The Geometry of Siegel Modular Varieties

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Introduction

Siegel modular varieties are interesting because they arise as moduli spaces for abelian varieties with a polarization and a level structure, and also because of their concrete analytic realization as locally symmetric varieties. Even in the early days of modern algebraic geometry the study of quartic surfaces led to some specific examples of these moduli spaces being studied in the context of projective geometry. Later advances in complex analytic and algebraic geometry and in number theory have given us many very effective tools for studying these varieties and their various compactifications, and in the last ten years a considerable amount of progress has been made in understanding the general picture. In this survey we intend to give a reasonably thorough account of the more recent work, though mostly without detailed proofs, and to describe sufficiently but not exhaustively the earlier work of, among others, Satake, Igusa, Mumford and Tai that has made the recent progress possible.

We confine ourselves to working over the complex numbers. This does not mean that we can wholly ignore number theory, since much of what is known depends on interpreting differential forms on Siegel modular varieties as Siegel modular forms. It does mean, though, that we are neglecting many important, interesting and difficult questions: in particular, the work of Faltings and Chai, who extended much of the compactification theory to $\operatorname{Spec}\mathbb{Z}$, will make only a fleeting appearance. To have attempted to cover this material would have greatly increased the length of this article and would have led us beyond the areas where we can pretend to competence.

The plan of the article is as follows.

In Section I we first give a general description of Siegel modular varieties as complex analytic spaces, and then explain how to compactify them and obtain projective varieties. There are essentially two related ways to do this.

In Section II we start to understand the birational geometry of these compactified varieties. We examine the canonical divisor and explain some results which calculate the Kodaira dimension in many cases and the Chow ring in a few. We also describe the fundamental group.

In Section III we restrict ourselves to the special case of moduli of abelian surfaces (Siegel modular threefolds), which is of particular interest. We describe a rather general lifting method, due to Gritsenko in the form we use, which produces Siegel modular forms of low weight by starting from their behaviour near the boundary of the moduli space. This enables us to get more precise results about the Kodaira dimension in a few interesting special cases, due to Gritsenko and others. Then we describe some results, including a still unpublished theorem of L. Borisov, which tend to show that in most cases the compactified varieties are of general type. In the last part of this section we examine some finite covers and quotients of moduli spaces of polarized abelian surfaces, some of which can be interpreted as moduli of Kummer surfaces. The lifting method gives particularly good results for these varieties.

In Section IV we examine three cases, two of them classical, where a Siegel modular variety (or a near relative) has a particularly good projective description. These are the Segre cubic and the Burkhardt quartic, which are classical, and the Nieto quintic, which is on the contrary a surprisingly recent discovery. There is a huge body of work on the first two and we cannot do more than summarize enough of the results to enable us to highlight the similarities among the three cases.

In Section V we examine the moduli spaces of $(1,t)$-polarized abelian surfaces (sometimes with level structure) for small t. We begin with the famous Horrocks-Mumford case, $t = 5$, and then move on to the work of Manolache and Schreyer on $t = 7$ and Gross and Popescu on other cases, especially $t = 11$.

In Section VI we return to the compactification problems and describe very recent improvements brought about by Alexeev and Nakamura, who (building on earlier work by Nakamura, Namikawa, Tai and Mumford) have shed some light on the question of whether there are compactifications of the moduli space that are really compactifications of moduli, that is, support a proper universal family.

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I Siegel modular varieties

In this section we give the basic definitions in connection with Siegel modular varieties and sketch the construction of the Satake and toroidal compactifications.

I.1 Arithmetic quotients of the Siegel upper half plane

To any point τ in the *upper half plane*

$$\mathbb{H}_1 = \{\tau \in \mathbb{C}\,;\ \ \mathrm{Im}\ \tau > 0\}$$

one can associate a lattice

$$L_\tau = \mathbb{Z}\tau + \mathbb{Z}$$

and an *elliptic curve*

$$E_\tau = \mathbb{C}/L_\tau .$$

Since every elliptic curve arises in this way one obtains a surjective map

$$\mathbb{H}_1 \to \{\text{elliptic curves}\}/ \text{ isomorphism}.$$

The group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{H}_1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau + b}{c\tau + d}$$

and

$$E_\tau \cong E_{\tau'} \Leftrightarrow \tau \sim \tau' \mod \mathrm{SL}(2,\mathbb{Z}).$$

Hence there is a bijection

$$X^\circ(1) = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}_1 \xrightarrow{1:1} \{\text{elliptic curves}\}/ \text{ isomorphism}.$$

The j-function is an $\mathrm{SL}(2,\mathbb{Z})$-invariant function on $\mathbb{H}_1$ and defines an isomorphism of Riemann surfaces

$$j : X^\circ(1) \cong \mathbb{C}.$$

An *abelian variety* (over the complex numbers $\mathbb{C}$) is a g-dimensional complex torus $\mathbb{C}^g/L$ which is a projective variety, i.e. can be embedded into some projective space $\mathbb{P}^n$. Whereas every 1-dimensional torus $\mathbb{C}/L$ is an algebraic curve, it is no longer true that every torus $X = \mathbb{C}^g/L$ of dimension $g \geq 2$ is projective. This is the case if and only if X admits a *polarization*. There are several ways to define polarizations. Perhaps the most common definition is that using Riemann forms. A *Riemann form* on $\mathbb{C}^g$ with respect to the lattice L is a hermitian form $H \geq 0$ on $\mathbb{C}^g$ whose imaginary part $H' = \mathrm{Im}(H)$ is integer-valued on L, i.e. defines an alternating bilinear form

$$H' : L \otimes L \to \mathbb{Z}.$$

The $\mathbb{R}$-linear extension of H' to $\mathbb{C}^g$ satisfies $H'(x,y) = H'(ix,iy)$ and determines H by the relation

$$H(x,y) = H'(ix,y) + iH'(x,y).$$

H is positive definite if and only if H' is non-degenerate. In this case H (or equivalently H') is called a *polarization*. By the elementary divisor theorem there exists then a basis of L with respect to which H' is given by the form

$$\Lambda = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \quad , \quad E = \begin{pmatrix} e_1 & & \\ & \ddots & \\ & & e_g \end{pmatrix},$$

where the $e_1, \dots, e_g$ are positive integers such that $e_1|e_2 \dots |e_g$. The g-tuple $(e_1, \dots, e_g)$ is uniquely determined by H and is called the *type of the polarization*. If $e_1 = \dots = e_g = 1$ one speaks of a *principal polarization*. A *(principally) polarized abelian variety* is a pair (A, H) consisting of a torus A and a (principal) polarization H.

Assume we have chosen a basis of the lattice L. If we express each basis vector of L in terms of the standard basis of $\mathbb{C}^g$ we obtain a matrix $\Omega \in M(2g \times g, \mathbb{C})$ called a *period matrix* of A. The fact that H is hermitian and positive definite is equivalent to

$$^t\Omega\Lambda^{-1}\Omega = 0, \text{ and } i\ {}^t\Omega\Lambda^{-1}\bar{\Omega} > 0.$$

These are the *Riemann bilinear relations*. We consider vectors of $\mathbb{C}^g$ as row vectors. Using the action of $\mathrm{GL}(g, \mathbb{C})$ on row vectors by right multiplication we can transform the last g vectors of the chosen basis of L to be $(e_1, 0, \dots, 0), (0, e_2, 0, \dots, 0), \dots, (0, \dots, 0, e_g)$. Then Ω takes on the form

$$\Omega = \Omega_\tau = \begin{pmatrix} \tau \\ E \end{pmatrix}$$

and the Riemann bilinear relations translate into

$$\tau = {}^t\tau, \quad \operatorname{Im} \tau > 0.$$

In other words, the complex $(g \times g)$-matrix τ is an element of the *Siegel space of degree g*

$$\mathbb{H}_g = \{\tau \in M(g \times g, \mathbb{C}); \tau = {}^t\tau, \operatorname{Im} \tau > 0\}.$$

Conversely, given a matrix $\tau \in \mathbb{H}_g$ we can associate to it the period matrix Ω_τ and the lattice $L = L_\tau$ spanned by the rows of Ω_τ. The complex torus $A = \mathbb{C}^g/L_\tau$ carries a Riemann form given by

$$H(x, y) = x \operatorname{Im}(\tau)^{-1}\ {}^t\bar{y}.$$

This defines a polarization of type $(e_1, \dots, e_g)$. Hence for every given type of polarization we have a surjection

$$\mathbb{H}_g \to \{(A, H); (A, H) \text{ is an } (e_1, \dots, e_g)\text{-polarized ab.var.}\}/ \text{ isom.}$$

To describe the set of these isomorphism classes we have to see what happens when we change the basis of L. Consider the *symplectic group*

$$\mathrm{Sp}(\Lambda, \mathbb{Z}) = \{h \in \mathrm{GL}(2g, \mathbb{Z});\ h\Lambda^t h = \Lambda\}.$$

As usual we write elements $h \in \mathrm{Sp}(\Lambda, \mathbb{Z})$ in the form

$$h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \quad A, \dots, D \in M(g \times g, \mathbb{Z}).$$

It is useful to work with the "right projective space P of $\mathrm{GL}(g,\mathbb{C})$" i.e. the set of all $(2g \times g)$-matrices of rank g divided out by the equivalence relation

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \sim \begin{pmatrix} M_1 M \\ M_2 M \end{pmatrix} \text{ for any } M \in \mathrm{GL}(g,\mathbb{C}).$$

Clearly P is isomorphic to the Grassmannian $G = \mathrm{Gr}(g, \mathbb{C}^{2g})$. The group $\mathrm{Sp}(\Lambda,\mathbb{Z})$ acts on P by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} AM_1 + BM_2 \\ CM_1 + DM_2 \end{bmatrix}$$

where $[\ ]$ denotes equivalence classes in P. One can embed $\mathbb{H}_g$ into P by $\tau \mapsto \begin{bmatrix} \tau \\ E \end{bmatrix}$. Then the action of $\mathrm{Sp}(\Lambda,\mathbb{Z})$ restricts to an action on the image of $\mathbb{H}_g$ and is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{bmatrix} \tau \\ E \end{bmatrix} = \begin{bmatrix} A\tau + BE \\ C\tau + DE \end{bmatrix} = \begin{bmatrix} (A\tau + BE)(C\tau + DE)^{-1}E \\ E \end{bmatrix}.$$

In other words, $\mathrm{Sp}(\Lambda,\mathbb{Z})$ acts on $\mathbb{H}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tau \mapsto (A\tau + BE)(C\tau + DE)^{-1}E.$$

We can then summarize our above discussion with the observation that for a given type $(e_1, \ldots, e_g)$ of a polarization the quotient

$$\mathcal{A}_{e_1,\ldots,e_g} = \mathrm{Sp}(\Lambda,\mathbb{Z}) \backslash \mathbb{H}_g$$

parametrizes the isomorphism classes of $(e_1, \ldots, e_g)$-polarized abelian varieties, i.e. $\mathcal{A}_{e_1,\ldots,e_g}$ is the coarse moduli space of $(e_1, \ldots, e_g)$-polarized abelian varieties. (Note that the action of $\mathrm{Sp}(\Lambda,\mathbb{Z})$ on $\mathbb{H}_g$ depends on the type of the polarization.) If we consider principally polarized abelian varieties, then the form Λ is the standard symplectic form

$$J = \begin{pmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{pmatrix}$$

and $\mathrm{Sp}(\Lambda,\mathbb{Z}) = \mathrm{Sp}(2g,\mathbb{Z})$ is the standard symplectic integer group. In this case we use the notation

$$\mathcal{A}_g = \mathcal{A}_{1,\ldots,1} = \mathrm{Sp}(2g,\mathbb{Z}) \backslash \mathbb{H}_g.$$

This clearly generalizes the situation which we encountered with elliptic curves. The space $\mathbb{H}_1$ is just the ordinary upper half plane and $\mathrm{Sp}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})$. We also observe that multiplying the type of a polarization by a common factor does not change the moduli space. Instead of the group $\mathrm{Sp}(\Lambda, \mathbb{Z})$ one can also use a suitable conjugate which is a subgroup of $\mathrm{Sp}(J, \mathbb{Q})$. One can then work with the standard symplectic form and the usual action of the symplectic group on Siegel space, but the elements of the conjugate group will in general have rational and no longer just integer entries.

One is often interested in polarized abelian varieties with extra structures, the so-called *level structures*. If L is a lattice equipped with a non-degenerate form Λ the *dual lattice* $L^\vee$ of L is defined by

$$L^\vee = \{y \in L \otimes \mathbb{Q};\ \Lambda(x, y) \in \mathbb{Z} \text{ for all } x \in L\}.$$

Then $L^\vee/L$ is non-canonically isomorphic to $(\mathbb{Z}_{e_1} \times \ldots \times \mathbb{Z}_{e_g})^2$. The group $L^\vee/L$ carries a skew form induced by Λ and the group $(\mathbb{Z}_{e_1} \times \ldots \times \mathbb{Z}_{e_g})^2$ has a $\mathbb{Q}/\mathbb{Z}$-valued skew form which with respect to the canonical generators is given by

$$\begin{pmatrix} 0 & E^{-1} \\ -E^{-1} & 0 \end{pmatrix}.$$

If (A, H) is a polarized abelian variety, then a *canonical level structure* on (A, H) is a symplectic isomorphism

$$\alpha : L^\vee/L \to (\mathbb{Z}_{e_1} \times \ldots \times \mathbb{Z}_{e_g})^2$$

where the two groups are equipped with the forms described above. Given Λ we can define the group.

$$\mathrm{Sp}^{\mathrm{lev}}(\Lambda, \mathbb{Z}) := \{h \in \mathrm{Sp}(\Lambda, \mathbb{Z});\ h|_{L^\vee/L} = \mathrm{id}_{\ L^\vee/L}\}.$$

The quotient space

$$\mathcal{A}^{\mathrm{lev}}_{e_1,\ldots,e_g} := \mathrm{Sp}^{\mathrm{lev}}(\Lambda, \mathbb{Z}) \backslash \mathbb{H}_g$$

has the interpretation

$$\begin{aligned}\mathcal{A}^{\mathrm{lev}}_{e_1,\ldots,e_g} = \{(A, H, \alpha);\ (A, H) &\text{ is an } (e_1, \ldots, e_g)\text{-polarized abelian} \\ &\text{variety}, \alpha \text{ is a canonical level structure}\}/\text{ isom.}\end{aligned}$$

If Λ is a multiple nJ of the standard symplectic form then $\mathrm{Sp}(nJ, \mathbb{Z}) = \mathrm{Sp}(J, \mathbb{Z})$ but

$$\Gamma_g(n) := \mathrm{Sp}^{\mathrm{lev}}(nJ, \mathbb{Z}) = \{h \in \mathrm{Sp}(J, \mathbb{Z});\ h \equiv \mathbf{1} \bmod n\}.$$

This group is called the *principal congruence subgroup* of level n. A *level-n structure* on a principally polarized abelian variety (A, H) is a canonical level structure in the above sense for the polarization nH. The space

$$\mathcal{A}_g(n) := \Gamma_g(n) \backslash \mathbb{H}_g$$

is the moduli space of principally polarized abelian varieties with a level-n structure.

The groups $\mathrm{Sp}(\Lambda,\mathbb{Z})$ act properly discontinuously on the Siegel space $\mathbb{H}_g$. If $e_1 \geq 3$ then $\mathrm{Sp}^{\mathrm{lev}}(\Lambda,\mathbb{Z})$ acts freely and consequently the spaces $\mathcal{A}^{\mathrm{lev}}_{e_1,\dots,e_g}$ are smooth in this case. The finite group $\mathrm{Sp}(\Lambda,\mathbb{Z})/\mathrm{Sp}^{\mathrm{lev}}(\Lambda,\mathbb{Z})$ acts on $\mathcal{A}^{\mathrm{lev}}_{e_1,\dots,e_g}$ with quotient $\mathcal{A}_{e_1,\dots,e_g}$. In particular, these spaces have at most finite quotient singularities.

A torus $A = \mathbb{C}^g/L$ is projective if and only if there exists an ample line bundle $\mathcal{L}$ on it. By the Lefschetz theorem the first Chern class defines an isomorphism

$$c_1 : \mathrm{NS}(A) \cong H^2(A,\mathbb{Z}) \cap H^{1,1}(A,\mathbb{C}).$$

The natural identification $H_1(A,\mathbb{Z}) \cong L$ induces isomorphisms

$$H^2(A,\mathbb{Z}) \cong \mathrm{Hom}(\bigwedge^2 H_1(A,\mathbb{Z}),\mathbb{Z}) \cong \mathrm{Hom}(\bigwedge^2 L,\mathbb{Z}).$$

Hence given a line bundle $\mathcal{L}$ the first Chern class $c_1(\mathcal{L})$ can be interpreted as a skew form on the lattice L. Let $H' := -c_1(\mathcal{L}) \in \mathrm{Hom}(\bigwedge^2 L,\mathbb{Z})$. Since $c_1(\mathcal{L})$ is a $(1,1)$-form it follows that $H'(x,y) = H'(ix,iy)$ and hence the associated form H is hermitian. The ampleness of $\mathcal{L}$ is equivalent to positive definiteness of H. In this way an ample line bundle defines, via its first Chern class, a hermitian form H. Reversing this process one can also associate to a Riemann form an element in $H^2(A,\mathbb{Z})$ which is the first Chern class of an ample line bundle $\mathcal{L}$. The line bundle $\mathcal{L}$ itself is only defined up to translation. One can also view level structures from this point of view. Consider an ample line bundle $\mathcal{L}$ representing a polarization H. This defines a map

$$\begin{array}{cccc} \lambda: & A & \to & \hat{A} = \mathrm{Pic}^0 A \\ & x & \mapsto & t_x^*\mathcal{L} \otimes \mathcal{L}^{-1} \end{array}$$

where t_x is translation by x. The map λ depends only on the polarization, not on the choice of the line bundle $\mathcal{L}$. If we write $A = \mathbb{C}^g/L$ then we have $\mathrm{Ker}\,\lambda \cong L^\vee/L$ and this defines a skew form on $\mathrm{Ker}\,\lambda$, the *Weil pairing*. This also shows that $\mathrm{Ker}\,\lambda$ and the group $(\mathbb{Z}_{e_1} \times \ldots \times \mathbb{Z}_{e_g})^2$ are (non-canonically) isomorphic. We have already equipped the latter group with a skew form. From this point of view a canonical level structure is then nothing but a symplectic isomorphism

$$\alpha : \mathrm{Ker}\,\lambda \cong (\mathbb{Z}_{e_1} \times \ldots \times \mathbb{Z}_{e_g})^2.$$

I.2 Compactifications of Siegel modular varieties

We have already observed that the j-function defines an isomorphism of Riemann surfaces

$$j : X^\circ(1) = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}_1 \cong \mathbb{C}.$$

Clearly this can be compactified to $X(1) = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. It is, however, important to understand this compactification more systematically. The action of the group $\mathrm{SL}(2,\mathbb{Z})$ extends to an action on

$$\overline{\mathbb{H}}_1 = \mathbb{H}_1 \cup \mathbb{Q} \cup \{i\infty\}.$$

The extra points $\mathbb{Q} \cup \{i\infty\}$ form one orbit under this action and we can set

$$X(1) = \mathrm{SL}(2,\mathbb{Z}) \backslash \overline{\mathbb{H}}_1.$$

To understand the structure of $X(1)$ as a Riemann surface we have to consider the stabilizer

$$P(i\infty) = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}; n \in \mathbb{Z} \right\}$$

of the point $i\infty$. It acts on $\mathbb{H}_1$ by $\tau \mapsto \tau + n$. Taking the quotient by $P(i\infty)$ we obtain the map

$$\begin{array}{ccl} \mathbb{H}_1 & \to & D_1^* = \{z \in \mathbb{C}; \quad 0 < |z| < 1\} \\ \tau & \mapsto & t = e^{2\pi i \tau}. \end{array}$$

Adding the origin gives us the "partial compactification" D_1 of D_1^*. For ε sufficiently small no two points in the punctured disc D_ε^* of radius ε are identified under the map from D_1^* to the quotient $\mathrm{SL}(2,\mathbb{Z}) \backslash \mathbb{H}_1$. Hence we obtain $X(1)$ by

$$X(1) = X^\circ(1) \cup_{D_\varepsilon^*} D_\varepsilon.$$

This process is known as "adding the cusp $i\infty$". If we take an arbitrary arithmetic subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ then $\mathbb{Q} \cup \{i\infty\}$ will in general have several, but finitely many, orbits. However, given a representative of such an orbit we can always find an element in $\mathrm{SL}(2,\mathbb{Z})$ which maps this representative to $i\infty$. We can then perform the above construction once more, the only difference being that we will, in general, have to work with a subgroup of $P(i\infty)$. Using this process we can always compactify the quotient $X^\circ(\Gamma) = \Gamma \backslash \mathbb{H}_1$, by adding a finite number of cusps, to a compact Riemann surface $X(\Gamma)$.

The situation is considerably more complicated for higher genus g where it is no longer the case that there is a unique compactification of a quotient $\mathcal{A}(\Gamma) = \Gamma \backslash \mathbb{H}_g$. There have been many attempts to construct suitable compactifications of $\mathcal{A}(\Gamma)$. The first solution was given by Satake [Sa] in the case of $\mathcal{A}_g$. Satake's compactification $\bar{\mathcal{A}}_g$ is in some sense minimal. The boundary $\bar{\mathcal{A}}_g \backslash \mathcal{A}_g$ is set-theoretically the union of the spaces $\mathcal{A}_i, i \leq g-1$. The projective variety $\bar{\mathcal{A}}_g$ is normal but highly singular along the boundary. Satake's compactification was later generalized by Baily and Borel to arbitrary quotients of symmetric domains by arithmetic groups. By blowing up along the boundary, Igusa [I3] constructed a partial desingularization of Satake's compactification. The boundary of Igusa's compactification has codimension 1.

