mathcal{F}$. We say that the type of a monad is determined if the sheaves $\mathcal{K}^i$ are determined.

In [23] $X$ is assumed to be $\mathbb{P}^n$. Below we shall always have $X = \mathbb{P}^4$ in practice. Every monad (1.2) has an associated commutative diagram with exact rows and columns which is called the display of the monad, shown below.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & K & \longrightarrow & E & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{a} & B & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle b} & & \downarrow & & \\
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

**Remark 3.** *In cohomology we learn that simplest bundles over $\mathbb{P}^n$ are bundles $\Omega^i_{\mathbb{P}^4}(j)$ of twisted $i-$forms. We can think of these bundles as a foundation or, as Eisenbud et al. [22] put it, "as building blocks for more complicated bundles". The next theorems have been used to exploit this fact.*

**Theorem 5** (Beilinson 1978, Monad Version)**.** *(see [19] or [22])*

*For any coherent sheaf $\mathcal{J}_{\widehat{X}}$ on $\mathbb{P}^4$ there is a complex $\mathcal{K}^\bullet$ with*

$$\mathcal{K}^i \cong \bigoplus_j H^{i-j}\left(\mathbb{P}^4, \mathcal{J}_{\widehat{X}}(j)\right) \otimes \Omega^{-j}(-j)$$

*such that*

$$H^i(\mathcal{K}^\bullet) = \begin{cases} \mathcal{J}_{\widehat{X}} & i = 0 \\ 0 & i \neq 0 \end{cases} \tag{1.4}$$

**Remark 4.** *The differentials of $\mathcal{K}^\bullet$ are given by matrices with entries in the exterior algebra $\bigwedge V$ over the underlying vector space $V$ of $\mathbb{P}^4$*

**Definition 7.** $\mathcal{K}^\bullet$ above is called the Beilinson monad for $\mathcal{J}$.

**Theorem 6** (Beilinson 1978, Spectral Sequence Version)**.** *(see [22])*

*For any coherent sheaf $\mathcal{J}$ on $\mathbb{P}^n$ there is a spectral sequence with $E_1-$term*

$$E_1^{ji} = H^i(\mathbb{P}^n, \mathcal{J}(j)) \otimes \Omega^{-j}(-j)$$

*converging to $\mathcal{J}$, that is, $E_\infty^{ji} = 0$ for $j + i \neq 0$ and $\oplus E_\infty^{-j,j}$ is the associated graded sheaf of a suitable filtration of $\mathcal{J}$.*

**Definition 8.** The sequence above is called the Beilinson spectral sequence.

Decker et al. [19] advise that one should pick a twist $m$ carefully and apply Beilinson's theorem to the twisted ideal sheaf $\mathcal{J}(m)$ instead of $\mathcal{J}$ itself, and experience will always guide one on which is a suitable twist. In the last section of this chapter we will see that one has some way of choosing a suitable twist for very simple monads with only two nonzero terms on $\mathbb{P}^4$.

All the $E_1$-terms are in the second quadrant and only finitely many of them are different from zero. The higher differentials are

$$d_r^{ji} : E_r^{ji} \longrightarrow E_r^{j+r,i-r+1},\ r \geq 2.$$

### 1.3.1. Hand calculations on differentials

We use the first example, appearing in Chapter 4, to illustrate the steps and the $E_i$-diagrams involved in the spectral sequence. In practice our use of computer algebra system renders this exercise unnecessary. This is the only time we ever do an example by hand this way: for more examples done this way consult [40, Chapter II, section 3].

The $E_1$-diagram is shown in Table 1.1.

The differential $d_1^{j,i} : E_1^{j,i} \to E_1^{j+1,i}$ gives the complex

$$0 \longrightarrow 5\Omega^1(1) \xrightarrow{d_1^{-1,1}} 9\mathcal{O} \longrightarrow 0$$

So that the $E_2$-diagram is as in Table 1.2

The differential $d_2^{j,i} : E_2^{j,i} \to E_2^{j+2,i-1}$ gives the complex

$$0 \longrightarrow \Omega^3(3) \xrightarrow{d_2^{-3,2}} \ker d_1^{-1,1} \longrightarrow 0$$

Hence the $E_3$-diagram is as in Table 1.3

The differential $d_3^{j,i} : E_3^{j,i} \to E_3^{j+3,i-2}$ gives the complex

$$0 \longrightarrow 6\mathcal{O}(-1) \xrightarrow{d_3^{-4,3}} \operatorname{coker} d_2^{-3,2} \longrightarrow 0$$

Table 1.1: $E_1$- Diagram I

| | | | | | |
|---|---|---|---|---|---|
| 3 | $6\mathcal{O}(-1)$ | | | | |
| 2 | | $\Omega^3(3)$ | | | |
| 1 | | | | $5\Omega^1(1)$ | $9\mathcal{O}$ |
| 0 | | | | | |
| | $-4$ | $-3$ | $-2$ | $-1$ | 0 |

Therefore the $E_\infty = E_4$-diagram is as in Table 1.4
In this case the display of our monad looks as follows.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 6\Omega^4(4)\oplus\Omega^3(3) & \longrightarrow & K & \longrightarrow & E & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 6\Omega^4(4)\oplus\Omega^3(3) & \xrightarrow{a} & 5\Omega^1(1) & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle b} & & \downarrow & & \\
 & & & & 9\mathcal{O} & = & 9\mathcal{O} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

**Lemma 2.** *If $E$ is the cohomology of the monad*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

*then the rank $E$ is given by*

$$\operatorname{rk} E = \operatorname{rk} B - \operatorname{rk} A - \operatorname{rk} C$$

Table 1.2: $E_2$-Diagram I

| | | | | |
|---|---|---|---|---|
| $6\mathcal{O}(-1)$ | | | | |
| | $\Omega^3(3)$ | | | |
| | | | $\ker d_1^{-1,1}$ | $\operatorname{coker} d_1^{-1,1}$ |
| | | | | |
| $-4$ | $-3$ | $-2$ | $-1$ | $0$ |

And in this case we get

$$\operatorname{rk} E = 5 \times \binom{4}{1} - \left(6 \times \binom{4}{4} + 1 \times \binom{4}{3} + 9 \times \binom{4}{0}\right) = 1$$

For more details on spectral sequences and $E_i-$diagrams consult [40].

The shape of the monad (spectral sequence) for $\mathcal{J}_{\widehat{X}}$ is determined by the dimensions $h^i\mathcal{J}(j) := H^i(\mathbb{P}^n, \mathcal{J}(j+n))$, $0 \leq i \leq n$ in the range $-n \leq j \leq 0$.

When $n = 4$ we know that for $i = 1, 2$ only finitely many of the $h^i\mathcal{J}(j), j \in \mathbb{Z}$, are different from zero. That is, the Hartshorne-Rao modules of $\widehat{X}$, the graded $R-$modules

$$H^i_*\mathcal{J}_{\widehat{X}} = \bigoplus_{j\in\mathbb{Z}} H^i(\mathbb{P}^4, \mathcal{J}_{\widehat{X}}(j)),\ i = 1, 2,$$

are of finite length.

Table 1.5 show a typical Beilinson cohomology table in $\mathbb{P}^4$: the blanks represent zeros.

The longest complex we can have in $\mathbb{P}^4$ is

$$0 \longrightarrow \mathcal{K}^{-2} \longrightarrow \mathcal{K}^{-1} \longrightarrow \mathcal{K}^0 \longrightarrow \mathcal{K}^1 \longrightarrow \mathcal{K}^2 \longrightarrow \mathcal{K}^3 \longrightarrow 0$$

or where $\mathcal{K}^{-2} = A\Omega^4(4)$, $\mathcal{K}^{-1} = B_1\Omega^4(4) \oplus B_2\Omega^3(3) \oplus B_3$ and so on.

Table 1.3: $E_3$-Diagram I

| | | | | |
|---|---|---|---|---|
| $6\mathcal{O}(-1)$ | | | | |
| | $\ker d_2^{-3,2}$ | | | |
| | | | $\operatorname{coker} d_2^{-3,2}$ | $\operatorname{coker} d_1^{-1,1}$ |
| | | | | |
| $-4$ | $-3$ | $-2$ | $-1$ | $0$ |

## 1.4. Weighted projective spaces

**Definition 9.** Weighted projective space $\mathbb{P}^n(w)$, with $w = (w_0, \ldots, w_n)$ nonzero positive integers, is the quotient

$$\mathbb{P}^n(w) = \left(\mathbb{A}^{n+1} \setminus 0\right) / \mathbb{G}_m$$

of $\mathbb{A}^{n+1}$ under the equivalence relation

$$(x_0, \ldots, x_n) \sim (\lambda^{w_0} x_0, \ldots, \lambda^{w_n} x_n) \text{ for } \lambda \in \mathbb{G}_m$$

In our case $n = 4$, and $\mathbb{G}_m$ is $\mathbb{C}^*$.

**Definition 10.** We will say that a weighted projective space is well-formed if no $n-1$ of $w_0, w_1, \ldots, w_n$ have a common factor.

In this thesis we may assume that we have a well-formed space. The above definition comes from [42]: compare with [34, Definition 5.11].

**Remark 5.** *For a general polynomial $f$ in a weighted projective space, $f(\mathbf{x}) = a$ does not make sense for $a \in \mathbb{C}$. For example, take $\mathbb{P}^2(1,2,3), f = xy + y - z, a = -8$ then $f(1,2,4) = 0$ but $f(2,8,32) = -8$, yet the points $(1,2,4)$ and $(2,8,32)$ represent the same point of $\mathbb{P}^2(1,2,3)$ in weighted homogeneous coordinates. We can never make $f(\mathbf{x}) = a$ make sense for $a \neq 0$ but we can make sure that the set of points in $\mathbb{P}^n(w)$ at which $f$ vanishes, $V(f)$, is always well-defined by restricting to only weighted homogeneous polynomials (see Lemma 3).*

Table 1.4: $E_4$-Diagram I

| | | | | |
|---|---|---|---|---|
| $\ker d_3^{-4,3}$ | | | | |
| | $\ker d_2^{-3,2}$ | | | |
| | | | $\operatorname{coker} d_3^{-4,3}$ | $\operatorname{coker} d_1^{-1,1}$ |
| | | | | |
| $-4$ | $-3$ | $-2$ | $-1$ | $0$ |

Table 1.5: General Beilinson cohomology table in $\mathbb{P}^4$

| | | | | |
|---|---|---|---|---|
| $B_1$ | $C_1$ | $D_1$ | $E_1$ | $F$ |
| $A$ | $B_2$ | $C_2$ | $D_2$ | $E_2$ |
| | | $B_3$ | $C_3$ | $D_3$ |
| | | | | $C_4$ |

Thus, the only polynomials worth looking at in weighted projective space are the weighted homogeneous polynomials, as defined below.

**Definition 11.** A polynomial $f \in \mathbb{P}^n(w)$ is weighted homogeneous of degree $d$ if each term appearing in $f$ has total degree $d$, where $d = \alpha_0 w_0 + \alpha_1 w_1 + \ldots + \alpha_n w_n$ for a term in $f$ of the form $c x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $c \in \mathbb{C}$.

A weighted homogeneous ideal is an ideal generated by weighted homogeneous polynomials.

### 1.4.1. Quasismoothness

Let $X$ be a closed subscheme of a weighted space $\mathbb{P}^n(w)$ and $\rho : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n(w)$ be the canonical projection. The Zariski closure $C_X$ of $\rho^{-1}(X)$ in $\mathbb{A}^{r+1}$ is called the

affine quasicone over X. The point $0 \in C_X$ is called the vertex of $C_X$.

**Definition 12.** A closed subscheme $X \subset \mathbb{P}^n(w)$ is called *quasismooth* (with respect to the embedding $X \to \mathbb{P}^n(w)$) if its affine quasicone is smooth outside its vertex.

### 1.4.2. Some observations

More details can be found in [20].

**Proposition 2.** *(a) If $d$ is a factor of all the $w_i$ then*

$$\mathbb{P}^n(w_0, \ldots, w_n) = \mathbb{P}^n(\tfrac{w_0}{d}, \ldots, \tfrac{w_n}{d})$$

*(b) Suppose that $w_0, w_1, \ldots, w_n$ have no common factors and that $d$ is a common factor of all $w_i$ for $i \neq j$ (and therefore coprime to $w_j$ [42]). Then*

$$\mathbb{P}^n(w_0, \ldots, w_n) = \mathbb{P}^n(\tfrac{w_0}{d}, \ldots, \tfrac{w_{j-1}}{d}, w_j, \tfrac{w_{j+1}}{d}, \ldots, \tfrac{w_n}{d})$$

$\mathbb{P}^n(w)$ is singular: see Chapter 3.

**Lemma 3.** *Let $w = (w_0, \ldots, w_n)$, be the weights, where $w_i \in \mathbb{N}$. Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be a weighted homogeneous polynomial. If $f$ vanishes on any set of homogeneous coordinates for a point $p \in \mathbb{P}^n(w)$, then $f$ vanishes for all homogeneous coordinates of $p$. In particular*

$$V(f) = \{p \in \mathbb{P}^n(w) : f(p) = 0\}$$

*is a well-defined subset of $\mathbb{P}^n(w)$.*

*Proof.* Let $(a_0, \ldots, a_n)$ and $(\lambda^{w_0} a_0, \ldots, \lambda^{w_n} a_n)$ be homogeneous coordinates of $p \in \mathbb{P}^n(w)$ and assume that $f(a_0, \ldots, a_n) = 0$. If $f$ is homogeneous of degree $d$ then every term in $f$ has the form

$$c x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \tag{1.5}$$

where $\alpha_0 w_0 + \alpha_1 w_1 + \ldots + \alpha_n w_n = d$ and $c \in \mathbb{C}$. When we substitute $\lambda^{w_i} a_i$ into (1.5), we obtain

$$\lambda^d c a_0^{\alpha_0} a_1^{\alpha_1} \cdots a_n^{\alpha_n}.$$

Summing over terms of $f$, we find a common factor $\lambda^d$, and hence

$$f(\lambda^{w_0} a_0, \ldots, \lambda^{w_n} a_n) = \lambda^d f(a_0, \ldots, a_n) = 0$$

□

## 1.5. Betti table

The main purpose of this section is to summarise the information one reads from a typical Betti table: see [23, page 22]). Consider a Betti Table 1.6 produced by Macaulay2.

Table 1.6: An example of a Betti table produced by Macaulay2

| | 0 | 1 | 2 |
|---|---|---|---|
| total: | 54 | 28 | 5 |
| 1: | 54 | 27 | 4 |
| 2: | . | 1 | 1 |

Interpretation: the top row denotes the subscripts of the free modules in the resolution of our twisted ideal sheaf $\mathcal{J}_{\widehat{X}}(4)$: we will call them $G_0$, $G_1$ and $G_2$. The maps go from right to left, so that we can write

$$G_0 \longleftarrow G_1 \longleftarrow G_2 \longleftarrow 0$$

The second row tells the number of generators in each module, so $G_0$ has 54 generators, $G_1$ has 28 generators and $G_2$ has five generators. Now look at the third and fourth rows, ignoring the first column headed by the word "total" for now. Then notice that the sum of the numbers in a column in these two rows is exactly the number in the same column in row two. This gives a breakdown of the generators in different degrees, and this is how it works: $G_0$ has all fifty-four generators in degree one ( $T_{1,0}$, where one comes from the first column and zero comes from the first row ), $G_1$ has 27 of its generators in degree two $T_{1,1}$, one generator in degree three $T_{2,1}$ and $G_2$ has four of its generators in degree three $T_{1,2}$, one generator in degree four $(2+2)$.

## 1.6. Construction method of Decker, Ein and Schreyer

This section is a summary of construction method for smooth non general type surfaces in $\mathbb{P}^4$. We will mostly cover material from [19] and [18], but an interested reader would probably want to read these sources.

Appendix A of [18] on remarks about computations is really worth looking at because it explains why computations done entirely over a finite prime field should be

trusted to carry over to the field of complex numbers. The largest prime that *Macaulay2 version 1.1* can work with is 32749. It is not obvious why an example obtained in characteristic $p$ should lift to characteristic zero.

However, the authors of [18] eventually want to say that their results lift to the complex numbers. First they argue that it is to their advantage that Macaulay2 works over a finite prime field because this helps reduce the accumulation of denominators during the calculations. The next argument is to deduce that a surface exists over the complex numbers by saying that their constructions work over a Zariski open subset of Spec $\mathbb{Z}$. The explicit equations obtained may be regarded as the reduction modulo $p$ of a surface over Spec $\mathbb{Z}$, which then implies that the surface exists over the complex numbers by applying openness of smoothness.

At some point in the computations they need to pick a sufficiently general element of some space. The trick here is to make a random choice and then verify that the point chosen is a good point for the current situation. They say that the reason explicit examples can be computed easily is that the components of the Hilbert scheme of (nearly all) of their examples are unirational, and the unirationality is inherent in the construction method.

The main references here are [1, 4, 18, 19, 22] and [23, pages 215–247].

The method makes frequent use of Riemann-Roch in the following form.

**Proposition 3** (Riemann-Roch)**.** *Let $S \subset \mathbb{P}^4$ be a smooth surface of degree $d$, sectional genus $\pi$. Then*

$$\chi\left(\mathcal{J}_S(j)\right) = \chi\left(\mathcal{O}_{\mathbb{P}^4}(j)\right) - \binom{j+1}{2} d + j(\pi - 1) - \chi(\mathcal{O}_S) \tag{1.6}$$

Note that $\chi\left(\mathcal{O}_{\mathbb{P}^4}(j)\right) = \binom{j+4}{4}$ For a detailed account of the steps in this construction method, consult [19, 5.10]. The main idea is that the existence of a family of smooth non general type surfaces in $\mathbb{P}^4$, with prescribed invariants, is verified by constructing an explicit example. The authors of [18] construct vector bundles $\mathcal{F}$ and $\mathcal{G}$. With a bit of patience they can find simple $\mathcal{F}$ and $\mathcal{G}$ such that rank $\mathcal{G}$ exceeds rank $\mathcal{F}$ by one and a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ drops rank along the desired surface $S$. If $S$ really has the expected codimension 2, then $S$ is locally Cohen-Macaulay and the Eagon-Northcott complex (1) defined by the minors of $\varphi$ identifies $\operatorname{coker}\varphi$ with a suitable twisted ideal sheaf of $S$,

$$0 \longleftarrow \mathcal{J}_S(m) \longleftarrow \mathcal{G} \xleftarrow{\varphi} \mathcal{F} \longleftarrow 0.$$

This will generally describe a smooth $S$.

Riemann-Roch gives $\chi(\mathcal{J}(j))$ for each $0 \leq j \leq 4$. Filling out a plausible cohomology

table for the ideal sheaf of $S$ carefully is also central to finding $S$. Notice that choosing a suitable twist $m$ is part of the exercise. Then from the table as we explained earlier one reads off the shape of the monad. It is possible that for surfaces of low degree, the monad is just a sequence

$$0 \longrightarrow \mathcal{K}^{-1} \longrightarrow \mathcal{K}^{0} \longrightarrow \mathcal{J}_S(m) \longrightarrow 0.$$

Note that when $m = 4$ one obtains this simple monad if and only if $S$ is regular and $\mathcal{J}_S$ is $5-$regular. Then in this case one just takes $\mathcal{F} = \mathcal{K}^{-1}$ and $\mathcal{G} = \mathcal{K}^0$ and a generic $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ will yield a smooth surface, or else no smooth surface with such a cohomology table exists.

If a smooth surface is obtained then the corresponding family of surfaces is unirational.

