Fundamental groups of toroidal compactifications

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Abstract

We compute the fundamental group of a toroidal compactification of a Hermitian locally symmetric space D/Γ, without assuming either that Γ is neat or that it is arithmetic. We also give bounds for the first Betti number.

Many important complex algebraic varieties can be described as locally symmetric varieties. Examples include modular curves $\mathbb{H}/\Gamma$, where $\mathbb{H}$ is the upper half-plane and $\Gamma < \mathrm{PSL}(2, \mathbb{Z})$; classifying spaces for Hodge structures or (in cases where a Torelli theorem holds) moduli spaces of polarised varieties, such as moduli of abelian varieties and of K3 surfaces; and special surfaces, such as Hilbert modular surfaces.

Locally symmetric varieties are in general non-compact, and we want to be able to compactify them and to study the geometry of the compactifications, especially the birational geometry, which does not depend on the choice of compactification. We work with toroidal compactifications as described in [AMRT].

Two basic birational invariants of a compact complex manifold X are the Kodaira dimension $\kappa(X)$ and the fundamental group $\pi_1(X)$. There is an extensive literature on computing Kodaira dimensions of specific locally symmetric varieties, which is usually very difficult.

Computing the fundamental group is easier, but there are some gaps in the literature which we aim to fill. We study the fundamental group of a toroidal compactification $(D/\Gamma)'_\Sigma$ of a non-compact, not necessarily arithmetic quotient D/Γ by a lattice Γ. In general this is not a manifold, but

*Research partially supported by Contract 015/9.04.2014 with the the Scientific Foundation of Kliment Ohridski University of Sofia.

it is normal and can be chosen to have only quotient singularities. By [Ko, Sect 7] these do not affect the fundamental group.

The main result of the article is Theorem 4.3, describing $\pi_1((D/\Gamma)'_\Sigma)$ as a quotient of the lattice Γ.

Acknowledgements

The first author is grateful to a referee for pointing out a flaw in a previous version and for bringing to her attention several references. We also thank Klaus Hulek and Xiuping Su for useful remarks, and Maximilian Rössler for correcting an error.

1 Background

In this section we explain the background to the problem and establish some terminology and notation. A *symmetric space* of non-compact type is a quotient $D = G/K$ of a connected non-compact semisimple Lie group G, assumed to be the real points of a linear algebraic group defined over $\mathbb{Q}$, by a connected maximal compact subgroup K of G. If the centre of K is not discrete, then D carries a Hermitian structure and hence the structure of a complex manifold, in fact a Kähler manifold [He, Theorem VIII.6.1.].

By a *lattice* in G we mean a discrete subgroup of G of finite covolume with respect to Haar measure. A lattice Γ is said to be *arithmetic* if $\Gamma \cap G(\mathbb{Z})$ is of finite index in both Γ and $G(\mathbb{Z})$. It is said to be *neat* if the subgroup of $\mathbb{C}^*$ generated by all eigenvalues of elements of Γ is torsion free.

A *locally symmetric variety* is the quotient of a Hermitian symmetric space D by a lattice $\Gamma < G$. If D/Γ is compact then Γ is said to be *cocompact* or *uniform*. Non-uniform lattices are very common, however, and it is this case that we are concerned with. By [BB] (for Γ arithmetic) and [Mok] these quotients are always algebraic varieties, not just complex analytic spaces.

Toroidal compactifications of arithmetic quotients of Hermitian symmetric spaces are constructed and described in [AMRT]. Margulis [Ma] showed that an irreducible lattice Γ in a semisimple Lie group G is necessarily arithmetic if G has real rank $\operatorname{rk}_{\mathbb{R}} G > 1$, and Garland and Raghunathan [GR] had earlier considered the case of $\operatorname{rk}_{\mathbb{R}} G = 1$ and constructed a fundamental domain for the action of any lattice Γ on $D = G/K$.

A parabolic subgroup $Q < G$ is said to be Γ-*rational*, for Γ a lattice (arithmetic or not) in G, if $\Gamma \cap N_Q$ is a lattice in the unipotent radical N_Q of Q. An analytic boundary component F of D is said to be a Γ-rational boundary component if its stabiliser is a Γ-rational parabolic subgroup. The construction in [GR] shows that D/Γ can be compactified by adjoining the Γ-quotients of the Γ-rational boundary components, and this was later used by Mok [Mok] to extend the construction of [AMRT] to the case of non-arithmetic quotients.

The only Hermitian symmetric spaces whose automorphism groups have real rank 1 are the complex balls $\mathbb{B}^n = \mathrm{SU}(n,1)/\,\mathrm{S}(\mathrm{U}(n)\times \mathrm{U}(1))$, so Mok's generalisation is needed only for ball quotients, but they are of great interest. The construction of [AMRT] becomes much simpler when $\mathrm{rk}_{\mathbb{R}}\, G = 1$, as we explain below.

An extensive reference on toroidal and other compactifications of locally symmetric spaces D/Γ is the monograph [BJ] of Borel and Ji.

The first results on the fundamental group of a smooth compactification of a locally symmetric space concerned Siegel modular 3-folds, where $G = \mathrm{Sp}(2,\mathbb{R})$. These are moduli spaces of abelian surfaces and some such cases were studied in [HK], in [Kn] and in [HSa]. More generally, the fundamental group of a toroidal compactification of an arbitrary Hermitian locally symmetric variety is studied in [Sa].

For simplicity, we assume throughout this paper that D is irreducible: that is, that G does not decompose into a product of factors.