The ideas of Igusa together with work of Hirzebruch on Hilbert modular surfaces were the starting point for Mumford's general theory of toroidal compactifications of quotients of bounded symmetric domains [Mu3]. A detailed description of this theory can be found in [AMRT]. Namikawa showed in [Nam2] that Igusa's compactification is a toroidal compactification in Mumford's sense. Toroidal compactifications depend on the choice of cone decompositions and are, therefore, not unique. The disadvantage of this is that this makes it difficult to give a good modular interpretation for these compactifications. Recently, however, Alexeev and Nakamura [AN],[Ale1], partly improving work of Nakamura and Namikawa [Nak1],[Nam1], have made progress by showing that the toroidal compactification $\mathcal{A}_g^*$ which is given by the second Voronoi decomposition represents a good functor. We shall return to this topic in chapter VI of our survey article.

This survey article is clearly not the right place to give a complete exposition of the construction of compactifications of Siegel modular varieties. Nevertheless we want to sketch the basic ideas behind the construction of the Satake compactification and of toroidal compactifications. We shall start with the *Satake compactification*. For this we consider an arithmetic subgroup Γ of $\mathrm{Sp}(2g, \mathbb{Q})$ for some $g \geq 2$. (This is no restriction since the groups $\mathrm{Sp}(\Lambda, \mathbb{Z})$ which arise for non-principal polarizations are conjugate to subgroups of $\mathrm{Sp}(2g, \mathbb{Q})$). A *modular form* of *weight* k with respect to the group Γ is a holomorphic function

$$F : \mathbb{H}_g \longrightarrow \mathbb{C}$$

with the following transformation behaviour with respect to the group Γ:

$$F(M\tau) = \det(C\tau + D)^k F(\tau) \quad \text{for all } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma.$$

(For $g = 1$ one has to add the condition that F is holomorpic at the cusps, but this is automatic for $g \geq 2$). If Γ acts freely then the automorphy factor $\det(C\tau + D)^k$ defines a line bundle L^k on the quotient $\Gamma\backslash\mathbb{H}_g$. In general some elements in Γ will have fixed points, but every such element is torsion and the order of all torsion elements in Γ is bounded (see e.g. [LB, p.120]). Hence, even if Γ does not act freely, the modular forms of weight nk_0 for some suitable integer k_0 and $n \geq 1$ are sections of a line bundle L^{nk_0}. The space $M_k(\Gamma)$ of modular forms of fixed weight k with respect to Γ is a finite-dimensional vector space and the elements of $M_{nk_0}(\Gamma)$ define a rational map to some projective space $\mathbb{P}^N$. If n is sufficiently large it turns out that this map is actually an immersion and the Satake compactification $\mathcal{A}(\Gamma)$ can be defined as the projective closure of the image of this map.

There is another way of describing the Satake compactification which also leads us to toroidal compactifications. The *Cayley transformation*

$$\begin{array}{rcl} \Phi : \mathbb{H}_g & \to & \mathrm{Sym}(g, \mathbb{C}) \\ \tau & \mapsto & (\tau - i\mathbf{1})(\tau + i\mathbf{1})^{-1} \end{array}$$

realizes $\mathbb{H}_g$ as the symmetric domain

$$\mathcal{D}_g = \{Z \in \mathrm{Sym}(g,\mathbb{C});\ \mathbf{1} - Z\bar{Z} > 0\}.$$

Let $\bar{\mathcal{D}}_g$ be the topological closure of $\mathcal{D}_g$ in $\mathrm{Sym}(g,\mathbb{C})$. The action of $\mathrm{Sp}(2g,\mathbb{R})$ on $\mathbb{H}_g$ defines, via the Cayley transformation, an action on $\mathcal{D}_g$ which extends to $\bar{\mathcal{D}}_g$. Two points in $\bar{\mathcal{D}}_g$ are called equivalent if they can be connected by finitely many holomorphic curves. Under this equivalence relation all points in $\mathcal{D}_g$ are equivalent. The equivalence classes of $\bar{\mathcal{D}}_g \backslash \mathcal{D}_g$ are called the proper boundary components of $\bar{\mathcal{D}}_g$. Given any point $Z \in \bar{\mathcal{D}}_g$ one can associate to it the real subspace $U(Z) = \ker \psi(Z)$ of $\mathbb{R}^{2g}$ where

$$\psi(Z) : \mathbb{R}^{2g} \to \mathbb{C}^g, \nu \mapsto \nu \begin{pmatrix} i(\mathbf{1}+Z) \\ \mathbf{1}-Z \end{pmatrix}.$$

Then $U(Z)$ is an isotropic subspace of $\mathbb{R}^{2g}$ equipped with the standard symplectic form J. Moreover $U(Z) \neq 0$ if and only if $Z \in \bar{\mathcal{D}}_g \backslash \mathcal{D}_g$ and $U(Z_1) = U(Z_2)$ if and only if Z_1 and Z_2 are equivalent. This defines a bijection between the proper boundary components of $\bar{\mathcal{D}}_g$ and the non-trivial isotropic subspaces of $\mathbb{R}^{2g}$.

For any boundary component F we can define its stabilizer in $\mathrm{Sp}(2g,\mathbb{R})$ by

$$\mathcal{P}(F) = \{h \in \mathrm{Sp}(2g,\mathbb{R});\ h(F) = F\}.$$

If $U = U(F)$ is the associated isotropic subspace, then

$$\mathcal{P}(F) = \mathcal{P}(U) = \{h \in \mathrm{Sp}(2g,\mathbb{R});\ Uh^{-1} = U\}.$$

A boundary component F is called *rational* if $\mathcal{P}(F)$ is defined over the rationals or, equivalently, if $U(F)$ is a rational subspace, i.e. can be generated by rational vectors. Adding the rational boundary components to $\mathcal{D}_g$ one obtains the *rational closure* $\mathcal{D}_g^{\mathrm{rat}}$ of $\mathcal{D}_g$. This can be equipped with either the Satake topology or the cylindrical topology. The Satake compactification, as a topological space, is then the quotient $\Gamma\backslash\mathcal{D}_g^{\mathrm{rat}}$. (The Satake topology and the cylindrical topology are actually different, but the quotients turn out to be homeomorphic.) For $g = 1$ the above procedure is easily understood: the Cayley transformation ψ maps the upper half plane $\mathbb{H}_1$ to the unit disc D_1. Under this transformation the rational boundary points $\mathbb{Q} \cup \{i\infty\}$ of $\mathbb{H}_1$ are mapped to the rational boundary points of D_1. The relevant topology is the image under ψ of the *horocyclic topology* on $\overline{\mathbb{H}}_1 = \mathbb{H}_1 \cup \mathbb{Q} \cup \{i\infty\}$.

Given two boundary components F and F' with $F \neq F'$ we say that F is *adjacent* to F' (denoted by $F' \succ F$) if $F \subset \overline{F'}$. This is the case if and only if $U(F') \subsetneqq U(F)$. In this way we obtain two partially ordered sets, namely

$$\begin{aligned}(X_1, <) &= (\{\text{proper rational boundary components } F \text{ of } \mathcal{D}_g\}, \succ)\\ (X_2, <) &= (\{\text{non-trivial isotropic subspaces } U \text{ of } \mathbb{Q}^g\}, \subsetneqq).\end{aligned}$$

The group $\mathrm{Sp}(2g,\mathbb{Q})$ acts on both partially ordered sets as a group of automorphisms and the map $f: X_1 \to X_2$ which associates to each F the isotropic subspace $U(F)$ is an $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant isomorphism of partially ordered sets. To every partially ordered set $(X,<)$ one can associate its *simplicial realization* $\mathrm{SR}(X)$ which is the simplicial complex consisting of all simplices $(x_0,\dots,x_n)$ where $x_0,\dots,x_n \in X$ and $x_0 < x_1 < \dots < x_n$. The *Tits building* $\mathcal{T}$ of $\mathrm{Sp}(2g,\mathbb{Q})$ is the simplicial complex $\mathcal{T} = \mathrm{SR}(X_1) = \mathrm{SR}(X_2)$. If Γ is an arithmetic subgroup of $\mathrm{Sp}(2g,\mathbb{Q})$, then the Tits building of Γ is the quotient $\mathcal{T}(\Gamma) = \Gamma\backslash\mathcal{T}$.

The group $\mathcal{P}(F)$ is a maximal parabolic subgroup of $\mathrm{Sp}(2g,\mathbb{R})$. More generally, given any flag $U_1 \subsetneqq \dots \subsetneqq U_l$ of isotropic subspaces, its stabilizer is a parabolic subgroup of $\mathrm{Sp}(2g,\mathbb{R})$. Conversely any parabolic subgroup is the stabilizer of some isotropic flag. The maximal length of an isotropic flag in $\mathbb{R}^{2g}$ is g and the corresponding subgroups are the minimal parabolic subgroups or *Borel subgroups* of $\mathrm{Sp}(2g,\mathbb{R})$. We have already remarked that a boundary component F is rational if and only if the stabilizer $\mathcal{P}(F)$ is defined over the rationals, which happens if and only if $U(F)$ is a rational subspace. More generally an isotropic flag is rational if and only if its stabilizer is defined over $\mathbb{Q}$. This explains how the Tits building $\mathcal{T}$ of $\mathrm{Sp}(2g,\mathbb{Q})$ can be defined using parabolic subgroups of $\mathrm{Sp}(2g,\mathbb{R})$ which are defined over $\mathbb{Q}$. The Tits building of an arithmetic subgroup Γ of $\mathrm{Sp}(2g,\mathbb{Q})$ can, therefore, also be defined in terms of conjugacy classes of groups $\Gamma \cap \mathcal{P}(F)$.

As an example we consider the integer symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$. There exists exactly one maximal isotropic flag modulo the action of $\mathrm{Sp}(2g,\mathbb{Z})$, namely

$$\{0\} \subsetneqq U_1 \subsetneqq U_2 \subsetneqq \dots \subsetneqq U_g; \quad U_i = \mathrm{span}(e_1,\dots,e_i).$$

Hence the Tits building $\mathcal{T}(\mathrm{Sp}(2g,\mathbb{Z}))$ is a $(g-1)$-simplex whose vertices correspond to the space U_i. This corresponds to the fact that set-theoretically

$$\bar{\mathcal{A}}_g = \mathcal{A}_g \amalg \mathcal{A}_{g-1} \amalg \dots \amalg \mathcal{A}_1 \amalg \mathcal{A}_0.$$

With these preparations we can now sketch the construction of a toroidal compactification of a quotient $\mathcal{A}(\Gamma) = \Gamma\backslash\mathbb{H}_g$ where Γ is an arithmetic subgroup of $\mathrm{Sp}(2g,\mathbb{Q})$. We have to compactify $\mathcal{A}(\Gamma)$ in the direction of the cusps, which are in 1-to-1 correspondence with the vertices of the Tits building $\mathcal{T}(\Gamma)$. We shall first fix one cusp and consider the associated boundary component F, resp. the isotropic subspace $U = U(F)$. Let $\mathcal{P}(F)$ be the stabilizer of F in $\mathrm{Sp}(2g,\mathbb{R})$. Then there is an exact sequence of Lie groups

$$1 \to \mathcal{P}'(F) \to \mathcal{P}(F) \to \mathcal{P}''(F) \to 1$$

where $\mathcal{P}'(F)$ is the centre of the unipotent radical $R_u(\mathcal{P}(F))$ of $\mathcal{P}(F)$. Here $\mathcal{P}'(F)$ is a real vector space isomorphic to $\mathrm{Sym}(g',\mathbb{R})$ where $g' = \dim U(F)$. Let $P(F) = \mathcal{P}(F) \cap \Gamma, P'(F) = \mathcal{P}'(F) \cap \Gamma$ and $P''(F) = P(F)/P'(F)$.

The group $P'(F)$ is a lattice of maximal rank in $\mathcal{P}'(F)$. To F one can now associate a torus bundle $\mathcal{X}(F)$ with fibre $T = P'(F) \otimes_{\mathbb{Z}} \mathbb{C}/P'(F) \cong (\mathbb{C}^*)^{g'}$ over the base $S = F \times V(F)$ where $V(F) = R_u(\mathcal{P}(F))/\mathcal{P}'(F)$ is an affine abelian Lie group and hence a vector space. To construct a partial compactification of $\mathcal{A}(\Gamma)$ in the direction of the cusp corresponding to F, one then proceeds as follows:

(1) Consider the partial quotient $X(F) = P'(F)\backslash\mathbb{H}_g$. This is a torus bundle with fibre $(\mathbb{C}^*)^{\frac{1}{2}g'(g'+1)}$ over some open subset of $\mathbb{C}^{\frac{1}{2}(g(g+1)-g'(g'+1))}$ and can be regarded as an open subset of the torus bundle $\mathcal{X}(F)$.

(2) Choose a fan Σ in the real vector space $\mathcal{P}'(F) \cong \mathrm{Sym}(g', \mathbb{R})$ and construct a trivial bundle $\mathcal{X}_\Sigma(F)$ whose fibres are torus embeddings.

(3) If Σ is chosen compatible with the action of $P''(F)$, then the action of $P''(F)$ on $\mathcal{X}(F)$ extends to an action of $P''(F)$ on $\mathcal{X}_\Sigma(F)$.

(4) Denote by $X_\Sigma(F)$ the interior of the closure of $X(F)$ in $\mathcal{X}_\Sigma(F)$. Define the partial compactification of $\mathcal{A}(\Gamma)$ in the direction of F as the quotient space $Y_\Sigma(F) = P''(F)\backslash X_\Sigma(F)$.

To be able to carry out this programme we may not choose the fan Σ arbitrarily, but we must restrict ourselves to *admissible* fans Σ (for a precise definition see [Nam2, Definition 7.3]). In particular Σ must define a cone decomposition of the cone $\mathrm{Sym}_+(g', \mathbb{R})$ of positive definite symmetric $(g' \times g')$-matrices. The space $Y_\Sigma(F)$ is called the partial compactification in the direction F.

The above procedure describes how to compactify $\mathcal{A}(\Gamma)$ in the direction of one cusp F. This programme then has to be carried out for each cusp in such a way that the partial compactifications glue together and give the desired toroidal compactification. For this purpose we have to consider a collection $\tilde{\Sigma} = \{\Sigma(F)\}$ of fans $\Sigma(F) \subset \mathcal{P}'(F)$. Such a collection is called an *admissible collection of fans* if

(1) Every fan $\Sigma(F) \subset \mathcal{P}'(F)$ is an admissible fan.

(2) If $F = g(F')$ for some $g \in \Gamma$, then $\Sigma(F) = g(\Sigma(F'))$ as fans in the space $\mathcal{P}'(F) = g(\mathcal{P}'(F'))$.

(3) If $F' \succ F$ is a pair of adjacent rational boundary components, then equality $\Sigma(F') = \Sigma(F) \cap \mathcal{P}'(F')$ holds as fans in $\mathcal{P}'(F') \subset \mathcal{P}'(F)$.

The conditions (2) and (3) ensure that the compactifications in the direction of the various cusps are compatible and can be glued together. More precisely we obtain the following:

(2′) If $g \in \Gamma$ with $F = g(F')$, then there exists a natural isomorphism $\tilde{g} : X_{\Sigma(F')}(F') \to X_{\Sigma(F)}(F)$.

($3'$) Suppose $F' \succ F$ is a pair of adjacent rational boundary components. Then $P'(F') \subset P'(F)$ and there exists a natural quotient map $\pi_0(F', F) : X(F') \to X(F)$. Because of (3) this extends to an étale map: $\pi(F', F) : X_{\Sigma(F')}(F') \to X_{\Sigma(F)}(F)$.

We can now consider the disjoint union

$$X = \coprod_F X_{\Sigma(F)}(F)$$

over all rational boundary components F. One can define an equivalence relation on X as follows: if $x \in X_{\Sigma(F)}(F)$ and $x' \in X_{\Sigma(F')}(F')$, then

(a) $x \sim x'$ if there exists $g \in \Gamma$ such that $F = g(F')$ and $x = \tilde{g}(x')$.

(b) $x \sim x'$ if $F' \succ F$ and $\pi(F', F)(x') = x$.

The *toroidal compactification* of $\mathcal{A}(\Gamma)$ defined by the admissible collection of fans $\tilde{\Sigma}$ is then the space

$$\mathcal{A}(\Gamma)^* = X/\sim .$$

Clearly $\mathcal{A}(\Gamma)^*$ depends on $\tilde{\Sigma}$. We could also have described $\mathcal{A}(\Gamma)^*$ as $Y/\sim$ where $Y = \amalg\, Y_{\Sigma(F)}(F)$ and the equivalence relation $\sim$ on Y is induced from that on X. There is a notion of a *projective* admissible collection of fans (see [Nam2, Definition 7.22]) which ensures that the space $\mathcal{A}(\Gamma)^*$ is projective.

For every toroidal compactification there is a natural map $\pi : \mathcal{A}(\Gamma)^* \to \bar{\mathcal{A}}(\Gamma)$ to the Satake compactification. Tai, in [AMRT], showed that if $\mathcal{A}(\Gamma)^*$ is defined by a projective admissible collection of fans, then π is the normalization of the blow-up of some ideal sheaf supported on the boundary of $\bar{\mathcal{A}}(\Gamma)$.

There are several well known cone decompositions for $\mathrm{Sym}_+(g', \mathbb{R})$: see e.g. [Nam2, section 8]. The *central cone decomposition* was used by Igusa [I1] and leads to the *Igusa compactification.* The most important decomposition for our purposes is the *second Voronoi decomposition.* The corresponding compactification is simply called the *Voronoi compactification.* The Voronoi compactification $\mathcal{A}(\Gamma)^* = \mathcal{A}_g^*$ for $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$ is a projective variety [Ale1]. For $g = 2$ all standard known cone decompositions coincide with the *Legendre decomposition.*

II Classification theory

Here we discuss known results about the Kodaira dimension of Siegel modular varieties and about canonical and minimal models. We also report on some work on the fundamental group of Siegel modular varieties.

II.1 The canonical divisor

If one wants to prove results about the Kodaira dimension of Siegel modular varieties, one first has to understand the canonical divisor. For an element $\tau \in \mathbb{H}_g$ we write

$$\tau = \left(\begin{array}{ccc|c} \tau_{11} & \cdots & \tau_{1,g-1} & \tau_{1g} \\ \vdots & & & \vdots \\ \tau_{1,g-1} & \cdots & \tau_{g-1,g-1} & \tau_{g-1,g} \\ \hline \tau_{1g} & \cdots & \tau_{g-1,g} & \tau_{gg} \end{array}\right) = \left(\begin{array}{c|c} \tau' & {}^t z \\ \hline z & \tau_{gg} \end{array}\right).$$

Let

$$d\tau = d\tau_{11} \wedge d\tau_{12} \wedge \ldots \wedge d\tau_{gg}.$$

If F is a modular form of weight $g+1$ with respect to an arithmetic group Γ, then it is easy to check that the form $\omega = F d\tau$ is Γ-invariant. Hence, if Γ acts *freely*, then

$$K_{\mathcal{A}(\Gamma)} = (g+1)L$$

where L is the line bundle of modular forms, i.e. the line bundle given by the automorphy factor $\det(C\tau + D)$. If Γ does not act freely, let $^\circ\!\mathcal{A}(\Gamma) = \mathcal{A}(\Gamma)\backslash R$ where R is the branch locus of the quotient map $\mathbb{H}_g \to \mathcal{A}(\Gamma)$. Then by the above reasoning it is still true that

$$K_{^\circ\!\mathcal{A}(\Gamma)} = (g+1)L|_{^\circ\!\mathcal{A}(\Gamma)}.$$

In order to describe the canonical bundle on a toroidal compactification $\mathcal{A}(\Gamma)^*$ we have to understand the behaviour of the differential form ω at the boundary. To simplify the exposition, we shall first consider the case $\Gamma_g = \mathrm{Sp}(2g, \mathbb{Z})$. Then there exists, up to the action of Γ, exactly one maximal boundary component F. We can assume that $U(F) = U = \mathrm{span}(e_g)$. The stabilizer $P(F) = P(U)$ of U in Γ_g is generated by elements of the form

$$g_1 = \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} \mathbf{1}_{g-1} & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{g-1} & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} \mathbf{1}_{g-1} & 0 & 0 & {}^tN \\ M & 1 & N & 0 \\ 0 & 0 & \mathbf{1}_{g-1} & -{}^tM \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_4 = \begin{pmatrix} \mathbf{1}_{g-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & S \\ 0 & 0 & \mathbf{1}_{g-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{g-1}$, $M, N \in \mathbb{Z}^{g-1}$ and $S \in \mathbb{Z}$.