**Remark 6.** *(i) A subvariety $X$ has expected codimension at $x$ if locally it resembles a complete intersection. See [25, page 158] or [26, page 2].*

*(ii) An ideal $I$ is $r-$regular if the $j$th syzygy module of $I$ is generated in degrees $\leq r + j$, for $j \geq 0$. Consult [8].*

*(iii) We recall that a variety $X$ is rational if, equivalently,*

- *(a) $X$ is birational to $\mathbb{P}^n$;*
- *(b) $K(X) \cong K(x_1, \ldots, x_n)$; or*
- *(c) $X$ possesses an open subset $U$ isomorphic to an open subset of $\mathbb{A}^n$. If these conditions do not hold, the variety is called irrational.*

*(iv) Recall that a variety $X$ is is unirational if there exists a dominant rational map*

$$\phi : \mathbb{P}^n \dashrightarrow X$$

*for some $n$.*

In other words, in this construction one aims to compute $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ and then see if a general $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ yields a smooth surface. But one may fail to obtain a smooth surface this way, and then one should try and see why this misfortune occurred. Possible causes are

1. $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$.
2. A general $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is not injective.

3. A general $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ does not vanish in expected codimension.

4. A general $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ defines a surface but always a singular one.

Depending on the outcome of the analysis one decides on the course of action; this will usually be constructing $\mathcal{F}$ or $\mathcal{G}$ or both, such that either $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ becomes bigger or a surface in an entirely different family is obtained.

The final step in this construction is to identify the constructed surface given by explicit equations within the Enriques-Kodaira classification. This is achieved by applying adjunction: consult [19, Theorem 8.1]. In [19, Remark 8.3], the authors say that they do not know if adjunction theory holds over a finite field. It is not known if adjunction theory applies but Proposition 8.3 and Corollary 8.4 of [19] provide sufficient conditions to carry out the process to spot the surface in the Enriques-Kodaira classification, for a surface given by explicit equations over a finite field. The authors describe in detail how this is done in [19, 8.6], for a surface in $\mathbb{P}^4$.

We end this section by recording two smoothness checking criteria.

**Notation** Let $f_1, \ldots, f_N$ be the generators of $\mathcal{J}_S$ and $I := \langle f_1, \ldots, f_N \rangle$. Then

$$J := \langle \frac{\partial f_i}{\partial x_j} | 1 \leq i \leq N, 0 \leq j \leq 4 \rangle$$

is the Jacobian ideal of $f_1, \ldots, f_N$ and $I_k(J)$ the ideal of $k \times k$ minors of $J$. Moreover, if $f = f_i$ is one of the generators, then we write $I_k(f)$ for the ideal of $k \times k$ minors of $J$ which involves the row corresponding to $f$ and $J(f)$ for the Jacobian matrix of $f$.

**Remark 7.** *If an ideal $I$ is primary, i.e., if $fg \in I \implies$ either $f \in I$ or $g^m \in I$ for some $m > 0$, and if $\sqrt{I} = P$, then we say that $I$ is $P-$primary.(Consult [21, page 94] for more details.)*

Let us recall the following definition (obtained from [28], page 137.)

**Definition 13.** $S$ has pure dimension $n$ if every irreducible component of $S$ has the same dimension $n$.

**Theorem 7** (Jacobian Criterion)**.** *A pure 2-dimensional subscheme $S \subset \mathbb{P}^4$ is smooth if and only if*

$$S \cap V(I_2(J)) = \emptyset,$$

*that is, if and only if $I_2(J) + I$ is $\langle x_0, \ldots, x_4 \rangle-$primary.*

That is, to check smoothness it is enough to check that the ideal generated by the $2 \times 2$-minors of the Jacobian matrix together with the original generators defines

something in codimension 5. Checking smoothness is the most time and memory consuming step in the computation.

It is known that checking smoothness by the above criterion is expensive because it requires computing the codimension of $I_2(J) + I$. This involves two very large computations:

1. computation of the ideal $I_2(J)$
2. computation of a Gröbner basis of $I_2(J) + I$.

The authors of [18, 19] provide an alternative smoothness checking criterion which is more efficient.

**Theorem 8.** *Let $S \subset \mathbb{P}^4$ be a locally Cohen-Macaulay surface of degree $d$ and sectional genus $\pi$. Let $f = f_i$ be one of the generators of $\mathcal{J}_S$ as above and write $e := \deg f$. Suppose that*

1. *$V((I_1(J)_{<e} + I) = \emptyset$*
2. *$V(I_2(f))$ is finite and*

$$\deg V(I_2(f) + I = \deg V(J(f) + I) = d^2 + e(e-4)d - 2e(\pi - 1)$$

*Then $S$ is smooth.*

Since checking (2) involves computing a Gröbner basis of $I_2(f)$, it is easiest if $f$ is chosen as a generator of lowest possible degree.

Chapter 2

# Algebraic surfaces

**Definition 14.** An algebraic surface $X$ is minimal if it does not contain any exceptional curve $E$ of the first kind (i.e. $E \cong \mathbb{P}^1, E^2 = -1$). An algebraic surface $X$ is a minimal model of a surface $Y$ if there exists a birational morphism $Y \to X$ such that $X$ is minimal (see [25, page 70]).

**Remark 8.** *(see [7]) Every surface can be obtained from a minimal one (its "minimal model") after a finite sequence of blowings up of smooth points; this model is moreover unique if $\kappa(S) \geq 0$. (See Definition 15.) This means that the minimal models of the rational and ruled surfaces are not unique.*

**Theorem 9** (Serre Duality)**.** *For any coherent sheaf $\mathscr{E}$ on a smooth complete algebraic variety $X$ of dimension $n$*

$$H^q(X, \mathscr{E})^* \cong H^{n-q}(X, \mathscr{E}^* \otimes \mathcal{O}(K_X))$$

**Theorem 10.** *Let $A$ be a noetherian ring, $S = A[x_0, \ldots, x_r]$ and let $Y = \mathbb{P}^r_A$ be the projective space over $A$, with $r \geq 1$. Then:*

*(a) the natural map $S \longrightarrow \bigoplus_{n\in\mathbb{Z}} H^0(Y, \mathcal{O}_Y(n))$ is an isomorphism of graded $S$-modules;*

*(b) $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < r$ and all $n \in \mathbb{Z}$;*

*(c) $H^r(Y, \mathcal{O}_Y(-r-1)) \cong A$;*

*(d) The natural map*

$$H^0(Y, \mathcal{O}_Y(n)) \times H^r(Y, \mathcal{O}_Y(-n-r-1)) \longrightarrow H^r(Y, \mathcal{O}_Y(-r-1)) \cong A$$

*is a perfect pairing of finitely generated free $A$-modules, for each $n \in Z$.*

For the proof see [29, Chapter III, Theorem 5.1.].

Notice in particular that Theorem 10, part $(b)$ says that the cohomology groups $H^i(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(n))$ vanish for $0 < i < 4$ and all $n \in \mathbb{Z}$.

**Theorem 11** (Noether's formula)**.** *Let $X$ be a smooth projective surface. Then*

$$\chi(\mathcal{O}_X) = \frac{1}{12}\left(K_X^2 + c_2(X)\right)$$

**Theorem 12** (The Kodaira Vanishing Theorem)**.** *If $Y$ is a projective nonsingular variety of dimension $n$ over $\mathbb{C}$, and if $\mathscr{L}$ is an ample line bundle on $Y$, then:*

*(a) $H^i(Y, \mathscr{L} \otimes \mathcal{O}(K_Y)) = 0$ for $i > 0$;*

*(b) $H^i(Y, \mathscr{L}^{-1}) = 0$ for $i < n$.*

## 2.1. Enriques–Kodaira classification of algebraic surfaces

Table 2.1 summarises Enriques–Kodaira classification of algebraic surfaces.

**Definition 15.** (see [9, page 86]) Let $S$ be a smooth projective variety, $K$ a canonical divisor of $S$, $\phi_{nK}$ the rational map from $S$ to the projective space determined by the linear system $|nK|$. The Kodaira dimension of $S$, written $\kappa(S)$ or just $\kappa$, is the maximum dimension of the images $\phi_{nK}(S)$, for $n \geq 1$.

So for example curves can have Kodaira dimension $-\infty$, 0 or 1, surfaces can have Kodaira dimension $-\infty$, 0, 1 or 2.

Recall that by definition $P_n(S) = h^0(S, K^{\otimes n})$

For more details see [6, chapters V and VI] or [9].

### 2.1.1. Surfaces with $\kappa = -\infty$

**Definition 16.** A surface $X$ is ruled if it is birationally equivalent to $C \times \mathbb{P}^1$, where $C$ is a smooth curve.

**Definition 17.** A rational surface is a surface that is birationally equivalent to $\mathbb{P}^2$.

For example $C \times \mathbb{P}^1$ itself is ruled. In particular rational surfaces are ruled. [9, Chapter IV] provides many examples of rational surfaces.

**Definition 18.** Let $C$ be a smooth curve. A geometrically ruled surface over $C$ is a surface $S$, together with a smooth morphism $p : S \to C$ whose fibres are isomorphic to $\mathbb{P}^1$.

It is not immediately clear that a geometrically ruled surface is ruled. Theorem III.4 in [9] establishes this.

The following characterisation [9, Proposition III.21] of ruled surfaces is worth noting.

**Proposition 4.** *Let $S$ be a ruled surface over a curve $C$. Then $q(S) = g(S)$; $p_g(S) = 0$; $P_n(S) = 0$ for all $n \geq 2$.*

*If $S$ is geometrically ruled, then $K_s^2 = 8(1 - g(C))$, $b_2(S) = 2$.*

### 2.1.2. Surfaces with $\kappa = 0$

For a thorough treatment of this kind of surfaces consult [9, Chapter VIII]. Note that all surfaces of $\kappa = 0$ to be discussed below are minimal by definition.

**Definition 19.** A torus $T$ is a surface, isomorphic to the quotient of $\mathbb{C}^2$ by a lattice of real rank four. If $T$ admits an embedding into projective space, we say it is an abelian surface.

Note that in this case we have $K_X^2 = c_1^2 = 0$, and this is true for all minimal surfaces with $\kappa = 0$. We also know that $p_g(X) = 1$ and $q(X) = 2$. From $\chi(\mathcal{O}_X) = 1 - q(X) + p_g(X)$ and Noether's formula (Theorem 11), it follows that the topological Euler–Poincaré characteristic, $c_2(X)$ of a minimal abelian surface is 0.

**Definition 20.** An Enriques surface is a surface $X$ with $q(X) = 0$, for which $K_X^{\otimes 2} \cong \mathcal{O}_X$, but $K_X \neq \mathcal{O}_X$.

We also know that $p_g = 0$: hence a similar argument to the one given for the abelian surfaces implies that the topological Euler–Poincaré characteristic $c_2(X)$ of a minimal Enriques surface is 12.

**Definition 21.** A hyperelliptic surface is a surface with $q(X) = 1$, admitting a holomorphic, locally trivial fibration over an elliptic curve with an elliptic curve as typical fibre. In particular every hyperelliptic surface is algebraic.

Here $p_g(X) = 0$, hence the topological Euler–Poincaré characteristic $c_2(X)$ of a minimal hyperelliptic surface is 0.

**Definition 22.** A K3 surface is a surface $X$ with $q(X) = 0$ for which $K_X = \mathcal{O}_X$.

Note that this definition means that $K_X^2 = c_1^2 = 0$. We also know that $p_g = 1$: in fact much more is true, namely $P_n = 1$ for all $n \geq 1$. From $\chi(\mathcal{O}_X) = 1 - q + p_g$ and $\chi(\mathcal{O}_X) = \frac{1}{12}\left(c_1^2(X) + c_2(X)\right)$ it follows that the topological Euler characteristic

or the Euler number of a minimal K3 surface is 24. Examples of these surfaces include the complete intersections $S_4 \subset \mathbb{P}^3$, $S_{2,3} \subset \mathbb{P}^4$ and $S_{2,2,2} \subset \mathbb{P}^5$. In 1979 Reid found 95 families of K3 hypersurfaces which are now referred to as the famous 95 and a recent proof by Johnson and Kollár [35][1] reaffirms that this list is complete, i.e. these are the only K3 hypersurfaces in weighted $\mathbb{P}^3$. But they are singular.

### 2.1.3. Surfaces with $\kappa = 1$

**Definition 23.** A properly elliptic surface is an elliptic surface with $\kappa = 1$.

**Remark 9.** *For more examples see [9, Examples IX.4, page 109]. In particular note that every Enriques surface is elliptic. We recall also that an elliptic fibration of a surface $X$ means a proper, connected holomorphic map $f : X \to S$, such that the general fibre $X_s = f^{-1}(s)$ is non-singular elliptic (the holomorphic structure may depend on $s \in S$.)*

### 2.1.4. Surfaces with $\kappa = 2$

These are called surfaces of general type. For some recent progress see [7]. Although classification of these type of surfaces is far from being fully understood, a lot is known about them. In particular we mention the well-known theorem of Gieseker:

**Theorem 13.** *There exists a quasi-projective coarse moduli scheme for the minimal surfaces of general type $X$ with fixed Chern numbers $c_1^2(X)$ and $c_2(X)$.*

Notice that since the numbers $c_1^2(X)$ and $c_2(X)$ are non-negative integers, we read this theorem to say that any given pair would define a moduli scheme, in particular it does not tell us to exclude the empty ones. A lot of work has been done to try and see which numbers are worth looking at.

**Remark 10.** *Enriques classified surfaces according to the plurigenus $P_{12}$. It seems that the terminology of Kodaira dimension was introduced by Shafarevich in his 1965 seminar. We have*

- $\kappa = -\infty \iff P_{12} = 0$.
- $\kappa = 0 \iff P_{12} = 1$
- $\kappa = 1 \iff P_{12} \geq 2$ *and* $K^2 = 0$
- $\kappa = 2 \iff P_{12} \geq 2$ *and* $K^2 > 0$

[1] reference kindly shown to us by Gavin Brown

Here we collect some classical inequalities which a minimal surface of general type must satisfy. Consult [7].

**Lemma 4.** *Let $X$ be a minimal general type surface. Then*

1. $K_X^2 \geq 1$; $\chi(\mathcal{O}_X) \geq 1$
2. *(Noether)* $K_X^2 \geq 2p_g - 4$ *or the weaker* $K_X^2 \geq 2\chi(\mathcal{O}_X) - 6$
3. *(Debarre) if* $q > 0$, $K_X^2 \geq 2p_g$ *or the weaker if* $q > 0$, $K_X^2 \geq 2\chi(\mathcal{O}_X)$
4. *(Bogomolov-Miyaoka-Yau)* $K_X^2 \leq 9\chi$
5. $\chi(\mathcal{O}_X) \leq \frac{1}{2}c_1^2 + 3$ $(c_1^2(X)$ *even*$)$
6. $\chi(\mathcal{O}_X) \leq \frac{1}{2}c_1^2 + \frac{5}{2}$ $(c_1^2(X)$ *odd*$)$
7. $c_2 > 0$
8. $c_1^2(X) + c_2(X) \equiv 0 \mod 12$

We also record Corollary (3.3) from [6].

**Corollary 1.** *If $X$ is a minimal surface of general type with*

$$5c_1^2(X) - c_2 + 36 = 0 \ (c_1^2(X) \text{ even})$$

*or*

$$5c_1^2(X) - c_2 + 30 = 0 \ (c_1^2(X) \text{ odd})$$

*then $q(X) = 0$.*

In the geography diagram (Figure 2-1) of minimal surfaces of general type, the coloured circles indicate the approximate position of the examples we found. The picture is not drawn to scale. Note that we mimic [7] and draw the Severi line $K^2 = 4\chi$, the discussion of which can be found in [7], section 1.4. We have also inserted the Konno and Horikawa lines (see Chapter 6.)

Beyond these numerical restrictions, the study of surfaces of general type largely consists of studying examples and goes under two names: Botany and Geography (see [25]).

It is customary to summarise the Enriques–Kodaira classification in a table.

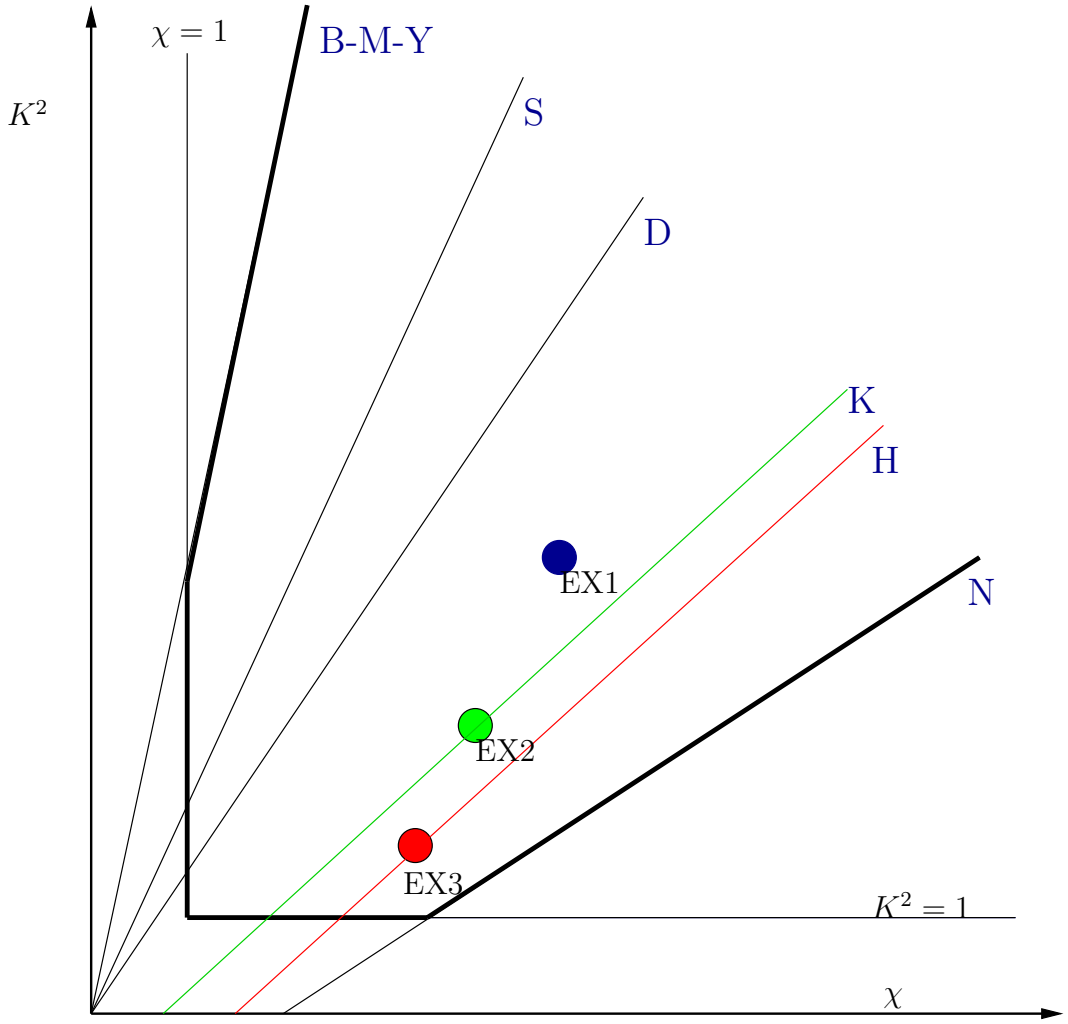


Figure 2-1: The geography diagram for minimal surfaces of general type.

Table 2.1: Enriques–Kodaira classification

| $\kappa$ | $p_g$ | $q$ | $c_1^2$ | $c_2$ | surface class |
|---|---|---|---|---|---|
| $-\infty$ | 0 | 0 | 8 or 9 | 4 or 3 | Rational |
| $-\infty$ | 0 | $\pi$ | $8(1-\pi)$ | $4(1-\pi)$ | Ruled |
| 0 | 1 | 0 | 0 | 24 | $K3$ |
| 0 | 0 | 0 | 0 | 12 | Enriques |
| 0 | 1 | 2 | 0 | 0 | Abelian |
| 0 | 0 | 1 | 0 | 0 | Hyperelliptic |
| 1 | | | 0 | $\geq 0$ | Elliptic |
| 2 | | | $> 0$ | $> 0$ | General Type. |

Chapter 3

# Boundedness for surfaces in weighted $\mathbb{P}^4$

In this chapter we consider bounds on the degree of quasismooth non general type surfaces in weighted projective 4–space. We show that such a bound in terms of the weights exists, and compute an explicit bound in simple cases.