Denote by MPar_Γ the set of Γ-rational maximal parabolic subgroups of G. It is shown in [Sa] that if $\Gamma < G$ is a neat arithmetic non-uniform lattice, then a toroidal compactification $(D/\Gamma)'_\Sigma$ satisfies $\pi_1(D/\Gamma)'_\Sigma = \Gamma/\Upsilon$, where Υ is the subgroup of Γ generated by the centres of the unipotent radicals of all $Q \in \mathrm{MPar}_\Gamma$. Moreover, in [GHS] it is shown that there is a surjective group homomorphism $\Gamma \to \pi_1((D/\Gamma)'_\Sigma)$, whose kernel contains all $\gamma \in \Gamma$ with a fixed point on D.

If $Q \in \mathrm{MPar}_\Gamma$ then we denote the unipotent radical of Q by N_Q, the centre of N_Q by U_Q and the solvable radical of Q by R_Q. Let Λ be the subgroup of Γ generated by all $\gamma \in \Gamma \cap Q$ (for some $Q \in \mathrm{MPar}_\Gamma$) for which some power $\gamma^k \in R_Q$, and if $\gamma^k \in N_Q \subset R_Q$ then $\gamma^k \in U_Q$. From this definition, Λ is normal in Γ.

Our main result, Theorem 4.3, is that $\pi_1((D/\Gamma)'_\Sigma) = \Gamma/\Lambda\Upsilon$. This is a much more precise version of a result of the second author from [Sa], stated here as Theorem 4.1. It is also slightly more general than the results of [Sa] because, in the light of [Mok], we may now also allow non-arithmetic lattices.

Here is a synopsis of the paper. Section 2 introduces some notation and terminology and describes the structure of maximal Γ-rational parabolic subgroups, largely following [BJ]. Section 3 describes the toroidal compactifications $(D/\Gamma)'_\Sigma$ and their coverings $(D/\Gamma_o)'_\Sigma$ for normal subgroups (not necessarily lattices) $\Gamma_o \lhd \Gamma$ containing Υ. Section 4 comprises the main results of the article.

We show in Proposition 4.4 that any element $\gamma\Upsilon \in \Gamma/\Upsilon$ with a fixed point on $(D/\Upsilon)'_\Sigma$ has a representative $\gamma \in \Gamma \cap Q$ with $\gamma^k \in R_Q$ for some $Q \in \mathrm{MPar}_\Gamma$ and $k \in \mathbb{N}$, and that $\gamma^k \in U_Q$ if $\gamma^k \in N_Q$. This suffices to prove Theorem 4.3, and from that we deduce bounds on the first Betti numbers in Subsection 4.2.

2 Parabolic subgroups

We collect here some properties of Hermitian symmetric spaces $D = G/K$ of non-compact type and maximal parabolic subgroups Q of G. For more details see [BJ, Chapter 1].

2.1 Langlands decomposition of a parabolic subgroup

Any parabolic subgroup Q of G has a *Langlands decomposition* [BJ, Equation (I.1.10)]

$$Q = N_Q A_Q M_Q$$

where N_Q is the *unipotent radical* of Q. We write $L_Q = A_Q \times M_Q$, the *Levi subgroup* of Q, and $R_Q = N_Q \rtimes A_Q$, the *solvable radical* of Q. The subgroup A_Q is called the *split component* of Q, and M_Q is a semisimple complement of R_Q. All these groups are uniquely defined once we choose a maximal compact subgroup K of G.

We can refine this further if we assume, as we henceforth do, that Q is a maximal parabolic subgroup of G. We denote by U_Q the centre of the unipotent radical N_Q of Q. Since N_Q is a 2-step nilpotent group, i.e. $[[N_Q, N_Q], N_Q] = 0$, we have $U_Q = [N_Q, N_Q]$, the commutator subgroup. We may identify U_Q with its Lie algebra $\mathrm{Lie}(U_Q) \cong \mathbb{R}^m$, for $m = \dim_{\mathbb{R}} U_Q$. The quotient $V_Q = N_Q/U_Q$ is also an abelian group, naturally isomorphic to $\mathbb{C}^n$ [BJ, (III.7.10)] and $N_Q = U_Q \rtimes V_Q$ is a semi-direct product of U_Q and V_Q.

The semi-simple complement M_Q of the solvable radical R_Q of Q is a product $M_Q = G'_{Q,l} \times G_{Q,h}$ of semisimple groups $G'_{Q,l}$, $G_{Q,h}$ of noncompact type [BJ, (III.7.8)].

This gives us the *refined Langlands decomposition*

$$Q = (U_Q \rtimes V_Q) \rtimes (A_Q \times G'_{Q,l} \times G_{Q,h}) \tag{1}$$

of an arbitrary maximal parabolic subgroup Q of G. Note also that $G = QK$.

Let us describe briefly the group laws on N_Q and Q. For this, and later, it will be convenient to use superfix notation for conjugation: $g^h = hgh^{-1}$. However, the letters j and k will always denote integers, so g^k means the k-th power of g.

There is an $\mathbb{R}$-linear embedding $\eta\colon \mathrm{Lie}(V_Q) \to \mathrm{Lie}(N_Q)$, whose image is the orthogonal complement of $\mathrm{Lie}(U_Q)$ with respect to the Killing form. The group law in N_Q is

$$\exp(u_1+\eta(v_1))\exp(u_2+\eta(v_2)) = \exp\Big((u_1+u_2)+\eta(v_1+v_2)+\frac{1}{2}[\eta(v_1),\eta(v_2)]\Big). \tag{2}$$

For the group law in Q we write $q_j = (n_j, l_j) \in Q = N_Q \rtimes L_Q$, further decomposed as $l_j = (a_j, g_j, h_j) \in A_Q \times G'_{Q,l} \times G_{Q,h}$ and $n_j = \exp(u_j + \eta(v_j))$

with $u_j \in \mathrm{Lie}(U_Q)$ and $v_j \in \mathrm{Lie}(V_Q)$. Then

$$q_1 q_2 = (n_1 n_2^{l_1}, l_1 l_2) = \Big(\exp\big(u + \eta(v)\big), a_1 a_2, g_1 g_2, h_1 h_2 \Big), \tag{3}$$

where $u = u_1 + u_2^{a_1 g_1} + \frac{1}{2}[\eta(v_1), \eta(v_2^{l_1})]$ and $v = v_1 + v_2^{l_1}$.