The group $P'(F)$ is the rank 1 lattice generated by g_4, and the partial quotient with respect to $P'(F)$ is given by

$$\begin{array}{rcl} e(F): \ \mathbb{H}_g & \longrightarrow & \mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \mathbb{C}^* \\ \tau & \longmapsto & (\tau', z, t = e^{2\pi i \tau_{gg}}). \end{array}$$

Here $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \mathbb{C}^*$ is a rank 1 torus bundle over $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} = F \times V(F)$. Partial compactification in the direction of F consists of adding $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \{0\}$ and then taking the quotient with respect to $P''(F)$. Since $d\tau_{gg} = (2\pi i)^{-1} dt/t$ it follows that

$$\omega = (2\pi i)^{-1} F \frac{d\tau_{11} \wedge \ldots \wedge d\tau_{g-1,g} \wedge dt}{t}$$

has a pole of order 1 along the boundary, unless F vanishes there. Moreover, since $F(g_4(\tau)) = F(\tau)$ it follows that F has a Fourier expansion

$$F(\tau) = \sum_{n \geq 0} F_n(\tau', z) t^n.$$

A modular form F is a *cusp form* if $F_0(\tau', z) = 0$, i.e. if F vanishes along the boundary. (If Γ is an arbitrary arithmetic subgroup of $\mathrm{Sp}(2g, \mathbb{Q})$ we have in general several boundary components and then we require vanishing of F along each of these boundary components.) The above discussion can be interpreted as follows. First assume that Γ is neat (i.e. the subgroup of $\mathbb{C}^*$ generated by the eigenvalues of all elements of Γ is torsion free) and that $\mathcal{A}(\Gamma)^*$ is a smooth compactification with the following property: for every point in the boundary there exists a representative $x \in X_{\Sigma(F)}(F)$ for some boundary component such that $X_{\Sigma(F)}(F)$ is smooth at x and $P''(F)$ acts freely at x. (Such a toroidal compactification always exists if Γ is neat.) Let D be the boundary divisor of $\mathcal{A}(\Gamma)^*$. Then

$$K_{\mathcal{A}(\Gamma)^*} = (g+1)L - D.$$

Here L is the extension of the line bundle on modular forms on $\mathcal{A}(\Gamma)$ to $\mathcal{A}(\Gamma)^*$. This makes sense since by construction the line bundle extends to the Satake compactification $\bar{\mathcal{A}}(\Gamma)$ and since there is a natural map $\pi : \mathcal{A}(\Gamma)^* \to \bar{\mathcal{A}}(\Gamma)$. We use the same notation for L and $\pi^* L$. If Γ does not act freely we can define the open set ${}^{\circ}\!\mathcal{A}(\Gamma)^*$ consisting of ${}^{\circ}\!\mathcal{A}(\Gamma)$ and those points in the boundary which have a representative $x \in X_{\Sigma(F)}(F)$ where $P''(F)$ acts freely at x. In this case we still have

$$K_{{}^{\circ}\!\mathcal{A}(\Gamma)^*} = ((g+1)L - D)|_{{}^{\circ}\!\mathcal{A}(\Gamma)^*}.$$

This shows in particular that every cusp form F of weight $g+1$ with respect to Γ defines via $\omega = F d\tau$ a differential N-form on ${}^{\circ}\!\mathcal{A}(\Gamma)^*$ where $N = \frac{g(g+1)}{2}$ is the dimension of $\mathcal{A}(\Gamma)$. It is a non-trivial result of Freitag that every such form can be extended to any smooth projective model of $\mathcal{A}(\Gamma)$. If we denote by $S_k(\Gamma)$ the space of cusp forms of weight k with respect to Γ, then we can formulate Freitag's result as follows.

Theorem II.1.1 ([F]) *Let $\tilde{\mathcal{A}}(\Gamma)$ be a smooth projective model of $\mathcal{A}(\Gamma)$. Then every cusp form F of weight $g+1$ with respect to Γ defines a differential form $\omega = F d\tau$ which extends to $\tilde{\mathcal{A}}(\Gamma)$. In particular, there is a natural isomorphism*

$$\Gamma(\tilde{\mathcal{A}}(\Gamma), \omega_{\tilde{\mathcal{A}}(\Gamma)}) \cong S_{g+1}(\Gamma)$$

and hence $p_g(\tilde{\mathcal{A}}(\Gamma)) = \dim S_{g+1}(\Gamma)$.

Proof. See [F, Satz III.2.6] and the remark following this. □

Similarly a form of weight $k(g+1)$ which vanishes of order k along the boundary defines a k-fold differential form on $^{\circ}\!\mathcal{A}(\Gamma)^*$. In general, however, such a form does not extend to a smooth model $\tilde{\mathcal{A}}(\Gamma)$ of $\mathcal{A}(\Gamma)$.

II.2 The Kodaira dimension of $\mathcal{A}_g(n)$

By the *Kodaira dimension* of a Siegel modular variety $\mathcal{A}(\Gamma)$ we mean the Kodaira dimension of a smooth projective model of $\mathcal{A}(\Gamma)$. Such a model always exists and the Kodaira dimension is independent of the specific model chosen. It is a well known result that $\mathcal{A}_g$ is of general type for $g \geq 7$. This was first proved by Tai for $g \geq 9$ [T1] and then improved to $g \geq 8$ by Freitag [F] and to $g \geq 7$ by Mumford [Mu4]. In this section we want to discuss the proof of the following result.

Theorem II.2.1 ([T1],[F],[Mu4],[H2]) *$\mathcal{A}_g(n)$ is of general type for the following values of g and $n \geq n_0$:*

g	2	3	4	5	6	≥ 7
n_0	4	3	2	2	2	1

We have already seen that the construction of differential forms is closely related to the existence of cusp forms. Using Mumford's extension of Hirzebruch proportionality to the non-compact case and the Atiyah-Bott fixed point theorem it is not difficult to show that the dimension of the space of cusp forms of weight k grows as follows:

$$\dim S_k(\Gamma_g) \sim 2^{-N-g} k^N V_g \pi^{-N}$$

where

$$N = \frac{g(g+1)}{2} = \dim \mathcal{A}_g(n)$$

and V_g is Siegel's symplectic volume

$$V_g = 2^{g^2+1}\pi^N \prod_{j=1}^{g} \frac{(j-1)!}{2j!} B_j.$$

Here B_j are the Bernoulli numbers.

Every form of weight $k(g+1)$ gives rise to a k-fold differential form on $\mathring{\mathcal{A}}_g(n)$. If $k=1$, we have already seen that these forms extend by Freitag's extension theorem to every smooth model of $\mathcal{A}_g(n)$. This is no longer automatically the case if $k \geq 2$. Then one encounters two types of obstructions: one is extension to the boundary (since we need higher vanishing order along D), the other type of obstruction comes from the singularities, or more precisely from those points where $\Gamma_g(n)$ does not act freely. These can be points on $\mathcal{A}_g(n)$ or on the boundary. If $n \geq 3$, then $\Gamma_g(n)$ is neat and in particular it acts freely. Moreover we can choose a suitable cone decomposition such that the corresponding toroidal compactification is smooth. In this case there are no obstructions from points where $\Gamma_g(n)$ does not act freely. If $n=1$ or 2 we shall, however, always have such points. It is one of the main results of Tai [T1, Section 5] that for $g \geq 5$ all resulting singularities are *canonical*, i.e. give no obstructions to extending k-fold differential forms to a smooth model. The remainder of the proof of Tai then consists of a careful analysis of the obstructions to the extension of k-forms to the boundary. These obstructions lie in a vector space which can be interpreted as a space of Jacobi forms on $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1}$. Tai gives an estimate of this space in [T1, Section 2] and compares it with the dimension formula for $S_k(\Gamma_g)$.

The approach developed by Mumford in [Mu4] is more geometric in nature. First recall that

$$K|_{\mathring{\mathcal{A}}_g^*(n)} = ((g+1)L - D)|_{\mathring{\mathcal{A}}_g^*(n)}. \tag{1}$$

Let $\bar{\Theta}_{\mathrm{null}}$ be the closure of the locus of pairs (A, Θ) where A is an abelian variety and Θ is a symmetric divisor representing a principal polarization such that Θ has a singularity at a point of order 2. Then one can show that for the class of $\bar{\Theta}_{\mathrm{null}}$ on $\mathcal{A}_g^*(n)$:

$$[\bar{\Theta}_{\mathrm{null}}] = 2^{g-2}(2^g+1)L - 2^{2g-5}D. \tag{2}$$

One can now use (2) to eliminate the boundary D in (1). Since the natural quotient $\mathcal{A}_g^*(n) \to \mathcal{A}_g^*$ is branched of order n along D one finds the following formula for K:

$$K|_{\mathring{\mathcal{A}}_g^*(n)} = \left((g+1) - \frac{2^{g-2}(2^g+1)}{n2^{2g-5}}\right)L + \frac{1}{n2^{2g-5}}[\bar{\Theta}_{\mathrm{null}}]. \tag{3}$$

In view of Tai's result on the singularities of $\mathcal{A}_g^*(n)$ this gives general type whenever the factor in front of L is positive and $n \geq 3$ or $g \geq 5$. This gives all cases in the list with two exceptions, namely $(g,n) = (4,2)$ and $(7,1)$. In the first case the factor in front of L is still positive, but one cannot immediately invoke Tai's result on canonical singularities. As Salvetti Manni has pointed out, one can, however, argue as follows. An easy calculation shows that for every element $\sigma \in \Gamma_g(2)$ the square $\sigma^2 \in \Gamma_g(4)$. Hence if σ has a fixed point then $\sigma^2 = 1$ since $\Gamma_g(4)$ acts freely. But now one can again use Tai's extension result (see [T1, Remark after Lemma 4.5] and [T1, Remark after Lemma 5.2]).

This leaves the case $(g,n) = (7,1)$ which is the main result of [Mu4]. Mumford considers the locus

$$N_0 = \{(A,\Theta)\ ;\ \operatorname{Sing}\Theta \neq \emptyset\}$$

in $\mathcal{A}_g$. Clearly this contains Θ_{null}, but is bigger than Θ_{null} if $g \geq 4$. Mumford shows that the class of the closure $\bar{N}_0$ on $\mathcal{A}_g^*$ is

$$[\bar{N}_0] = \left(\frac{(g+1)!}{2} + g!\right) L - \frac{(g+1)!}{12} D \tag{4}$$

and hence one finds for the canonical divisor:

$$K|_{{}^\circ\!\mathcal{A}_g^*(n)} = \frac{12(g^2-4g-17)}{g+1} L + \frac{12}{(g+1)!}[\bar{N}_0].$$

Since the factor in front of L is positive for $g = 7$ one can once more use Tai's extension result to prove the theorem for $(g,n) = (7,1)$.

The classification of the varieties $\mathcal{A}_g(n)$ with respect to the Kodaira dimension is therefore now complete with the exception of one important case:

Problem Determine the Kodaira dimension of $\mathcal{A}_6$.

All other varieties $\mathcal{A}_g(n)$ which do not appear in the above list are known to be either rational or unirational. Unirationality of $\mathcal{A}_5$ was proved by Donagi [D] and independently by Mori and Mukai [MM] and Verra [V]. Unirationality of $\mathcal{A}_4$ was shown by Clemens [Cl] and unirationality of $\mathcal{A}_g, g \leq 3$ is easy. For $g = 3$ there exists a dominant map from the space of plane quartics to $\mathcal{M}_3$ which in turn is birational to $\mathcal{A}_3$. For $g = 2$ one can use the fact that $\mathcal{M}_2$ is birational to $\mathcal{A}_2$ and that every genus 2 curve is a 2:1 cover of $\mathbb{P}^1$ branched in 6 points. Rationality of these spaces is a more difficult question. Igusa [I1] showed that $\mathcal{A}_2$ is rational. The rationality of $\mathcal{M}_3$, and hence also of $\mathcal{A}_3$, was proved by Katsylo [K]. The space $\mathcal{A}_3(2)$ is rational by the work of van Geemen [vG] and Dolgachev and Ortland [DO]. The variety $\mathcal{A}_2(3)$ is birational to the Burkhardt quartic in $\mathbb{P}^4$ and hence also rational. This was proved by Todd in 1936 [To] and Baker in 1942 (see [Ba2]), but see also the thesis of Finkelnberg [Fi]. The variety $\mathcal{A}_2(2)$ is birational to the Segre cubic (cf. [vdG1]) in $\mathbb{P}^4$ and hence also rational. The latter two cases are examples of Siegel modular varieties which have very interesting projective models. We will come back to this more systematically in chapter IV. It should also be noted that Yamazaki [Ya] was the first to prove that $\mathcal{A}_2(n)$ is of general type for $n \geq 4$.

All the results discussed above concern the case of principal polarization. The case of non-principal polarizations of type $(e_1, \dots, e_g)$ was also studied by Tai.

Theorem II.2.2 ([T2]) *The moduli space $\mathcal{A}_{e_1,\dots,e_g}$ of abelian varieties with a polarization of type $(e_1, \dots, e_g)$ is of general type if either $g \geq 16$ or $g \geq 8$ and all e_i are odd and sums of two squares.*

The essential point in the proof is the construction of sufficiently many cusp forms with high vanishing order along the boundary. These modular forms are obtained as pullbacks of theta series on Hermitian or quaternionic upper half spaces.

More detailed results are known in the case of abelian surfaces ($g = 2$). We will discuss this separately in chapters III and V.

By a different method, namely using symmetrization of modular forms, Gritsenko has shown the following:

Theorem II.2.3 ([Gr1]) *For every integer t there is an integer $g(t)$ such that the moduli space $\mathcal{A}_{1,\dots,1,t}$ is of general type for $g \geq g(t)$. In particular $\mathcal{A}_{1,\dots,1,2}$ is of general type for $g \geq 13$.*

Proof. See [Gr1, Satz 1.1.10], where an explicit bound for $g(t)$ is given. $\square$

Once one has determined that a variety is of general type it is natural to ask for a minimal or canonical model. For a given model this means asking whether the canonical divisor is nef or ample. In fact one can ask more generally what the nef cone is. The Picard group of $\mathcal{A}_g^*, g = 2, 3$ is generated (modulo torsion) by two elements, namely the ($\mathbb{Q}$-) line bundle L given by modular forms of weight 1 and the boundary D.

In [H2] one of us computed the nef cone of $\mathcal{A}_g^*, g = 2, 3$. The result is given by the theorem below. As we shall see one can give a quick proof of this using known results about $\overline{\mathcal{M}}_g$ and the Torelli map. However this approach cannot be generalized to higher genus since the Torelli map is then no longer surjective, nor to other than principal polarizations. For this reason an alternative proof was given in [H2] making essential use of a result of Weissauer [We] on the existence of cusp forms of small slope which do not vanish on a given point in Siegel space.

Theorem II.2.4 *Let $g = 2$ or 3. Then a divisor $aL - bD$ on $\mathcal{A}_g^*$ is nef if and only if $b \geq 0$ and $a - 12b \geq 0$.*

Proof. First note that the two conditions are necessary. In fact let C be a curve which is contracted under the natural map $\pi : \mathcal{A}_g^* \to \bar{A}_g$ onto the Satake compactification. The divisor $-D$ is π-ample (cf. also [Mu4]) and L is the pull-back of a line bundle on $\bar{A}_g$. Hence $(aL - bD).C \geq 0$ implies $b \geq 0$. Let C be the closure of the locus given by split abelian varieties $E \times A'$ where E is an arbitrary elliptic curve and A' is a fixed abelian variety of dimension $g-1$. Then C is a rational curve with $D.C = 1$ and $L.C = 1/12$. This shows that $a - 12b \geq 0$ for every nef divisor D.

To prove that the conditions stated are sufficient we consider the Torelli map $t : \mathcal{M}_g \to \mathcal{A}_g$ which extends to a map $\bar{t} : \overline{\mathcal{M}}_g \to \mathcal{A}_g^*$. This map is surjective for $g = 2, 3$. Here $\overline{\mathcal{M}}_g$ denotes the compactification of $\mathcal{M}_g$ by stable curves. It follows that for every curve C in $\mathcal{A}_g^*$ there exists a curve C' in $\overline{\mathcal{M}}_g$ which is finite over C. Hence a divisor on $\mathcal{A}_g^*, g = 2, 3$ is nef if and only if this is true for its pull-back to $\overline{\mathcal{M}}_g$. We can now use Faber's paper [Fa]. Then $\bar{t}^* L = \lambda$ where λ is the Hodge bundle and $\bar{t}^* D = \delta_0$. Here δ_0 is the boundary ($g = 2$), resp. the closure of the locus of genus 2 curves with one node ($g = 3$). The result now follows from [Fa] since $a\lambda - b\delta_0$ is nef on $\overline{\mathcal{M}}_g$, $g = 2, 3$ if $b \geq 0$ and $a - 12b \geq 0$. □

Corollary II.2.5 *The canonical divisor on $\mathcal{A}_2^*(n)$ is nef but not ample for $n = 4$ and ample for $n \geq 5$. In particular $\mathcal{A}_2^*(4)$ is a minimal model and $\mathcal{A}_2^*(n)$ is a canonical model for $n \geq 5$.*

This was first observed, though not proved in detail, by Borisov in an early version of [Bori].

Corollary II.2.6 *The canonical divisor on $\mathcal{A}_3^*(n)$ is nef but not ample for $n = 3$ and ample for $n \geq 4$. In particular $\mathcal{A}_3^*(3)$ is a minimal model and $\mathcal{A}_3^*(n)$ is a canonical model for $n \geq 4$*

Proof of the corollaries. Nefness or ampleness of K follows immediately from Theorem II.2.4 since

$$(g+1) - \frac{12}{n} \geq 0 \Leftrightarrow \begin{cases} n \geq 4 & \text{if} \quad g = 2 \\ n \geq 3 & \text{if} \quad g = 3. \end{cases}$$

To see that K is not ample on $\mathcal{A}_2^*(4)$ nor on $\mathcal{A}_3^*(3)$ we can again use the curves C coming from products $E \times A'$ where A' is a fixed abelian variety of dimension $g - 1$. For these curves $K.C = 0$. □

For $g \geq 4$ it is, contrary to what was said in [H2], no longer true that the Picard group is generated by L and D. Here we simply state the

Problem Describe the nef cone of $\mathcal{A}_g^*$.

In [H3] the methods of [H2] were used to prove ampleness of K in the case of $(1,p)$-polarized abelian surfaces with a canonical level structure and a level-n structure, for p prime and $n \geq 5$, provided p does not divide n.

Finally we want to mention some results concerning the Chow ring of $\mathcal{A}_g^*$. The Chow groups considered here are defined as the invariant part of the Chow ring of $\mathcal{A}_g^*(n)$. The Chow ring of $\overline{\mathcal{M}}_2$ was computed by Mumford [Mu5]. This gives also the Chow ring of $\mathcal{A}_2^*$, which was also calculated by a different method by van der Geer in [vdG3].

Theorem II.2.7 ([Mu5],[vdG3]) *Let $\lambda_1 = \lambda$ and λ_2 be the tautological classes on $\mathcal{A}_2^*$. Let σ_1 be the class of the boundary. Then*

$$\mathrm{CH}_{\mathbb{Q}}(\mathcal{A}_2^*) \cong \mathbb{Q}[\lambda_1, \lambda_2, \sigma_1]/I$$

where I is the ideal generated by the relations

$$\begin{gathered}(1+\lambda_1+\lambda_2)(1-\lambda_1+\lambda_2) = 1,\\ \lambda_2\sigma_1 = 0,\\ \sigma_1^2 = 22\sigma_1\lambda_1 - 120\lambda_1^2.\end{gathered}$$

The ranks of the Chow groups are $1, 2, 2, 1$.

Van der Geer also computed the Chow ring of $\mathcal{A}_3^*$.

Theorem II.2.8 ([vdG3]) *Let $\lambda_1, \lambda_2, \lambda_3$ be the tautological classes in $\mathcal{A}_3^*$ and σ_1, σ_2 be the first and second symmetric functions in the boundary divisors (viewed as an invariant class on $\mathcal{A}_g^*(n)$). Then*

$$\mathrm{CH}_{\mathbb{Q}}(\mathcal{A}_3^*) \cong \mathbb{Q}[\lambda_1, \lambda_2, \lambda_3, \sigma_1, \sigma_2]/J$$

where J is the ideal generated by the relations

$$\begin{array}{lcl}(1+\lambda_1+\lambda_2+\lambda_3)(1-\lambda_1+\lambda_2-\lambda_3) = 1 & & ,\\ \lambda_3\sigma_1 & = & \lambda_3\sigma_2 = \lambda_1^2\sigma_2 = 0,\\ \sigma_1^3 & = & 2016\lambda_3 - 4\lambda_1^2\sigma_1 - 24\lambda_1\sigma_2 + \frac{11}{3}\sigma_2\sigma_1,\\ \sigma_2^2 & = & 360\lambda_1^3\sigma_1 - 45\lambda_1^2\sigma_1^2 + 15\lambda_1\sigma_2\sigma_1.\end{array}$$

The ranks of the Chow groups are $1, 2, 4, 6, 4, 2, 1$.

Proof. See [vdG3]. The proof uses in an essential way the description of the Voronoi compactification $\mathcal{A}_3^*$ given by Nakamura [Nak1] and Tsushima [Ts]. □

II.3 Fundamental groups

The fundamental group of a smooth projective model $\widetilde{\mathcal{A}}(\Gamma)$ of $\mathcal{A}(\Gamma)$ is independent of the specific model chosen. We assume in this section that $g \geq 2$, so that the dimension of $\mathcal{A}(\Gamma)$ is at least 3.

The first results about the fundamental group of $\widetilde{\mathcal{A}}(\Gamma)$ were obtained by Heidrich and Knöller [HK], [Kn] and concern the principal congruence subgroups $\Gamma(n) \subset \mathrm{Sp}(2g, \mathbb{Z})$. They proved the following result.

Theorem II.3.1 ([HK],[Kn]) *If $n \geq 3$ or if $n = g = 2$ then $\widetilde{\mathcal{A}}_g(n)$ is simply-connected.*

As an immediate corollary (first explicitly pointed out by Heidrich-Riske) one has

Corollary II.3.2 ([H–R]) *If Γ is an arithmetic subgroup of* $\mathrm{Sp}(2g,\mathbb{Q})$, *then the fundamental group of $\widetilde{\mathcal{A}}(\Gamma)$ is finite.*

Corollary II.3.2 follows from Theorem II.3.1 because any subgroup of $\mathrm{Sp}(2g,\mathbb{Z})$ of finite index contains a principal congruence subgroup of some level.