Ellingsrud and Peskine [24] proved that there exists an integer $d_0$ such that all smooth non general type surfaces in $\mathbb{P}^4$ have degree less than or equal to $d_0$. This motivated a search for such surfaces, partly by computational methods, and also an effort to find an effective bound on $d_0$, begun by Braun and Fløystad in [12]. As far as we know the smallest proven bound is 52 by Decker and Schreyer [19]. It is generally believed that the true bound is 15. Examples in degree 15: see for instance [4, 41].

Some of the methods used to find such surfaces are also applicable to surfaces in weighted projective spaces $\mathbb{P}^4(w)$. It is therefore natural to ask whether a similar bound can be found for the degree of quasismooth non general type surfaces in a weighted projective space with given weights. In this chapter we show that such a readily computable bound (of course depending on the weights) does exist, and we compute it in some cases.

To show that a bound exists all we need is a fairly simple adaptation of the way in which the results of [24] (or [12]) are applied. For a computable bound we use the results of [12] together with some information about the contribution from the singularities of the surface in $\mathbb{P}^4(w)$.

Our procedure is to exploit the representation of $\mathbb{P}^4(w)$ as a quotient of $\mathbb{P}^4$ by a finite group action. Starting with a quasismooth non general type surface $X$ in weighted projective 4–space $\mathbb{P}^4(w)$, we take its cover in $\mathbb{P}^4$. This will (usually) be of general type, but it will have invariants bounded in terms of those of $X$, and the results of [12] still apply in this situation.

## 3.1. Bounding the degrees

We fix weights $w = (w_0, w_1, w_2, w_3, w_4)$ with $w_i \in \mathbb{N}$: unless otherwise stated, $i$ and $j$ always denote indices in the range $0 \leq i,\ j \leq 4$. We may assume that any four of the $w_i$ are coprime: see for example [42, Definition 3.5 and Proposition 3.6]. Such weights are called well-formed: see Definition 10. Later we shall look in more detail at the case where the $w_i$ are pairwise coprime. We also order the weights so that $w_i \leq w_{i+1}$: in particular, the largest weight is $w_4$. We write $|w|$ for the sum of the weights, and $m$ for their product.

We recall that the weighted projective space $\mathbb{P}^4(w)$ of dimension 4 is defined to be the quotient $(\mathbb{C}^5 \setminus \{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts by

$$t\colon (x_0, \ldots, x_4) \to (t^{w_0}x_0, \ldots, t^{w_4}x_4).$$

Recall from Definition 12 that a surface $X \subset \mathbb{P}^4(w)$ is said to be quasismooth if its punctured affine cone $X^*$ is smooth: that is, if $X^* = q^{-1}(X)$ is smooth, where $q\colon (\mathbb{C}^5 \setminus \{0\})/\mathbb{C}^* \to \mathbb{P}^4(w)$ is the quotient map.

Alternatively we may regard $\mathbb{P}^4(w)$ as a quotient of $\mathbb{P}^4$ under an action of the group $G_w = \prod_i \mathbb{Z}/w_i\mathbb{Z}$ of order $m$. A generator $g_i$ of the $i$th factor acts by $x_i \mapsto x_i^{w_i}$. We denote the quotient map $\mathbb{P}^4 \to \mathbb{P}^4(w)$ by $\phi_w$.

Suppose that $X$ is a quasismooth surface, not of general type, in $\mathbb{P}^4(w)$. Denote by $\widehat{X}$ the cover of $X$ in $\mathbb{P}^4$ under the $m$-to-1 map $\phi_w$: then $\widehat{X}$ is smooth. We always assume that $X$ and $\widehat{X}$ are nondegenerate: that is, $\widehat{X}$ is not contained in any hyperplane in $\mathbb{P}^4$.

Let $f\colon \widetilde{X} \to X$ be the minimal resolution of $X$ (note that $\widetilde{X}$ need not be a minimal surface).

$$\begin{array}{ccc} \widehat{X} & \subset & \mathbb{P}^4 \\ {\scriptstyle \phi_w}\downarrow & & \downarrow{\scriptstyle \phi_w} \\ \widetilde{X} & \xrightarrow{\ f\ } & X \subset \mathbb{P}^4(w) \end{array}$$

Further let $d$ be the degree of $X \subset \mathbb{P}^4(w)$ and $\pi$ the sectional genus of $X$. These are defined as follows: $\mathbb{P}^4(w)$ and $X$ are $\mathbb{Q}$-factorial varieties and there are $\mathbb{Q}$-line bundles $\mathcal{O}_{\mathbb{P}^4(w)}(1)$, $\mathcal{O}_X(1)$ and $K_X$. Writing $H$ for the class of $\mathcal{O}_X(1)$ in $\operatorname{Pic} X \otimes \mathbb{Q}$ and using the intersection form on $\operatorname{Pic} X$ we have $d = H^2$ and $2\pi - 2 = H \cdot (H + K_X)$, so $d,\ \pi \in \mathbb{Q}$.

We let $\widehat{d}$ be the degree of $\widehat{X}$ and $\widehat{\pi}$ the sectional genus of $\widehat{X}$. We put

$$\widehat{s} = \min\left\{k | h^0\mathcal{I}_{\widehat{X}}(k) \neq 0\right\}$$

and denote by $\sigma_f$ the number of irreducible exceptional curves of $f$.

We first collect the facts about these invariants of the smooth surface $\widehat{X} \subset \mathbb{P}^4$.

**Proposition 5.** *If $\widehat{X} \subset \mathbb{P}^4$ is a smooth surface (possibly of general type), and $r \leq \widehat{s}$ and $r^2 < \widehat{d}$, then*

$$2\widehat{\pi} \leq \frac{\widehat{d}^2}{r} + (r-4)\widehat{d} + 1. \tag{3.1}$$

*Moreover*

$$\widehat{d}^2 - 5\widehat{d} - 10(\widehat{\pi} - 1) + 12\chi\left(\mathcal{O}_{\widehat{X}}\right) - 2K_{\widehat{X}}^2 = 0. \tag{3.2}$$

*Finally, if $\widehat{d} > \widehat{s}(\widehat{s}-1)$ we have the lower bound for $\chi\left(\mathcal{O}_{\widehat{X}}\right)$*

$$\begin{aligned} \chi\left(\mathcal{O}_{\widehat{X}}\right) \quad \geq \quad & \frac{\widehat{d}^3}{6\widehat{s}} + \widehat{d}^2\left(\frac{\widehat{s}-5}{4\widehat{s}}\right) + \widehat{d}\left(\frac{3\widehat{s}^2 - 30\widehat{s} + 71}{24}\right) \\ & - \frac{\widehat{s}^4 - 5\widehat{s}^3 - \widehat{s}^2 + 5\widehat{s}}{24} - \frac{\gamma^2}{2} - \gamma\left(\frac{\widehat{d}}{\widehat{s}} + \widehat{s} - \frac{5}{2}\right) \end{aligned} \tag{3.3}$$

*where $0 \leq \gamma \leq \widehat{d}(\widehat{s}-1)^2/2\widehat{s}$.*

*Proof:* The inequality (3.1) is a consequence of [24, (B), (C), page 2]. Let $\widehat{H}$ denote a general hyperplane section of $\widehat{X}$, so that $\widehat{\pi} = g(\widehat{H})$. According to [43] (as quoted in [24, (C), page 2]), if $\widehat{s} > r$ and $\widehat{d} > r^2$ then $\widehat{H} \subset \mathbb{P}^3$ does not lie on any surface of degree $< r$. Therefore, according to [24, (B), page 2], we have $r(2\widehat{\pi} - 2) \leq \widehat{d}^2 + r(r-4)\widehat{d}$. If $\widehat{s} = r$ then (again by [24, (B), page 2]) we have the same inequality because then $\widehat{H}$ does lie on a surface of degree $r$.

Equation (3.2) is the double point formula as stated in [12] and [24]: see [29, page 434]. The estimate (3.3) is [12, (1.1)(e)].

A more precise version of (3.1), valid under certain conditions, is given in [12, (1.1)]. In order to bound the degree of smooth surfaces in $\mathbb{P}^4$ what is needed is not the precise form of (3.3) but an estimate of the form $\chi\left(\mathcal{O}_{\widehat{X}}\right) \geq a(\widehat{s})\widehat{d}^3 + o(\widehat{d}^3)$, where $a(\widehat{s})$ is some positive constant depending on $\widehat{s}$ only. Ellingsrud and Peskine proved the existence of such a bound in [24] but did not give an explicit one.

It will be convenient to work with the invariants $c_1^2(S) = K_S^2$ and $c_2(S)$ (which is the topological Euler number $e(S)$) of a smooth projective surface $S$: these are connected by Noether's formula (Theorem (11))

$$12\chi(\mathcal{O}_S) = c_1^2(S) + c_2(S) \tag{3.4}$$

Since we are assuming that $\widetilde{X}$ is not of general type we have (as in [12] and [24] ) that $K^2_{\widetilde{X}} \leq 9$. Moreover, unless $\widetilde{X}$ is a rational surface with $K^2_{\widetilde{X}} \geq 6$ we also have $6\chi\left(\mathcal{O}_{\widetilde{X}}\right) \geq K^2_{\widetilde{X}}$ (i.e. $c_2(\widetilde{X}) - c_1^2(\widetilde{X}) \geq 0$). If $\widetilde{X}$ is a rational surface then $\chi\left(\mathcal{O}_{\widetilde{X}}\right) = 1$ so $c_2(\widetilde{X}) - c_1^2(\widetilde{X}) = 12\chi\left(\mathcal{O}_{\widetilde{X}}\right) - 2K^2_{\widetilde{X}} = 12 - 2K^2_{\widetilde{X}} \geq -6$. So in any case if $X$ is not of general type we have

$$c_1^2(\widetilde{X}) - c_2(\widetilde{X}) \leq 6. \tag{3.5}$$

So we need to estimate $\widehat{d}$ and $\widehat{\pi}$ in terms of $d$ and $\pi$, and $K^2_{\widehat{X}}$ and $\chi\left(\mathcal{O}_{\widehat{X}}\right)$ in terms of $K^2_{\widetilde{X}}$ and $\chi\left(\mathcal{O}_{\widetilde{X}}\right)$. We shall show the two propositions below.

**Proposition 6.** *Suppose $X$ is a quasismooth normal surface in $\mathbb{P}^4(w)$. Then*

$$c_1^2(\widehat{X}) \leq m c_1^2(\widetilde{X}) + \theta_1 \tag{3.6}$$

*where*

$$\theta_1 = k_0 + k_1\widehat{d} + k_2\widehat{\delta} \tag{3.7}$$

*for suitable $k_0,\ k_1,\ k_2$ depending only on the weights $w_i$. Moreover*

$$c_2(\widehat{X}) \geq m c_2(\widetilde{X}) - \theta_2, \tag{3.8}$$

*and*

$$\theta_1 + \theta_2 = k_0' + k_1'\widehat{d} + k_2'\widehat{\delta} \tag{3.9}$$

*for suitable $k_0',\ k_1',\ k_2'$ depending only on the weights, and $k_2' > -5$.*

This proposition will be proved in Sections 3.3 and 3.4, below.

Our main qualitative result is then the following.

**Theorem 14.** *There exists $d_w \in \mathbb{N}$ depending only on the weights $w_i$ such that any quasismooth normal surface $X \in \mathbb{P}^4(w)$ of degree $d > d_w$ is of general type.*

*Proof:* We have seen that $\widehat{X} \to X$ is $m$-to-1, so

$$\widehat{d} = md \tag{3.10}$$

so it is sufficient to show that if $X$ is not of general type then $\widehat{d}$ is bounded by a function of the weights.

Suppose then that $X$ is not of general type. We have, by adjunction, $2\widehat{\pi} - 2 = \widehat{H} \cdot (\widehat{H} + K_{\widehat{X}}) = \widehat{d} + \widehat{\delta}$, where $\widehat{H}$ is a hyperplane section of $\widehat{X}$. Therefore by the estimate

(3.1) we obtain

$$\widehat{\delta} \leq \frac{1}{r}\widehat{d}^2 + (r-5)\widehat{d} - 1 \tag{3.11}$$

as long as $r \leq \widehat{s}$ and $r^2 < \widehat{d}$. We may also write the double point formula as

$$\widehat{d}^2 - 10\widehat{d} - 5\widehat{\delta} + c_2(\widehat{X}) - c_1^2(\widehat{X}) = 0. \tag{3.12}$$

By Proposition 6 and the inequality (3.5) we have

$$c_2(\widehat{X}) - c_1^2(\widehat{X}) \geq -6m - (\theta_1 + \theta_2), \tag{3.13}$$

so

$$\begin{aligned} 0 &\geq \widehat{d}^2 - 10\widehat{d} - 5\widehat{\delta} - 6m - (\theta_1 + \theta_2) \\ &= \widehat{d}^2 - (10 + k_1')\widehat{d} - (6m + k_0') - (5 + k_2')\widehat{\delta}. \end{aligned} \tag{3.14}$$

Combining this with (3.11) gives (since $5 + k_2' > 0$)

$$\begin{aligned} 0 &\geq \widehat{d}^2 - (10 + k_1')\widehat{d} - (6m + k_0') - (5 + k_2')(\frac{1}{r}\widehat{d}^2 + (r-5)\widehat{d}) \\ &= \left(1 - \frac{5 + k_2'}{r}\right)\widehat{d}^2 - ((10 + k_1' + (5 + k_2')(r-5))\widehat{d} - (6m + k_0'). \end{aligned}$$

So if $\widehat{s} > k_2' + 5$ we may take $r = k_2' + 6$ and this bounds $\widehat{d}$ in that case.

On the other hand, suppose that $X$ is not of general type and $\widehat{s} \leq k_2' + 5$. Then using Noether's formula, the double point formula (3.12), and (3.3) we have

$$\begin{aligned} 0 &= \widehat{d}^2 - 10\widehat{d} - 5\widehat{\delta} + 12\chi\left(\mathcal{O}_{\widehat{X}}\right) - 2c_1^2(\widehat{X}) \\ &\geq -2c_1^2(\widehat{X}) + \frac{2}{\widehat{s}}\widehat{d}^3 + O(\widehat{d}^2) \\ &\geq -2mc_1^2(\widetilde{X}) - \theta_1 + \frac{2}{\widehat{s}}\widehat{d}^3 + O(\widehat{d}^2) \\ &\geq \frac{2}{\widehat{s}}\widehat{d}^3 + O(\widehat{d}^2) - 18m - k_0 - k_1\widehat{d} - k_2\widehat{\delta} \\ &= \frac{2}{\widehat{s}}\widehat{d}^3 + O(\widehat{d}^2) \end{aligned}$$

by (3.7) and (3.11): the constants depend on $\widehat{s}$ but this is now bounded in terms of the weights. So again we obtain a bound for $\widehat{d}$ in terms of the $w_i$.

## 3.2. Singularities of $\mathbb{P}^4(w)$ and of $X$

In this section we collect some preliminary information about the action of $G_w$ on $\mathbb{P}^4$ and on $\widehat{X}$. We choose an isomorphism $G_w \tilde{\to} \prod \mathbb{Z}/w_i\mathbb{Z}$ by choosing generators $g_i \in G_w$ of order $w_i$. The singularities arise at fixed points of the $G_w$-action, so let us consider those.

Suppose that $x = (x_0 : \ldots : x_4) \in \mathbb{P}^4$ is fixed by $g = g_0^{a_0} \ldots g_4^{a_4}$. Without loss of generality we take $x_0 = 1$: then for $j \neq 0$ we have $\zeta_0^{-a_0}\zeta_j^{a_j} = 1$, where $\zeta_j = e^{2\pi i/w_j}$.

**Lemma 5.** *If $x \in \mathbb{P}^4$ is fixed by a non-trivial element of $G_w$, then $x$ lies in a coordinate linear subspace $\mathbb{P}_J$ given by $\mathbb{P}_J = \{x_j = 0 \mid j \in J \subset \{0, \ldots, 4\}\}$. The stabiliser of a general point of $\mathbb{P}_J$ is the group $\Gamma_J$ generated by the $g_j$ for $j \in J$ and the element $g_J = \prod_{i \notin J} g_i^{w_i/r_J}$, where $r_J = \mathrm{hcf}(a_i \mid i \notin J)$.*

This is immediate from the description of the action above. By a general point in $\mathbb{P}_J$ is meant, in this case, a point that is not in $\mathbb{P}_{J'}$ for any $J' \supset J$.

**Lemma 6.** *The singularities of $X$ are cyclic quotient singularities whose order divides one of the weights.*

*Proof:* At a fixed point $x \in \mathbb{P}^4$, the elements $g_j \in \Gamma_J$ act on the tangent space by quasi-reflections: the $j$th eigenvalue is $\zeta_j^{a_j}$ and the others are 1. So the quotient by the subgroup $\Gamma'_J$ generated by those elements is smooth, and the singularity of $\mathbb{P}_w$ or of $X$ at $z = \phi_w(x)$ is a quotient by the action of the cyclic group generated by $g_J$. The order of this element, or of its image in $\Gamma_J/\Gamma'_J$, is $r_J$, which divides $w_i$ for $i \notin J$.

**Remark 11.** *If $\#J = 1$ then $r_J = 1$ since the weights are well-formed, so the general point of a coordinate hyperplane in $\mathbb{P}^4(w)$ is smooth. For each $i$, the number of singular points of $X$ with $z_i = 0$ is at most $\widehat{d}$.*

**Remark 12.** *If the weights are pairwise coprime then the singularities occur at the points $P_0 = (1 : 0 : \ldots : 0), \ldots, P_4 = (0 : \ldots : 0 : 1) \in \mathbb{P}^4(w)$, and the singularity of $\mathbb{P}_w$ at $P_i$ has order exactly $w_i$. If $X \ni P_i$ then $X$ also has a cyclic quotient singularity of order $w_i$ at $P_i$.*

**Lemma 7.** *Suppose that $(Y, 0)$ is a nondegenerate smooth surface germ in $(\mathbb{A}^4, 0)$ with coordinates $t_1, \ldots, t_4$ at $0 \in \mathbb{A}^4$. Let $\gamma$ be the quasi-reflection $\gamma(t_1) = \xi t_1$, where $\xi$ is a primitive $n$th root of unity, and that $Y$ is $\gamma$-invariant. Then $Y$ meets $A = (t_1 = 0)$ transversely.*

*Proof:* Suppose not: then $T_{Y,0} \subset A$. Therefore the ideal $\mathcal{I}_{Y,0} \subset \mathcal{O}_{\mathbb{A}^4,0}$ contains an element $f$ of the form $f = t_1 + h$ with $h \in \mathfrak{m}^2 \subset \mathcal{O}_{\mathbb{A}^4,0}$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{\mathbb{A}^4,0}$.

We write $h = \sum_{\nu=0}^{n-1} h_\nu$, where $\gamma(h_\nu) = \xi^\nu(h_\nu)$: if we write $h = \sum_r a_r(t_2,t_3,t_4)t_1^r$ as a polynomial in $t_1$ we have $h_\nu = \sum_{r \equiv \nu \bmod n} a_r t_1^r$. Then

$$\mathcal{I}_{Y,0} \ni (1 + \gamma + \gamma^2 + \cdots + \gamma^{n-1})(f) = nh_0$$

so $\mathcal{I}_{Y,0} \ni f - h_0 = t_1 + \sum_{\nu \neq 0} h_\nu$. But $t_1$ divides the right-hand side, so since $h \in \mathfrak{m}^2$ we have $f - h_0 = t_1(1+b)$, where $b \in \mathfrak{m}$. Since $\mathcal{I}_{Y,0}$ is a prime ideal contained in $\mathfrak{m}$ this implies $t_1 \in \mathcal{I}_{Y,0}$, contradicting the nondegeneracy.