Since Q is maximal and D is irreducible, the group $A_Q \cong (\mathbb{R}_{>0}, .)$ is a 1-dimensional real torus of G (see, for instance, [BJ, Section I.1.10]).

The symmetric space D has an embedding in a space $\check{D}$, the compact dual, on which G acts. The topological boundary of D then decomposes into complex analytic boundary components corresponding to parabolic subgroups Q: namely, Q is the normaliser of the boundary component $F(Q)$. See [BJ, Proposition I.5.28] or [AMRT, Proposition III.3.9.] for details.

If $F(P) \subseteq \overline{F(Q)}$ then $U(P) \supseteq U(Q)$ by [AMRT, Theorem III.4.8(i)].

2.2 Horospherical decomposition

For any parabolic subgroup we have $G = QK$ [BJ, (I.1.20)], so Q acts transitively on D; moreover, $Q \cap K = M_Q \cap K$. As a result, the refined Langlands decomposition (1) of Q induces the *refined horospherical decomposition*

$$D = U_Q \times V_Q \times A_Q \times D'_{Q,l} \times D_{Q,h} \tag{4}$$

of D with $D'_{Q,l} = G'_{Q,l}/G'_{Q,l} \cap K$ and $D_{Q,h} = G_{Q,h}/G_{Q,h} \cap K$: see [BJ, Lemma III.7.9] and the discussion there. The equality in (4) is a real analytic diffeomorphism. The factors $D_{Q,h} \cong F(Q)$ and $D'_{Q,l}$ are respectively Hermitian and Riemannian symmetric spaces of noncompact type.

The group law in Q induces a Q-action on the corresponding horospherical decomposition. If $(n, l) = (\exp(u + \eta(v)), a, g, h) \in Q = N_Q \rtimes L_Q$ and $y = (n_y, x_y) = (u_y, v_y, a_y, z'_y, z_y) \in D$ then

$$(n, l)(y) = (n n_y^l, l x_y) = \Big(\exp\big(u + u_y^{ag} + \frac{1}{2}[\eta(v), \eta(v_y^l)] + \eta(v + v_y^l)\big), a a_y, g z'_y, h z_y \Big).$$

2.3 Siegel domains

In [P-S] Pyatetskii-Shapiro realises the Hermitian symmetric spaces $D = G/K$ of noncompact type as Siegel domains of third kind. These are families of open cones, parametrised by products of complex Euclidean spaces and Hermitian symmetric spaces of noncompact type.

In the refined horospherical decomposition (4), the A_Q-orbit

$$C_Q = A_Q D'_{Q,l} = \{(a, z') \mid a \in A_Q, z' \in D'_{Q,l}\} \tag{5}$$

of the Riemannian symmetric space $D'_{Q,l}$ is an open, strongly convex cone in $U_Q \cong \mathbb{R}^m$ [BJ, Lemma III.7.7]. Note that the reductive group $G_{Q,l} = A_Q \rtimes G'_{Q,l}$ acts transitively on C_Q since $C_Q = G_{Q,l}/G_{Q,l} \cap K = G_{Q,l}/G'_{Q,l} \cap K$.

We embed C_Q in the complexification $U_Q \otimes_{\mathbb{R}} \mathbb{C} \cong (\mathbb{C}^m, +)$ of U_Q as a subset $iC_Q \subset iU_Q$ with pure imaginary components. Combining with (4), one obtains a real analytic diffeomorphism of D onto the product

$$(U_Q + iC_Q) \times V_Q \times D_{Q,h}. \tag{6}$$

which will be called the Siegel domain realisation of D associated with Q. See [AMRT] for the relation between (6) and the classical Siegel domain presentation of D.

In these coordinates, with notation as above, the action of Q is given by

$$\begin{aligned}(u + \eta(v), a, g, h)&(u_y + i(a_y, z'_y), v_y, z_y) \\ &= \Big(u + u_y^{ag} + \frac{1}{2}[\eta(v), \eta(v_y^l)] + i(aa_y, gz'_y), v + v_y^l, hz_y\Big)\end{aligned} \tag{7}$$

where $(u_y + i(a_y, z'_y), v_y, z_y) \in (U_Q + iC_Q) \times V_Q \times D_{Q,h}$.

3 Toroidal compactifications

We recall briefly enough detail on toroidal compactification for our immediate purposes: for full details we refer to [AMRT].

3.1 Admissible fans and collections

Suppose that $Q \in \mathrm{MPar}_\Gamma$: then $\Upsilon_Q = \Gamma \cap U_Q \cong \mathbb{Z}^m$ is a lattice in $U_Q \cong \mathbb{R}^m$.

We say that a closed polyhedral cone $\sigma \subset U_Q$ is *Υ_Q-rational* if $\sigma = \mathbb{R}_{\geq 0} u_1 + \cdots + \mathbb{R}_{\geq 0} u_s$ for some $u_i \in \Upsilon_Q$.