*Proof.*The proof of Theorem II.3.1 uses the fact that there is, up to the action of the group $\mathrm{Sp}(2g,\mathbb{Z}_n)$, only one codimension 1 boundary component F in the Igusa compactification $\mathcal{A}_g^*(n)$. Suppose for simplicity that $n \geq 4$, so that $\Gamma(n)$ is neat. A small loop passing around this component can be identified with a loop in the fibre $\mathbb{C}^*$ of $\mathcal{X}(F)$ and hence with a generator u_F of the 1-dimensional lattice $P'(F)$. This loop determines an element γ_F, usually non-trivial, of $\pi_1\big(\mathcal{A}_g(n)\big)$ (which is simply $\Gamma(n)$, since $\Gamma(n)$ is torsion-free and hence acts freely on $\mathbb{H}_g$). The element γ_F is in the kernel of the map $\pi_1\big(\mathcal{A}_g(n)\big) \to \pi_1\big(\widetilde{\mathcal{A}}_g(n)\big)$, so u_F is in the kernel of $\Gamma(n) \to \pi_1\big(\widetilde{\mathcal{A}}_g(n)\big)$. But it turns out that the normalizer of $P'(F)$ in $\Gamma(n)$ is the whole of $\Gamma(n)$, as was shown by Mennicke [Me] by a direct calculation. $\square$

We (the authors of the present article) applied this method in [HS2] to the case of $\mathcal{A}_{1,p}^{\mathrm{lev}}$ for $p \geq 5$ prime, where there are many codimension 1 boundary components. A minor extra complication is the presence of some singularities in $\Gamma\backslash\mathbb{H}_2$, but they are easily dealt with. In [S1] one of us also considered the case of $\mathcal{A}_{1,p}$. We found the following simple result.

Theorem II.3.3 ([HS2],[S1]) *If $p \geq 5$ is prime then $\tilde{\mathcal{A}}_{1,p}^{\mathrm{lev}}$ and $\tilde{\mathcal{A}}_{1,p}$ are both simply-connected.*

In some other cases one knows that $\widetilde{\mathcal{A}}(\Gamma)$ is rational and hence simply-connected. In all these cases, as F. Campana pointed out, it follows that the Satake compactification, and any other normal model, is also simply-connected.

By a more systematic use of these ideas, one of us [S1] gave a more general result, valid in fact for all locally symmetric varieties over $\mathbb{C}$. From it several results about Siegel modular varieties can be easily deduced, of which Theorem II.3.4 below is the most striking.

Theorem II.3.4 ([S1]) *For any finite group G there exists a $g \geq 2$ and an arithmetic subgroup $\Gamma \subset \mathrm{Sp}(2g,\mathbb{Z})$ such that $\pi_1\big(\widetilde{\mathcal{A}}(\Gamma)\big) \cong G$.*

Proof. We choose an $l \geq 4$ and a faithful representation $\rho : G \to \mathrm{Sp}(2g, \mathbb{F}_p)$ for some prime p not dividing $2l|G|$. The reduction mod p map $\phi_p : \Gamma(l) \to \mathrm{Sp}(2g, \mathbb{F}_p)$ is surjective and we take $\Gamma = \phi_p^{-1}\big(\rho(G)\big)$. As this is a subgroup of $\Gamma(l)$ it is neat, and under these circumstances the fundamental group of the corresponding smooth compactification of $\mathcal{A}(\Gamma)$ is Γ/Υ, where Υ is a certain subgroup of Γ generated by unipotent elements (each unipotent element corresponds to a loop around a boundary component). From this it follows that $\Upsilon \subset \operatorname{Ker} \phi_p = \Gamma(pl)$. Then from Theorem II.3.1 applied to level pl it follows that $\Upsilon = \Gamma(pl)$ and hence that the fundamental group is $\Gamma/\Gamma(pl) \cong G$. □

For $G = D_8$ we may take $g = 2$; in particular, the fundamental group of a smooth projective model of a Siegel modular threefold need not be abelian. Apart from the slightly artificial examples which constitute Theorem II.3.4, it is also shown in [S1] that a smooth model of the double cover $\tilde{\mathcal{N}}_5$ of Nieto's threefold $\mathcal{N}_5$ has fundamental group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The space $\tilde{\mathcal{N}}_5$ will be discussed in Section IV below: it is birational with the moduli space of abelian surfaces with a polarization of type $(1, 3)$ and a level-2 structure.

III Abelian surfaces

In the case of abelian surfaces the moduli spaces $\mathcal{A}_{1,t}$ and $\mathcal{A}_{1,t}^{\mathrm{lev}}$ of abelian surfaces with a $(1,t)$-polarization, resp. with a $(1,t)$-polarization and a canonical level structure were investigated by a number of authors. One of the starting points for this development was the paper by Horrocks and Mumford [HM] which established a connection between the Horrocks-Mumford bundle on $\mathbb{P}^4$ and the moduli space $\mathcal{A}_{1,5}^{\mathrm{lev}}$.

III.1 The lifting method

Using a version of Maaß lifting Gritsenko has proved the existence of a weight 3 cusp forms for almost all values of t. Before we can describe his lifting result recall the *paramodular group* $\mathrm{Sp}(\Lambda, \mathbb{Z})$ where

$$\Lambda = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \; E = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$$

for some integer $t \geq 1$, with respect to a basis (e_1, e_2, e_3, e_4). This group is conjugate to the (rational) paramodular group

$$\Gamma_{1,t} = R^{-1} \,\mathrm{Sp}(\Lambda, \mathbb{Z}) R, \qquad R = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & t \end{pmatrix}.$$

It is straightforward to check that

$$\Gamma_{1,t} = \left\{ g \in \mathrm{Sp}(4,\mathbb{Q});\ g \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t\mathbb{Z} \\ t\mathbb{Z} & \mathbb{Z} & t\mathbb{Z} & t\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t\mathbb{Z} \\ \mathbb{Z} & t^{-1}\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}.$$

Then $\mathcal{A}_{1,t} = \Gamma_{1,t}\backslash\mathbb{H}_2$ is the moduli space of $(1,t)$-polarized abelian surfaces. In this chapter we shall denote the elements of $\mathbb{H}_2$ by

$$\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathbb{H}_2 .$$

The Tits building of $\Gamma_{1,t}$, and hence the combinatorial structure of the boundary components of the Satake or the Voronoi (Igusa) compactification of $\mathcal{A}_{1,t}$ are known, at least if t is square free: see [FrS], where Tits buildings for some other groups are also calculated. There are exactly $\mu(t)$ corank 1 boundary components, where $\mu(t)$ denotes the number of prime divisors of t [Gr1, Folgerung 2.4]. If t is square free, then there exists exactly one corank 2 boundary component [Fr, Satz 4.7]. In particular, if $t > 1$ is a prime number then there exist two corank 1 boundary components and one corank 2 boundary component. These boundary components belong to the isotropic subspaces spanned by e_3 and e_4, resp. by $e_3 \wedge e_4$. In terms of the Siegel space the two corank 1 boundary components correspond to $\tau_1 \to i\infty$ and $\tau_3 \to i\infty$. For $t = 1$ these two components are equivalent under the group $\Gamma_{1,1} = \mathrm{Sp}(4,\mathbb{Z})$.

Gritsenko's construction of cusp forms uses a version of Maaß lifting. In order to explain this, we first have to recall the definition of *Jacobi forms.* Here we restrict ourselves to the case of $\Gamma_{1,1} = \mathrm{Sp}(4,\mathbb{Z})$. The stabilizer of $\mathbb{Q}e_4$ in $\mathrm{Sp}(4,\mathbb{Z})$ has the structure

$$P(e_4) \cong \mathrm{SL}(2,\mathbb{Z}) \ltimes H(\mathbb{Z})$$

where $\mathrm{SL}(2,\mathbb{Z})$ is identified with

$$\left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \right\}$$

and

$$H(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & r \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix};\lambda,\mu,r \in \mathbb{Z} \right\}$$

is the *integral Heisenberg group.*

Every modular form $F \in M_k(\mathrm{Sp}(4,\mathbb{Z}))$ of weight k with respect to $\mathrm{Sp}(4,\mathbb{Z})$ has a Fourier extension with respect to τ_3 which is of the following form

$$F(\tau) = \sum_{m\geq 0} f_m(\tau_1,\tau_2)e^{2\pi i m\tau_3}.$$

The same is true for modular forms with respect to $\Gamma_{1,t}$, the only difference is that the factor $\exp(2\pi i m\tau_3)$ has to be replaced by $\exp(2\pi i m t\tau_3)$. The coefficients $f_m(\tau_1,\tau_2)$ are examples of *Jacobi forms*. Formally Jacobi forms are defined as follows:

Definition A *Jacobi form* of *index* m and *weight* k is a holomorphic function

$$\Phi = \Phi(\tau, z) : \mathbb{H}_1 \times \mathbb{C} \to \mathbb{C}$$

which has the following properties:

(1) It has the *transformation behaviour*

(a) $\Phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi i c m z^2}{c\tau+d}}\Phi(\tau,z), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$

(b) $\Phi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau+2\lambda z)}\Phi(\tau,z), \quad \lambda,\mu\in\mathbb{Z}.$

(2) It has a Fourier expansion

$$\Phi(\tau,z) = \sum_{\substack{n,l\in\mathbb{Z},n\geq 0\\ 4nm\geq l^2}} f(n,l)e^{2\pi i(n\tau+lz)}.$$

A Jacobi form is called a *cusp form* if one has strict inequality $4nm > l^2$ in the Fourier expansion.

Note that for $z=0$ the transformation behaviour described by (1)(a) is exactly that of a modular form. For fixed τ the transformation law (1)(b) is, up to a factor 2 in the exponent, the transformation law for theta functions. One can also summarize (1)(a) and (1)(b) by saying that $\Phi = \Phi(\tau,z)$ is a modular form with respect to the *Jacobi group* $\mathrm{SL}(2,\mathbb{Z}) \ltimes H(\mathbb{Z})$. (Very roughly, Jacobi forms can be thought of as sections of a suitable $\mathbb{Q}$-line bundle over the universal elliptic curve, which doesn't actually exist.) The Jacobi forms of weight k and index m form a vector space $J_{k,m}$ of finite dimension. The standard reference for Jacobi forms is the book by Eichler and Zagier [EZ].

As we have said before, Jacobi forms arise naturally as coefficients in the Fourier expansion of modular forms. These coefficients are functions, or more precisely sections of a suitable line bundle, on a boundary component of the Siegel modular threefold. The idea of lifting is to reverse this process.

Starting with a Jacobi form one wants to construct a Siegel modular form where this Jacobi form appears as a Fourier coefficient. This idea goes back to Maaß [Ma2] and has in recent years been refined in several ways by Gritsenko, Borcherds and others: see e.g.[Gr1], [Gr3], [GrN] and [Borc]. The following lifting result is due to Gritsenko.

Theorem III.1.1 ([Gr1]) *There is a lifting, i.e. an embedding*

$$\mathrm{Lift} : J_{k,t} \longrightarrow M_k(\Gamma_{1,t})$$

of the space of Jacobi forms of weight k and index t into the space of modular forms of weight k with respect to the paramodular group $\Gamma_{1,t}$. The lifting of a Jacobi cusp form is again a cusp form.

Proof. For details see [Gr1, Hauptsatz 2.1] or [Gr2, Theorem 3]. For a Jacobi form $\Phi = \Phi(\tau z)$ with Fourier expansion

$$\Phi(\tau, z) = \sum_{\substack{n,l \in \mathbb{Z} \\ 4nt \geq l^2}} f(n,l) e^{2\pi i(n\tau + lz)}$$

the lift can be written down explicitly as

$$\mathrm{Lift}\,\Phi(\tau) = \sum_{4tmn \geq l^2} \sum_{a|(n,l,m)} a^{k-1} f\left(\frac{nm}{a^2}, \frac{l}{a}\right) e^{2\pi i(n\tau_1 + l\tau_2 + mt\tau_3)}.$$

□

Since one knows dimension formulae for Jacobi cusp forms one obtains in this way lower bounds for the dimension of the space of modular forms and cusp forms with respect to the paramodular group. Using this together with Freitag's extension theorem it is then easy to obtain the following corollaries.

Corollary III.1.2 *Let $p_g(t)$ be the geometric genus of a smooth projective model of the moduli space $\mathcal{A}_{1,t}$ of $(1,t)$-polarized abelian surfaces. Then*

$$p_g(t) \geq \sum_{j=1}^{t-1} \left(\{2j+2\}_{12} - \left\lfloor \frac{j^2}{12} \right\rfloor \right)$$

where

$$\{m\}_{12} = \begin{cases} \lfloor \frac{m}{12} \rfloor & \text{if} \quad m \not\equiv 2 \mod 12 \\ \lfloor \frac{m}{12} \rfloor - 1 & \text{if} \quad m \equiv 2 \mod 12 \end{cases}$$

and $\lfloor x \rfloor$ denotes the integer part of x.

This corollary also implies that $p_g(t)$ goes to infinity as t goes to infinity.

Corollary III.1.3 *The Kodaira dimension of $\mathcal{A}_{1,t}$ is non-negative if $t \geq 13$ and $t \neq 14, 15, 16, 18, 20, 24, 30$ or 36. In particular these spaces are not unirational.*

Corollary III.1.4 *The Kodaira dimension of $\mathcal{A}_{1,t}$ is positive if $t \geq 29$ and $t \neq 30, 32, 35, 36, 40, 42, 48$ or 60.*

On the other hand one knows that $\mathcal{A}_{1,t}$ is rational or unirational for small values of t. We have already mentioned that Igusa proved rationality of $\mathcal{A}_{1,1} = \mathcal{A}_2$ in [I1]. Rationality of $\mathcal{A}_{1,2}$ and $\mathcal{A}_{1,3}$ was proved by Birkenhake and Lange [BL]. Birkenhake, Lange and van Straten [BLvS] also showed that $\mathcal{A}_{1,4}$ is unirational. It is a consequence of the work of Horrocks and Mumford [HM] that $\mathcal{A}^{\mathrm{lev}}_{1,5}$ is rational. The variety $\mathcal{A}^{\mathrm{lev}}_{1,7}$ is birational to a Fano variety of type V_{22} [MS] and hence also rational. The following result of Gross and Popescu was stated in [GP1] and is proved in the series of papers [GP1]–[GP4].

Theorem III.1.5 ([GP1],[GP2],[GP3],[GP4]) $\mathcal{A}^{\mathrm{lev}}_{1,t}$ *is rational for $6 \leq t \leq 10$ and $t = 12$ and unirational, but not rational, for $t = 11$. Moreover the variety $\mathcal{A}_{1,t}$ is unirational for $t = 14, 16, 18$ and 20.*

We shall return to some of the projective models of the modular varieties $\mathcal{A}_{1,t}$ in chapter V. Altogether this gives a fairly complete picture as regards the question which of the spaces $\mathcal{A}_{1,t}$ can be rational or unirational. In fact there are only very few open cases.

Problem Determine whether the spaces $\mathcal{A}_{1,t}$ for $t = 15, 24, 30$ or 36 are unirational.

III.2 General type results for moduli spaces of abelian surfaces

In the case of moduli spaces of abelian surfaces there are a number of concrete bounds which guarantee that the moduli spaces $\mathcal{A}_{1,t}$, resp. $\mathcal{A}^{\mathrm{lev}}_{1,t}$ are of general type. Here we collect the known results and comment on the different approaches which enable one to prove these theorems.

Theorem III.2.1 ([HS1],[GrH1]) *Let p be a prime number. The moduli spaces $\mathcal{A}^{\mathrm{lev}}_{1,p}$ are of general type if $p \geq 37$.*

*Proof.*This theorem was first proved in [HS1] for $p \geq 41$ and was improved in [GrH1] to $p = 37$. The two methods of proof differ in one important point. In [HS1] we first estimate how the dimension of the space of cusp forms grows with the weight k and find that

$$\dim S_{3k}\left(\Gamma^{\mathrm{lev}}_{1,p}\right) = \frac{p(p^4-1)}{640}k^3 + O(k^2). \tag{5}$$

These cusp forms give rise to k-fold differential forms on ${}^{\circ}\!\mathcal{A}^{\mathrm{lev}}_{1,p}$ and we have two types of obstruction to extending them to a smooth projective model of $\mathcal{A}^{\mathrm{lev}}_{1,p}$: one comes from the boundary and the other arises from the elliptic fixed points. To calculate the number of obstructions from the boundary we used the description of the boundary of the Igusa compactification (which is equal to the Voronoi decomposition) given in [HKW2]. We found that the number of obstructions to extending k-fold differentials is bounded by

$$H_B(p,k) = \frac{(p^2-1)}{144}(9p^2+2p+11)k^3 + O(k^2). \tag{6}$$

The singularities of the moduli spaces $\mathcal{A}^{\mathrm{lev}}_{1,p}$ and of the Igusa compactification were computed in [HKW1]. This allowed us to calculate the obstructions arising from the fixed points of the action of the group $\Gamma^{\mathrm{lev}}_{1,p}$. The result is that the number of these obstructions is bounded by

$$H_S(p,k) = \frac{1}{12}(p^2-1)(\frac{7}{18}p-1)k^3 + O(k^2). \tag{7}$$

The result then follows from comparing the leading terms of (6) and (7) with that of (5).

The approach in [GrH1] is different. The crucial point is to use Gritsenko's lifting result to produce non-zero cusp forms of weight 2. The first prime where this works is $p=37$, but it also works for all primes $p>71$. Let G be a non-trivial modular form of weight 2 with respect to $\Gamma_{1,37}$. Then we can consider the subspace

$$V_k = G^k M_k\left(\Gamma^{\mathrm{lev}}_{1,37}\right) \subset M_{3k}\left(\Gamma^{\mathrm{lev}}_{1,37}\right).$$

The crucial point is that the elements of V_k vanish by construction to order k on the boundary. This ensures that the extension to the boundary imposes no further conditions. The only possible obstructions are those coming from the elliptic fixed points. These obstructions were computed above. A comparison of the leading terms again gives the result. $\square$

The second method described above was also used in the proof of the following two results.

Theorem III.2.2 ([OG],[GrS]) *The moduli space $\mathcal{A}^{\mathrm{lev}}_{1,p^2}$ is of general type for every prime $p \geq 11$.*

This was proved in [GrS] and improves a result of O'Grady [OG] who had shown this for $p \geq 17$. The crucial point in [GrS] is that, because of the square p^2, there is a covering $\mathcal{A}_{1,p^2} \to \mathcal{A}_{1,1}$. The proof in [GrS] then also uses the existence of a weight 2 cusp form with respect to the group Γ_{1,p^2} for $p \geq 11$. The only obstructions which have to be computed explicitly are those coming from the elliptic fixed points. The essential ingredient in

O'Grady's proof is the existence of a map from a partial desingularization of a toroidal compactification to the space $\overline{\mathcal{M}}_2$ of semi-stable genus 2 curves.

A further result in this direction is

Theorem III.2.3 ([S2]) *The moduli spaces $\mathcal{A}_{1,p}$ are of general type for all primes $p \geq 173$.*

It is important to remark that in this case there is no natural map from $\mathcal{A}_{1,p}$ to the moduli space $\mathcal{A}_{1,1} = \mathcal{A}_2$ of principally polarized abelian surfaces. A crucial ingredient in the proof of the above theorem is the calculation of the singularities of the spaces $\mathcal{A}_{1,p}$ which was achieved by Brasch [Br]. Another recent result is

Theorem III.2.4 ([H3]) *The moduli spaces of $(1, d)$-polarized abelian surfaces with a full level-n structure are of general type for all pairs (d, n) with $(d, n) = 1$ and $n \geq 4$.*

A general result due to L. Borisov is

Theorem III.2.5 ([Bori]) *There are only finitely many subgroups H of $\mathrm{Sp}(4, \mathbb{Z})$ such that $\mathcal{A}(H)$ is not of general type.*

Note that this result applies to the groups $\Gamma_{1,p}^{\mathrm{lev}}$ and Γ_{1,p^2} which are both conjugate to subgroups of $\mathrm{Sp}(4, \mathbb{Z})$, but does not apply to the groups $\Gamma_{1,p}$, which are not. (At least for $p \geq 7$: the subgroup of $\mathbb{C}^*$ generated by the eigenvalues of non-torsion elements of $\Gamma_{1,p}$ contains pth roots of unity, as was shown by Brasch in [Br], but the corresponding group for $\mathrm{Sp}(4, \mathbb{Z})$ has only 2- and 3-torsion.)

We shall give a rough outline of the proof of this result. For details the reader is referred to [Bori]. We shall mostly comment on the geometric aspects of the proof. Every subgroup H in $\mathrm{Sp}(4, \mathbb{Z})$ contains a principal congruence subgroup $\Gamma(n)$. The first reduction is the observation that it is sufficient to consider only subgroups H which contain a principal congruence subgroup $\Gamma(p^t)$ for some prime p. This is essentially a group theoretic argument using the fact that the finite group $\mathrm{Sp}(4, \mathbb{Z}_p)$ is simple for all primes $p \geq 3$. Let us now assume that H contains $\Gamma(n)$ (we assume $n \geq 5$). This implies that there is a finite morphism $\mathcal{A}_2(n) \to \mathcal{A}(H)$. The idea is to show that for almost all groups H there are sufficiently many pluricanonical forms on the Igusa (Voronoi) compactification $X = \mathcal{A}_2^*(n)$ which descend to a smooth projective model of $\mathcal{A}(H)$. For this it is crucial to get a hold on the possible singularities of the quotient Y. We have already observed in Corollary II.2.6 that the canonical divisor on X is ample for $n \geq 5$. The finite group $\bar{H} = \Gamma_2(n)/H$ acts on X and the quotient $Y = \bar{H}\backslash X$ is a (in general singular) projective model of $\mathcal{A}(H)$. Since X is smooth and H is finite, the variety Y is normal and has log-terminal singularities, i.e. if $\pi : Z \to Y$ is a desingularization whose exceptional divisor $E = \sum_i E_i$ has simple normal crossings, then

$$K_Z = \pi^* K_Y + \sum_i (-1+\delta_i) E_i \quad \text{with } \delta_i > 0.$$

Choose $\delta > 0$ such that $-1+\delta$ is the minimal discrepancy. By L_X, resp. L_Y we denote the $\mathbb{Q}$-line bundle whose sections are modular forms of weight 1. Then $L_X = \mu^* L_Y$ where $\mu : X \to Y$ is the quotient map.