**Corollary 2.** *If $w_i \neq 1$, then $\widehat{X}$ meets the ramification divisor $\mathbb{P}_{\{i\}}$ transversely and the curve $\widehat{C}_i = \widehat{X} \cap \mathbb{P}_{\{i\}}$ is a smooth curve of genus $\widehat{\pi}$.*

*Proof:* The second part follows immediately from the first, which is immediate from Lemma 7.

## 3.3. Comparing $c_1^2$.

In this section we prove (3.6) and (3.7) from Proposition 6, and give values for the constants $k_0$, $k_1$ and $k_2$.

Let $\Delta = \sum_{1 \leq \nu \leq \sigma_f} a_\nu E_\nu$ be the discrepancy of $f$, so that $a_\nu \in \mathbb{Q}$ and $K_{\widetilde{X}} = f^*K_X + \Delta$. Then $f^*K_X \cdot \Delta$ vanishes and $(f^*K_X)^2 = K_X^2$, so $K_{\widetilde{X}}^2 = K_X^2 + \Delta^2$.

**Lemma 8.** *If $f_0 \colon \tilde{Y} \to Y$ is the minimal resolution of a isolated cyclic quotient $(Y,0)$ of order $n$ and the discrepancy of $f_0$ is $\Delta_0$, then $0 > \Delta_0^2 \geq -n$.*

*Proof.* This (which is not a sharp bound) is most easily seen by toric methods. If the singularity is $\frac{1}{n}(1,a)$ with $(n,a) = 1$ then the minimal resolution is described by taking the decomposition given by the convex hull of $\mathbb{Z}^2 + \frac{1}{n}(1,a)\mathbb{Z}$ in the first quadrant of $\mathbb{R}^2$. The exceptional curves $E_\nu$, $0 < \nu < k$, correspond to primitive vectors $P_\nu = (x_\nu, y_\nu)$ of this lattice: put $\ell_\nu = x_\nu + y_\nu$, and write $E_0$ and $E_k$ for the toric curves corresponding to the rays spanned by $(1,0)$ and $(0,1)$. Then we have $E_\nu E_{\nu\pm1} = 1$ and $E_\mu E_\nu = 0$ if $\mu \neq \nu,\ \nu \pm 1$. Moreover on $\tilde{Y}$ we have $\sum_{0 \leq \nu \leq k} \ell_\nu E_\nu \equiv 0$ (linear equivalence), and

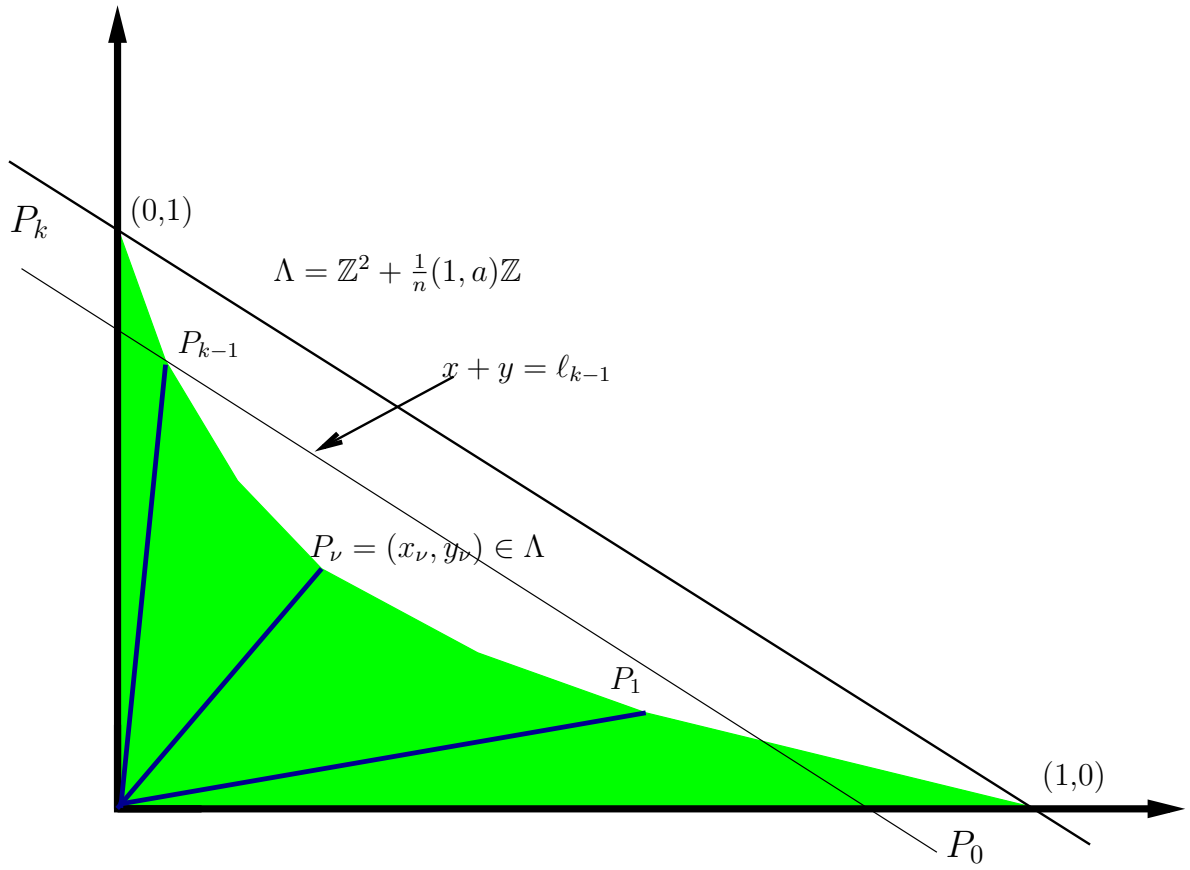


Figure 3-1: The toric resolution.

$\Delta = -\sum_{0<\nu<k} E_\nu$. Therefore

$$\begin{aligned}
\Delta_0^2 &= \sum_{0<\nu<k} E_\nu(\sum_{0<\mu<k} E_\mu) \\
&= \sum_{0<\nu<k} E_\nu\Big((\sum_{\mu\neq 0,\ \nu,\ k} E_\mu) + E_\nu\Big) \\
&= \sum_{0<\nu<k} E_\nu(-E_0 - E_k + \sum_{\mu\neq\nu}(1 - \frac{\ell_\mu}{\ell_\nu})E_\mu) \\
&= -2 - \sum_{0<\nu<k}((\frac{\ell_{\nu-1}}{\ell_\nu} - 1) + (\frac{\ell_{\nu+1}}{\ell_\nu} - 1))
\end{aligned}$$

Suppose for definiteness that $\ell_{nu+1} > \ell_\nu$. Then $\frac{\ell_{\nu+1}}{\ell_\nu} - 1$ is twice the area (relative

to the lattice $\Lambda = \mathbb{Z}^2 + \frac{1}{n}(1,a)\mathbb{Z})$ of the triangle $T_\nu^+ = P_\nu Q_\nu P_{\nu+1}$, where $Q_\nu = \frac{\ell_{\nu+1}}{\ell_\nu} P_\nu$, since $\mathrm{Area}(OP_\nu P_{\nu+1}) = \frac{1}{2}$ relative to $\Lambda$: see Figure 3-2. So

$$\begin{aligned} -\frac{1}{2}\Delta_0^2 &\leq -1 - \sum_{0<\nu<k} \mathrm{Area}(T_\nu^+) \\ &= -\mathrm{Area}(OP_0P_1) - \mathrm{Area}(OP_{k-1}P_k) - \sum_{0<\nu<k} \mathrm{Area}(T_\nu^+). \end{aligned}$$

But these triangles do not overlap and they are contained in the unit triangle $OP_0P_k$, which has area $\frac{n}{2}$ relative to $\Lambda$.

□

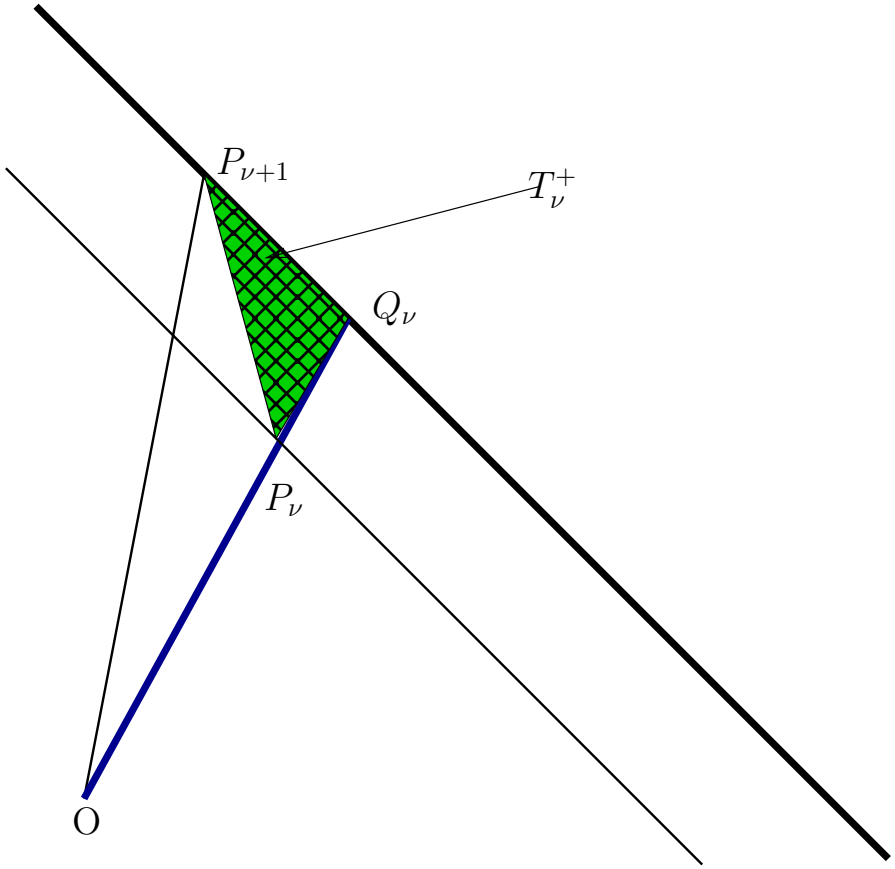


Figure 3-2: $\Delta_0^2$.

Now we compute $K_{\widehat{X}}^2$ from $K_{\widehat{X}} = \phi_w^*(K_X) + \sum(w_i - 1)\widehat{H}_i$, where $\widehat{H}_i = \mathbb{P}_{\{i\}} \cap \widehat{X} = (x_i = 0)$ and so we get

$$K^2_{\widehat{X}} = mK^2_X + 2\sum(w_i-1)\widehat{\delta} - \sum(w_i-1)(w_j-1)\widehat{d} \tag{3.15}$$

since $\phi^*_w(K_X)^2 = mK^2_X$.

**Proposition 7.** *We have $c_1^2(\widehat{X}) \le mc_1^2(\widetilde{X}) + \theta_1$, where (recall that $w_4$ is the largest weight)*

$$\theta_1 = \Big(10mw_4 - \sum_{0\le i,j\le 4}(w_i-1)(w_j-1)\Big)\widehat{d} + 2(|w|-5)\widehat{\delta}. \tag{3.16}$$

*Proof:* For a singular point $z \in \mathrm{Sing}(X)$ we denote the discrepancy at $z$ by $\Delta_z$. If $z \in H_J = \phi_w(\mathbb{P}_J) \cap X$ then the order of the singularity is $r_J = \mathrm{hcf}(w_i \mid i \notin J)$. There are at most $\frac{1}{2}\binom{5}{2}\widehat{d}$ distinct points on the $\widehat{H}_{\{ij\}}$ altogether, so the total number of singular points is at most $10\widehat{d}$.

Each singular point has order $r_J$ dividing some of the $w_i$, so $\Delta_z^2 \ge -r_J \ge -w_4$. Then

$$\begin{aligned} c_1^2(\widetilde{X}) = K^2_{\widetilde{X}} &= K^2_X + \Delta^2 \\ &= K^2_X + \sum_{z\in\mathrm{Sing}(X)} \Delta_x^2 \\ &\ge K^2_X - 10w_4\widehat{d}. \end{aligned}$$

Now, using (3.15), we get

$$\begin{aligned} c_1^2(\widehat{X}) &= mK^2_X + 2\widehat{\delta}\big(|w|-5\big) - \widehat{d}\sum_{0\le i,j\le 4}(w_i-1)(w_j-1) \\ &\le mc_1^2(\widetilde{X}) + 2\widehat{\delta}\big(|w|-5\big) + \widehat{d}\Big(10mw_4 - \sum_{0\le i,j\le 4}(w_i-1)(w_j-1)\Big) \end{aligned}$$

as required.

If the $w_i$ are pairwise coprime we can do slightly better. In that case the only singularities are at the points $P_i$ if they are in $X$. Therefore we have

$$c_1^2(\widetilde{X}) = K^2_X + \sum_{P_i\in X}\Delta_i^2 \ge K^2_X - \sum_i q_i w_i, \tag{3.17}$$

where $\Delta_i$ is the discrepancy at $P_i$ and $q_i = 1$ if $P_i \in X$, $q_i = 0$ if $P_i \notin X$. This gives

$$c_1^2(\widehat{X}) \le mc_1^2(\widetilde{X}) + m\sum q_i w_i + 2(|w|-5)\widehat{\delta} - \widehat{d}\sum_{0\le i,j\le 4}(w_i-1)(w_j-1). \tag{3.18}$$

## 3.4. Comparing $c_2$

Recall that if $x \in \widehat{X} \cap \mathbb{P}_J$ then $\Gamma_J$ stabilises $x$. We put $\widehat{X}_J = \widehat{X} \cap (\mathbb{P}_J \setminus \bigcup_{J' \supsetneq J} \mathbb{P}_{J'})$. That is, at the points $x \in \widehat{X}_J$ we have $x_i = 0$ if and only if $i \in J$. On $\widehat{X}_J$ the stabiliser is precisely $\Gamma_J$. The order of $\Gamma_J$ is $h_J = r_J \prod_{j \in J} w_j$: in particular, $h_\emptyset = 1$ and $h_{\{i\}} = w_i$.

$\widehat{X}_{\{i\}}$ is the complement of up to $4\widehat{d}$ points on a smooth curve of genus $\widehat{\pi}$, by Corollary 2. Those points lie in some $\widehat{X}_J$ with $\#J \geq 2$: in particular they all lie on $\widehat{H}_j$ for some $j \neq i$, and there are $\widehat{d}$ such points for each such $j$. They may not all be distinct, however. Therefore

$$2 - 2\widehat{\pi} > e(\widehat{X}_{\{i\}}) \geq 2 - 2\widehat{\pi} - 4\widehat{d}. \tag{3.19}$$

Denote by $\mathcal{Q}$ the set of points of $\widehat{X}$ lying in at least two coordinate hyperplanes of $\mathbb{P}^4$: thus $\mathcal{Q} = \widehat{X} \cap \bigcup_{\#J \geq 2} \mathbb{P}_J$ as a set. The set $\mathcal{Q}$ is finite, of cardinality $q \leq 10\widehat{d}$, and $\widehat{X} = \widehat{X}_\emptyset \coprod \bigcup_i \widehat{X}_{\{i\}} \coprod \mathcal{Q}$.

We put $X_J = \phi_w(\widehat{X}_J)$, for $J \subset \{0, \ldots, 4\}$, so that

$$\phi_w|_{\widehat{X}_J} \colon \widehat{X}_J \longrightarrow X_J$$

is unramified and its degree is $|G_w : \Gamma_J| = m/r_J$.

**Lemma 9.** *For each $x \in \mathcal{Q}$, let $r_x$ be the order of the singularity of $z = \phi_w(x) \in X$, so $r_x = r_J$ if $x \in \widehat{X}_J$. Then*

$$c_2(\widetilde{X}) \leq e(X) + \sum_{x \in \mathcal{Q}} (r_x - 1).$$

*Proof:* The resolution $f \colon \widetilde{X} \to X$, in a neighbourhood of $z$, consists of a sequence of at most $r_x - 1$ blow-ups, needed to resolve the quotient singularity of order $r_x$ at $z \in X$. Therefore $\sigma_f \leq \sum_{x \in \mathcal{Q}} (r_x - 1)$. Each blow-up contracts a smooth rational curve: topologically, therefore, $f$ contracts $\sigma_f$ 2–spheres to points, and each of these contractions reduces the Euler characteristic by 1, so $e(\widetilde{X}) = e(X) + \sigma_f \leq e(X) + \sum_{x \in \mathcal{Q}} (r_x - 1)$.

**Proposition 8.** *We have $c_2(\widehat{X}) \geq m c_2(\widetilde{X}) - \theta_2$, where (recall that $w_4$ is the largest weight)*

$$\theta_2 = \Big(10 m w_4 - (|w| - 5)\Big)\widehat{d} - (|w| - 5)\widehat{\delta}. \tag{3.20}$$

*Proof:* By the additiviity of Euler characteristic we have

$$\begin{aligned} c_2(\widehat{X}) &= \sum_J e(\widehat{X}_J) \\ &= \sum_J |G_w : \Gamma_J| e(X_J) \\ &= me(X_\emptyset) + \sum_{J\neq\emptyset} |G_w : \Gamma_J| e(X_J) \\ &= m\Big(e(X) - \sum_{J\neq\emptyset} e(X_J)\Big) + \sum_{J\neq\emptyset} |G_w : \Gamma_J| e(X_J) \\ &= me(X) + \sum_{J\neq\emptyset} (1-h_J)|G_w : \Gamma_J| e(X_J) \\ &= me(X) + \sum_{J\neq\emptyset} (1-h_J) e(\widehat{X}_J). \end{aligned}$$

Write $h_x = h_J$ if $x \in \widehat{X}_J$. Using $\widehat{X} = \widehat{X}_\emptyset \coprod \bigcup_i \widehat{X}_{\{i\}} \coprod \mathcal{Q}$ and Lemma 9, this gives

$$\begin{aligned} c_2(\widehat{X}) &= me(X) - \sum_i (w_i - 1) e(\widehat{X}_{\{i\}}) - \sum_{x\in\mathcal{Q}} (h_x - 1) \\ &\geq mc_2(\widetilde{X}) - \sum_i (w_i - 1) e(\widehat{X}_{\{i\}}) - m\sum_{x\in\mathcal{Q}} (r_x - 1) - \sum_{x\in\mathcal{Q}} (h_x - 1) \\ &\geq mc_2(\widetilde{X}) - (|w| - 5)(2 - 2\widehat{\pi}) - m\sum_{x\in\mathcal{Q}} (r_x - 1) - \sum_{x\in\mathcal{Q}} (h_x - 1) \\ &= mc_2(\widetilde{X}) + (|w| - 5)(\widehat{d} + \widehat{\delta}) - m\sum_{x\in\mathcal{Q}} (r_x - 1) - \sum_{x\in\mathcal{Q}} (h_x - 1) \\ &\geq mc_2(\widetilde{X}) + (|w| - 5)(\widehat{d} + \widehat{\delta}) - 10mw_4\widehat{d}, \end{aligned}$$

as claimed, since $q \leq 10\widehat{d}$, $r_x \leq w_4$ and $h_x \leq m$.

We can now complete the proof of Proposition 6 and hence of Theorem 14, by remarking that from Propositions 7 and 8 we get

$$\theta_1 + \theta_2 = \Big(20mw_4 - (|w| - 5) - \sum (w_i - 1)(w_j - 1)\Big)\widehat{d} + (|w| - 5)\widehat{\delta}$$

so $k_2' = |w| - 5 > -5$.

## 3.5. Examples

It would of course be possible to obtain an explicit bound as in Theorem 14 from the argument above. However, such a bound would be likely to be rather poor. In specific cases it is possible to obtain a bound better than the general one implied above. Although we still do not expect such a bound to be good, in the sense that we expect that in fact all non general type surfaces will be of much lower degree, in some cases it is not absurdly big.