A *fan* (see [Fu]) $\Sigma(Q)$ is a collection of closed polyhedral cones in U_Q such that any face of a cone in $\Sigma(Q)$ is also in $\Sigma(Q)$ and any two cones in $\Sigma(Q)$ intersect in a common face. It is *Υ_Q-rational* if all cones in $\Sigma(Q)$ are Υ_Q-rational.

The fan $\Sigma(Q)$ in U_Q is said to be *Γ-admissible* if it is Υ_Q-rational, it decomposes C_Q (that is, $C_Q \subseteq \bigcup_{\sigma \in \Sigma(Q)} \sigma$) and $\Gamma_{Q,l} = \Gamma \cap G_{Q,l}$ acts on $\Sigma(Q)$ with only finitely many orbits.

The lattice Γ acts on MPar_Γ by conjugation. We say that a family $\Sigma = \{\Sigma(Q)\}_{Q \in \mathrm{MPar}_\Gamma}$ of Γ-admissible fans $\Sigma(Q)$ is a *Γ-admissible family* if:

(i) $\gamma\Sigma(Q) = \Sigma(Q^\gamma)$, where $Q^\gamma = \gamma Q \gamma^{-1}$, for all $\gamma \in \Gamma$ and $Q \in \mathrm{MPar}_\Gamma$; and

(ii) $\Sigma(Q) = \{\sigma \cap U_Q \mid \sigma \in \Sigma(P)\}$ whenever $F(P) \subseteq \overline{F(Q)}$, for P, $Q \in \mathrm{MPar}_\Gamma$.

3.2 Partial compactification at a cusp

For $Q \in \mathrm{MPar}_\Gamma$, the quotient $\mathbb{T}(Q) = (U_Q \otimes_{\mathbb{R}} \mathbb{C})/\Upsilon_Q \cong (\mathbb{C}^*)^m$ is an algebraic torus over $\mathbb{C}$.

A Γ-admissible fan $\Sigma(Q)$ determines a toric variety $X_{\Sigma(Q)}$ that includes $\mathbb{T}(Q)$ as a dense Zariski-open subset. More precisely, $X_{\Sigma(Q)}$ is the disjoint union of all the quotients $\mathbb{T}(Q)_\sigma$ of $\mathbb{T}(Q)$ by the complex algebraic tori $\mathrm{Span}_{\mathbb{C}}(\sigma)/(\mathrm{Span}_{\mathbb{C}}(\sigma) \cap \Upsilon_Q)$, corresponding to the cones $\sigma \in \Sigma(Q)$. In particular, the dense torus in $X_{\Sigma(Q)}$ is $\mathbb{T}(Q) = \mathbb{T}(Q)_{\{0\}}$.

Bearing in mind that $(U_Q + iC_Q)/\Upsilon_Q$ is an open subset of $\mathbb{T}(Q)$, we take the closure $\overline{(U_Q + iC_Q)/\Upsilon_Q}$ of $(U_Q + iC_Q)/\Upsilon_Q$ in $X_{\Sigma(Q)}$ and define $Y_{\Sigma(Q)}$ as the interior of $\overline{(U_Q + iC_Q)/\Upsilon_Q}$ in $X_{\Sigma(Q)}$. Here things are simpler in the case of real rank 1: then $m = 1$ and $\Sigma(Q) = \{\{0\}, \mathbb{R}_{\geq 0}\}$ (there is no choice of fan to be made), so $X_{\Sigma(Q)} = \mathbb{C}$ and $Y_{\Sigma(Q)}$ is a disc.

The Siegel domain presentation (6) of D associated with Q provides a real analytic diffeomorphism

$$D/\Upsilon_Q = (U_Q + iC_Q)/\Upsilon_Q \times V_Q \times D_{Q,h},$$

and the Γ-admissible fan $\Sigma(Q)$ defines a partial compactification

$$(D/\Upsilon_Q)_{\Sigma(Q)} = Z_{\Sigma(Q)} = Y_{\Sigma(Q)} \times V_Q \times D_{Q,h}. \tag{8}$$

By subdividing $\Sigma(Q)$ we may, and henceforth do, assume that $X_{\Sigma(Q)}$ and $Y_{\Sigma(Q)}$ are smooth: see [Fu].

To describe the Q-action on $Z_{\Sigma(Q)}$, consider the Υ_Q-covering map

$$\epsilon_Q \colon D = (U_Q + iC_Q) \times V_Q \times D_{Q,h} \longrightarrow D/\Upsilon_Q = (U_Q + iC_Q)/\Upsilon_Q \times V_Q \times D_{Q,h},$$

given in the notation of (5) and (7) by

$$\epsilon_Q(u + i(a, \zeta'),\, v,\, \zeta) = \left(\mathbf{e}_Q(u + i(a, \zeta')),\, v,\, \zeta\right),$$

where $\mathbf{e}_Q \colon U_Q \otimes \mathbb{C} \to \mathbb{T}(Q)$ is the canonical map with kernel Υ_Q. If we identify Υ_Q with $\mathbb{Z}^m$ then we can identify $\mathbf{e}_Q$ with exponentiation, i.e. $\mathbf{e}_Q(z_1, \ldots, z_m) = (e^{2\pi i z_1}, \ldots, e^{2\pi i z_m})$ for $(z_1, \ldots, z_m) \in \mathbb{C}^m$.