The next reduction is that it suffices to construct a non-trivial section $s \in H^0(m(K_Y - L_Y))$ such that $s_y \in \mathcal{O}_Y\left(m(K_Y - L_Y)\mathfrak{m}_y^{m(1-\delta)}\right)$ for all $y \in Y$ where Y has a non-canonical singularity. This is enough because $\pi^*(sH^0(mL_Y)) \subset H^0(mK_Z)$ and the dimension of the space $H^0(mL_Y))$ grows as m^3.

The idea is to construct s as a suitable $\bar{H}$-invariant section

$$s \in H^0\left(\mu^*(m(K_Y - L_Y))\right)^{\bar{H}}$$

satisfying vanishing conditions at the branch locus of the finite map $\mu : X \to Y$. For this one has to understand the geometry of the quotient map μ. First of all one has branching along the boundary $D = \sum D_i$ of X. We also have to look at the Humbert surfaces

$$\mathcal{H}_1 = \left\{\tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix}; \tau_1, \tau_3 \in \mathbb{H}_1\right\} = \operatorname{Fix}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

and

$$\mathcal{H}_4 = \left\{\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}; \tau_1 = \tau_3\right\} = \operatorname{Fix}\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}.$$

Let

$$\mathcal{F} = \bigcup_{g \in \mathrm{Sp}(4,\mathbb{Z})} g(\mathcal{H}_1) \quad , \quad \mathcal{G} = \bigcup_{g \in \mathrm{Sp}(4,\mathbb{Z})} g(\mathcal{H}_2)$$

and let

$$F = \overline{\pi(\mathcal{F})}, \quad G = \overline{\pi(\mathcal{G})}$$

where $\pi : \mathbb{H}_2 \to \Gamma(n)\backslash\mathbb{H}_2 \subset X$ is the quotient map. One can then show that the branching divisor of the map $\mathcal{A}(\Gamma_2(n)) \to \mathcal{A}(H)$ is contained in $F \cup G$ and that all singularities in $\mathcal{A}(H)$ which lie outside $\mu(F \cup G)$ are canonical. Moreover the stabilizer subgroups in $\mathrm{Sp}(4,\mathbb{Z})$ of points in $\mathcal{F} \cup \mathcal{G}$ are solvable groups of bounded order. Let $F = \sum F_i$ and $G = \sum G_i$ be the decomposition of the surfaces F and G into irreducible components. We denote by d_i, f_i and g_i the ramification order of the quotient map $\mu : X \to Y$ along D_i, F_i and G_i. The numbers f_i and g_i are equal to 1 or 2. One has

$$\begin{aligned}\mu^*(m(K_Y - L_Y)) &= m(K_X - L_X) - \sum_i m(d_i - 1)D_i - \sum_i m(f_i - 1)F_i \\ &\quad - \sum_i m(g_i - 1)G_i.\end{aligned}$$

Recall that the finite group $\bar{H}$ is a subgroup of the group $\bar{G} = \Gamma/\Gamma(n) = \mathrm{Sp}(4, \mathbb{Z}_n)$. The crucial point in Borisov's argument is to show, roughly speaking, that the index $[\bar{G} : \bar{H}]$ can be bounded from above in terms of the singularities of Y. There are several such types of bounds depending on whether one considers points on the branch locus or on one or more boundary components. We first use this bound for the points on X which lie on 3 boundary divisors. Using this and the fact that Y has only finite quotient singularities one obtains the following further reduction: if R is the ramification divisor of the map $\mu : X \to Y$, then it is enough to construct a non-zero section in $H^0(m(K_X - L_X - R))$ for some $m > 0$ which lies in $\mathfrak{m}_x^{mk(\mathrm{Stab}^H x)}$ for all points x in X which lie over non-canonical points of Y and which are not on the intersection of 3 boundary divisors. Here $k(\mathrm{Stab}^H x)$ is defined as follows. First note that $\mathrm{Stab}^H x$ is solvable and consider a series

$$\{0\} = H_0 \triangleleft H_1 \triangleleft \ldots \triangleleft H_t = \mathrm{Stab}^H x$$

with H_i/H_{i-1} abelian of exponent k_i. Take $k' = k_1 \cdot \ldots \cdot k_t$. Then $k(\mathrm{Stab}^H x)$ is the minimum over all k' which are obtained in this way. To obtain an invariant section one can then take the product with respect to the action of the finite group $\bar{H}$. Now recall that all non-canonical points on $\mathcal{A}(H)$ lie in $\mu(F \cup G)$. The subgroup $Z\,\mathrm{Stab}^H D_i$ of $\mathrm{Stab}^H D_i$ which acts trivially on D_i is cyclic of order d_i. Moreover if x lies on exactly one boundary divisor of X then the order of the group $\mathrm{Stab}^H x / Z\,\mathrm{Stab}^H D_i$ is bounded by 6 and if x lies on exactly 2 boundary divisors, then the order of this group is bounded by 4. Using this one can show that there is a constant c (independent of H) such that it is sufficient to construct a non-zero section in $m(K_X - L_X - cR)$ for some positive m. By results of Yamazaki [Ya] the divisor $mK_X - 2mL_X$ is effective. It is, therefore, sufficient to prove the existence of a non-zero section in $m(K_X - 2cR)$. The latter equals

$$mK_X - 2c\sum_i m(d_i - 1)D_i - 2c\sum_i m(f_i - 1)F_i - 2c\sum_i m(g_i - 1)G_i.$$

We shall now restrict ourselves to obstructions coming from components F_i; the obstructions coming from G_i, D_i can be treated similarly. Since

$h^0(mK_X) > c_1 n^{10} m^3$ for some $c_1 > 0$, $m \gg 0$ one has to prove the following result: let $\varepsilon > 0$; then for all but finitely many subgroups H one has

$$\sum_{f_i=2} (h^0(mK_X) - h^0(mK_X - 2cmf_iF_i)) \leq \varepsilon n^{10} m^3 \quad \text{for} \quad m \gg 0$$

and all n. This can finally be derived from the following boundedness result. Let $\varepsilon > 0$ and assume that

$$\frac{\#\{F_i\,;\ f_i = 2\}}{\#\{F_i\}} \geq \varepsilon,$$

then the index $[\bar{G} : \bar{H}]$ is bounded by an (explicitly known) constant depending only on ε. The proof of this statement is group theoretic and the idea is as follows. Assume the above inequality holds: then H contains many involutions and these generate a subgroup of $\mathrm{Sp}(4,\mathbb{Z})$ whose index is bounded in terms of ε.

III.3 Left and right neighbours

The paramodular group $\Gamma_{1,t} \subset \mathrm{Sp}(4,\mathbb{Q})$ is (for $t > 1$) not a maximal discrete subgroup of the group of analytic automorphisms of $\mathbb{H}_2$. For every divisor $d \| t$ (i.e. $d|t$ and $(d, t/d) = 1$) one can choose integers x and y such that

$$xd - yt_d = 1, \quad \text{where } t_d = t/d.$$

The matrix

$$V_d = \frac{1}{\sqrt{d}} \begin{pmatrix} dx & -1 & 0 & 0 \\ -yt & d & 0 & 0 \\ 0 & 0 & d & yt \\ 0 & 0 & 1 & dx \end{pmatrix}$$

is an element of $\mathrm{Sp}(4,\mathbb{R})$ and one easily checks that

$$V_d^2 \in \Gamma_{1,t}, \qquad V_d \Gamma_{1,t} V_d^{-1} = \Gamma_{1,t}.$$

The group generated by $\Gamma_{1,t}$ and the elements V_d, i.e.

$$\Gamma_{1,t}^{\ddagger} = \langle \Gamma_{1,t},\ V_d;\ d\|t \rangle$$

does not depend on the choice of the integers x, y. It is a normal extension of $\Gamma_{1,t}$ with

$$\Gamma_{1,t}^{\ddagger}/\Gamma_{1,t} \cong (\mathbb{Z}_2)^{\mu(t)}$$

where $\mu(t)$ is the number of prime divisors of t. If t is square-free, it is known that $\Gamma_{1,t}^{\ddagger}$ is a maximal discrete subgroup of $\mathrm{Sp}(4,\mathbb{R})$ (see [Al],[Gu]). The coset $\Gamma_{1,t}V_t$ equals $\Gamma_{1,t}V_t'$ where

$$V_d = \begin{pmatrix} 0 & \sqrt{t}^{-1} & 0 & 0 \\ \sqrt{t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{t} \\ 0 & 0 & \sqrt{t}^{-1} & 0 \end{pmatrix}.$$

This generalizes the Fricke involution known from the theory of elliptic curves. The geometric meaning of the involution $\bar{V}_t : \mathcal{A}_{1,t} \to \mathcal{A}_{1,t}$ induced by V_t is that it maps a polarized abelian surface (A, H) to its dual. A similar geometric interpretation can also be given for the involutions V_d (see [GrH2, Proposition 1.6] and also [Br, Satz (1.11)] for the case $d = t$). We also consider the degree 2 extension

$$\Gamma^+_{1,t} = \langle \Gamma_{1,t}, V_t \rangle$$

of $\Gamma_{1,t}$. If $t = p^n$ for a prime number p, then $\Gamma^+_{1,t} = \Gamma^\ddagger_{1,t}$. The groups $\Gamma^\ddagger_{1,t}$ and $\Gamma^+_{1,t}$ define Siegel modular threefolds

$$\mathcal{A}^\ddagger_{1,t} = \Gamma^\ddagger_{1,t} \setminus \mathbb{H}_2, \qquad \mathcal{A}^+_{1,t} = \Gamma^+_{1,t} \setminus \mathbb{H}_2.$$

Since $\Gamma^\ddagger_{1,t}$ is a maximal discrete subgroup for t square free the space $\mathcal{A}^\ddagger_{1,t}$ was called a *minimal Siegel modular threefold.* This should not be confused with minimal models in the sense of Mori theory.

The paper [GrH2] contains an interpretation of the varieties $\mathcal{A}^\ddagger_{1,t}$ and $\mathcal{A}^+_{1,t}$ as moduli spaces. We start with the spaces $\mathcal{A}^\ddagger_{1,t}$.

Theorem III.3.1 ([GrH2]) (i) *Let A, A' be two $(1,t)$-polarized abelian surfaces which define the same point in $\mathcal{A}^\ddagger_{1,t}$. Then their (smooth) Kummer surfaces X, X' are isomorphic.*

(ii) *Assume that the Néron-Severi group of A and A' is generated by the polarization. Then the converse is also true: if A and A' have isomorphic Kummer surfaces, then A and A' define the same point in $\mathcal{A}^\ddagger_{1,t}$.*

The proof of this theorem is given in [GrH2, Theorem 1.5]. The crucial ingredient is the Torelli theorem for K3 surfaces. The above theorem says in particular that an abelian surface and its dual have isomorphic Kummer surfaces. This implies a negative answer to a problem posed by Shioda, who asked whether it was true that two abelian surfaces whose Kummer surfaces are isomorphic are necessarily isomorphic themselves. In view of the above result, a general $(1,t)$-polarized surface with $t > 1$ gives a counterexample: the surface A and its dual $\hat{A}$ have isomorphic Kummer surfaces, but A and $\hat{A}$ are not isomorphic as polarized abelian surfaces. If the polarization generates the Néron-Severi group this implies that A and $\hat{A}$ are not isomorphic as algebraic surfaces. In view of the above theorem one can interpret $\mathcal{A}^\ddagger_{1,t}$ as the space of Kummer surfaces associated to $(1,t)$-polarized abelian surfaces.

The space $\mathcal{A}^+_{1,t}$ can be interpreted as a space of lattice-polarized $K3$-surfaces in the sense of [N3],[Dol]. As usual let E_8 be the even, unimodular, positive definite lattice of rank 8. By $E_8(-1)$ we denote the lattice which arises from E_8 by multiplying the form with -1. Let $\langle n \rangle$ be the rank 1 lattice $\mathbb{Z}l$ with the form given by $l^2 = n$.

Theorem III.3.2 ([GrH2]) *The moduli space $\mathcal{A}_{1,t}^{+}$ is isomorphic to the moduli space of lattice polarized $K3$-surfaces with a polarization of type $\langle 2t\rangle \oplus 2E_8(-1)$.*

For a proof see [GrH2, Proposition 1.4]. If

$$L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4,$$

then $\bigwedge^2 L$ carries a symmetric bilinear form $(\ ,\ )$ given by

$$x \wedge y = (x,y)e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in \bigwedge^4 L.$$

If $w_t = e_1 \wedge e_3 + te_2 \wedge e_4$, then the group

$$\tilde{\Gamma}_{1,t} = \{g : L \to L;\ \bigwedge^2 g(w_t) = w_t\}$$

is isomorphic to the paramodular group $\Gamma_{1,t}$. The lattice $L_t = w_t^{\perp}$ has rank 5 and the form $(\ ,\ )$ induces a quadratic form of signature (3,2) on L_t. If $\mathrm{O}(L_t)$ is the orthogonal group of isometries of the lattice L_t, then there is a natural homomorphism

$$\bigwedge^2 : \quad \Gamma_{1,t} \cong \tilde{\Gamma}_{1,t} \longrightarrow \mathrm{O}(L_t).$$

This homomorphism can be extended to $\Gamma_{1,t}^{\ddagger}$ and

$$\Gamma_{1,t}^{\ddagger}/\Gamma_{1,t} \cong \mathrm{O}(L_t^{\vee}/L_t) \cong (\mathbb{Z}_2)^{\mu(t)}$$

where $L_t^{\vee}$ is the dual lattice of L_t. This, together with Nikulin's theory ([N2], [N3]) is the crucial ingredient in the proof of the above theorems.

The varieties $\mathcal{A}_{1,t}^{+}$ and $\mathcal{A}_{1,t}^{\ddagger}$ are quotients of the moduli space $\mathcal{A}_{1,t}$ of $(1,t)$-polarized abelian surfaces. In [GrH3] there is an investigation into an interesting class of Galois coverings of the spaces $\mathcal{A}_{1,t}$. These coverings are called *left neighbours*, and the quotients are called *right neighbours*. To explain the coverings of $\mathcal{A}_{1,t}$ which were considered in [GrH3], we have to recall a well known result about the commutator subgroup $\mathrm{Sp}(2g,\mathbb{Z})'$ of the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$. Reiner [Re] and Maaß [Ma1] proved that

$$\mathrm{Sp}(2g,\mathbb{Z})/\,\mathrm{Sp}(2g,\mathbb{Z})' = \begin{cases} \mathbb{Z}_{12} & \text{for } g=1 \\ \mathbb{Z}_2 & \text{for } g=2 \\ 1 & \text{for } g \geq 3 \end{cases}.$$

The existence of a character of order 12 of $\mathrm{Sp}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})$ follows from the Dedekind η-function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad q = e^{2\pi i\tau}.$$

This function is a modular form of weight 1/2 with a multiplier system of order 24. Its square η^2 has weight 1 and is a modular form with respect to a character v_η of order 12. For $g = 2$ the product

$$\Delta_5(\tau) = \prod_{(m,m')\,\text{even}} \Theta_{mm'}(\tau, 0)$$

of the 10 even theta characteristics is a modular form for $\mathrm{Sp}(4,\mathbb{Z})$ of weight 5 with respect to a character of order 2.

In [GrH3] the commutator subgroups of the groups $\Gamma_{1,t}$ and $\Gamma_{1,t}^+$ were computed. For $t \geq 1$ we put

$$t_1 = (t, 12), \qquad t_2 = (2t, 12).$$

Theorem III.3.3 ([GrH3]) *For the commutator subgroups $\Gamma'_{1,t}$ of $\Gamma_{1,t}$ and $(\Gamma^+_{1,t})'$ of $\Gamma^+_{1,t}$ one obtains*

(i) $\Gamma_{1,t}/\Gamma'_{1,t} \cong \mathbb{Z}_{t_1} \times \mathbb{Z}_{t_2}$

(ii) $\Gamma^+_{1,t}/(\Gamma^+_{1,t})' \cong \mathbb{Z}_2 \times \mathbb{Z}_{t_2}$.

This was shown in [GrH3, Theorem 2.1].

In [Mu1] Mumford pointed out an interesting application of the computation of $\mathrm{Sp}(2,\mathbb{Z})'$ to the Picard group of the moduli stack $\underline{\mathcal{A}}_1$. He showed that

$$\mathrm{Pic}(\underline{\mathcal{A}}_1) \cong \mathbb{Z}_{12}.$$

In the same way the above theorem implies that

$$\mathrm{Pic}(\underline{\mathcal{A}}_2) = \mathrm{Pic}(\underline{\mathcal{A}}_{1,1}) \cong \mathbb{Z} \times \mathbb{Z}_2$$

and

$$\mathrm{Tors}\,\mathrm{Pic}(\underline{\mathcal{A}}_{1,t}) = \mathbb{Z}_{t_1} \times \mathbb{Z}_{t_2}.$$

The difference between the cases $\underline{\mathcal{A}}_{1,1}$ and $\underline{\mathcal{A}}_{1,t}$, $t > 1$ is that one knows that the rank of the Picard group of $\underline{\mathcal{A}}_2 = \underline{\mathcal{A}}_{1,1}$ is 1, whereas the rank of the Picard group of $\underline{\mathcal{A}}_{1,t}$, $t > 1$ is unknown. One only knows that it is positive. This is true for all moduli stacks of abelian varieties of dimension $g \geq 2$, since the bundle L of modular forms of weight 1 is non-trivial. The difference from the genus 1 case lies in the fact that there the boundary of the Satake compactification is a divisor.

Problem Determine the rank of the Picard group $\mathrm{Pic}(\underline{\mathcal{A}}_{1,t})$.

We have already discussed Gritsenko's result which gives the existence of weight 3 cusp forms for $\Gamma_{1,t}$ for all but finitely many values of t. We call these values

$$t = 1, 2, \ldots, 12, 14, 15, 16, 18, 20, 24, 30, 36$$

the *exceptional* polarizations. In many cases the results of Gross and Popescu show that weight 3 cusp forms indeed cannot exist. The best possible one can hope for is the existence of weight 3 cusp forms with a character of a small order. The following result is such an existence theorem.

Theorem III.3.4 ([GrH3]) *Let t be exceptional.*

(i) *If $t \neq 1, 2, 4, 5, 8, 16$ then there exists a weight 3 cusp form with respect to $\Gamma_{1,t}$ with a character of order 2.*

(ii) *For $t = 8, 16$ there exists a weight 3 cusp form with a character of order 4.*

(iii) *For $t \equiv 0 \mod 3, t \neq 3, 9$ there exists a weight 3 cusp form with a character of order 3.*

To every character $\chi : \Gamma_{1,t} \to \mathbb{C}^*$ one can associate a Siegel modular variety

$$\mathcal{A}(\chi) = \operatorname{Ker} \chi \setminus \mathbb{H}_2 .$$

The existence of a non-trivial cusp form of weight 3 with a character χ then implies by Freitag's theorem the existence of a differential form on a smooth projective model $\tilde{\mathcal{A}}(\chi)$ of $\mathcal{A}(\chi)$. In particular the above result proves the existence of abelian covers $\mathcal{A}(\chi) \to \mathcal{A}_{1,t}$ of small degree with $p_g(\tilde{\mathcal{A}}(\chi)) > 0$.

The proof is again an application of Gritsenko's lifting techniques. To give the reader an idea we shall discuss the case $t = 11$ which is particularly interesting since by the result of Gross and Popescu $\mathcal{A}_{1,11}$ is unirational, but not rational. In this case $\Gamma_{1,11}$ has exactly one character χ_2. This character has order 2. By the above theorem there is a degree 2 cover $\mathcal{A}(\chi_2) \to \mathcal{A}_{1,11}$ with positive geometric genus. In this case the lifting procedure gives us a map

$$\text{Lift: } J^{\text{cusp}}_{3,\frac{11}{2}}(v_\eta^{12} \times v_H) \to S_3(\Gamma_{1,11}, \chi_2).$$

Here v_η is the multiplier system of the Dedekind η-function and v_η^{12} is a character of order 2. The character v_H is a character of order 2 of the integer Heisenberg group $H = H(\mathbb{Z})$. By $J^{\text{cusp}}_{3,\frac{11}{2}}(v_\eta^2 \times v_H)$ we denote the Jacobi cusp forms of weight 3 and index $11/2$ with a character $v_\eta^{12} \times v_H$. Similarly $S_3(\Gamma_{1,11}, \chi_2)$ is the space of weight 3 cusp form with respect to the group $\Gamma_{1,11}$ and the character χ_2. Recall the Jacobi theta series

$$\vartheta(\tau, z) = \sum_{m \in \mathbb{Z}} \left(-\frac{4}{m} \right) q^{m^2/8} r^{m/2} \qquad (q = e^{2\pi i \tau},\, r = e^{2\pi i z})$$

where

$$\left(-\frac{4}{m} \right) = \begin{cases} \pm 1 & \text{if } m \equiv \pm 1 \mod 4 \\ 0 & \text{if } m \equiv 0 \mod 2. \end{cases}$$

This is a Jacobi form of weight $1/2$, index $3/2$ and multiplier system $v_\eta^3 \times v_H$. For an integer a we can consider the Jacobi form

$$\vartheta_a(\tau, z) = \vartheta(\tau, az) \in J_{\frac{1}{2},\frac{1}{2}a^2}(v_\eta^3 \times v_H^a).$$

One then obtains the desired Siegel cusp form by taking

$$F = \mathrm{Lift}(\eta^3\vartheta^2\vartheta_3) \in S_3(\Gamma_{1,11}, \chi_2).$$

Finally we want to consider the maximal abelian covering of $\mathcal{A}_{1,t}$, namely the Siegel modular threefold

$$\mathcal{A}_{1,t}^{\mathrm{com}} = \Gamma'_{1,t} \setminus \mathbb{H}_2.$$

By $\tilde{\mathcal{A}}_{1,t}^{\mathrm{com}}$ we denote a smooth projective model of $\mathcal{A}_{1,t}^{\mathrm{com}}$.