### 3.5.1. Weights $(1,1,1,1,2)$

We calculate a bound for the case of weights $(1,1,1,1,2)$. In this case there is at most one singular point of $X$ and if there is a singular point it is an ordinary double point. We let $q$ be the number of singularities of $X$, so $q=0$ or $q=1$.

In this case the singularity, if any, is canonical and blowing up once gives a crepant resolution, so $\Delta^2=0$ and $c_1^2(\widetilde{X})=K_X^2$. Moreover $K_{\widehat{X}}=\phi^*K_X+\widehat{H}$, so

$$\begin{aligned} c_1^2(\widehat{X}) &= (\phi^*K_X+\widehat{H})^2 \\ &= 2K_X^2+2\phi^*K_X\widehat{H}+\widehat{H}^2 \\ &= 2c_1^2(\widetilde{X})+2(K_{\widehat{X}}-\widehat{H})\widehat{H}+\widehat{H}^2 \\ &= 2c_1^2(\widetilde{X})-\widehat{d}+2\widehat{\delta}. \end{aligned}$$

We also have $c_2(\widetilde{X})=e(X)+q$ and

$$\begin{aligned} c_2(\widehat{X}) &= 2e(X)-\sum_i (w_i-1)e(\widehat{X}_{\{i\}})-\sum_{x\in\mathcal{Q}}(h_x-1) \\ &= 2e(X)-(2-2\widehat{\pi})-q \\ &= 2c_2(\widetilde{X})+\widehat{d}+\widehat{\delta}-3q. \end{aligned}$$

Thus $\theta_1=-\widehat{d}+2\widehat{\delta}$ and $\theta_2=3q-\widehat{d}-\widehat{\delta}$. Therefore $k_0'=3q$, $k_1'=-2$ and $k_2'=1$, and (3.14) and the formula below it give

$$0\geq\left(1-\frac{6}{r}\right)\widehat{d}^2-(6r-22)\widehat{d}-(12+3q)$$

as long as $r\geq\widehat{s}\geq 7$ and $r^2<\widehat{d}$. Taking $r=7$, we see that $\widehat{d}\leq 140$ in this case. (By taking $r=9$ we can obtain $\widehat{d}\leq 96$, but as we shall see that will not yield a better

bound in the end. Clearly taking $r \geq 10$ we cannot do better than $d \leq 100$ because for this case we need $r^2 < \widehat{d}$.)

We must also deal with the cases $\widehat{s} < 7$: if we use $r = 9$ we must also handle $\widehat{s} = 7$ and $\widehat{s} = 8$ separately. But now we have, using $c_1^2(\widetilde{X}) \leq 9$, the estimate (3.1) for $\widehat{\delta}$ with $r = \widehat{s}$, the bounds on $\chi\left(\mathcal{O}_{\widehat{X}}\right)$ and $\gamma$ from Proposition 5, and the double point formula

$$\begin{aligned}
0 &= \widehat{d}^2 - 10\widehat{d} + 12\chi\left(\mathcal{O}_{\widehat{X}}\right) - 2c_1^2(\widehat{X}) \\
&= \widehat{d}^2 - 10\widehat{d} + 12\chi\left(\mathcal{O}_{\widehat{X}}\right) - 4c_1^2(\widetilde{X}) + 2\widehat{d} = 4\widehat{\delta} \\
&\geq \widehat{d}^2 - 8\widehat{d} + 12\chi\left(\mathcal{O}_{\widehat{X}}\right) - 36 - \frac{4}{\widehat{s}}\widehat{d}^2 - 4(\widehat{s}-5)\widehat{d} + 4 \\
&\geq 12\Big[\frac{\widehat{d}^3}{6\widehat{s}} + \widehat{d}^2\Big(\frac{\widehat{s}-5}{4\widehat{s}}\Big) + \widehat{d}\Big(\frac{3\widehat{s}^2 - 30\widehat{s} + 71}{24}\Big) - \frac{\widehat{s}^4 - 5\widehat{s}^3 - \widehat{s}^2 + 5\widehat{s}}{24} \\
&\quad -\frac{1}{2}\Big(\frac{(\widehat{s}-1)^4}{4\widehat{s}^2}\Big)\widehat{d}^2 - \Big(\frac{(\widehat{s}-1)^2}{2\widehat{s}^2}\Big)\widehat{d}^2 - \Big(\frac{(\widehat{s}-5/2)(\widehat{s}-1)^2}{2\widehat{s}}\Big)\widehat{d}\Big] \\
&\quad +\widehat{d}^2(1 - \frac{4}{\widehat{s}}) + \widehat{d}(-8 - 4(\widehat{s}-5)) - 32 \\
&= \frac{2}{\widehat{s}}\widehat{d}^3 - \frac{3\widehat{s}^4 - 12\widehat{s}^3 + 22\widehat{s}^2 + 2\widehat{s} + 15}{2\widehat{s}^2}\widehat{d}^2 \\
&\quad -\frac{9\widehat{s}^3 - 16\widehat{s}^2 - 23\widehat{s} - 30}{2\widehat{s}}\widehat{d} - \frac{\widehat{s}^4 - 5\widehat{s}^3 - \widehat{s}^2 + 5\widehat{s} + 64}{2}
\end{aligned}$$

(for $\widehat{s} = 2$ the $-\Big(\frac{(\widehat{s}-5/2)(\widehat{s}-1)^2}{2\widehat{s}}\Big)\widehat{d}$ term should be omitted). It is easy to compute that this implies $\widehat{d} \leq 91$ for $\widehat{s} \leq 6$, but $\widehat{s} = 7$ we obtain only $\widehat{d} \leq 153$, so taking $r = 9$ does not improve the overall bound. Taking $r = 7$, we find the overall bound $\widehat{d} \leq 140$.

Generally we see from (3.3) that for large weights, and hence large $\widehat{s}$, the two biggest terms in absolute value in the cubic will be the $\widehat{d}^3$ term and a term $-\frac{\widehat{s}^2}{4}\widehat{d}^2$. Therefore the bound on $\widehat{d}$ will be around $\widehat{s}^3/8$, i.e. around $|w|^3/8$.

### 3.5.2. Weights $(1, 1, 1, 2, 6)$

As a further example, we calculate a bound for the case of weights $(1, 1, 1, 2, 6)$. In this case the possible singularities are: up to $\widehat{d}$ order 2 singularities (ordinary nodes) along $x_0 = x_1 = x_2 = 0$, with $r_x = h_x = 2$, and one singularity of order 6 at $(0 : 0 : 0 : 0 : 1)$, with $r_x = 6$, $h_x = 12$. At the double points, $\Delta_x^2 = 0$, and at the 6-fold point one has in fact $\Delta_x^2 \geq -\frac{8}{3}$.

In this case we have $K_{\widehat{X}}^2 = 12K_x^2 - 36\widehat{d} + 12\widehat{\delta}$, and $c_1^2(\widetilde{X}) = K_{\widetilde{X}}^2 - \sum_x \Delta_x^2 \geq K_x^2 - \frac{8}{3}$,

so $\theta_1 = 32 - 36\widehat{d} + 12\widehat{\delta}$. We also have

$$\begin{aligned} c_2(\widehat{X}) &= 12c_2(\widetilde{X}) + 6(\widehat{d}+\widehat{\delta}) - 12\sum_{x\in\mathcal{Q}}(r_x-1) - \sum_{x\in\mathcal{Q}}(h_x-1) \\ &\geq 12c_2(\widetilde{X}) + 6(\widehat{d}+\widehat{\delta}) - 12\widehat{d} - 60 - \widehat{d} - 11 \\ &= 2c_2(\widetilde{X}) - 71 - 7\widehat{d} + 6\widehat{\delta}. \end{aligned}$$

Thus $\theta_2 = 71 + 7\widehat{d} - 6\widehat{\delta}$. Therefore $k_0' = 103$, $k_1' = -29$ and $k_2' = 6$, and the quadratic is

$$0 \geq \left(1 - \frac{11}{r}\right)\widehat{d}^2 - (11r - 274)\widehat{d} - 175$$

as long as $r \geq \widehat{s} \geq 12$ and $r^2 < \widehat{d}$. Taking $r = 12$, we see that $\widehat{d} \leq 699$ in this case.

We must also deal with the cases $\widehat{s} < 12$ by using the cubic. For $\widehat{s} = 11$ we obtain $\widehat{d} \leq 710$: as this is already bigger than 699 it is no use looking at other choices for $r$. Smaller values of $\widehat{s}$ give smaller bounds, so the overall bound remains $\widehat{d} \leq 710$.

Chapter 4

# On Enriques surfaces

We illustrate the main ideas in our construction method by concentrating on the particular case of the weights (1,1,1,1,2). In this chapter any mention of $w$ shall be understood to mean $w = (1, 1, 1, 1, 2)$ which is the next simplest one after $\mathbb{P}^4$. The idea here is to predict some numerical invariants of a double cover of a non general type surface, if such a surface exists. Once we have the numerical invariants we use methods of sheaf cohomology and the Beilinson monad to find explicit generators of the twisted ideal sheaf of $\widehat{X}$. The next task is to prove that the corresponding variety is smooth.

The main result of this chapter is construction of a general type surface which possibly arises as a double cover of an Enriques surface blown up in three points in $\mathbb{P}^4(w)$. Although we manage to find a general type example that has a quotient that lives in $\mathbb{P}^4(w)$ we must point out that this in itself does not necessarily mean the quotient is an Enriques surface, nor even non general type for that matter. The point is that a quotient of a general type surface can very well be also of general type. Nevertheless, we still obtain an interesting example which can be studied further. We conjecture that it arises as a double cover of an Enriques surface but we do not know how to show this at the moment.

## 4.1. Fixing numerical invariants

The space $\mathbb{P}^4(w)$ has a single ordinary double point at the point $P_0 = (0 : 0 : 0 : 0 : 1)$.

Suppose $X$ is a quasismooth surface that passes through $P_0$, so it is singular only at $P_0$ and the singularity is inherited from the ambient space. As a first attempt, assume that $X$ is a minimal Enriques surface, that is, an Enriques surface that does not contain any $-1$-curves. Obviously we are abusing notation a little here, by using minimal for

a surface that still has a node. Used correctly this term, minimal, means a *smooth* algebraic surface that cannot be obtained from another smooth algebraic surface by blowing up a point. For us we are happy to use it in this case where we allow only one node and everywhere away from the node things are smooth and we cannot do any blowing down to smooth points.

First, we take an automorphism $\psi$ on $\mathbb{P}^4$ defined by

$$\psi : \mathbb{P}^4 \longrightarrow \mathbb{P}^4;$$

$$(z_0 : z_1 : z_2 : z_3 : z_4) \mapsto (-z_0 : -z_1 : -z_2 : -z_3 : z_4) = (z_0 : z_1 : z_2 : z_3 : -z_4)$$

whose fixed points are $(0 : 0 : 0 : 0 : 1)$ and $(z_4 = 0)$.

Now we take a map $\varphi$ defined by

$$\varphi : \mathbb{P}^4 \longrightarrow \mathbb{P}^4(w);$$

$$(z_0 : z_1 : z_2 : z_3 : z_4) \mapsto (z_0 : z_1 : z_2 : z_3 : z_4^2).$$

This map $\varphi$ is $2 : 1$. Therefore $\widehat{X}$ is a double cover of $X$, with branching along the hyperplane section $Q_0 = (z_4 = 0)$ and ramified at the singular point $P_0 = (0 : 0 : 0 : 0 : 1)$ of $\mathbb{P}^4$.

Recall that if this $\widehat{X}$ exists then it is a smooth surface, and $\widehat{X}$ is of general type because $K_{\widehat{X}} = \widehat{H}$ mod torsion. We study the numerical invariants of $\widehat{X}$. A general linear hyperplane section of $X$ will pass through the singular point of the ambient space (any linear section will not contain a term in $z_4$ in its defining equations because $z_4$ is already in degree 2.)

The simplest invariant first: from (3.10) in Chapter 3 we have $\widehat{d} = 2d$.

Since $X$ is not smooth by assumption, we can blow up at the singular point to obtain the minimal resolution $\sigma : \widetilde{X} \longrightarrow X$. Then $\widetilde{X}$ being a minimal smooth Enriques surface (by assumption) has $c_2(\widetilde{X}) = 12$. (see Chapter 2, section 2.1.2.)

We will prove that it is not possible to get any useful construction without assuming any $(-1)$-curves. Formally, we prove

**Proposition 9.** *There is no minimal nodal Enriques surface in* $\mathbb{P}^4(w)$ *with weights* $w = (1, 1, 1, 1, 2)$.

*Proof.* The proof is very simple. All we do is find all the numerical invariants of the required smooth surface in $\mathbb{P}^4$; having obtained these numbers we put them into the double point formula and observe that we do not get an integer solution. So we proceed as follows.

Let $E_0 \subset \widetilde{X}$ be the exceptional curve corresponding to $P_0$. That is, $E_0 = \sigma^{-1}(P_0)$ and let $Q_0 = (z_4 = 0)$. Then $X_0 = X \setminus P_0 = \widetilde{X} \setminus E_0$. Then the Euler number $e(X_0)$ of $X_0$ is given by $e(X_0) = e(\widetilde{X}) - e(E_0)$, because the Euler number is additive and we have already seen that $e(\widetilde{X}) = 12$ and we know that $e(E_0) = 2$, hence we obtain $e(X_0) = 10$.

From the virtual genus formula $-\chi(Q_0 \cap \widehat{X}) = C \cdot (C + K_{\widehat{X}})$, where $C = Q_0 \cap \widehat{X}$ and we know that $C^2 = \widehat{d}$, so we obtain $-\chi(Q_0 \cap \widehat{X}) = 2\widehat{d}$. That is, $-e(Q_0) = 2\widehat{d}$

We need some more notation. Let $X_{00} = X_0 \setminus Q_0$. Then $e(X_{00}) = 10 + 2\widehat{d}$. From $\widehat{X} = 2X_{00} \cup Q_0 \cup P_0$, and additivity of the Euler number we arrive easily at $e(\widehat{X}) = 21 + 2\widehat{d}$, which is actually $c_2(\widehat{X})$.

Now, recall that $K_X^2 = 0$ by assumption, so that together with the ramification formula $K_{\widehat{X}} = \varphi^* K_X + \widehat{H}$, it follows that $c_1^2(\widehat{X}) = K_{\widehat{X}}^2 = \widehat{H}^2 = \widehat{d}$. Let $\widehat{\pi}$ denote the sectional genus of $\widehat{X}$. Then from the adjunction formula $2\widehat{\pi} - 2 = \widehat{H} \cdot (\widehat{H} + K_{\widehat{X}})$ we obtain $\widehat{\pi} = \widehat{d} + 1$. Now, recall that the numerical invariants of a smooth surface in $\mathbb{P}^4$ must satisfy the double point formula (3.2) in Chapter 3.

Substituting all the values in the double point formula we obtain $\widehat{d}^2 - 14\widehat{d} + 21 = 0$, which clearly has no integer solutions for $\widehat{d}$. This completes the proof of Proposition 9. □

## 4.2. Nodal Enriques surfaces

Since we had no luck by assuming that we have a minimal surface to start with, let us modify our assumption a little and assume existence of some $(-1)$-curves on $X$.

So, let us begin: suppose instead that $X$ is a nodal Enriques surface that has $k$ disjoint $(-1)$-curves none of which passes through the node. Denote these curves by $E_1, \ldots, E_k$ and let $E_j \cdot H = \lambda_j$ be the degree of the curve $E_j$, where $H$ is a general hyperplane section in $X$. Then we note:

**Lemma 10.**

$$E_j^2 = -1,\ \widehat{E}_j^2 = 2E_j^2 = -2 \text{ and } \widehat{E}_i \cdot \widehat{E}_j = 0 \text{ whenever } i \neq j$$

*Proof.* Follows from definitions. □

Let us introduce some more notation before moving on. We let $\lambda = \sum_{j=1}^{k} \lambda_j$ denote the sum of the degrees of the $(-1)$-curves. Let $Q$ denote a general quadric section of $\widehat{X}$.

**Lemma 11.**

$$-\chi(Q \cap \widehat{X}) = 2\widehat{d} + 2\lambda$$

*Proof.* Let $Q \cap \widehat{X} = \widehat{C}$. Then by direct application of the virtual genus formula $-\chi(\widehat{C}) = \widehat{C} \cdot (\widehat{C} + K_{\widehat{X}})$ and Lemma 13 below we obtain the result. □

Now $\sigma : \widetilde{X} \to X$ is a resolution of $X$ but $\widetilde{X}$ is no longer minimal: rather, it is an Enriques surface with $k$ $(-1)$-curves on it, so we get

**Lemma 12.** $c_2(\widetilde{X}) = 12 + k$

**Lemma 13.** *Let $\widehat{H}$ denote the hyperplane section of $\widehat{X}$. Then the canonical class $K_{\widehat{X}}$ is given by $K_{\widehat{X}} = \widehat{H} + \sum_{j=1}^{k} \widehat{E}_j$.*

*Proof.* By the ramification formula $K_{\widehat{X}} = \widehat{H} + \varphi^* K_X$ and because we assumed an Enriques surface with $k$ $(-1)$-curves, we get $K_{\widehat{X}} = \sum_{j=1}^{k} \mu_j \widehat{E}_j + \widehat{H}$. So to complete the proof we need to show that $\mu_j = 1$ for all $1 \leq j \leq k$. We achieve this by computing the genus of the curve $\widehat{E}_j$ in two ways.

First: from adjunction formula we get

$$\begin{aligned} 2g(\widehat{E}_j) - 2 &= \widehat{E}_j \cdot \left( \widehat{E}_j + K_{\widehat{X}} \right) \\ &= -2 + \widehat{E}_j \cdot \left( \widehat{H} + \sum_{i=1}^{k} \mu_i \widehat{E}_i \right) \\ &= -2 - 2\mu_j + 2\lambda_j \end{aligned} \tag{4.1}$$

Second: Using the fact that our map $\varphi$ is 2:1; that is $\widehat{X} \supset \widehat{E}_j \xrightarrow{2:1} E_j \subset X$ and the Hurwitz formula (see [29],page 301) we obtain

$$\begin{aligned} 2g(\widehat{E}_j) - 2 &= 2(2g(E_j) - 2) + 2\lambda_j \\ &= -4 + 2\lambda_j \end{aligned} \tag{4.2}$$

Equating (4.1) and (4.2) gives the result. □

The following example on $\mathbb{P}^2(1,1,2)$ might make things clearer. Take $E = (f = 0)$ with $f(x,y,z) = x^2 - y^2 + z$ then

$$\begin{aligned} E \cap (x = 0) &= \{(0 : y : y^2) \mid y \neq 0\} \\ &= \{(0 : 1 : 1)\} \end{aligned}$$

so the meeting happens at one point as expected. On the other hand if we take

$$E \cap (z = 0) = \{(1 : \pm 1 : 0)\}$$

and this time they meet at two distinct points.

**Lemma 14.** *The self intersection of the canonical class of $\widehat{X}$ is given by*

$$K_{\widehat{X}}^2 = \widehat{d} + 2\lambda - 2k$$

*Proof.* The result follows by direct computation as follows:

$$\begin{aligned} K_{\widehat{X}}^2 &= \left(\widehat{H} + \sum_{j=1}^{k} \widehat{E}_j\right)^2 \\ &= \widehat{d} + 2\lambda - 2k \end{aligned}$$

□

**Lemma 15.** *The Euler number of $X_0$ is given by $e(X_0) = 10 + k$.*

*Proof.* Follows easily from $X_0 = \widetilde{X} \setminus E_0$. □

**Lemma 16.** *The Euler number of $X_{00}$ is given by $e(X_{00}) = 10 + k + 2\widehat{d} + 2\lambda$.*

*Proof.* Again this follows by easy calculation from $X_{00} = X_0 \setminus Q_0$. □

Hence, the Euler number of $\widehat{X} = 2X_{00} \cup Q_0 \cup P_0$ is easily computed.