Identifying N_Q with $\mathrm{Lie}(N_Q)$ and using (7), one expresses the action of $\gamma = (u + \eta(v), a, g, h) \in \mathrm{Lie}(N_Q) \rtimes (A_Q \times G'_{Q,l} \times G_{Q,h})$ on $y = (\mathbf{e}_Q(u_y + i(a_y, z'_y)), v_y, z_y) \in (U_Q + iC_Q)/\Upsilon_Q \times V_Q \times D_{Q,h}$ by

$$\gamma y = \Big(\mathbf{e}_Q\big(u + u_y^{ag} + \frac{1}{2}[\eta(v), \eta(v_y^l)] + i(a a_y, g z'_y)\big), v + v_y^l, h z_y\Big). \tag{9}$$

This Q-action extends by continuity to $Z_{\Sigma(Q)}$.

3.3 The gluing maps

For P, $Q \in \mathrm{MPar}_\Gamma$ with $F(P) \subseteq \overline{F(Q)}$, we are going to describe explicitly the holomorphic map $\mu_P^Q \colon Z_{\Sigma(Q)} \longrightarrow Z_{\Sigma(P)}$ of [AMRT, Lemma III.5.4].

According to [AMRT, Theorem III.4.8], U_Q is an $\mathbb{R}$-linear subspace of U_P. Therefore $\Upsilon_Q < \Upsilon_P$ and the identity map

$$\mathrm{id}_D \colon D = (U_Q + iC_Q) \times V_Q \times D_{Q,h} \longrightarrow D = (U_P + iC_P) \times V_P \times D_{P,h}$$

induces a holomorphic covering

$$\mu_P^Q \colon D/\Upsilon_Q = (U_Q + iC_Q)/\Upsilon_Q \times V_Q \times D_{Q,h} \longrightarrow D/\Upsilon_P$$

given by $\mu_P^Q(\Upsilon_Q x) = \Upsilon_P x$.

The inclusions $U_Q \otimes \mathbb{C} \subset U_P \otimes \mathbb{C}$ and $\Upsilon_Q < \Upsilon_P$ induce a homomorphism $\mu_{P,1}^Q \colon \mathbb{T}(Q) \longrightarrow \mathbb{T}(P)$, which extends to $\mu_{P,1}^Q \colon X_{\Sigma(Q)} \longrightarrow X_{\Sigma(P)}$, mapping $Y_{\Sigma(Q)}$ into $Y_{\Sigma(P)} \subset \overline{(U_P + iC_P)/\Upsilon_P}$. In this way, one obtains a holomorphic gluing map

$$\mu_P^Q \colon Z_{\Sigma(Q)} = Y_{\Sigma(Q)} \times V_Q \times D_{Q,h} \longrightarrow Y_{\Sigma(P)} \times V_P \times D_{P,h} = Z_{\Sigma(P)},$$

given by

$$\mu_P^Q \left(\lim_{t\to\infty} (y_t, v, z) \right) = \lim_{t\to\infty} \mu_P^Q(y_t, v, z) = \lim_{t\to\infty} (y_t + \Upsilon_P/\Upsilon_Q, v, z)$$

where $y_t \in (U_Q + iC_Q)/\Upsilon_Q$ for $t \in \mathbb{R}$ tends to some point $\lim_{t\to\infty} y_t \in Y_{\Sigma(Q)}$. From this definition, μ_Q^Q is the identity on $Z_{\Sigma(Q)} = (D/\Upsilon_Q)_{\Sigma(Q)}$, whether Γ is arithmetic or not. Again the real rank 1 case (in particular, the non-arithmetic case) is simpler: then $F(Q)$ is a point and there are no nontrivial inclusions $F(P) \subseteq \overline{F(Q)}$.

3.4 Toroidal compactifications and coverings

We recall the construction of a toroidal compactification $(D/\Gamma)'_\Sigma$ of a locally symmetric variety D/Γ, associated with a Γ-admissible family $\Sigma = \{\Sigma(Q)\}_{Q\in\mathrm{MPar}_\Gamma}$ of fans $\Sigma(Q)$ in U_Q. In the notation of subsection 3.2, consider the disjoint union $\coprod_{Q\in\mathrm{MPar}_\Gamma} Z_{\Sigma(Q)}$.

We denote by Υ the subgroup of Γ generated by Υ_Q for all $Q \in \mathrm{MPar}_\Gamma$. Suppose that Γ_o is a normal subgroup of Γ containing Υ. Its action on D induces an equivalence relation $\sim_{\Gamma_o}$ on $\coprod_{Q\in\mathrm{MPar}_\Gamma} Z_{\Sigma(Q)}$, as in the proof of [Sa, Theorem 2.1]: for $\Gamma_o = \Gamma$ it is described in [AMRT, III.5]. Let $z_1 \in Z_{\Sigma(Q_1)}$ and $z_2 \in Z_{\Sigma(Q_2)}$: then $z_1 \sim_{\Gamma_o} z_2$ if there exist $\gamma \in \Gamma_o$, $Q \in \mathrm{MPar}_\Gamma$ and $z \in Z_{\Sigma(Q)}$, such that $\overline{F(Q)} \supseteq F(Q_1)$, $\overline{F(Q)} \supseteq F(Q_2^\gamma)$, $\mu_{Q_1}^Q(z) = z_1$ and $\mu_{Q_2^\gamma}^Q(z) = \gamma z_2$. In the non-arithmetic case, $z_1 \sim_{\Gamma_o} z_2$ simply reduces to the usual Γ_o-equivalence: $z_2 = \gamma z_1$ for some $\gamma \in \Gamma_o$.

Then we put

$$(D/\Gamma_o)'_\Sigma = \left(\coprod_{Q \in \mathrm{MPar}_\Gamma} Z_{\Sigma(Q)} \right) / \sim_{\Gamma_o} .$$

In [Sa] this is used to construct the (Γ/Υ)-Galois covering $(D/\Upsilon)'_\Sigma$ of $(D/\Gamma)'_\Sigma$ and show that $(D/\Upsilon)'_\Sigma$ is a simply connected complex analytic space. Notice that Γ_o is not required to be a lattice, and that $(D/\Upsilon)'_\Sigma$ is not compact.