Theorem III.3.5 ([GrH3]) (i) *The geometric genus of* $\tilde{\mathcal{A}}_{1,t}^{\mathrm{com}}$ *is* 0 *if and only if* $t = 1, 2, 4, 5$.

(ii) *The geometric genus of* $\mathcal{A}_{1,3}^{\mathrm{com}}$ *and* $\mathcal{A}_{1,7}^{\mathrm{com}}$ *is* 1.

The proof can be found as part of the proof of [GrH3, Theorem 3.1].

At this point we should like to remark that all known construction methods fail when one wants to construct modular forms of small weight with respect to the groups $\Gamma_{1,t}^{+}$ or $\Gamma_{1,t}^{\ddagger}$. We therefore pose the

Problem Construct modular forms of small weight with respect to the groups $\Gamma_{1,t}^{+}$ and $\Gamma_{1,t}^{\ddagger}$.

IV Projective models

In this section we describe some cases in which a Siegel modular variety is or is closely related to an interesting projective variety. Many of the results are very old.

IV.1 The Segre cubic

Segre's cubic primal, or the *Segre cubic*, is the subvariety $\mathcal{S}_3$ of $\mathbb{P}^5$ given by the equations

$$\sum_{i=0}^{5} x_i = \sum_{i=0}^{5} x_i^3 = 0$$

in homogeneous coordinates $(x_0 : \ldots : x_5)$ on $\mathbb{P}^5$. Since it lies in the hyperplane $\left(\sum x_i = 0\right) \subset \mathbb{P}^5$ it may be thought of as a cubic hypersurface in $\mathbb{P}^4$, but the equations as given here have the advantage of showing that there is an action of the symmetric group $\mathrm{Sym}(6)$ on $\mathcal{S}_3$.

These are the equations of $\mathcal{S}_3$ as they are most often given in the literature but there is another equally elegant formulation: $\mathcal{S}_3$ is given by the equations

$$\sigma_1(x_i) = \sigma_3(x_i) = 0$$

where $\sigma_k(x_i)$ is the kth elementary symmetric polynomial in the x_i,

$$\sigma_k(x_i) = \sum_{\#I=k} \prod_{i\in I} x_i.$$

To check that these equations do indeed define $\mathcal{S}_3$ it is enough to notice that

$$3\sigma_3(x_i) = \left(\sum x_i\right)^3 - 3\left(\sum x_i\right)\left(\sum x_i^2\right) - \sum x_i^3.$$

Lemma IV.1.1 *$\mathcal{S}_3$ is invariant under the action of* $\mathrm{Sym}(6)$ *and has ten nodes, at the points equivalent to* $(1:1:1:-1:-1:-1)$ *under the* $\mathrm{Sym}(6)$*-action. This is the maximum possible for a cubic hypersurface in $\mathbb{P}^4$, and any cubic hypersurface with ten nodes is projectively equivalent to $\mathcal{S}_3$.*

Many other beautiful properties of the Segre cubic and related varieties were discovered in the nineteenth century.

The dual variety of the Segre cubic is a quartic hypersurface $\mathcal{I}_4 \subset \mathbb{P}^4$, the *Igusa quartic.* If we take homogeneous coordinates $(y_0 : \ldots : y_5)$ on $\mathbb{P}^5$ then it was shown by Baker [Ba1] that $\mathcal{I}_4$ is given by

$$\sum_{i=0}^{5} y_i = a^2 + b^2 + c^2 - 2(ab + bc + ca) = 0$$

where

$$a = (y_1 - y_5)(y_4 - y_2), \quad b = (y_2 - y_3)(y_5 - y_0) \text{ and } c = (y_0 - y_4)(y_3 - y_1).$$

This can also be written in terms of symmetric functions in suitable variables as

$$\sigma_1(x_i) = 4\sigma_4(x_i) - \sigma_2(x_i)^2 = 0.$$

This quartic is singular along $\binom{6}{2} = 15$ lines ℓ_{ij}, $0 \le i,j \le 5$, and $\ell_{ij} \cap \ell_{mn} = \emptyset$ if and only if $\{i,j\} \cap \{m,n\} \neq \emptyset$. There are $\frac{1}{2}\binom{6}{3} = 10$ smooth quadric surfaces Q_{ijk} in $\mathcal{I}_4$, such that, for instance, ℓ_{01}, ℓ_{12} and ℓ_{20} lie in one ruling of $Q_{012} = Q_{345}$ and ℓ_{34}, ℓ_{45} and ℓ_{53} lie in the other ruling. The birational map $\mathcal{I}_4 \dashrightarrow \mathcal{S}_3$ given by the duality blows up the 15 lines ℓ_{ij}, which resolves the singularities of $\mathcal{I}_4$, and blows down the proper transform of each Q_{ijk} (still a smooth quadric) to give the ten nodes of $\mathcal{S}_3$.

It has long been known that if $H \subset \mathbb{P}^4 = \left(\sum_{i=0}^{5} y_i\right)$ is a hyperplane which is tangent to $\mathcal{I}_4$ then $H \cap \mathcal{I}_4$ is a Kummer quartic surface. This fact provides a connection with abelian surfaces and their moduli. The Igusa quartic can be seen as a moduli space of Kummer surfaces. In this case, because the polarization is principal, two abelian surfaces giving the same Kummer surface are isomorphic and the (coarse) moduli space of abelian surfaces is the same as the moduli space of Kummer surfaces. This will fail in the non-principally polarized case, in IV.3, below.

Theorem IV.1.2 *$\mathcal{S}_3$ is birationally equivalent to a compactification of the moduli space $\mathcal{A}_2(2)$ of principally polarized abelian surfaces with a level-2 structure.*

The Segre cubic is rational. An explicit birational map $\mathbb{P}^3 \dashrightarrow \mathcal{S}_3$ was given by Baker [Ba1] and is presented in more modern language in [Hun].

Corollary IV.1.3 *$\mathcal{A}_2^*(2)$ is rational.*

A much more precise description of the relation between $\mathcal{S}_3$ and $\mathcal{A}_2(2)$ is given by this theorem of Igusa.

Theorem IV.1.4 ([I2]) *The Igusa compactification $\mathcal{A}_2^*(2)$ of the moduli space of principally polarized abelian surfaces with a level-2 structure is isomorphic to the the blow-up $\tilde{\mathcal{S}}_3$ of $\mathcal{S}_3$ in the ten nodes. The Satake compactification $\bar{\mathcal{A}}_2(2)$ is isomorphic to $\mathcal{I}_4$, which is obtained from $\tilde{\mathcal{S}}_3$ by contracting 15 rational surfaces to lines.*

Proof. The Satake compactification is $\operatorname{Proj} \mathcal{M}\big(\Gamma_2(2)\big)$, where $\mathcal{M}(\Gamma)$ is the ring of modular forms for the group Γ. The ten even theta characteristics determine ten theta constants $\theta_{m_0}(\tau), \dots, \theta_{m_9}(\tau)$ of weight $\frac{1}{2}$ for $\Gamma_2(2)$, and $\theta^4_{m_i}(\tau)$ is a modular form of weight 2 for $\Gamma_2(2)$. These modular forms determine a map $f : \mathcal{A}_2(2) \to \mathbb{P}^9$ whose image actually lies in a certain $\mathbb{P}^4 \subset \mathbb{P}^9$. The integral closure of the subring of $\mathcal{M}\big(\Gamma_2(2)\big)$ generated by the $\theta^4_{m_i}$ is the whole of $\mathcal{M}\big(\Gamma_2(2)\big)$ and there is a quartic relation among the $\theta^4_{m_i}$ (as well as five linear relations defining $\mathbb{P}^4 \subset \mathbb{P}^9$) which, with a suitable choice of basis, is the quartic $a^2 + b^2 + c^2 - 2(ab + bc + ca) = 0$. Furthermore, f is an embedding and the closure of its image is normal, so it is the Satake compactification. $\square$

The Igusa compactification is, in this context, the blow-up of the Satake compactification along the boundary, which here consists of the fifteen lines ℓ_{ij}. The birational map $\mathcal{I}_4 \dashrightarrow \mathcal{S}_3$ does this blow-up and also blows down the ten quadrics Q_{ijk} to the ten nodes of $\mathcal{S}_3$.

For full details of the proof see [I2]; for a more extended sketch than we have given here and some further facts, see [Hun]. We mention that the surfaces Q_{ijk}, considered as surfaces in $\mathcal{A}_2(2)$, correspond to principally polarized abelian surfaces which are products of two elliptic curves.

Without going into details, we mention also that $\mathcal{I}_4$ may be thought of as the natural compactification of the moduli of ordered 6-tuples of distinct points on a conic in $\mathbb{P}^2$. Such a 6-tuple determines 6 lines in $\check{\mathbb{P}}^2$ which are all tangent to some conic, and the Kummer surface is the double cover of $\check{\mathbb{P}}^2$ branched along the six lines. The order gives the level-2 structure (note that $\Gamma_2/\Gamma_2(2) \cong \mathrm{Sp}(4,\mathbb{Z}_2) \cong \mathrm{Sym}(6)$.) The abelian surface is the Jacobian of the double cover of the conic branched at the six points. On the other hand, $\mathcal{S}_3$ may be thought of as the natural compactification of the moduli of ordered 6-tuples of points on a line: for this, see [DO].

The topology of the Segre cubic and related spaces has been studied by van der Geer [vdG1] and by Lee and Weintraub [LW1], [LW2]. The method in [LW1] is to show that the isomorphism between the open parts of $\mathcal{S}_3$ and $\mathcal{A}_2(2)$ is defined over a suitable number field and use the Weil conjectures.

Theorem IV.1.5 ([LW1],[vdG1]) *The homology of the Igusa compactification of $\mathcal{A}_2(2)$ is torsion-free. The Hodge numbers are $h^{0,0} = h^{3,3} = 1$, $h^{1,1} = h^{2,2} = 16$ and $h^{p,q} = 0$ otherwise.*

By using the covering $\mathcal{A}_2(4) \to \mathcal{A}_2(2)$, Lee and Weintraub [LW3] also prove a similar result for $\mathcal{A}_2(4)$.

IV.2 The Burkhardt quartic

The *Burkhardt quartic* is the subvariety $\mathcal{B}_4$ of $\mathbb{P}^4$ given by the equation

$$y_0^4 - y_0(y_1^3 + y_2^3 + y_3^3 + y_4^3) + 3y_1y_2y_3y_4 = 0.$$

This form of degree 4 was found by Burkhardt [Bu] in 1888. It is the invariant of smallest degree of a certain action of the finite simple group $\mathrm{PSp}(4,\mathbb{Z}_3)$ of order 25920 on $\mathbb{P}^4$, which arises in the study of the 27 lines on a cubic surface. In fact this group is a subgroup of index 2 in the Weyl group $W(E_6)$ of E_6, which is the automorphism group of the configuration of the 27 lines. The 27 lines themselves can be recovered by solving an equation whose Galois group is $W(E_6)$ or, after adjoining a square root of the discriminant, $\mathrm{PSp}(4,\mathbb{Z}_3)$.

Lemma IV.2.1 *$\mathcal{B}_4$ has forty-five nodes. Fifteen of them are equivalent to $(1:-1:0:0:0:0)$ under the action of $\mathrm{Sym}(6)$ and the other thirty are equivalent to $(1:1:\xi_3:\xi_3:\xi_3^2:\xi_3^2)$, where $\xi_3 = e^{2\pi i/3}$. This is the greatest number of nodes that a quartic hypersurface in $\mathbb{P}^4$ can have and any quartic hypersurface in $\mathbb{P}^4$ with 45 nodes is projectively equivalent to $\mathcal{B}_4$.*

This lemma is an assemblage of results of Baker [Ba2] and de Jong, Shepherd-Barron and Van de Ven [JSV]: the bound on the number of double points is the Varchenko (or spectral) bound [Va], which in this case is sharp.

We denote by $\theta_{\alpha\beta}(\tau)$, $\alpha,\beta\in\mathbb{Z}_3$, the theta constants

$$\theta_{\alpha\beta}(\tau)=\theta\left[\begin{array}{cc}0&0\\ \alpha&\beta\end{array}\right](\tau,0)=\sum_{n\in\mathbb{Z}^2}\exp\{\pi i\,{}^tn\tau n+2\pi i(\alpha n_1+\beta n_2)\}$$

where $\tau\in\mathbb{H}_2$. Here we identify $\alpha\in\mathbb{Z}_3$ with $\alpha/3\in\mathbb{Q}$. The action of $\Gamma_2(1)=\mathrm{Sp}(4,\mathbb{Z})$ on $\mathbb{H}_2$ induces a linear action on the space spanned by these $\theta_{\alpha\beta}$, and $\Gamma_2(3)$ acts trivially on the corresponding projective space. Since $-1\in\Gamma_2(1)$ acts trivially on $\mathbb{H}_2$, this gives an action of $\mathrm{PSp}(4,\mathbb{Z})/\Gamma_2(3)\cong\mathrm{PSp}(4,\mathbb{Z}_3)$ on $\mathbb{P}^8$. The subspace spanned by the $y_{\alpha\beta}=\frac{1}{2}(\theta_{\alpha\beta}+\theta_{-\alpha,-\beta})$ is invariant. Burkhardt studied the ring of invariants of this action. We put $y_0=-y_{00}$, $y_1=2y_{10}$, $y_2=2y_{01}$, $y_3=2y_{11}$ and $y_4=2y_{1,-1}$.

Theorem IV.2.2 ([Bu],[vdG2]) *The quartic form $y_0^4-y_0(y_1^3+y_2^3+y_3^3+y_4^3)+3y_1y_2y_3y_4$ is an invariant, of lowest degree, for this action. The map*

$$\tau\longmapsto(y_0:y_1:y_2:y_3:y_4)$$

defines a map $\mathbb{H}_2/\Gamma_2(3)\to\mathcal{B}_4$ which extends to a birational map $\mathcal{A}_2^*(3)\dashrightarrow\mathcal{B}_4$.

This much is fairly easy to prove, but far more is true: van der Geer, in [vdG2], gives a short modern proof as well as providing more detail. The projective geometry of $\mathcal{B}_4$ is better understood by embedding it in $\mathbb{P}^5$, as we did for $\mathcal{S}_3$. Baker [Ba2] gives explicit linear functions $x_0,\ldots,x_5$ of $y_0,\ldots,y_4$ such that $\mathcal{B}_4\subset\mathbb{P}^5$ is given by

$$\sigma_1(x_i)=\sigma_4(x_i)=0.$$

The details are reproduced in [Hun].

Theorem IV.2.3 ([To],[Ba2]) *$\mathcal{B}_4$ is rational: consequently $\mathcal{A}_2^*(3)$ is rational.*

This was first proved by Todd [To]; later Baker [Ba2] gave an explicit birational map from $\mathbb{P}^3$ to $\mathcal{B}_4$.

To prove Theorem IV.2.2 we need to say how to recover a principally polarized abelian surface and a level-3 structure from a general point of $\mathcal{B}_4$. The linear system on a principally polarized abelian surface given by three times the polarization is very ample, so the theta functions $\theta_{\alpha\beta}(\tau,z)$ determine an embedding of $A_\tau=\mathbb{C}^2/\mathbb{Z}^2+\mathbb{Z}^2\tau$ $(\tau\in\mathbb{H}_2)$ into $\mathbb{P}^8$. Moreover the extended Heisenberg group G_3 acts on the linear space spanned by the $\theta_{\alpha\beta}$. The Heisenberg group of level 3 is a central extension

$$0\longrightarrow\mu_3\longrightarrow H_3\longrightarrow\mathbb{Z}_3^2\longrightarrow 0$$

and G_3 is an extension of this by an involution ι. The involution acts by $z \mapsto -z$ and $\mathbb{Z}_3^2$ acts by translation by 3-torsion points. The space spanned by the $y_{\alpha\beta}$ is invariant under the normalizer of the Heisenberg group in $\mathrm{PGL}(4,\mathbb{C})$, which is isomorphic to $\mathrm{PSp}(4,\mathbb{Z}_3)$, so we get an action of this group on $\mathbb{P}^4$ and on $\mathcal{B}_4 \subset \mathbb{P}^4$.

For a general point $p \in \mathcal{B}_4$ the hyperplane in $\mathbb{P}^4$ tangent to $\mathcal{B}_4$ at p meets $\mathcal{B}_4$ in a quartic surface with six nodes, of a type known as a Weddle surface. Such a surface is birational to a unique Kummer surface (Hudson [Hud] and Jessop [Je] both give constructions) and this is the Kummer surface of A_τ.

It is not straightforward to see the level-3 structure in this picture. One method is to start with a principally polarized abelian surface (A,Θ) and embed it in $\mathbb{P}^8$ by $|3\Theta|$. Then there is a projection $\mathbb{P}^8 \to \mathbb{P}^3$ under which the image of A is the Weddle surface, so one identifies this $\mathbb{P}^3$ with the tangent hyperplane to $\mathcal{B}_4$. The Heisenberg group acts on $\mathbb{P}^8$ and on $H^0\big(\mathbb{P}^8,\mathcal{O}_{\mathbb{P}^8}(2)\big)$, which has dimension 45. In $\mathbb{P}^8$, A is cut out by nine quadrics in $\mathbb{P}^8$. The span of these nine quadrics is determined by five coefficients $\alpha_0,\ldots,\alpha_4$ which satisfy a homogeneous Heisenberg-invariant relation of degree 4. As the Heisenberg group acting on $\mathbb{P}^4$ has only one such relation this relation must again be the one that defines $\mathcal{B}_4$. Thus the linear space spanned by nine quadrics, and hence A with its polarization and Heisenberg action, are determined by a point of $\mathcal{B}_4$. The fact that the two degree 4 relations coincide is equivalent to saying that $\mathcal{B}_4$ has an unusual projective property, namely it is self-Steinerian.

It is quite complicated to say what the level-3 structure means for the Kummer surface. It is not enough to look at the Weddle surface: one also has to consider the image of A in another projection $\mathbb{P}^8 \to \mathbb{P}^4$, which is again a birational model of the Kummer surface, this time as a complete intersection of type $(2,3)$ with ten nodes. More details can be found in [Hun].

The details of this proof were carried out by Coble [Cob], who also proved much more about the geometry of $\mathcal{B}_4$ and the embedded surface $A_\tau \subset \mathbb{P}^8$. The next theorem is a consequence of Coble's results.

Theorem IV.2.4 ([Cob]) *Let $\pi : \tilde{\mathcal{B}}_4 \to \mathcal{B}_4$ be the blow-up of $\mathcal{B}_4$ in the 45 nodes. Then $\tilde{\mathcal{B}}_4 \cong \mathcal{A}_2^*(3)$; the exceptional surfaces in $\tilde{\mathcal{B}}_4$ correspond to the Humbert surfaces that parametrize product abelian surfaces. The Satake compactification is obtained by contracting the preimages of 40 planes in $\mathcal{B}_4$, each of which contains 9 of the nodes.*

One should compare the birational map $\mathcal{A}_2^*(3) \dashrightarrow \mathcal{B}_4$ with the birational map $\mathcal{I}_4 \dashrightarrow \mathcal{S}_3$ of the previous section.

By computing the zeta function of $\tilde{\mathcal{B}}_4$ over $\mathbb{F}_q$ for $q \equiv 1 \pmod 3$, Hoffman and Weintraub [HoW] calculated the cohomology of $\mathcal{A}_2^*(3)$.

Theorem IV.2.5 ([HoW]) *$H^i(\mathcal{A}_2^*(3),\mathbb{Z})$ is free: the odd Betti numbers are zero and $b_2 = b_4 = 61$.*

In fact [HoW] gives much more detail, describing the mixed Hodge structures, the intersection cohomology of the Satake compactification, the $\mathrm{PSp}(4,\mathbb{Z}_3)$-module structure of the cohomology and some of the cohomology of the group $\Gamma_2(3)$. The cohomology of $\Gamma_2(3)$ was also partly computed, by another method, by MacPherson and McConnell [McMc], but neither result contains the other.

IV.3 The Nieto quintic

The *Nieto quintic* $\mathcal{N}_5$ is the subvariety of $\mathbb{P}^5$ given in homogeneous coordinates $x_0,\ldots,x_5$ by

$$\sigma_1(x_i)=\sigma_5(x_i)=0.$$

This is conveniently written as $\sum x_i=\sum\frac{1}{x_i}=0$. As in the cases of $\mathcal{S}_3$ and $\mathcal{B}_4$, this form of the equation displays the action of $\mathrm{Sym}(6)$ and is preferable for most purposes to a single quintic equation in $\mathbb{P}^4$. Unlike $\mathcal{S}_3$ and $\mathcal{B}_4$, which were extensively studied in the nineteenth century, $\mathcal{N}_5$ and its relation to abelian surfaces was first studied only in the 1989 Ph.D. thesis of Nieto [Ni] and the paper of Barth and Nieto [BN].