**Lemma 17.** $e(\widehat{X}) = 21 + 2k + 2\widehat{d} + 2\lambda$

**Lemma 18.** *The sectional genus of $\widehat{X}$ is $\widehat{\pi} = \widehat{d} + \frac{\lambda}{2} + 1$.*

*Proof.* Application of the formula $2\widehat{\pi} - 2 = \widehat{H} \cdot (\widehat{H} + K_{\widehat{X}})$ and Lemma 13 yield the result. □

**Lemma 19.** *The Euler-Poincaré characteristic of the structure sheaf, $\chi(\mathcal{O}_{\widehat{X}})$ is equal to $\dfrac{1}{12}(3\widehat{d} + 4\lambda + 21)$*

*Proof.* We apply Lemma 14 and 17 and Noether's formula $\chi(\mathcal{O}_{\widehat{X}}) = \dfrac{1}{12}(K_{\widehat{X}}^2 + c_2(\widehat{X}))$ and obtain the result. □

Putting all the numerical values into the double point formula (3.2), we get

$$\widehat{d}^2 - 14\widehat{d} - 5\lambda + 4k + 21 = 0 \tag{4.3}$$

We look for plausible values of $k$ for which the discriminant is a perfect square and $\widehat{d}$ is a positive integer. That is, $28 + 5\lambda - 4k$ must be a square number.

In particular

**Lemma 20.** *$k$ must be congruent to $1, 2$ or $3 \mod 5$. Also (from Lemma 18), $\lambda$ must be even.*

*Proof.* Straightforward. □

Table 4.1 shows a few of the plausible numbers. We only include the values that also satisfy the classical inequalities in Lemma 4.

Table 4.1: Plausible numbers I

| $\widehat{d}$ | $\widehat{\pi}$ | $\chi$ | $c_1^2$ | $c_2$ | $\lambda$ | $k$ |
|---|---|---|---|---|---|---|
| 17 | 26 | 12 | 49 | 91 | 16 | 2 |
| 17 | 28 | 15 | 57 | 109 | 20 | 7 |
| 21 | 40 | 20 | 93 | 141 | 36 | 3 |
| 21 | 42 | 23 | 101 | 159 | 40 | 8 |

Unfortunately we were unable to find an example from any of these numbers. We tried several simple ways of completing the cohomology table 2 subject to Riemann-Roch constraints, and they seemed plausible until it came to finding a surjective $\beta$ in the monad. This was not due to a failure of a random matrix (see [19, Remark 5.13]), but rather to monad inconsistency.

## 4.3. On non nodal Enriques surfaces

The notation is still the same as in the previous section. This section presents the second main result of this thesis. Suppose $X$ is a an Enriques surface which does not

pass through the node $P_0$. Further, assume $X$ has $k$ $(-1)$-curves on it. To save time and space we just record the results because the necessary details are similar to those of the previous section with obvious changes.

**Lemma 21.** *(i)* $c_1^2\left(\widehat{X}\right) = \widehat{d} + 4\lambda - 2k = K_{\widehat{X}}^2$

*(ii)* $c_2\left(\widehat{X}\right) = 2k + 2\widehat{d} + 2\lambda + 24$

*(iii)* $\widehat{\pi} = \widehat{d} + \lambda + 1$

*(iv)* $\chi\left(\mathcal{O}_{\widehat{X}}\right) = \dfrac{1}{12}\left(3\widehat{d} + 6\lambda + 24\right)$

*The double point formula in this case is*

$$\widehat{d}^2 - 14\widehat{d} - 12\lambda + 4k + 24 = 0$$

Clearly for the quadratic of the double point formula to have an integer solution for $\widehat{d}$, $25 + 12\lambda - 4k$ must be a square.

**Remark 13.** *Notice that here the case $k = 0 = \lambda$ gives two possibilities. The first with $\widehat{d} = 2$ is rejected because it results in the sum $c_2(\widehat{X}) + c_1^2(\widehat{X})$ not being divisible by 12, that is, it gives a non-integer value for $\chi\left(\mathcal{O}_{\widehat{X}}\right)$.*

*The second, with $\widehat{d} = 12$ is a possibility and satisfies all the classical inequalities in Lemma 4. However, we observe that in this case $p_g = 4$ and so $\widehat{X} \subset \mathbb{P}^4$ is canonical; that is, $\mathcal{O}_{\widehat{X}}(1) = K_{\widehat{X}}$ because of $k = 0$, so in fact $\widehat{X} \subset \mathbb{P}^3$: this is not our case. Note that surfaces with $p_g = 4$ are fully described in [7, Section 3].*

## 4.4. Example EX1

We perform a computer search for values of $k$ and $\lambda$. Table 4.2 gives some possible values for $k = 3$ for all possible $1 \leq \lambda \leq 100$.

We pick the first row of Table 4.2 and we are ready to state the main result of this chapter:

**Proposition 10** (Main result of this chapter)**.** *There exists a smooth general type surface $\widehat{X}$ in $\mathbb{P}^4$, of degree $\widehat{d} = 14$, sectional genus $\widehat{\pi} = 18$, topological Euler characteristic $c_2(\widehat{X}) = 64$, first Chern number $c_1^2(\widehat{X}) = 20$, Euler characteristic of the structure sheaf $\chi\left(\mathcal{O}_{\widehat{X}}\right) = 7$, irregularity $q = 0$ and geometric genus $p_g = 6$.*

Table 4.2: Plausible numbers II

| $\widehat{d}$ | $\widehat{\pi}$ | $\chi$ | $c_1^2$ | $c_2$ | $\lambda$ |
|---|---|---|---|---|---|
| 14 | 18 | 7 | 20 | 64 | 3 |
| 18 | 28 | 11 | 48 | 84 | 9 |
| 30 | 74 | 31 | 196 | 176 | 43 |
| 38 | 118 | 51 | 348 | 264 | 79 |

Moreover, $\mathbb{P}^4$ has an involution under which $\widehat{X}$ is invariant, giving a quotient $X \subset \mathbb{P}^4(w)$.

**Remark 14.** *We believe that $\widehat{X}$ arises as a double cover of an Enriques surface in $\mathbb{P}^4(w)$, blown up in three points; and we make no claims at all about our conviction except that we arrive at this family of general type surfaces guided by invariants we fixed by initially assuming that we are taking a double cover of an Enriques surface. The nature of our methods is such that we are not yet able to tell if we have a double cover we were looking for, even though the family of general type is nonempty, and symmetric under $\psi$.*

On substituting all these values into the Riemann-Roch formula, we obtain numerical values of $\chi\left(\mathcal{J}_{\widehat{X}}(j)\right)$ for all $0 \leq j \leq 4$. These are shown in Table 4.3.

Table 4.3: Euler characteristic EX1

| $j$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\chi\left(\mathcal{J}_{\widehat{X}}(j)\right)$ | $-6$ | 1 | 0 | $-5$ | $-9$ |

**Lemma 22.** *Using values from Table 4.3 we present a plausible Beilinson cohomology Table 4.4.*

Table 4.4: Cohomology table EX1

| | | | | |
|---|---|---|---|---|
| 6 | | | | |
| | 1 | | | |
| | | | 5 | 9 |
| | | | | |

*The resulting Beilinson monad is as shown below:*

$$0 \longrightarrow 6\Omega^4(4) \oplus \Omega^3(3) \xrightarrow{\alpha} 5\Omega^1(1) \xrightarrow{\beta} 9\mathcal{O} \longrightarrow 0$$

**Remark 15.** *Observe that all our computer computations in this thesis are done over* $\mathbb{F}_{17}$*. All the invariants of each of our surfaces lift to characteric zero: that a lift exists can be seen from [18, Appendix A, page 213]. We will say more about this at the end of Chapter 5, in section 5.3. In all cases the computations were checked over other primes such as 101 and for example EX2 (because it runs the fastest), we checked them over 32749, the largest* Macaulay2 *can work with and everthing remains the same except that the equations are not pretty for printing.*

### 4.4.1. Macaulay2 implementation I

**Remark 16.** *Each example requires its own program. However a lot of the code is being reused. There are two implications here as far as errors are concerned: although errors have been eliminated it is possible ( but unlikely) that there are systematic errors that prevail in every program. On the other hand, we use our code more, and therefore more systematic checks have been done thereby significantly reducing chances of unseen errors. The reader who is more interested in general discusion of errors in algebraic calculations may consult Birch and Swinnerton-Dyer [10, page 18–19]*[1]*: but our results do have to meet their standard of independently programmed computations.*

What follows is a very brief summary and is not intended to replace the program; ExampleQues002.m2. The code used to determines the maps $\alpha$ and $\beta$ is long but very simple to follow. We do not reproduce it here. We keep the programs in the Appendix

[1] reference kindly shown to us by James Davenport.

to this thesis and provide the DVD that contains them.

Now, $\alpha$ consists of

$$\alpha_{11} \in \mathrm{Hom}(\Omega^4(4), \Omega^1(1)) \cong \bigwedge^3 V^*$$

and

$$\alpha_{21} \in \mathrm{Hom}(\Omega^3(3), \Omega^1(1)) \cong \bigwedge^2 V^*.$$

On the other hand, $\beta$ is just one block

$$\beta_{11} \in \mathrm{Hom}(\Omega^1(1), \mathcal{O}) \cong \bigwedge^1 V^* \cong V.$$

We normally write the matrices representing this maps in blocks as shown below where we have just omitted the $V^*$.

$$\alpha = \begin{array}{c} \\ 6 \\ 1 \end{array} \begin{array}{c} 5 \\ \begin{pmatrix} \bigwedge^3 \\ \bigwedge^2 \end{pmatrix} \end{array}$$

$$\beta = \begin{array}{c} \\ 5 \end{array} \begin{array}{c} 9 \\ \left(\bigwedge^1\right) \end{array}$$

Recall that $V$ is a vector space with basis $\{e_0, e_1, e_2, e_3, e_4\}$ therefore $\bigwedge^2 V^*$ has basis $e_i \wedge e_j$, and $\bigwedge^3 V^*$ has basis $e_i \wedge e_j \wedge e_k$, etc.

So $\bigwedge^m V^* = 0$, for all $m > 5$ and $m < 0$; and $\bigwedge^0 V^* = \mathbb{F}$ where $\mathbb{F}$ is our field of scalars. So the dimension of $\bigwedge^m V^* = \binom{5}{m}$.

We ask Macaulay2 for $\alpha_{21}$ and solve for $\alpha_{11}$ and $\beta$ from two systems:

- We start first by finding $\beta$ from $\alpha_{21} \times \beta = 0$, this yields $1 \times 9 \times 10 = 90$ equations in $5 \times 9 \times 5 = 225$ unknowns.
- then we find $\alpha_{11}$ using $\alpha_{11} \times \beta = 0$, giving $6 \times 9 \times 5 = 270$ equations in $6 \times 5 \times 10 = 300$ unknowns.

Once we have our $\alpha$ and $\beta$ the next task is to determine the homology of the monad using Macaulay2. The following code does precisely that and we take full advantage of scripts written by Eisenbud et al. (see [23]). We choose to work over the prime field of

characteristic 17. We have also verified the results with other primes such as 101 and 32749; here we choose a small prime for ease of printing.

```
ii76 : S=ZZ/17[x,y,z,u,t];

ii77 : Afinal = transpose(Afinal);

               5       7
oo77 : Matrix E  <--- E

ii78 : Bfinal = transpose(Bfinal);

               9       5
oo78 : Matrix E  <--- E

ii79 : beta=map(E^{9:1},E^{5:-1},Bfinal);

               9       5
oo79 : Matrix E  <--- E

ii80 : alpha=map( E^{5:-1},E^{6:-4,1:-3},Afinal);

               5       7
oo80 : Matrix E  <--- E

ii81 : needs"BeilinSon.m2"

ii82 : beta1=beilinson(beta,S);

oo82 : Matrix

ii83 : alpha1=beilinson(alpha,S);

oo83 : Matrix

ii84 : G6nj2 =prune homology(beta1,alpha1);
```

```
ii85 :
       betti res G6nj2
```

The last command gives the Betti Table 4.5.

Table 4.5: Betti table HEX1

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| total: | 45 | 49 | 25 | 5 |
| 1: | 45 | 49 | 25 | 5 |

The next piece of code gives us a random codimension two subvariety in $\mathbb{P}^4$. This is in practice general: see discussion in [18, Appendix A]. Note that we guess the number 28 and if did not work we would try another one, in fact some potential examples were discarded because we could not even get a number like the 28 appearing below in `ii87`.

```
ii86 :  jj=numRows(presentation G6nj2);

ii87 : twenty8= random(S^28,S^jj)*presentation G6nj2;

                28       49
oo87 : Matrix S   <--- S

ii88 : IX = trim coker twenty8;

ii89 : --numgens IX

       codim IX  --can take some time so see it once
        and comment out.

oo89 = 2

ii90 :
       time betti res IX--see previous comment.
     -- used 23.7214 seconds
```

The last command produces Betti Table 4.6 for the twisted ideal sheaf $\mathcal{J}_{\widehat{X}}(4)$. The original Macaulay2 (with just a few semicolons removed) output is provided in exampleQues02.out

Table 4.6: Betti table JX4EX1

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| total: | 28 | 49 | 26 | 5 |
| 0: | 28 | 49 | 25 | 5 |
| 1: | . | | | |
| 2: | . | . | . | |
| .: | . | . | . | |
| .: | . | . | . | |
| 11: | . | . | . | |
| 12: | . | . | 1 | |

It gives a free resolution of the twisted ideal sheaf as follows

```
ii90 : ResIX = res IX

        28      49      26      5
oo90 = S   <-- S   <-- S   <-- S  <-- 0

       0       1       2       3      4

oo90 : ChainComplex
```

The next command is supposed to yield all the matrices involved together with the degrees, starting with the first $28 \times 49$, then $49 \times 26$, followed by the $26 \times 5$ and finally the zero matrix.

```
ii91 : ResIX.dd
```

Yet if we extract them one at a time we do see them and we have included the first one in the output file whose truncated version is presented here

```
i116 : ResIX.dd_1
```

```
o116 = | x+y+4z+6u-5t -4y+6u-t     -2y-7z+3u-7t ...
       | -2y+5z-2u+6t x-5z+8u-5t   2y-6z+3u-5t  ...
       | -3y-7z-6u+5t -4y-z-8u-3t  x+6y-6z+2u   ...
       | 5y-7z-5u-t   8y+5z+3u     8y-3z-u      ...
       | -8y+8z+5u+t  -6y-3z+4u+2t -5y-2z+5u-t  ...
       | -4y-4z-u+8t  2z-7u-5t     y-z-8u+5t    ...
       | -4y+z+3u+7t  7y+6z-2u     -8y-5z-8u-8t ...
       | 4y+6z+4u+t   2y+3z-5u+6t  8y+8u-2t     ...
       | 8y-8z+u+8t   5y+5z-2u-2t  6z+2u+2t     ...
       | -y+3u+2t     -y-z-5u+5t   -y-3z+3u+5t  ...
       | y-3z+6u+8t   -5y+2u+4t    -6y-z-u-4t   ...
       .........................................
       -----------------------------------------
```

We end this example by extracting two of the generators from the twisted ideal sheaf $\mathcal{J}_{\widehat{X}}(4)$.

```
i115 : IIX_0
o115 = z + 3u + 4t
o115 : S
i116 : IIX_60

        13     12      11 2     10 3     9 4     8 5    6 7     5 8
o116 = y   - 4y  z - 4y  z  - 7y  z  + 5y z  - 3y z  - y z  - 5y z
       -------------------------------------------------------------

       +
       -.....................................................

o116 : S
```

the dots mean we have truncated the polynomial because it was too long.

Chapter 5

# On nodal K3 surfaces

Gavin Brown keeps a comprehensive database of K3 surfaces

(see http://malham.kent.ac.uk/grdb/K3Form.php or [13].)

We assume that $X$ is a K3 surface that passes and is singular at the point $P_0$ of $\mathbb{P}^4_w$. Again, any mention of $w$ in this chapter shall mean $w = (1, 1, 1, 1, 2)$. We further suppose $X$ has $k$ $(-1)$-curves, none of which passes through $P_0$. Then we have the following easy lemma

**Lemma 23.** *(i)* $c_2(\widetilde{X}) = 24 + k$

*(ii)* $K^2_{\widehat{X}} = \widehat{d}$

*(iii)* $e(\widehat{X}) = 2\widehat{d} + 2k + 45$

*(iv)* $\widehat{\pi} - 1 = 2\widehat{d}$

*(v)* $\chi\left(\mathcal{O}_{\widehat{X}}\right) = \dfrac{1}{12}(3\widehat{d} + 2k + 45)$

*Proof.* Part (i): just recall that we have shown in Chapter 2 that for a minimal K3 surface we have $c_2 = 24$ then add the fact that now we have $k$ $(-1)$-curves and the result is trivial.

Part (ii): for K3 surfaces $\varphi^*_w K_X = 0$ so $K_{\widehat{X}} = \varphi^*_w K_X + \widehat{H}$ just becomes $K_{\widehat{X}} = \widehat{H}$. Hence the result.

Part (iii): again follows by easy computation and part (i).

Part (iv) : follows by adjunction formula.

Part (v): this follows directly from (i) and (ii) by Noether's formula. □

Finally putting all these expressions from Lemma 23 into the double point formula and solving we obtain

**Lemma 24.**

$$\widehat{d} = 7 \pm \sqrt{4 - 2k}.$$

Which clearly means we must have $k = 0$ or $k = 2$ to get an integer $\widehat{d}$.

The case $k = 2$ cannot happen because it yields a non integer value for $\chi\left(\mathcal{O}_{\widehat{X}}\right)$.

The case $k = 0$ presents two possible examples, one of which we study first in detail. The other likely example giving $\widehat{d} = 5$, falls on a Horikawa line; these are general type surfaces low on the geography diagram that satisfy $c_1^2 = 3p_g - 7$ (see [30]). In fact, Horikawa carried out a detailed study over several papers with similar titles ([31, 32, 33]). Despite our persistence, we could not prove smoothness without Macaulay2 running out of memory for this example, but we will record what we have after the next example.

## 5.1. Example EX2

The first case when $k = 0$, corresponds to $\widehat{d} = 9$. We assume that $q = 0$. Using $\chi\left(\mathcal{O}_{\widehat{X}}\right) = 1 - q + p_g$ this gives $p_g = 5$. Perhaps one thing to note at once about this example is that it belongs to a family that satisfies $c_1^2 = 3p_g - 6$. Surfaces that have this property were studied by Konno [37][1].

**Proposition 11** (Main result of this chapter)**.** *There exists a smooth general type surface $\widehat{X}$ in $\mathbb{P}^4$ not lying on a cubic, of degree $\widehat{d} = 9$, sectional genus $\widehat{\pi} = 10$, topological Euler characteristic $c_2(\widehat{X}) = 63$, first Chern number $c_1^2(\widehat{X}) = 9$, Euler characteristic of the structure sheaf $\chi\left(\mathcal{O}_{\widehat{X}}\right) = 6$, irregularity $q = 0$ and geometric genus $p_g = 5$.*

Observe that $X_{3,3}$, the complete intersection of two general cubics in $\mathbb{P}^4$, also has these invariants. That $q(X_{3,3}) = 0$ follows from [37, Theorems 3.1 and 4.1]. But $\widehat{X}$ is not canonically embedded and is not $X_{3,3}$. So $\widehat{X}$ and $X_{3,3}$ are different as polarised surfaces. It is not immediately clear whether $\widehat{X}$ is isomorphic as an abstract variety to a complete intersection $X_{3,3}$. Also in [37], Konno shows that the moduli space in this case has several components, some of which are distinguished by the degree of the canonical map. At present we do not know which component $\widehat{X}$ is in, nor whether it is in a special subvariety of that component. The methods we use do not look directly at the canonical map, so they are not well suited to settling such questions.

We apply the Riemann-Roch formula and present the numerical values in Table 5.1.