In the proof of [Sa, Theorem 1.5] it is shown that $Z_{\Sigma(Q)}$, which are diffeomorphic to $Y_{\Sigma(Q)} \times V_Q \times D_{Q,h}$ for all $Q \in \mathrm{MPar}_\Gamma$, are simply connected. Further, the proof of [Sa, Theorem 2.1] establishes that the natural coverings $D/\Upsilon_Q \to D/\Upsilon$ extend to open holomorphic maps $\pi^U_{\Sigma(Q)} \colon Z_{\Sigma(Q)} \to (D/\Upsilon)'_\Sigma$, which are biholomorphic onto their images.

4 The fundamental group and first Betti number

4.1 The fundamental group

We begin by stating two theorems that summarise the results from [Sa] and [GHS] on the fundamental group of a toroidal compactification $(D/\Gamma)'_\Sigma$ of a quotient of $D = G/K$ by an arithmetic lattice $\Gamma < G$.

Theorem 4.1 *[Sa, Corollary 1.6, Theorem 2.1] Let $D = G/K$ be a Hermitian symmetric space and let Γ be a non-uniform arithmetic lattice in G. Then the fundamental group $\pi_1((D/\Gamma)'_\Sigma)$ of a toroidal compactification $(D/\Gamma)'_\Sigma$ of D/Γ is a quotient group of Γ/Υ. In particular, if Γ is a neat arithmetic non-uniform lattice then $\pi_1((D/\Gamma)'_\Sigma) = \Gamma/\Upsilon$.*

The above is a more carefully stated version of the results in [Sa]. Recall that Υ is the group generated by Υ_Q for all $Q \in \mathrm{MPar}_\Gamma$. In a similar style, we define Λ to be the subgroup of Γ generated by all $\gamma \in \Gamma \cap Q$ such that $\gamma^k \in R_Q$ for some $k \in \mathbb{N}$ and some $Q \in \mathrm{MPar}_\Gamma$, and $\gamma^k \in U_Q$ if $\gamma^k \in N_Q$. (Another way to say this is that $(\gamma N_Q)^k = \gamma^k N_Q$ is in $R_Q/N_Q = A_Q \subset L_Q = Q/N_Q$, and if it is the identity then $\gamma^k \in \Upsilon$.)

Theorem 4.2 *[GHS, Lemma 5.2, Proposition 5.3] Under the conditions of Theorem 4.1, there is a commutative diagram*

$$\begin{tikzcd}
\Gamma \arrow[r, "\psi"] \arrow[dr, "\varphi"'] & \pi_1(D/\Gamma) \arrow[d] \\
& \pi_1(D/\Gamma)'_\Sigma
\end{tikzcd}$$

of surjective group homomorphisms, such that $\operatorname{Ker}\varphi$ *and* $\operatorname{Ker}\psi$ *contain all* $\gamma \in \Gamma$ *with a fixed point on* D*.*

The following is the main result of the present paper. We use the notation from Subsection 2.1.

Theorem 4.3 *Let* $D = G/K$ *be an irreducible Hermitian symmetric space and let* Γ *be a non-uniform lattice in* G*. Then for any* Γ*-admissible family* Σ*, the toroidal compactification* $(D/\Gamma)'_\Sigma$ *has fundamental group*

$$\pi_1((D/\Gamma)'_\Sigma) = \Gamma/\Lambda\Upsilon.$$

Proof. According to [Sa], $(D/\Upsilon)'_\Sigma$ is a path connected simply connected locally compact topological space and Γ/Υ acts properly discontinuously on $(D/\Upsilon)'_\Sigma$ by homeomorphisms. More precisely, $\gamma\Upsilon\colon (\Upsilon q) \mapsto \Upsilon\gamma q$ defines a Γ/Υ-action on D/Υ, which extends continuously to $(D/\Upsilon)'_\Sigma$. The quotient space $(D/\Upsilon)'_\Sigma/(\Gamma/\Upsilon) = (D/\Gamma)'_\Sigma$ is the toroidal compactification of D/Γ associated with Σ. By a theorem of Armstrong [Ar]

$$\pi_1((D/\Gamma)'_\Sigma) = (\Gamma/\Upsilon)/(\Gamma/\Upsilon)^{\mathrm{Fix}}_o$$

where $(\Gamma/\Upsilon)^{\mathrm{Fix}}_o$ is the subgroup of Γ/Υ generated by elements $\gamma\Upsilon$ with a fixed point on $(D/\Upsilon)'_\Sigma$.

Theorem 4.3 therefore follows from Proposition 4.4, which establishes that $(\Gamma/\Upsilon)^{\mathrm{Fix}}_o = \Lambda\Upsilon/\Upsilon$. □

In order to describe the action of Γ/Υ on $(D/\Upsilon)'_\Sigma$ note that the Γ-action on MPar_Γ by conjugation determines holomorphic maps $\gamma\colon Z_{\Sigma(Q)} \to Z_{\Sigma(Q^\gamma)}$ for all $\gamma \in \Gamma$ and $Q \in \mathrm{MPar}_\Gamma$. Any $\gamma \in \Gamma$ transforms the $\sim_\Upsilon$-equivalence class of $z \in Z_{\Sigma(Q)}$ into the $\sim_\Upsilon$-equivalence class of γz, giving a biholomorphic map $\gamma\colon (D/\Upsilon)'_\Sigma \longrightarrow (D/\Upsilon)'_\Sigma$. By definition of $\sim_\Upsilon$, all $\gamma \in \Upsilon$ act trivially on $(D/\Upsilon)'_\Sigma$ and the Γ-action on $(D/\Upsilon)'_\Sigma$ reduces to a (Γ/Υ)-action

$$(\Gamma/\Upsilon) \times (D/\Upsilon)'_\Sigma \longrightarrow (D/\Upsilon)'_\Sigma,$$

given by

$$(\gamma\Upsilon)\pi^U_{\Sigma(Q)}(z) = \pi^U_{\Sigma(Q^\gamma)}(\gamma z) \tag{10}$$

for $\gamma\Upsilon \in \Gamma/\Upsilon$ and $z \in Z_{\Sigma(Q)}$.