We begin with a result of van Straten [vS].

Theorem IV.3.1 ([vS]) *$\mathcal{N}_5$ has ten nodes but (unlike $\mathcal{S}_3$ and $\mathcal{B}_4$) it also has some non-isolated singularities. However the quintic hypersurface in $\mathbb{P}^4$ given as a subvariety of $\mathbb{P}^5$ by*

$$\sigma_1(x_i)=\sigma_5(x_i)+\sigma_2(x_i)\sigma_3(x_i)=0.$$

has 130 *nodes and no other singularities.*

This threefold and the Nieto quintic are both special elements of the pencil

$$\sigma_1(x_i)=\alpha\sigma_5(x_i)+\beta\sigma_2(x_i)\sigma_3(x_i)=0$$

and the general element of this pencil has 100 nodes. Van der Geer [vdG2] has analysed in a similar way the pencil

$$\sigma_1(x_i)=\alpha\sigma_4(x_i)+\beta\sigma_2(x_i)^2=0$$

which contains $\mathcal{B}_4$ (45 nodes) and $\mathcal{I}_4$ (15 singular lines) among the special fibres, the general fibre having 30 nodes.

No example of a quintic 3-fold with more than 130 nodes is known, though the Varchenko bound in this case is 135.

$\mathcal{N}_5$, like $\mathcal{S}_3$ and $\mathcal{B}_4$, is related to abelian surfaces via Kummer surfaces. The Heisenberg group $H_{2,2}$, which is a central extension

$$0\to\mu_2\to H_{2,2}\to\mathbb{Z}_2^4\to 0$$

acts on $\mathbb{P}^3$ via the Schrödinger representation on $\mathbb{C}^4$. This is fundamental for the relation between $\mathcal{N}_5$ and Kummer surfaces.

Theorem IV.3.2 ([BN]) *The space of $H_{2,2}$-invariant quartic surfaces in $\mathbb{P}^3$ is 5-dimensional. The subvariety of this $\mathbb{P}^5$ which consists of those $H_{2,2}$-invariant quartic surfaces that contain a line is three-dimensional and its closure is projectively equivalent to $\mathcal{N}_5$. There is a double cover $\tilde{\mathcal{N}}_5 \to \mathcal{N}_5$ such that $\tilde{\mathcal{N}}_5$ is birationally equivalent to $\mathcal{A}^*_{1,3}(2)$.*

Proof. A general $H_{2,2}$-invariant quartic surface X containing a line ℓ will contain 16 skew lines (namely the $H_{2,2}$-orbit of ℓ). By a theorem of Nikulin [N1] this means that X is the minimal desingularization of the Kummer surface of some abelian surface A. The $H_{2,2}$-action on X gives rise to a level-2 structure on A, but the natural polarization on A is of type $(1,3)$. There is a second $H_{2,2}$-orbit of lines on X and they give rise to a second realization of X as the desingularized Kummer surface of another (in general non-isomorphic) abelian surface $\hat{A}$, which is in fact the dual of A. The moduli points of A and $\hat{A}$ (with their respective polarizations, but without level structures) in $\mathcal{A}_{1,3}$ are related by $V_3(A) = \hat{A}$, where V_3 is the Gritsenko involution described in III.3, above.

Conversely, given a general abelian surface A with a $(1,3)$-polarization and a level-2 structure, let $\widetilde{\mathrm{Km}}\,A$ be the desingularized Kummer surface and $\mathcal{L}$ a symmetric line bundle on A in the polarization class. Then the linear system $|\mathcal{L}^{\otimes 2}|^-$ of anti-invariant sections embeds $\widetilde{\mathrm{Km}}\,A$ as an $H_{2,2}$-invariant quartic surface and the exceptional curves become lines in this embedding. This gives the connection between $\mathcal{N}_5$ and $\mathcal{A}_{1,3}(2)$. □

The double cover $\tilde{\mathcal{N}}_5 \to \mathcal{N}_5$ is the inverse image of $\mathcal{N}_5$ under the double cover of $\mathbb{P}^5$ branched along the coordinate hyperplanes.

$\mathcal{N}_5$ is not very singular and therefore resembles a smooth quintic threefold in some respects. Barth and Nieto prove much more.

Theorem IV.3.3 ([BN]) *Both $\mathcal{N}_5$ and $\tilde{\mathcal{N}}_5$ are birationally equivalent to (different) Calabi-Yau threefolds. In particular, the Kodaira dimension of $\mathcal{A}^*_{1,3}(2)$ is zero.*

The fundamental group of a smooth projective model of $\mathcal{A}^*_{1,3}(2)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (see [S1] and II.3 above). Hence, as R. Livné has pointed out, there are four unramified covers of such a model which are also Calabi-Yau threefolds. In all other cases where the Kodaira dimension of a Siegel modular variety (of dimension > 1) is known, the variety is either of general type or uniruled.

It is a consequence of the above theorem that the modular group $\Gamma_{1,3}(2)$ which defines the moduli space $\mathcal{A}_{1,3}(2)$ has a unique weight-3 cusp form (up to a scalar). This cusp form was determined in [GrH4]. Recall that there is a weight-3 cusp form Δ_1 for the group $\Gamma_{1,3}$ with a character of order 6. The form Δ_1 has several interesting properties, in particular it admits an infinite product expansion and determines a generalized Lorentzian Kac-Moody superalgebra of Borcherds type (see [GrN]).

Theorem IV.3.4 ([GrH4]) *The modular form Δ_1 is the unique weight-3 cusp form of the group $\Gamma_{1,3}(2)$.*

Using this, it is possible to give an explicit construction of a Calabi-Yau model of $\mathcal{A}_{1,3}(2)$ which does not use the projective geometry of [BN].

Nieto and the authors of the present survey have investigated the relation between $\tilde{\mathcal{N}}_5$ and $\mathcal{A}^*_{1,3}(2)$ in more detail. $\mathcal{N}_5$ contains 30 planes which fall naturally into two sets of 15, the so-called S- and V-planes.

Theorem IV.3.5 ([HNS1]) *The rational map $\mathcal{A}^*_{1,3}(2) \dashrightarrow \mathcal{N}_5$ (which is generically 2-to-1) contracts the locus of product surfaces to the 10 nodes. The locus of bielliptic surfaces is mapped to the V-planes and the boundary of $\mathcal{A}^*_{1,3}(2)$ is mapped to the S-planes. Thus by first blowing up the singular points and then contracting the surfaces in $\tilde{\mathcal{N}}_5$ that live over the S-planes to curves one obtains the Satake compactification.*

In [HNS2] we gave a description of some of the degenerations that occur over the S-planes.

One of the open problems here is to give a projective description of the branch locus of this map. The projective geometry associated with the Nieto quintic is much less worked out than in the classical cases of the Segre cubic and the Burkhardt quartic.

Theorem IV.3.6 ([HSGS]) *The varieties $\mathcal{N}_5$ and $\tilde{\mathcal{N}}_5$ have rigid Calabi-Yau models. Both Calabi-Yaus are modular: more precisely, their L-function is equal (up to the Euler factors at bad primes) to the Mellin transform of the normalised weight 4 cusp form of level 6.*

V Non-principal polarizations

We have encountered non-principal polarizations and some of the properties of the associated moduli spaces already. For abelian surfaces, a few of these moduli spaces have good descriptions in terms of projective geometry, and we will describe some of these results for abelian surfaces below. We begin with the most famous case, historically the starting point for much of the recent work on the whole subject.

V.1 Type $(1,5)$ and the Horrocks-Mumford bundle

In this section we shall briefly describe the relation between the Horrocks-Mumford bundle and abelian surfaces. Since this material has been covered extensively in another survey article (see [H1] and the references quoted there) we shall be very brief here.

The existence of the Horrocks-Mumford bundle is closely related to abelian surfaces embedded in $\mathbb{P}^4$. Indeed, let $A \subset \mathbb{P}^4$ be a smooth abelian surface. Since $\omega_A = \mathcal{O}_A$ it follows that the determinant of the normal bundle of A in $\mathbb{P}^4$ is $\det N_{A/\mathbb{P}^4} = \mathcal{O}_A(5) = \mathcal{O}_{\mathbb{P}^4}(5)_{|A}$, i.e. it can be extended to $\mathbb{P}^4$. It then follows from the Serre construction (see e.g. [OSS, Theorem 5.1.1]) that the normal bundle $N_{A/\mathbb{P}^4}$ itself can be extended to a rank 2 bundle on $\mathbb{P}^4$. On the other hand the double point formula shows immediately that a smooth abelian surface in $\mathbb{P}^4$ can only have degree 10, so the hyperplane section is a polarization of type $(1,5)$. Using Reider's criterion (see e.g. [LB, chapter 10, §4]) one can nowadays check immediately that a polarization of type $(1,n), n \geq 5$ on an abelian surface with Picard number $\rho(A) = 1$ is very ample. The history of this subject is, however, quite intricate. Comessatti proved in 1916 that certain abelian surfaces could be embedded in $\mathbb{P}^4$. He considered a 2-dimensional family of abelian surfaces, namely those which have real multiplication in $\mathbb{Q}(\sqrt{5})$. His main tool was theta functions. His paper [Com] was later forgotten outside the Italian school of algebraic geometers. A modern account of Comessatti's results using, however, a different language and modern methods was later given by Lange [L] in 1986. Before that Ramanan [R] had proved a criterion for a $(1,n)$-polarization to be very ample. This criterion applies to all $(1,n)$-polarized abelian surfaces (A,H) which are cyclic n-fold covers of a Jacobian. In particular this also gives the existence of abelian surfaces in $\mathbb{P}^4$. The remaining cases not covered by Ramanan's paper were treated in [HL].

With the exception of Comessatti's essentially forgotten paper, none of this was available when Horrocks and Mumford investigated the existence of indecomposable rank 2 bundles on $\mathbb{P}^4$. Although they also convinced themselves of the existence of smooth abelian surfaces in $\mathbb{P}^4$ they then presented a construction of their bundle F in [HM] in cohomological terms, i.e. they constructed F by means of a *monad*. A monad is a complex

$$(M) \qquad A \xrightarrow{p} B \xrightarrow{q} C$$

where A, B and C are vector bundles, p is injective as a map of vector bundles, q is surjective and $q \circ p = 0$. The cohomology of (M) is

$$F = \operatorname{Ker} q / \operatorname{Im} p$$

which is clearly a vector bundle. The Horrocks-Mumford bundle can be given by a monad of the form

$$V \otimes \mathcal{O}_{\mathbb{P}^4}(2) \xrightarrow{p} 2 \bigwedge^2 T_{\mathbb{P}^4} \xrightarrow{q} V^* \otimes \mathcal{O}_{\mathbb{P}^4}(3)$$

where $V = \mathbb{C}^5$ and $\mathbb{P}^4 = \mathbb{P}(V)$. The difficulty is to write down the maps p

and q. The crucial ingredient here is the maps

$$\begin{array}{rcl} f^+ : V \longrightarrow \bigwedge^2 V, & f^+(\sum v_i e_i) & = & \sum v_i e_{i+2} \wedge e_{i+3} \\ f^- : V \longrightarrow \bigwedge^2 V, & f^-(\sum v_i e_i) & = & \sum v_i e_{i+1} \wedge e_{i+4} \end{array}$$

where $(e_i)_{i \in \mathbb{Z}_5}$ is the standard basis of $V = \mathbb{C}^5$ and indices have to be read cyclically. The second ingredient is the Koszul complex on $\mathbb{P}^4$, especially its middle part

$$\begin{tikzcd} \bigwedge^2 V \otimes \mathcal{O}_{\mathbb{P}^4}(1) \arrow[rr, "\wedge s"] \arrow[dr, "p_0"'] & & \bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}^4}(2) \\ & \bigwedge^2 T_{\mathbb{P}^4}(-1) \arrow[ur, "q_0"'] & \end{tikzcd}$$

where $s : \mathcal{O}_{\mathbb{P}^4}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^4}$ is the tautological bundle map. The maps p and q are then given by

$$\begin{array}{llll} p : V \otimes \mathcal{O}_{\mathbb{P}^4}(2) & \xrightarrow{(f^+, f^-)} & 2 \bigwedge^2 V \otimes \mathcal{O}_{\mathbb{P}^4}(2) & \xrightarrow{2p_0(1)} & 2 \bigwedge^2 T_{\mathbb{P}^4} \\ q : 2 \bigwedge^2 T_{\mathbb{P}^4} & \xrightarrow{2q_0(1)} & 2 \bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}^4}(3) & \xrightarrow{(-{}^t f^-, {}^t f^+)} & V^* \otimes \mathcal{O}_{\mathbb{P}^4}(3). \end{array}$$

Once one has come up with these maps it is not difficult to check that p and q define a monad. Clearly the cohomology F of this monad is a rank 2 bundle and it is straightforward to calculate its Chern classes to be

$$c(F) = 1 + 5h + 10h^2$$

where h denotes the hyperplane section. Since this polynomial is irreducible over the integers it follows that F is indecomposable.

One of the remarkable features of the bundle F is its symmetry group. The *Heisenberg group* of level n is the subgroup H_n of $\mathrm{SL}(n, \mathbb{C})$ generated by the automorphisms

$$\sigma : \quad e_i \mapsto e_{i-1}, \quad \tau : e_i \mapsto \varepsilon^i e_i \quad (\varepsilon = e^{2\pi i/n}).$$

Since $[\sigma, \tau] = \varepsilon \cdot \mathrm{id}_V$ the group H_n is a central extension

$$0 \to \mu_n \to H_n \to \mathbb{Z}_n \times \mathbb{Z}_n \to 0.$$

Let N_5 be the normalizer of the Heisenberg group H_5 in $\mathrm{SL}(5, \mathbb{C})$. Then $N_5/H_5 \cong \mathrm{SL}(2, \mathbb{Z}_5)$ and N_5 is in fact a semi-direct product

$$N_5 \cong H_5 \rtimes \mathrm{SL}(2, \mathbb{Z}_5).$$

Its order is $|N_5| = |H_5| \cdot |\,\mathrm{SL}(2, \mathbb{Z}_5)| = 125 \cdot 120 = 15{,}000$. One can show that N_5 acts on the bundle F and that it is indeed its full symmetry group [De].

The Horrocks-Mumford bundle is *stable*. This follows since $F(-1) = F \otimes \mathcal{O}_{\mathbb{P}^4}(-1)$ has $c_1(F(-1)) = 3$ and $h^0(F(-1)) = 0$. Indeed F is the unique stable rank 2 bundle with $c_1 = 5$ and $c_2 = 10$ [DS]. The connection with abelian surfaces is given via sections of F. Since $F(-1)$ has no sections every section $0 \neq s \in H^0(F)$ vanishes on a surface whose degree is $c_2(F) = 10$.

Proposition V.1.1 *For a general section $s \in H^0(F)$ the zero-set $X_s = \{s = 0\}$ is a smooth abelian surface of degree* 10.

Proof. [HM, Theorem 5.1]. The crucial point is to prove that X_s is smooth. The vector bundle F is globally generated outside 25 lines L_{ij} in $\mathbb{P}^4$. It therefore follows from Bertini that X_s is smooth outside these lines. A calculation in local coordinates then shows that for general s the surface X_s is also smooth where it meets the lines L_{ij}. It is then an easy consequence of surface classification to show that X_s is abelian. □

In order to establish the connection with moduli spaces it is useful to study the space of sections $H^0(F)$ as an N_5-module. One can show that this space is 4-dimensional and that the Heisenberg group H_5 acts trivially on $H^0(F)$. Hence $H^0(F)$ is an $\mathrm{SL}(2, \mathbb{Z}_5)$-module. It turns out that the action of $\mathrm{SL}(2, \mathbb{Z}_5)$ on $H^0(F)$ factors through an action of $\mathrm{PSL}(2, \mathbb{Z}_5) \cong A_5$ and that as an A_5-module $H^0(F)$ is irreducible. Let $U \subset \mathbb{P}^3 = \mathbb{P}(H^0(F))$ be the open set parametrising *smooth* Horrocks-Mumford surfaces X_s. Then X_s is an abelian surface which is fixed under the Heisenberg group H_5. The action of H_5 on X_s defines a canonical level-5 structure on X_5. Let $\mathcal{A}_{1,5}^{\mathrm{lev}}$ be the moduli space of triples (A, H, α) where (A, H) is a $(1,5)$-polarized abelian surface and α a canonical level structure and denote by ${}^\circ\mathcal{A}_{1,5}^{\mathrm{lev}}$ the open part where the polarization H is very ample. Then the above discussion leads to

Theorem V.1.2 ([HM]) *The map which associates to a section s the Horrocks-Mumford surface $X_s = \{s = 0\}$ induces an isomorphism of U with ${}^\circ\mathcal{A}_{1,5}^{\mathrm{lev}}$. Under this isomorphism the action of* $\mathrm{PSL}(2, \mathbb{Z}_5) = A_5$ *on U is identified with the action of* $\mathrm{PSL}(2, \mathbb{Z}_5)$ *on $\mathcal{A}_{1,5}^{\mathrm{lev}}$ which permutes the canonical level structures on a $(1,5)$-polarized abelian surface. In particular $\mathcal{A}_{1,5}^{\mathrm{lev}}$ is a rational variety.*

Proof. [HM, Theorem 5.2]. □

The inverse morphism

$$\varphi : {}^\circ\mathcal{A}_{1,5}^{\mathrm{lev}} \to U \subset \mathbb{P}(H^0(F)) = \mathbb{P}^3$$

can be extended to a morphism

$$\tilde{\varphi} : (\mathcal{A}_{1,5}^{\mathrm{lev}})^* \to \mathbb{P}(H^0(F))$$

where $(\mathcal{A}_{1,5}^{\mathrm{lev}})^*$ denotes the Igusa (=Voronoi) compactification of $\mathcal{A}_{1,5}^{\mathrm{lev}}$. This extension can also be understood in terms of degenerations of abelian surfaces. Details can be found in [HKW2].

V.2 Type $(1,7)$

The case of type $(1,7)$ was studied by Manolache and Schreyer [MS] in 1993. We are grateful to them for making some private notes and a draft version of [MS] available to us and answering our questions. Some of their results have also been found by Gross and Popescu [GP1], [GP3] and by Ranestad: see also [S-BT].

Theorem V.2.1 ([MS]) *$\mathcal{A}^{\mathrm{lev}}_{1,7}$ is rational, because it is birationally equivalent to a Fano variety of type V_{22}.*

Proof. We can give only a sketch of the proof here. For a general abelian surface A with a polarization of type $(1,7)$ the polarization is very ample and embeds A in $\mathbb{P}^6$. In the presence of a canonical level structure the $\mathbb{P}^6$ may be thought of as $\mathbb{P}(V)$ where V is the Schrödinger representation of the Heisenberg group H_7. We also introduce, for $j \in \mathbb{Z}_7$, the representation V_j, which is the Schrödinger representation composed with the automorphism of H_7 given by $e^{2\pi i/7} \mapsto e^{6\pi ij/7}$. These can also be thought of as representations of the extended Heisenberg group G_7, the extension of H_7 by an extra involution coming from -1 on A. The representation S of G_7 is the character given by this involution (so S is trivial on H_7).

It is easy to see that $A \subset \mathbb{P}^6$ is not contained in any quadric, that is $H^0\big(\mathcal{I}_A(2)\big) = 0$, and from this it follows that there is an H_7-invariant resolution

$$\begin{aligned} 0 \leftarrow \mathcal{I}_A \leftarrow 3V_4 \otimes \mathcal{O}(-3) \leftarrow 7V_1 \otimes \mathcal{O}(-4) \leftarrow 6V_2 \otimes \mathcal{O}(-5) \\ \leftarrow 2V \otimes \mathcal{O}(-6) \oplus \mathcal{O}(-7) \leftarrow 2\mathcal{O}(-7) \leftarrow 0. \end{aligned}$$

By using this and the Koszul complex one obtains a symmetric resolution

$$0 \leftarrow \mathcal{O}_A \leftarrow \mathcal{O} \xleftarrow{\beta} 3V_4 \otimes \mathcal{O}(-3) \xleftarrow{\alpha} 2S \otimes \Omega^3 \xleftarrow{\alpha'} 3V_1 \otimes \mathcal{O}(-4) \xleftarrow{\beta'} \mathcal{O}(-7) \leftarrow 0.$$

This resolution is G_7-invariant. Because of the G_7-symmetry, α can be described by a 3×2 matrix X whose entries lie in a certain 4-dimensional space U, which is a module for $\mathrm{SL}(2, \mathbb{Z}_7)$. The symmetry of the resolution above amounts to saying that α' is given by the matrix $X' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^tX$, and the complex tells us that $\alpha\alpha' = 0$. The three 2×2 minors of X cut out a twisted cubic curve C_A in $\mathbb{P}(U^\vee)$ and because of the conditions on α the ideal I_A of this cubic is annihilated by the differential operators

$$\begin{aligned} \Delta_1 &= \frac{\partial^2}{\partial u_0 \partial u_1} - \tfrac{1}{2}\frac{\partial^2}{\partial u_2^2}, \\ \Delta_2 &= \frac{\partial^2}{\partial u_0 \partial u_2} - \tfrac{1}{2}\frac{\partial^2}{\partial u_3^2}, \\ \Delta_3 &= \frac{\partial^2}{\partial u_0 \partial u_3} - \tfrac{1}{2}\frac{\partial^2}{\partial u_1^2} \end{aligned}$$

where the u_i are coordinates on U.