**Lemma 25.** *Going through the now familiar exercise we see that the simplest Beilinson cohomology table we can choose is as depicted by Table 5.2.*

[1] reference kindly shown to us by Margarida Mendes Lopes

Table 5.1: Euler characteristic EX2

| $j$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\chi\left(\mathcal{J}_{\widehat{X}}(j)\right)$ | $-5$ | $-1$ | 0 | 2 | 10 |

Table 5.2: Cohomology table EX2

| | | | | |
|---|---|---|---|---|
| 5 | 6 | | | |
| | 5 | | 2 | |
| | | | | |
| | | | | 10 |

*The Beilinson monad that results is given by (5.1).*

$$0 \longrightarrow 5\Omega^4(4) \oplus 5\Omega^3(3) \xrightarrow{\alpha} 6\Omega^2(2) \oplus 10\mathcal{O} \xrightarrow{\beta} 2\Omega^1(1) \longrightarrow 0 \qquad (5.1)$$

### 5.1.1. Macaulay2 implementation II

The program for this example is ExampleK3n003.m2. We take a small sample.

```
ii2 : R=ZZ/17[e_0..e_4,a_0..a_149,SkewCommutative=>{e_0,e_1,e_2,e_3,e_4}];

ii3 : V11bss  =  matrix{{e_0,e_1,e_2,e_3,e_4}};

              1       5
oo3 : Matrix R  <--- R

ii4 : V22bss = matrix{{e_0*e_1,e_0*e_2,e_0*e_3,e_0*e_4,
e_1*e_2,e_1*e_3,e_1*e_4,e_2*e_3,e_2*e_4,e_3*e_4}};

              1       10
oo4 : Matrix R  <--- R
```

```
ii5 :
      A0 = V11bss*genericMatrix(R,a_0,5,6);--first row

               1       6
oo5 : Matrix R  <--- R

ii6 : A1 = V11bss*genericMatrix(R,a_30,5,6); --second row

               1       6
oo6 : Matrix R  <--- R

ii7 : A2 = V11bss*genericMatrix(R,a_60,5,6);

               1       6
oo7 : Matrix R  <--- R

ii8 : A3 = V11bss*genericMatrix(R,a_90,5,6);

               1       6
oo8 : Matrix R  <--- R

ii9 : A4 = V11bss*genericMatrix(R,a_120,5,6);

               1       6
oo9 : Matrix R  <--- R

ii10 : A11V= A0||A1||A2||A3||A4;

                5       6
oo10 : Matrix R  <--- R

ii11 : E = ZZ/17[e_0..e_4,SkewCommutative=>true];

ii12 : VEbss  =  matrix{{e_0,e_1,e_2,e_3,e_4}};

                1       5
```

```
oo12 : Matrix E  <--- E

ii13 : E2=ZZ/17;

ii14 : B11 = random(E^6,E^{2:-1});

               6       2
oo14 : Matrix E  <--- E

ii15 : B11A = sub(B11,R);-- for A21*B12=0

               6       2
oo15 : Matrix R  <--- R

ii16 : B21 = map(E^10,E^2,0);

               10       2
oo16 : Matrix E   <--- E

ii17 : A21 = map(E^5,E^6,0);

               5       6
oo17 : Matrix E  <--- E

ii18 : A12 = random(E^5,E^{10:-4});

               5       10
oo18 : Matrix E  <--- E

ii19 : A22 = random(E^5,E^{10:-3});

               5       10
oo19 : Matrix E  <--- E

ii20 : S1=A11V*B11A; --5*2 in e_ie_j:

               5       2
```

```
oo20 : Matrix R  <--- R

ii21 : IS1 = gens ideal(S1);

               1       10
oo21 : Matrix R  <--- R

ii22 : use R;

ii23 : H01=IS1//V22bss;

               10       10
oo23 : Matrix R   <--- R

ii24 :  --Now we produce the coefficient matrix
       IH1=gens ideal(H01);

               1       100
oo24 : Matrix R  <--- R

ii25 : mat1 =genericMatrix(R,a_0,1,150);

               1       150
oo25 : Matrix R  <--- R

ii26 : H1cols = IH1//mat1;

               150       100
oo26 : Matrix R    <--- R

ii27 : TraH1=transpose(H1cols);

               100       150
oo27 : Matrix R    <--- R

ii28 : TraH1 = sub(TraH1,E2);
```

```
                100        150
oo28 : Matrix E2    <--- E2

ii29 : ---Solving
       use E2;

ii30 : time psol1= gens ker TraH1;
     -- used 0.004999 seconds

                150        50
oo30 : Matrix E2    <--- E2

ii31 : nps1= numColumns(psol1);

ii32 : cho1 = psol1*random(E2^nps1,E2^1); --random choice

                150        1
oo32 : Matrix E2    <--- E2

ii33 : Amat3= flatten cho1;

                1        150
oo33 : Matrix E2  <--- E2

ii34 : Alist= entries Amat3;

ii35 : Amat1= flatten Alist;

ii36 : Af0= VEbss*(matrix table(5,6,(i,j)->Amat1#(0+i+5*j)));

               1       6
oo36 : Matrix E  <--- E

ii37 : Af1= VEbss*(matrix table(5,6,(i,j)->Amat1#(30+i+5*j)));

               1       6
oo37 : Matrix E  <--- E
```

```
ii38 : Af2= VEbss*(matrix table(5,6,(i,j)->Amat1#(60+i+5*j)));

               1       6
oo38 : Matrix E  <--- E

ii39 : Af3= VEbss*(matrix table(5,6,(i,j)->Amat1#(90+i+5*j)));

               1       6
oo39 : Matrix E  <--- E

ii40 : Af4= VEbss*(matrix table(5,6,(i,j)->Amat1#(120+i+5*j)));

               1       6
oo40 : Matrix E  <--- E

ii41 : Af11V =Af0||Af1||Af2||Af3||Af4;

               5       6
oo41 : Matrix E  <--- E

ii42 :  --A test
       if Af11V*B11==0 and A12*B21==0  and A21*B11==0 and A22*B21==0
        then print("OK") else print("\n  \n NO.")
OK

ii43 :  --Finally put A together, and B together
       Afinal= (Af11V||A21)|(A12||A22);

               10       16
oo43 : Matrix E   <--- E

ii44 : Bfinal= B11||B21;

               16       2
oo44 : Matrix E   <--- E
```

```
ii45 : if Afinal*Bfinal != 0 then print("\n Not GooD")
   else print("\n  \n Good")


 Good

ii46 : ------------------------------------------
       Afinal = toExternalString Afinal;

ii47 : Bfinal = toExternalString Bfinal;

ii48 : Atemp = openOut "Eg6Afinal";

ii49 : Btemp = openOut "Eg6Bfinal";

ii50 : Atemp << Afinal<< close;--keep

ii51 : Btemp << Bfinal<< close;--keep

ii52 : clearAll; --clear everything else

ii53 : E = ZZ/17[e_0..e_4,SkewCommutative=>true];

ii54 : Afinal = value get "Eg6Afinal";

               10       16
oo54 : Matrix E   <--- E

ii55 : Bfinal = value get "Eg6Bfinal";

               16       2
oo55 : Matrix E   <--- E

ii56 : ------------------------------------------------------
       removeFile"Eg6Afinal";

ii57 : removeFile"Eg6Bfinal";
```

```
ii58 : S=ZZ/17[x_0..x_4];

ii59 : Afinal = transpose(Afinal);

               16       10
oo59 : Matrix E   <--- E

ii60 : Bfinal = transpose(Bfinal);

               2       16
oo60 : Matrix E  <--- E

ii61 : beta=map(E^{2:-1},E^{6:-3,10:1},Bfinal)

oo61 = {1} | -3e_0-4e_1-4e_2-e_3+7e_4 8e_0+6e_1+8e_2+5e_3-e_4
       {1} | 3e_0-3e_1-7e_2-6e_3-2e_4 -8e_0-6e_1+5e_2+e_3-3e_4
   ------------------------------------------------------------------------
   -2e_1-5e_2+4e_3-6e_4 -e_0-5e_1+3e_2+e_3-2e_4 7e_0-8e_1-4e_2-6e_3+e_4
   -8e_0+6e_2+2e_4      6e_0-3e_1+2e_2+2e_4     -4e_0+6e_1-6e_2-7e_4
   ------------------------------------------------------------------------
   8e_0+4e_1+2e_2-8e_3+7e_4 0 0 0 0 0 0 0 0 0 0 |
   3e_0-3e_1+7e_2+7e_3-6e_4 0 0 0 0 0 0 0 0 0 0 |

               2       16
oo61 : Matrix E  <--- E



ii63 : needs"BeilinSon.m2";

ii64 : beta1=beilinson(beta,S);

oo64 : Matrix

ii65 : alpha1=beilinson(alpha,S);
```

```
oo65 : Matrix

ii66 : Gj2 =prune homology(beta1,alpha1);

ii67 : Gj2 = toExternalString Gj2;

ii68 : Gj2temp = openOut "Eg6gj2";

ii69 : Gj2temp << Gj2<< close;--keep

ii70 : clearAll;

ii71 : S=ZZ/17[x_0..x_4];

ii72 : Gj2 = value get "Eg6gj2";

ii73 : removeFile"Eg6gj2";

ii74 : betti res Gj2
```

The last command gives the Betti table 5.3.

Table 5.3: Betti table HEX2

| | 0 | 1 |
|---|---|---|
| total: | 25 | 6 |
| 1: | 25 | 6 |

For a change we check smoothness using the in-built smoothness (see `ii94`) checking algorithm in Macaulay2 in an example.

```
ii76 : fourty1= random(S^5,S^jj)*presentation Gj2;

              5       6
oo76 : Matrix S  <--- S

ii83 : IX = trim minors(5,fourty1);
```

```
oo83 : Ideal of S

ii84 :
       betti res IX
```

The last command produces the Betti table 5.4. Notice the use of `trim` to obtain a minimal presentation of $\mathcal{J}_{\widehat{X}}(4)$ (called `IX` in the programs).

Table 5.4: Betti table JX4EX2

| | 0 | 1 | 2 |
|---|---|---|---|
| total: | 1 | 6 | 5 |
| 0: | 1 | . | . |
| 1: | . | . | . |
| 2: | . | . | . |
| 3: | . | . | . |
| 4: | . | 6 | 5 |

```
ii87 : theo1 = res IX

        1      6      5
oo87 = S  <-- S  <-- S  <-- 0

       0      1      2      3

oo87 : ChainComplex
```

Since this resolution is minimal, the twisted ideal $\mathcal{J}_{\widehat{X}}(4)$ has six generators in degree five which can be viewed by running the next line without the semicolon.

```
ii89 : dIXX = gens IX;

                1       6
oo89 : Matrix S  <--- S
```

We can see the generators one at a time since the above matrix might appear unreadable. For example here is the truncated first generator

```
ii90 : dIXX_{0}^{0}

oo90 = | x_1^5-7x_0^4x_2-4x_0^3x_1x_2+3x_0^2x_1^2x_2
       ------------------------------------------
       ..........................................
       ------------------------------------------
```

We now just verify that we have a smooth surface in $\mathbb{P}^4$.

```
ii93 : codim IX -- usually see once then comment out!

oo93 = 2

ii94 :  time codim singularLocus IX
     -- used 6.24505 seconds

oo94 = 5
```

Now we turn to the example on the Horikawa line that did not finish.

## 5.2. Example EX3

**Conjecture 1.** *There exists a smooth general type surface $\widehat{X}$ in $\mathbb{P}^4$, of degree $\widehat{d} = 5$, sectional genus $\widehat{\pi} = 6$, topological Euler characteristic $c_2(\widehat{X}) = 55$, first Chern number $c_1^2\widehat{X}) = 5$, Euler characteristic of the structure sheaf $\chi\left(\mathcal{O}_{\widehat{X}}\right) = 5$, irregularity $q = 0$ and geometric genus $p_g = 4$.*

We have been unable to prove this so far only because the computation needed to check smoothness did not finish. We run out of memory before we can even built the equations of the surface we seek.

The results of the Riemann-Roch formula are recorded in Table 5.5.

**Lemma 26.** *A plausible Beilinson Cohomology table is as in Table 5.6. Hence the Beilinson monad is*

$$0 \longrightarrow 4\Omega^4(4) \oplus 11\Omega^2(2) \xrightarrow{\alpha} 16\Omega^2(2) \oplus 35\mathcal{O} \xrightarrow{\beta} 15\Omega^1(1) \longrightarrow 0 \quad (5.2)$$

Table 5.5: Euler characteristic EX3

| $j$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\chi\left(\mathcal{J}_{\widehat{X}}(j)\right)$ | $-4$ | 0 | 5 | 15 | 35 |

Table 5.6: Cohomology table EX3

| | | | | |
|---|---|---|---|---|
| 4 | | | | |
| | | 16 | 15 | |
| | | 11 | | |
| | | | | 35 |

### 5.2.1. Macaulay2 implementation III

The program for this example is ExampleK3n004.m2. We take a small sample.

```
ii53 : Gj2 =prune homology(beta1,alpha1);

ii54 :
       betti res Gj2
```

Table 5.7: Betti table HEG3

| | 0 | 1 | 2 |
|---|---|---|---|
| total: | 76 | 59 | 15 |
| 1: | 6 | . | . |
| 2: | 70 | 59 | 15 |

Macaulay2 runs out of memory when we try to find the value of `a` to go in the following command. We need this in order to produce equations for our surface to

continue the analysis.

```
fifty6= random(S^a,S^{jj})*presentation Gj2;
```

We believe the number we want is between 32 and 44 if it exists, that is if a smooth example is to be obtained by our method. If the reader can obtain this number then it should be easy to adapt the code of example one to complete this one.

The singular locus is cut out by many equations given by minors of a certain matrix. It is empty as soon as some subset of these equations defines the empty set. The number `a` is the size of a submatrix and we look only at the minors of that submatrix. Searching through all `a` takes too much computer memory but if we know `a` in advance it might work.

## 5.3. Further remarks about computations

Although we constructed our surfaces over $\mathbb{F}_{17}$, they lift to characteristic zero and the lifted surface has the same invariants. Take our constructed surface $X_p$ in characteristic $p = 17$, which is known to smooth (in particular reduced) and consider the invariants $q$, $p_g$, $c_1^2$, $c_2$, $\chi(\mathcal{O})$ and $\widehat{\pi}$ = sectional genus. Supose we have a lift $X_0$ to characteristic zero, got by lifting the equations to equations of the same degrees. We may assume that this is also smooth (see [18, Appendix A]. The question is, does it have the same invariants?

On $X_0$ we have line bundles $H = \mathcal{O}_{X_0}(1)$ and $K_{X_0}$, the canonical bundle, and we can compute $c_1^2$, $c_2$ and $\chi(\mathcal{O})$ in terms of their characteristic classes. The sectional genus is just $H \cdot (H + K)$. These do not change under good reduction mod $p$. So the only issue is whether we could have $p_g$ and $q$ change, in other words, whether $X_0$ could have $q > 0$.

Observe that for examples EX2 and EX3 that can be excluded immediately because such a surface would violate Debarre's inequality for irregular surfaces (Lemma 4(3)).

Unfortunately, for EX1 this inequality does not help us. But in any case $X_0$ cannot have $q > 0$. This would be requiring a differential form on $X_0$ that vanishes in characteristic $p$, but without the surface having bad (i.e. singular) reduction. It is of course absolutely crucial that $X_p$ is smooth: we cannot hope to get any useful information at a prime of bad reduction.

It is well-known that in this situation $X_p$ and $X_0$ have the same (étale) cohomology: see [39, Chapter 20] for details. This is sufficient for us.The real difficulty is not the cohomology but the existence of a lift at all (again see [39, Chapter 20]) Here, however, the situation is the same as in [18, Appendix A, p.213]. Frank Schreyer explained to

us that although lifting the equations can in general give a variety of smaller than expected dimension, the varieties here are defined as degeneracy loci and they always have at least the expected dimension.

Chapter 6

# Conclusions

In this final section we summarise the results of this thesis.

The main result of Chapter 3 proves that for given weights there are only finitely many families quasismooth non general type surfaces in weighted $\mathbb{P}^4$. This theorem was motivated by a similar result in straight $\mathbb{P}^4$. In 1989 Ellingsrud and Peskine proved that the degree of non general type surfaces in $\mathbb{P}^4$ are bounded. In 1994, Braun and Fløystad used Gröbner base techniques to prove that a bound is degree 105. Over a few attempts, Braun and Cook, and Cook alone, improved the bound to 66. Cook announced a bound of 46 but her proof was flawed, and Decker and Schreyer in 2000 put the bound back to 52. So the known bound is now 52. The conjectured bound is 15 and examples are known in all degrees up to 15.

If we consider weighted $\mathbb{P}^4$ as a quotient of straight $\mathbb{P}^4$, the techniques of Braun and Fløystad should apply. One quickly finds that in order to take advantage of their methods one should work in straight $\mathbb{P}^4$, rather than trying to mimic their proof directly with weights. But care now needs to be taken because the surfaces in $\mathbb{P}^4$ that are assumed to arise as covers of non general type surfaces in weighted $\mathbb{P}^4$ are of general type. In this case some of the standard theorems that work for non general type surfaces in straight $\mathbb{P}^4$ no longer apply. However, the general type surfaces that arise are special ones, with rather small invariants, so similar ideas still work.

In the rest of the thesis we construct two examples of general type surfaces in $\mathbb{P}^4$ that have the invariants one would expect if they were coming from non general type surfaces in weighted $\mathbb{P}^4$. We also give a third example, which remains conjectural because we could not finish all the computations. Thus we are able to predict plausible invariants for general type surfaces in $\mathbb{P}^4$ and show that in some cases such surfaces actually exist. This also gives some evidence that the corresponding non general type

surfaces in weighted $\mathbb{P}^4$ actually exist. Note that the examples we have are not complete intersections. Note also that an arbitrary surface will not in general have any embedding into $\mathbb{P}^4$: thus the surfaces we have constructed are special ones.

From a computational point of view, we see that the techniques used for finding non general type surfaces can be adapted so as to work for general type surfaces with sufficiently small invariants. One of our examples failed to complete, because we ran out of memory before we could produce the equations needed for further analysis. Thus we also see the current limitations of the method as we implemented it.

The computations exhibited in this thesis were carried out over $\mathbb{F}_{17}$ for ease of printing, though they were checked for some other primes (101 and 32749) also. Although the actual computational results were obtained over $\mathbb{F}_{17}$ they lift to characteristic zero. We used the computer algebra system *Macaulay2, version 1.1* for our computations.

There are many possible directions that this research could be carried further in. We did not attempt to exploit the results of Canonaco [14], which could be useful for a direct attempt to find surfaces in weighted $\mathbb{P}^4$. One could look at more complicated weights (we used only the simplest nontrivial case, $(1, 1, 1, 1, 2)$), or study the moduli of the surfaces we construct.