Proposition 4.4 *In the notations from Theorem 4.3, a coset* $\gamma_0\Upsilon \in \Gamma/\Upsilon$ *has a fixed point on* $(D/\Upsilon)'_\Sigma$ *if and only if for some* $k \in \mathbb{N}$ *and some* $Q \in \mathrm{MPar}_\Gamma$*, there is a representative* $\gamma \in \Gamma \cap Q$ *of* $\gamma_0\Upsilon = \gamma\Upsilon$ *with* $\gamma^k \in \Gamma \cap R_Q$ *and* $\gamma^k \in U_Q$ *if* $\gamma^k \in N_Q$*. Hence the subgroup* $(\Gamma/\Upsilon)^{\mathrm{Fix}}_o$ *of* Γ/Υ *satisfies* $(\Gamma/\Upsilon)^{\mathrm{Fix}}_o = \Lambda\Upsilon/\Upsilon$*.*

Proof. We first prove the "only if" part of the statement.

We claim that if $\gamma_0\Upsilon \in \Gamma/\Upsilon$ has a fixed point on

$$(D/\Upsilon)'_\Sigma = \left(\coprod_{P\in\mathrm{MPar}_\Gamma} Z_{\Sigma(P)}\right) / \sim_\Upsilon$$

then there exist $\gamma \in \gamma_0\Upsilon$ and $y \in Z_{\Sigma(Q)}$ for some $Q \in \mathrm{MPar}_\Gamma$, such that $\gamma y = y$. That is, if a coset of Υ has a fixed point mod Υ then some representative of that coset has a fixed point "on the nose".

To prove this, notice that if $z_0 \sim_\Upsilon \gamma_0 z_0$ for some $z_0 \in Z_{\Sigma(P)}$, then there exist $Q_1 \in \mathrm{MPar}_\Gamma$, $z_1 \in Z_{\Sigma(Q_1)}$ and $u_1 \in \Upsilon$ such that $F(P) \subseteq \overline{F(Q_1)}$ and $\mu_P^{Q_1}(z_1) = z_0$, and $F(P^{u_1\gamma_0}) \subseteq \overline{F(Q_1)}$ and $\mu^{Q_1}_{P^{u_1\gamma_0}}(z_1) = u_1\gamma_0 z_0$.

If $F(Q_1) = F(P)$, then $Q_1 = P$ and in (10) we have $\mu^{Q_1}_{P^{u_1\gamma_0}} = \mu_P^{Q_1} = \mathrm{id}_{Z_{\Sigma(P)}}$ and $z_0 = u_1\gamma_0 z_0$. Since Υ is a normal subgroup of Γ, we may take $\gamma = u_1\gamma_0 \in \Upsilon\gamma_0 = \gamma_0\Upsilon$. In particular, this shows that the claim is true if $F(P)$ is of maximal dimension.

Now we conclude the proof of the claim by induction on $\operatorname{codim} F(P)$: suppose that the claim holds for all $P' \in \mathrm{MPar}_\Gamma$ with $\dim F(P') > \dim F(P)$, and take Q_1 as above. If $F(Q_1) = F(P)$ we are done. If not, then $z_0 \sim_\Upsilon z_1$ and $\gamma_0 z_0 \sim_\Upsilon \gamma_0 z_1$, because $\mu^{Q_1^{\gamma_0}}_{P^{\gamma_0}}(\gamma_0 z_1) = \gamma_0 z_0$. On the other hand, $\gamma_0 z_0 \sim_\Upsilon z_1$, so $z_1 \sim_\Upsilon \gamma_0 z_1$ because $\sim_\Upsilon$ is an equivalence relation. Thus $\gamma_0\Upsilon$ has the fixed point $z_1 \in Z_{\Sigma(Q_1)}$ so the claim follows by taking $P' = Q_1$.

Suppose then that $\gamma \in \Gamma$ has a fixed point $y \in Z_{\Sigma(Q)}$ for some $Q \in \mathrm{MPar}_\Gamma$. Then $y = \gamma y \in Z_{\Sigma(Q^\gamma)}$ implies that $Q^\gamma = Q$; but the parabolic subgroup Q of G coincides with its normaliser in G, so $\gamma \in Q$. We may therefore use the Langlands decomposition of Q and write

$$\gamma = (\exp(u + \eta(v)), a, g, h) \in N_Q \rtimes (A_Q \times G'_{Q,l} \times G_{Q,h}).$$

As above we take

$$l = (a, g, h) \in L_Q = A_Q \times G'_{Q,l} \times G_{Q,h}.$$

Any element of $X_{\Sigma(Q)}$ may be written as a limit of elements of $\mathbb{T}(Q)$, that is, as $\lim_{t\to\infty}(\mathbf{e}_Q(u_t + ix_t))$, and if the element is in $Y_{\Sigma(Q)}$ then we may take $x_t \in C_Q$. So

$$y = (\lim_{t\to\infty} \mathbf{e}_Q(u_t + ix_t), v_y, z_y) \in Y_{\Sigma(Q)} \times V_Q \times D_{Q,h}.$$