This enables one to recover the abelian surface A from C_A. If we write $R = \mathbb{C}[u_0, u_1, u_2, u_3]$ then we have a complex (the Hilbert-Burch complex)

$$0 \longleftarrow R/I_A \longleftarrow R \longleftarrow R(-2)^{\oplus 3} \xleftarrow{X} R(-3)^{\oplus 2} \longleftarrow 0.$$

It is exact, because otherwise one can easily calculate the syzygies of I_A and see that they cannot be the syzygies of any ideal annihilated by the three Δ_i. So I_A determines α (up to conjugation) and the symmetric resolution of $\mathcal{O}_A$ can be reconstructed from α.

Let H_1 be the component of the Hilbert scheme parametrising twisted cubic curves. For a general net of quadrics $\delta \subset \mathbb{P}(U^\vee)$ the subspace $H(\delta) \subset H_1$ consisting of those cubics annihilated by δ is, by a result of Mukai [Muk], a smooth rational Fano 3-fold of genus 12, of the type known as V_{22}. To check that this is so in a particular case it is enough to show that $H(\delta)$ is smooth. We must do so for $\delta = \Delta = \operatorname{Span}(\Delta_1, \Delta_2\Delta_3)$. Manolache and Schreyer show that $H(\Delta)$ is isomorphic to the space $\operatorname{VSP}\big(\bar{X}(7), 6\big)$ of polar hexagons to the Klein quartic curve (the modular curve $\bar{X}(7)$):

$$\operatorname{VSP}\big(\bar{X}(7), 6\big) = \big\{\{l_1, \ldots, l_6\} \subset \operatorname{Hilb}^6(\check{\mathbb{P}}^2) \mid \textstyle\sum l_i^4 = x_0^3x_1 + x_1^3x_2 + x_2^3x_0\big\}.$$

(To be precise we first consider all 6-tuples $(l_1, \ldots, l_6)$ where the l_i are pairwise different with the above property and then take the Zariski-closure in the Hilbert scheme.) It is known that $\operatorname{VSP}\big(\bar{X}(7), 6\big)$ is smooth, so we are done. □

Manolache and Schreyer also give an explicit rational parametrization of $\operatorname{VSP}\big(\bar{X}(7), 6\big)$ by writing down equations for the abelian surfaces. They make the interesting observation that this rational parametrization is actually defined over the rational numbers.

V.3 Type $(1, 11)$

The spaces $\mathcal{A}_{1,d}^{\text{lev}}$ for small d are studied by Gross and Popescu, [GP1], [GP2], [GP3], [GP4]. In particular, in [GP2], they obtain a description of $\mathcal{A}_{1,11}^{\text{lev}}$.

Theorem V.3.1 ([GP2]) *There is a rational map $\Theta_{11} : \mathcal{A}_{1,11}^{\text{lev}} \dashrightarrow \operatorname{Gr}(2,6)$ which is birational onto its image. The closure of $\operatorname{Im}\Theta_{11}$ is a smooth linear section of $\operatorname{Gr}(2,6)$ in the Plücker embedding and is birational to the Klein cubic in $\mathbb{P}^4$. In particular $\mathcal{A}_{1,11}^{\text{lev}}$ is unirational but not rational.*

The Klein cubic is the cubic hypersurface in $\mathbb{P}^4$ with the equation

$$\sum_{i=0}^{4} x_i^2 x_{i+1} = 0$$

with homogeneous coordinates x_i, $i \in \mathbb{Z}_5$. It is smooth, and all smooth cubic hypersurfaces are unirational but not rational [CG], [IM].

The rational map Θ_{11} arises in the following way. For a general abelian surface A in $\mathcal{A}_{1,11}^{\mathrm{lev}}$, the polarization (which is very ample) and the level structure determine an H_{11}-invariant embedding of A into $\mathbb{P}^{10}$. The action of -1 on A lifts to $\mathbb{P}^{10} = \mathbb{P}(H^0(\mathcal{L}))$ and the (-1)-eigenspace of this action on $H^0(\mathcal{L})$ (where $\mathcal{L}$ is a symmetric bundle in the polarizing class) determines a $\mathbb{P}^4$, called $\mathbb{P}^- \subset \mathbb{P}^{10}$. We choose coordinates $x_0, \ldots, x_{10}$ on $\mathbb{P}^{10}$ with indices in $\mathbb{Z}_{11}$ such that $x_1, \ldots, x_5$ are coordinates on $\mathbb{P}^-$, so that on $\mathbb{P}^-$ we have $x_0 = 0$, $x_i = -x_{-i}$. The matrix T is defined to be the restriction of R to $\mathbb{P}^-$, where

$$R_{ij} = x_{j+i}x_{j-i}, \qquad 0 \le i, j \le 5$$

(This is part of a larger matrix which describes the action on $H^0\big(\mathcal{O}_{\mathbb{P}^{10}}(2)\big)$ of H_{11}.) The matrix T is skew-symmetric and non-degenerate at a general point of $\mathbb{P}^-$. However, it turns out that for a general $A \in \mathcal{A}_{1,11}^{\mathrm{lev}}$ the rank of T at a general point $x \in A \cap \mathbb{P}^-$ is 4. For a fixed A, the kernel of T is independent of the choice of x (except where the dimension of the kernel jumps), and this kernel is the point $\Theta_{11}(A) \in \mathrm{Gr}(2,6)$.

From the explicit matrix R, finally, Gross and Popescu obtain the description of the closure of $\operatorname{Im}\Theta_{11}$ as being the intersection of $\mathrm{Gr}(2,6)$ with five hyperplanes in Plücker coordinates. The equation of the Klein cubic emerges directly (as a 6×6 Pfaffian), but it is a theorem of Adler [AR] that the Klein cubic is the only degree 3 invariant of $\mathrm{PSL}(2,\mathbb{Z}_{11})$ in $\mathbb{P}^4$.

V.4 Other type $(1,t)$ cases

The results of Gross and Popescu for $t = 11$ described above are part of their more general results about $\mathcal{A}_{1,t}^{\mathrm{lev}}$ and $\mathcal{A}_{1,t}$ for $t \ge 5$. In the series of papers [GP1]–[GP4] they prove the following (already stated above as Theorem III.1.5).

Theorem V.4.1 ([GP1],[GP2],[GP3],[GP4]) *$\mathcal{A}_{1,t}^{\mathrm{lev}}$ is rational for $6 \le t \le 10$ and $t = 12$ and unirational, but not rational, for $t = 11$. Moreover the variety $\mathcal{A}_{1,t}$ is unirational for $t = 14, 16, 18$ and 20.*

The cases have a different flavour depending on whether t is even or odd. For odd $t = 2d+1$ the situation is essentially as described for $t = 11$ above: there is a rational map $\Theta_{2d+1} : \mathcal{A}_{1,t}^{\mathrm{lev}} \dashrightarrow \mathrm{Gr}(d-3, d+1)$, which can be described in terms of matrices or by saying that A maps to the H_t-subrepresentation $H^0\big(\mathcal{I}_A(2)\big)$ of $H^0\big(\mathcal{O}_A(2)\big)$. In other words, one embeds A in $\mathbb{P}^{t-1}$ and selects the H_t-space of quadrics vanishing along A.

Theorem V.4.2 ([GP1]) *If $t = 2d+1 \ge 11$ is odd then the homogeneous ideal of a general H_t-invariant abelian surface in $\mathbb{P}^{t-1}$ is generated by quadrics; consequently Θ_{2d+1} is birational onto its image.*

For $t = 7$ and $t = 9$ this is not true: however, a detailed analysis is still possible and is carried out in [GP3] for $t = 7$ and in [GP2] for $t = 9$. For $t \geq 13$ it is a good description of the image of Θ_t that is lacking. Even for $t = 13$ the moduli space is not unirational and for large t it is of general type (at least for t prime or a prime square).

For even $t = 2d$ the surface $A \subset \mathbb{P}^{t-1}$ meets $\mathbb{P}^- = \mathbb{P}^{d-2}$ in four distinct points (this is true even for many degenerate abelian surfaces). Because of the H_t-invariance these points form a $\mathbb{Z}_2 \times \mathbb{Z}_2$-orbit and there is therefore a rational map $\Theta_{2d} : \mathcal{A}_{1,t}^{\mathrm{lev}} \dashrightarrow \mathbb{P}^-/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Theorem V.4.3 ([GP1]) *If $t = 2d \geq 10$ is even then the homogeneous ideal of a general H_t-invariant abelian surface in $\mathbb{P}^{t-1}$ is generated by quadrics (certain Pfaffians) and Θ_{2d} is birational onto its image.*

To deduce Theorem V.4.1 from Theorem V.4.2 and Theorem V.4.3 a careful analysis of each case is necessary: for $t = 6, 8$ it is again the case that A is not cut out by quadrics in $\mathbb{P}^{t-1}$. In those cases when rationality or unirationality can be proved, the point is often that there are pencils of abelian surfaces in suitable Calabi-Yau 3-folds and these give rise to rational curves in the moduli spaces. Gross and Popescu use these methods in [GP2] ($t = 9, 11$), [GP3] ($t = 6, 7, 8$ and 10), and [GP4] ($t = 12$) to obtain detailed information about the moduli spaces $\mathcal{A}_{1,t}^{\mathrm{lev}}$. In [GP4] they also consider the spaces $\mathcal{A}_{1,t}$ for $t = 14, 16, 18$ and 20.

VI Degenerations

The procedure of toroidal compactification described in [AMRT] involves making many choices. Occasionally there is an obvious choice. For moduli of abelian surfaces this is usually the case, or nearly so, since one has the Igusa compactification (which is the blow-up of the Satake compactification along the boundary) and all known cone decompositions essentially agree with this one. But generally toroidal compactifications are not so simple. One has to make further modifications in order to obtain acceptably mild singularities at the boundary. Ideally one would like to do this in a way which is meaningful for moduli, so as to obtain a space which represents a functor described in terms of abelian varieties and well-understood degenerations. The model, of course, is the Deligne-Mumford compactification of the moduli space of curves.

VI.1 Local degenerations

The first systematic approach to the local problem of constructing degenerations of polarized abelian varieties is Mumford's paper [Mu2] (conveniently reprinted as an appendix to [FC]). Mumford specifies degeneration data which determine a family G of semi-abelian varieties over the spectrum S of a complete normal ring R. Faltings and Chai [FC] generalized this and also showed how to recover the degeneration data from such a family. This semi-abelian family can then be compactified: in fact, Mumford's construction actually produced the compactification first and the semi-abelian family as a subscheme. However, although G is uniquely determined, the compactification is non-canonical. We may as well assume that R is a DVR and that G_η, the generic fibre, is an abelian scheme: the compactification then amounts to compactifying the central fibre G_0 in some way.

Namikawa (see for instance [Nam3] for a concise account) and Nakamura [Nak1] used toroidal methods to construct natural compactifications in the complex-analytic category, together with proper degenerating families of so-called stable quasi-abelian varieties. Various difficulties, including non-reduced fibres, remained, but more recently Alexeev and Nakamura [Ale1], [AN], have produced a more satisfactory and simpler theory. We describe their results below, beginning with their simplified version of the constructions of Mumford and of Faltings and Chai. See [FC], [Mu2] or [AN] itself for more.

R is a complete DVR with maximal ideal I, residue field $k = R/I$ and field of fractions K. We take a split torus $\tilde{G}$ over $S = \operatorname{Spec} R$ with character group X and let $\tilde{G}(K) \cong (K^*)^g$ be the group of K-valued points of $\tilde{G}$. A set of periods is simply a subgroup $Y \subset \tilde{G}(K)$ which is isomorphic to $\mathbb{Z}^g$. One can define a polarization to be an injective map $\phi : Y \to X$ with suitable properties.

Theorem VI.1.1 ([Mu2],[FC]) *There is a quotient $G = \tilde{G}/Y$ which is a semi-abelian scheme over S: the generic fibre G_η is an abelian scheme over* $\operatorname{Spec} K$ *with a polarization (given by a line bundle $\mathcal{L}_\eta$ induced by ϕ).*

Mumford's proof also provides a projective degeneration, in fact a wide choice of projective degenerations, each containing G as an open subscheme.

Theorem VI.1.2 ([Mu2],[Ch],[FC],[AN]) *There is an integral scheme $\tilde{P}$, locally of finite type over S, containing $\tilde{G}$ as an open subscheme, with an ample line bundle $\tilde{\mathcal{L}}$ and an action of Y on $(\tilde{P}, \tilde{\mathcal{L}})$. There is an S-scheme $P = \tilde{P}/Y$, projective over S, with $P_\eta \cong G_\eta$ as polarized varieties, and G can be identified with an open subscheme of P.*

Many technical details have been omitted here. $\tilde{P}$ has to satisfy certain compatibility and completeness conditions: of these, the most complicated is a completeness condition which is used in [FC] to prove that each component of the central fibre P_0 is proper over k. Alexeev and Nakamura make a special choice of $\tilde{P}$ which, among other merits, enables them to dispense with this condition because the properness is automatic.

Mumford proved this result in the case of maximal degeneration, when G_0 is a torus over k. That condition which was dropped in [FC] and also in [AN] where $\tilde{G}$ is allowed to have an abelian part. Then $\tilde{G}$ and G_0 are Raynaud extensions, that is, extensions of abelian schemes by tori, over R and k respectively. The extra work entailed by this is carried out in [FC] but the results, though a little more complicated to state, are essentially the same as in the case of maximal degeneration.

In practice one often starts with the generic fibre G_η. According to the semistable reduction theorem there is always a semi-abelian family $G \to S$ with generic fibre G_η, and the aim is to construct a uniformization $G = \tilde{G}/Y$. It was proved in [FC] that this is always possible.

The proof of VI.1.1, in the version given by Chai [Ch] involves implicitly writing down theta functions on $\tilde{G}(K)$ in order to check that the generic fibre is the abelian scheme G_η. These theta functions can be written (analogously with the complex-analytic case) as Fourier power series convergent in the I-adic topology, by taking coordinates $w_1, \dots, w_g$ on $\tilde{G}(K)$ and setting

$$\theta = \sum_{x \in X} \sigma_x(\theta) w^x$$

with $\sigma_x(\theta) \in K$. In particular theta functions representing elements of $H^0(G_\eta, \mathcal{L}_\eta)$ can be written this way and the coefficients obey the transformation formula

$$\sigma_{x+\phi(y)}(\theta) = a(y)b(y,x)\sigma_x(\theta)$$

for suitable functions $a : Y \to K^*$ and $b : Y \times X \to K^*$.

For simplicity we shall assume for the moment that the polarization is principal: this allows us to identify Y with X via ϕ and also means that there is only one theta function, ϑ. The general case is only slightly more complicated.

These power series have K coefficients and converge in the I-adic topology but their behaviour is entirely analogous to the familiar complex-analytic theta functions. Thus there are cocycle conditions on a and b and it turns out that b is a symmetric bilinear form on $X \times X$ and a is an inhomogeneous quadratic form. Composing a and b with the valuation yields functions $A : X \to \mathbb{Z}$, $B : X \times X \to \mathbb{Z}$, and they are related by

$$A(x) = \frac{1}{2}B(x,x) + \frac{rx}{2}$$

for some $r \in \mathbb{N}$. We fix a parameter $s \in R$, so $I = sR$.

Theorem VI.1.3 ([AN]) *The normalization of* $\operatorname{Proj} R[s^{A(x)} w^x \theta; x \in X]$ *is a relatively complete model* $\tilde{P}$ *for the maximal degeneration of principally polarized abelian varieties associated with* G_η.

Similar results hold in general. The definition of $\tilde{P}$ has to be modified slightly if G_0 has an abelian part. If the polarization is non-principal it may be necessary to make a ramified base change first, since otherwise there may not be a suitable extension of $A : Y \to \mathbb{Z}$ to $A : X \to \mathbb{Z}$. Even for principal polarization it may be necessary to make a base change if we want the central fibre P_0 to have no non-reduced components.

The proof of Theorem VI.1.3 depends on the observation that the ring

$$R[s^{A(x)}w^x\theta; x \in X]$$

is generated by monomials. Consequently $\tilde{P}$ can be described in terms of toric geometry. The quadratic form B defines a *Delaunay decomposition* of $X \otimes \mathbb{R} = X_{\mathbb{R}}$. One of the many ways of describing this is to consider the paraboloid in $\mathbb{R}e_0 \oplus X_{\mathbb{R}}$ given by

$$x_0 = A(x) = \frac{1}{2}B(x,x) + \frac{rx}{2},$$

and the lattice $M = \mathbb{Z}e_0 \oplus X$. The convex hull of the points of the paraboloid with $x \in X$ consists of countably many facets and the projections of these facets on $X_{\mathbb{R}}$ form the Delaunay decomposition. This decomposition determines $\tilde{P}$. It is convenient to express this in terms of the *Voronoi decomposition* Vor_B of $X_{\mathbb{R}}$ which is dual to the Delaunay decomposition in the sense that there is a 1-to-1 inclusion-reversing correspondence between (closed) Delaunay and Voronoi cells. We introduce the map $dA : X_{\mathbb{R}} \to X^*_{\mathbb{R}}$ given by

$$dA(\xi)(x) = B(\xi, x) + \frac{rx}{2}.$$

Theorem VI.1.4 ([AN]) *$\tilde{P}$ is the torus embedding over R given by the lattice $N = M^* \subset \mathbb{R}e_0^* \oplus X^*_{\mathbb{R}}$ and the fan Δ consisting of $\{0\}$ and the cones on the polyhedral cells making up $\big(1, -dA(\mathrm{Vor}_B)\big)$.*

Using this description, Alexeev and Nakamura check the required properties of $\tilde{P}$ and prove Theorem VI.1.3. They also obtain a precise description of the central fibres $\tilde{P}_0$ (which has no non-reduced components if we have made a suitable base change) and P_0 (which is projective). The polarized fibres $(P_0, \mathcal{L}_0)$ that arise are called *stable quasi-abelian varieties*, as in [Nak1]. In the principally polarized case P_0 comes with a Cartier divisor Θ_0 and (P_0, Θ_0) is called a *stable quasi-abelian pair.* We refer to [AN] for a precise intrinsic definition, which does not depend on first knowing a degeneration that gives rise to the stable quasi-abelian variety. For our purposes all that matters is that such a characterization exists.

VI.2 Global degenerations and compactification

Alexeev, in [Ale1], uses the infinitesimal degenerations that we have just been considering to tackle the problem of canonical global moduli. For simplicity we shall describe results of [Ale1] only in the principally polarized case.

We define a *semi-abelic variety* to be a normal variety P with an action of a semi-abelian variety G having only finitely many orbits, such that the stabilizer of the generic point of P is a connected reduced subgroup of the torus part of G. If $G = A$ is actually an abelian variety then Alexeev refers to P as an *abelic variety*: this is the same thing as a torsor for the abelian variety A. If we relax the conditions by allowing P to be semi-normal then P is called a *stable semi-abelic variety* or SSAV.

A *stable semi-abelic pair* (P, Θ) is a projective SSAV together with an effective ample Cartier divisor Θ on P such that Θ does not contain any G-orbit. The degree of the corresponding polarization is $g!.h^0(\mathcal{O}_P(\Theta))$, and P is said to be principally polarized if the degree of the polarization is $g!$. If P is an abelic variety then (P, Θ) is called an *abelic pair.*

Theorem VI.2.1 ([Ale1]) *The categories $\underline{A}_g$ of g-dimensional principally polarized abelian varieties and $\underline{AP}_g$ of principally polarized abelic pairs are naturally equivalent. The corresponding coarse moduli spaces $\mathcal{A}_g$ and $\mathcal{AP}_g$ exist as separated schemes and are naturally isomorphic to each other.*

Because of this we may as well compactify $\mathcal{AP}_g$ instead of $\mathcal{A}_g$ if that is easier. Alexeev carries out this program in [Ale1]. In this way, he obtains a proper algebraic space $\overline{\mathcal{AP}}_g$ which is a coarse moduli space for stable semi-abelic pairs.

Theorem VI.2.2 ([Ale1]) *The main irreducible component of $\overline{\mathcal{AP}}_g$ (the component that contains $\mathcal{AP}_g = A_g$) is isomorphic to the Voronoi compactification $\mathcal{A}_g^*$ of $\mathcal{A}_g$. Moreover, the Voronoi compactification in this case is projective.*

The first part of Theorem VI.2.2 results from a careful comparison of the respective moduli stacks. The projectivity, however, is proved by elementary toric methods which, in view of the results of [FC], work over $\operatorname{Spec} \mathbb{Z}$.

In general $\overline{\mathcal{AP}}_g$ has other components, possibly of very large dimension. Alexeev has examined these components and the SSAVs that they parametrize in [Ale2]

Namikawa, in [Nam1], already showed how to attach a stable quasi-abelian variety to a point of the Voronoi compactification. Namikawa's families, however, have non-reduced fibres and require the presence of a level structure: a minor technical alteration (a base change and normalization) has to be made before the construction works satisfactorily. See [AN] for this and also for an alternative construction using explicit local families that were first written down by Chai [Ch]. The use of abelic rather than abelian varieties also seems to be essential in order to obtain a good family: this is rather more apparent over a non-algebraically closed field, when the difference between an abelian variety (which has a point) and an abelic variety is considerable.

Nakamura, in [Nak2], takes a different approach. He considers degenerating families of abelian varieties with certain types of level structure. In his case the boundary points correspond to *projectively stable quasi-abelian schemes* in the sense of GIT. His construction works over $\operatorname{Spec}\mathbb{Z}[\zeta_N, 1/N]$ for a suitable N. At the time of writing it is not clear whether Nakamura's compactification also leads to the second Voronoi compactification.

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