# Appendix A

## A.1. Program for example EX1

The the following Macaulay 2 program will also be found on the attached DVD under the name: ExampleQues002.m2.

```
--Strategy for Part I: determining the homogeneous matrices A and B.
--step1: Fix A21
--step2: solve for B from A21*B =0(225 vars,)(90 eqns)
--step3: use B from step 2 to solve for A11
      using A11*B=0(300 vars.)(280eqns)
-----------------------------------------------------
R1=ZZ/17[e_0..e_4,a_0..a_299,SkewCommutative=>{e_0,e_1,e_2,e_3,e_4}];
V11bss  =  matrix{{e_0,e_1,e_2,e_3,e_4}};
V33bss=matrix{{e_0*e_1*e_2,e_0*e_1*e_3,e_0*e_1*e_4,e_0*e_2*e_3,
e_0*e_2*e_4,e_0*e_3*e_4,e_1*e_2*e_3,
e_1*e_2*e_4,e_1*e_3*e_4,e_2*e_3*e_4}};

A0 = V33bss*genericMatrix(R1,a_0,10,5);--first row
A1 = V33bss*genericMatrix(R1,a_50,10,5); --second row
A2 = V33bss*genericMatrix(R1,a_100,10,5);
A3 = V33bss*genericMatrix(R1,a_150,10,5);
A4 = V33bss*genericMatrix(R1,a_200,10,5);
A5 = V33bss*genericMatrix(R1,a_250,10,5);
A11V= A0||A1||A2||A3||A4||A5; -- to solve for
B0 = V11bss*genericMatrix(R1,a_0,5,9);
B1 = V11bss*genericMatrix(R1,a_45,5,9);
B2 = V11bss*genericMatrix(R1,a_90,5,9);
```

```
B3 = V11bss*genericMatrix(R1,a_135,5,9);
B4 = V11bss*genericMatrix(R1,a_180,5,9);
B11V= B0||B1||B2||B3||B4;
E = ZZ/17[e_0..e_4,SkewCommutative=>true];
VEbss  =  matrix{{e_0,e_1,e_2,e_3,e_4}};
V3Ebss=matrix{{e_0*e_1*e_2,e_0*e_1*e_3,e_0*e_1*e_4,
e_0*e_2*e_3,e_0*e_2*e_4,e_0*e_3*e_4,e_1*e_2*e_3,
e_1*e_2*e_4,e_1*e_3*e_4,e_2*e_3*e_4}};
E2=ZZ/17;
A21 = random(E^1,E^{5:-2});
A21A = sub(A21,R1); -- for A21*B12=0
---------------step 2: Finding B
-------------------------------------------------------
S1=A21A*B11V;
------------------Factoring .----
IS1 = gens ideal(S1);
use R1
V3tbss=matrix{{e_0*e_1*e_2,e_0*e_1*e_3,
e_0*e_1*e_4,e_0*e_2*e_3,e_0*e_2*e_4,
e_0*e_3*e_4,e_1*e_2*e_3,e_1*e_2*e_4,
e_1*e_3*e_4,e_2*e_3*e_4}};
H01=IS1//V3tbss;
IH1=gens ideal(H01);
mat1 =genericMatrix(R1,a_0,1,225);
H1cols = IH1//mat1;
TraH1=transpose(H1cols);
TraH1 = sub(TraH1,E2);
-----------------Solving
use E2
time psol1= gens ker TraH1;
nps1= numColumns(psol1);
cho1 = psol1*random(E2^nps1,E2^1); --random choice
Bmat3= flatten cho1;
Blist= entries Bmat3;
Bmat1= flatten Blist;
Bf0= VEbss*(matrix table(5,9,(i,j)->Bmat1#(0+i+5*j)));
Bf1= VEbss*(matrix table(5,9,(i,j)->Bmat1#(45+i+5*j)));
```

```
Bf2= VEbss*(matrix table(5,9,(i,j)->Bmat1#(90+i+5*j)));
Bf3= VEbss*(matrix table(5,9,(i,j)->Bmat1#(135+i+5*j)));
Bf4= VEbss*(matrix table(5,9,(i,j)->Bmat1#(180+i+5*j)));
Bf11V =Bf0||Bf1||Bf2||Bf3||Bf4;
Bf11h = sub(Bf11V,R1);-- for A21*B11
-----------------------------------------------------
S2=A11V*Bf11h; --4*7 in e_ie_j:
IS2 = gens ideal(S2);
use R1
V4tbss=matrix{{e_0*e_1*e_2*e_3,e_0*e_1*e_2*e_4,
e_0*e_1*e_3*e_4,e_0*e_2*e_3*e_4,e_1*e_2*e_3*e_4}};
HO2=IS2//V4tbss;
IH2=gens ideal(HO2);
mat2 =genericMatrix(R1,a_0,1,300);
H2cols = IH2//mat2;
TraH2=transpose(H2cols);
TraH2 = sub(TraH2,E2);
use E2
time psol2= gens ker TraH2;
nps2= numColumns(psol2);
cho2 = psol2*random(E2^nps2,E2^1); --random choice
Amat3= flatten cho2;
Alist= entries Amat3;
Amat1= flatten Alist;
Af0= V3Ebss*(matrix table(10,5,(i,j)->Amat1#(0+i+10*j)));
Af1= V3Ebss*(matrix table(10,5,(i,j)->Amat1#(50+i+10*j)));
Af2= V3Ebss*(matrix table(10,5,(i,j)->Amat1#(100+i+10*j)));
Af3= V3Ebss*(matrix table(10,5,(i,j)->Amat1#(150+i+10*j)));
Af4= V3Ebss*(matrix table(10,5,(i,j)->Amat1#(200+i+10*j)));
Af5= V3Ebss*(matrix table(10,5,(i,j)->Amat1#(250+i+10*j)));
Af11V =Af0||Af1||Af2||Af3||Af4||Af5;
--A test
if Af11V*Bf11V==0   and A21*Bf11V==0  then print("OK")
else print("\n  \n NO.")
--Finally put A together, and B together
Afinal= Af11V||A21;
Bfinal= Bf11V;
```

```
if Afinal*Bfinal != 0 then print("\n Not GooD")
 else print("\n  \n Good")
S=ZZ/17[x,y,z,u,t];
--Now having all the maps we need it is time to determine the
   twisted ideal sheaf J_X(4)
Afinal = transpose(Afinal);
Bfinal = transpose(Bfinal);
beta=map(E^{9:1},E^{5:-1},Bfinal);
alpha=map( E^{5:-1},E^{6:-4,1:-3},Afinal);
needs"BeilinSon.m2"
beta1=beilinson(beta,S);
alpha1=beilinson(alpha,S);
G6nj2 =prune homology(beta1,alpha1);
betti res G6nj2
----------------------SMOOTHNESS-----------------------------
 --verifying smoothness in an example.

jj=numRows(presentation G6nj2);
twenty8= random(S^28,S^jj)*presentation G6nj2;
IX = trim coker twenty8;
--res IX
--numgens IX
--codim IX  --can take sometime so see it once and comment out.
--time betti res IX
-------------------------------------------------------------
peIX= presentation IX;
peIIX =gens gb peIX;
IIX = ideal();
IIX = sub(IIX,S);
for i from 0 to 60 do IIX = IIX +ideal sum(entries peIIX_i);
JJX = transpose jacobian IIX;
time Areunited = ideal flatten JJX + IIX;
if codim Areunited >= 5 then print("\n OK to continue.")
--part (i) of Theorem 7.3 completed
--time codim Areunited
-------------------------------------------------------------
--now I am taking suficiently  many 2by2  minors involving the
```

```
-- last generator, it is of degree 2.
-- I have 61 generators! I decide that 20 is sufficiently many.

--time JJ00 = submatrix(JJX,{60,40},);
--time minJJ00 = minors(2,JJ00);

--time JJ01 = submatrix(JJX,{60,41},);
--time minJJ01 = minors(2,JJ01);

--time JJ02 = submatrix(JJX,{60,42},);
--time minJJ02 = minors(2,JJ02);

--time JJ03 = submatrix(JJX,{60,43},);
--time minJJ03 = minors(2,JJ03);

--time JJ04 = submatrix(JJX,{60,44},);
--time minJJ04 = minors(2,JJ04);

--time JJ05 = submatrix(JJX,{60,45},);
--time minJJ05 = minors(2,JJ05);

--time JJ06 = submatrix(JJX,{60,46},);
--time minJJ06 = minors(2,JJ06);
--time JJ07 = submatrix(JJX,{60,47},);
--time minJJ07 = minors(2,JJ07);
--time JJ08 = submatrix(JJX,{60,48},);
--time minJJ08 = minors(2,JJ08);
--time JJ09 = submatrix(JJX,{60,49},);
--time minJJ09 = minors(2,JJ09);
--time JJ010 = submatrix(JJX,{60,50},);
--time minJJ010 = minors(2,JJ010);
--time JJ011 = submatrix(JJX,{60,51},);
--time minJJ011 = minors(2,JJ011);
--time JJ012 = submatrix(JJX,{60,52},);
--time minJJ012 = minors(2,JJ012);
--time JJ013 = submatrix(JJX,{60,53},);
--time minJJ013 = minors(2,JJ013);
```

```
--time JJ014 = submatrix(JJX,{60,54},);
--time minJJ014 = minors(2,JJ014);
--time JJ015 = submatrix(JJX,{60,55},);
--time minJJ015 = minors(2,JJ015);
--time JJ016 = submatrix(JJX,{60,56},);
--time minJJ016 = minors(2,JJ016);
--time JJ017 = submatrix(JJX,{60,57},);
--time minJJ017 = minors(2,JJ017);
--time JJ018 = submatrix(JJX,{60,58},);
--time minJJ018 = minors(2,JJ018);
--time JJ019 = submatrix(JJX,{60,59},);
--time minJJ019 = minors(2,JJ019);
--time MoreUnite = minJJ00+minJJ01+minJJ02+minJJ03+minJJ04
    +minJJ05+minJJ06+minJJ07+minJJ08+minJJ09+minJJ010+minJJ011
    +minJJ012+minJJ013+minJJ014+minJJ015+minJJ016+minJJ017
    +minJJ018+minJJ019+IIX;

-----------------------------------------------------------
--all of the above 20 lines can be replaced by
  the next four lines.
Morite=ideal();
Morite = sub(Morite,S);
time for j from 20 to 60 do
      Morite = Morite+ minors(2,submatrix(JJX,{60,j},));
MoreUnite = Morite +IIX;
---------------------------------------
time isfinite = dim MoreUnite;
time degI2f = degree MoreUnite;
time JJf = submatrix(JJX,{60},);
time IfunionIX = ideal flatten JJf + IIX;
time degUIX = degree IfunionIX;

if isfinite < infinity and degI2f==degUIX
   then print("\n SMOOTH");

 --__________________ SYMMETRY______________
```

```
symap = map(S,S,{t=>-t});
IXcopy = symap(IIX);
if IXcopy == IIX then print("\n \n  SYMMETRIC!!")
 else print("\n NOT SYMMETRIC.")
```

## A.2. Program for example EX2

The the following Macaulay 2 program will also be found on the attached DVD under the name: ExampleK3n003.m2.

```
--Strategy for Part I: determining the homogeneous matrices A and B.
--step1: Fix B,(B11, and B21=0)
--step2: Fix A12 and A22 ( they multiply 0)
--step3: solve for A11 using A21*B11=0(150 vars.)(100eqns)
-----------------------------------------------------------------
R=ZZ/17[e_0..e_4,a_0..a_149,SkewCommutative=>{e_0,e_1,e_2,e_3,e_4}];
V11bss  =  matrix{{e_0,e_1,e_2,e_3,e_4}};
V22bss = matrix{{e_0*e_1,e_0*e_2,e_0*e_3,
e_0*e_4,e_1*e_2,e_1*e_3,e_1*e_4,e_2*e_3,e_2*e_4,e_3*e_4}};
A0 = V11bss*genericMatrix(R,a_0,5,6);--first row
A1 = V11bss*genericMatrix(R,a_30,5,6); --second row
A2 = V11bss*genericMatrix(R,a_60,5,6);
A3 = V11bss*genericMatrix(R,a_90,5,6);
A4 = V11bss*genericMatrix(R,a_120,5,6);
A11V= A0||A1||A2||A3||A4;
E = ZZ/17[e_0..e_4,SkewCommutative=>true];
VEbss  =  matrix{{e_0,e_1,e_2,e_3,e_4}};
E2=ZZ/17;
B11 = random(E^6,E^{2:-1});
B11A = sub(B11,R);-- for A21*B12=0
B21 = map(E^10,E^2,0);
A21 = map(E^5,E^6,0);
A12 = random(E^5,E^{10:-4});
A22 = random(E^5,E^{10:-3});
-----------------------------------------------------
S1=A11V*B11A; --5*2 in e_ie_j:
```

```
IS1 = gens ideal(S1);
use R;
H01=IS1//V22bss;
--Now we produce the coefficient matrix
IH1=gens ideal(H01);
mat1 =genericMatrix(R,a_0,1,150);
H1cols = IH1//mat1;
TraH1=transpose(H1cols);
TraH1 = sub(TraH1,E2);
-----------------Solving-----------------------------
use E2;
time psol1= gens ker TraH1;
nps1= numColumns(psol1);
cho1 = psol1*random(E2^nps1,E2^1); --random choice
Amat3= flatten cho1;
Alist= entries Amat3;
Amat1= flatten Alist;
Af0= VEbss*(matrix table(5,6,(i,j)->Amat1#(0+i+5*j)));
Af1= VEbss*(matrix table(5,6,(i,j)->Amat1#(30+i+5*j)));
Af2= VEbss*(matrix table(5,6,(i,j)->Amat1#(60+i+5*j)));
Af3= VEbss*(matrix table(5,6,(i,j)->Amat1#(90+i+5*j)));
Af4= VEbss*(matrix table(5,6,(i,j)->Amat1#(120+i+5*j)));
Af11V =Af0||Af1||Af2||Af3||Af4;

--A test
if Af11V*B11==0 and A12*B21==0  and A21*B11==0
 and A22*B21==0 then print("OK") else print("\n  \n NO.")
--Finally put A together, and B together
Afinal= (Af11V||A21)|(A12||A22);
Bfinal= B11||B21;
if Afinal*Bfinal != 0 then print("\n Not GooD")
else print("\n  \n Good")
-------------------------------------------
Afinal = toExternalString Afinal;
Bfinal = toExternalString Bfinal;
Atemp = openOut "Eg6Afinal";
Btemp = openOut "Eg6Bfinal";
```

```
Atemp << Afinal<< close;--keep
Btemp << Bfinal<< close;--keep
clearAll; --clear everything else
E = ZZ/17[e_0..e_4,SkewCommutative=>true];
Afinal = value get "Eg6Afinal";
Bfinal = value get "Eg6Bfinal";
-------------------------------------------------------
removeFile"Eg6Afinal";
removeFile"Eg6Bfinal";
S=ZZ/17[x_0..x_4];
--Now having all the maps we need it is time to determine
the twisted ideal sheaf J_X(4)
Afinal = transpose(Afinal);
Bfinal = transpose(Bfinal);
beta=map(E^{2:-1},E^{6:-3,10:1},Bfinal)
alpha=map( E^{6:-3,10:1},E^{5:-4,5:-3},Afinal)
-------------------------------------------------------
needs"BeilinSon.m2";
beta1=beilinson(beta,S);
alpha1=beilinson(alpha,S);
Gj2 =prune homology(beta1,alpha1);
Gj2 = toExternalString Gj2;
Gj2temp = openOut "Eg6gj2";
Gj2temp << Gj2<< close;--keep
clearAll; --clear everything else
S=ZZ/17[x_0..x_4];
Gj2 = value get "Eg6gj2";
-------------------------------------------------------
removeFile"Eg6gj2";
time betti res Gj2
jj=numRows(presentation Gj2);
------------------SMOOTHNESS--------------------------
--The smoothness can be checked in an example using
--the built-in Jacobian criterion.
fourty1= random(S^5,S^jj)*presentation Gj2;
--fourty1 is a 5 by 6 matrix  of linear forms
-- which do not all vanish at the same time.
```

```
-------Linear Determinantal variety---------
geFo = gens ideal(fourty1);
geGo = genericMatrix(S,x_0,1,5);
intFo = geFo//geGo;
E2=ZZ/17;
use E2;
if  gens ker intFo !=0 then
 print("\n \n \n We have a linear determinantal variety ")
else error
-------------------------------------
--hence \wX is a linear determinantal variety.
IX = trim minors(5,fourty1);
betti res IX
hilbertSeries IX
-------------------
theo1 = res IX
theo2 = theo1_0
theo3 = theo1_1
theo4 = theo1_2
---------------
codim IX -- see once then comment out!
time codim singularLocus IX
--(in-built smoothness checking in  M2 acceptable here)
--Hence smooth.
--#flatten entries gens gb IX --to enable counting, flatten first!
--numgens IX  --should equal previous because minimal.
--isHomogeneous IX --true
--JX4=gens gb IX
```

## A.3. Program for example EX3

The the following Macaulay 2 program will also be found on the attached DVD under the name: ExampleK3n004.m2.

```
--step 1: Fix B (B11, and B21 =0)
--step 2: Fix A12 and A22
```

```
--step 3: solve for all A11 using A21*B11 =0

R = ZZ/17[e_0..e_4,a_0..a_639,SkewCommutative=>{e_0,e_1,e_2,e_3,e_4}];
V11bss = matrix{{e_0,e_1,e_2,e_3,e_4}};
V22bss = matrix{{e_0*e_1,e_0*e_2,e_0*e_3,e_0*e_4,e_1*e_2
,e_1*e_3,e_1*e_4,e_2*e_3,e_2*e_4,e_3*e_4}};
A0 = V22bss*genericMatrix(R,a_0,10,16);
A1 = V22bss*genericMatrix(R,a_160,10,16);
A2 = V22bss*genericMatrix(R,a_320,10,16);
A3 = V22bss*genericMatrix(R,a_480,10,16);

A11V= A0||A1||A2||A3;

E = ZZ/17[e_0..e_4,SkewCommutative=>true];
V2Ebss =  matrix{{e_0*e_1,e_0*e_2,e_0*e_3,
e_0*e_4,e_1*e_2,e_1*e_3,e_1*e_4,e_2*e_3,e_2*e_4,e_3*e_4}};
E2 = ZZ/17;
B11 = random(E^16,E^{15:-1});
B11A = sub(B11,R);
B21 = map(E^35,E^15,0);
A12 = random(E^4,E^{35:-4});
A22 = random(E^11,E^{35:-2});
A21 = map(E^11,E^16,0);

----------------------------------------A11

S1=A11V*B11A; --4*15 in e_ie_je_k:

------------------Factoring .----

IS1 = gens ideal(S1);

use R

V3tbss=matrix{{e_0*e_1*e_2,e_0*e_1*e_3,e_0*e_1*e_4,
e_0*e_2*e_3,e_0*e_2*e_4,e_0*e_3*e_4,e_1*e_2*e_3,
e_1*e_2*e_4,e_1*e_3*e_4,e_2*e_3*e_4}};
```

```
H01=IS1//V3tbss;

--Now we produce the coefficient matrix
IH1=gens ideal(H01);

mat1 =genericMatrix(R,a_0,1,640);
H1cols = IH1//mat1;

TraH1=transpose(H1cols);
TraH1 = sub(TraH1,E2);

-----------------Solving
use E2
time psol1= gens ker TraH1;
nps1= numColumns(psol1);

cho1 = psol1*random(E2^nps1,E2^1);
Amat3= flatten cho1;
Alist= entries Amat3;
Amat1= flatten Alist;


Af0= V2Ebss*(matrix table(10,16,(i,j)->Amat1#(0+i+10*j)));
Af1= V2Ebss*(matrix table(10,16,(i,j)->Amat1#(160+i+10*j)));
Af2= V2Ebss*(matrix table(10,16,(i,j)->Amat1#(320+i+10*j)));
Af3= V2Ebss*(matrix table(10,16,(i,j)->Amat1#(480+i+10*j)));


Af11V =Af0||Af1||Af2||Af3;


--A test
if Af11V*B11==0 and A12*B21==0  and A21*B11==0
and A22*B21==0 then print("OK") else print("\n  \n NO.")

--Finally put A together, and B together
Afinal= (Af11V||A21)|(A12||A22);
```

```
Bfinal= B11||B21;

if Afinal*Bfinal != 0 then print("\n Not GooD")
  else print("\n  \n Good")
S=ZZ/17[x,y,z,u,t];
--Now having all the maps we need it is time to determine
   -- the twisted ideal sheaf J_X(4)
Afinal = transpose(Afinal);
Bfinal = transpose(Bfinal);
beta=map(E^{15:-1},E^{16:-2,35:1},Bfinal);
alpha=map( E^{16:-2,35:1},E^{4:-4,11:-2},Afinal);
needs"BeilinSon.m2"
beta1=beilinson(beta,S);
alpha1=beilinson(alpha,S);
Gj2 =prune homology(beta1,alpha1);

betti res Gj2

jj=numRows(presentation Gj2);
--M2 ran out of memory when attempting to check
    smoothness in an example.
```

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