Then by (9) and the continuity of the Q-action on $(D/\Upsilon_Q)_{\Sigma(Q)}$, the condition $\gamma y = y$ is equivalent to

$$\lim_{t\to\infty} \mathbf{e}_Q\Big(u + u_t^{ag} + \frac{1}{2}[\eta(v), \eta(v_y^l)] + i(a,g)x_t\Big) = \lim_{t\to\infty} \mathbf{e}_Q(u_t + ix_t),$$

together with

$$v + v_y^l = v_y \text{ and } hz_y = z_y.$$

If γ has a fixed point y and $y \in D/\Upsilon_Q$ then γ belongs to the compact stabiliser of y in the isometry group G of D and hence γ is torsion. If $y \in Z_{\Sigma(Q)} \setminus (D/\Upsilon_Q)$ we need to look at g and h. For h what we need is immediate: it is in the stabiliser of $z_y \in D_{Q,h}$ and isotropy groups in symmetric spaces are always torsion, so h is of finite order: by replacing γ with a power we may assume that h is the identity.

For the Riemannian part g a little more work is needed. The Levi component $l = (a, g, \mathrm{Id})$ of the element γ has a fixed point $y' = \lim_{t\to\infty} \mathbf{e}_Q(u_t + ix_t) \in X_{\Sigma(Q)} \setminus \mathbb{T}(Q)$, so $y' \in \mathbb{T}(Q)_\sigma$ for some unique $\sigma \in \Sigma(Q)$. Therefore l preserves σ. If we assume, as we may do, that Q has been chosen so as to maximise $\dim F(Q)$ (see [AMRT, Lemma III.5.5]), then $\sigma \cap C_Q \neq \emptyset$ (remember that C_Q is an open cone but σ is closed): this follows from [AMRT, Theorem III.4.8(ii)].

Since l preserves σ, it permutes the top-dimensional cones of which σ is a face: there are finitely many of these as long as $\sigma \cap C_Q \neq \emptyset$. Therefore some power of l preserves a top-dimensional cone, so we may as well assume that σ is top-dimensional. The action of l is thus determined by its action on $\sigma = \sum_{i=1}^q \mathbb{R}_{\geq 0} u_i$.

Thus γ permutes the rays $\mathbb{R}_{\geq 0} u_i$ (it may not fix them pointwise) and therefore some power, in fact $l^{q!}$, fixes all the rays, so we may as well assume that l fixes all the rays. In particular it fixes a rational basis of U_Q up to scalars. Now consider the real subgroup of $G_{Q,l}$ that fixes that basis up to scalars. Its identity component is a torus, and because the u_i are defined over $\mathbb{Q}$ it is $\mathbb{R}$-split (in fact $\mathbb{Q}$-split) and therefore it is contained in the maximal $\mathbb{R}$-split torus in $G_{Q,l}$, which is A_Q. So some power of l is in A_Q, and some power of γ is in $N_Q \rtimes A_Q = R_Q$.

If $\gamma^k \in N_Q$ then $\gamma^k = \exp(u' + \eta(v'))$ and comparing the V_Q parts in $\gamma^k y = y$ yields $v' + v = v$. Therefore $v' = 0$ and $\gamma^k = \exp(u') \in U_Q$. This completes the proof of the "only if" part.

For the converse (the "if" part), for $\gamma \in \Gamma \cap Q$ we write

$$\gamma = (\exp(u + \eta(v)), a, g, h) \in Q = N_Q \rtimes (A_Q \times G'_{Q,l} \times G_{Q,h})$$

and as usual we write $l = (a, g, h) \in L_Q$. Suppose first of all that $\gamma \in \Lambda$ and that $l^k \in A_Q$ for some $Q \in \mathrm{MPar}_\Gamma$. Since $\gamma \in Q$ it preserves the cone $C = C_Q$. By the Brouwer fixed point theorem, γ preserves a ray ρ' in $\overline{C}_Q$.

We claim that there exists a boundary component $F(P)$, for some $P \in \mathrm{MPar}_\Gamma$, fixed by γ such that γ preserves a ray $\rho = \mathbb{R}_{\geq 0} u_\rho$ in the relative interior of C_P. This is trivial if $\dim C_Q = 1$. We shall proceed by induction on $\dim C_Q$.

The ray ρ' is preserved by γ, with eigenvalue λ say, and ρ' belongs to a unique real boundary component C' of C, since $\overline{C}$ is the disjoint union of its real boundary components by [AMRT, Proposition II.3.1]. Let H_λ be the λ-eigenspace of γ in U_Q. Then $H_\lambda \cap \overline{C}$ is a boundary component of C and contains ρ', so $H_\lambda \cap \overline{C} = C'$. Thus the normaliser of C' in $\mathrm{Aut}(C)$ is rational because it is the normaliser of the rational linear subspace H_λ. Therefore by [AMRT, Corollary II.3.22], C' is a rational boundary component. But γ preserves C', and $\dim C' < \dim C$. In particular $\gamma \in P$, so we may assume that such a fixed ray ρ exists already in C_Q, generated by an element $u_\rho \in U_Q$.

We claim that if $\gamma \in \Gamma$ and $\gamma^k \in (R_Q \setminus N_Q) \cup U_Q$ for some $Q \in \mathrm{MPar}_\Gamma$ and $k \in \mathbb{N}$, then γ has a fixed point

$$y_1 = (\lim_{t\to\infty} \mathbf{e}_Q(u_t + ix_t), v_1, z_1) \in Y_{\Sigma(Q)} \times V_Q \times D_{Q,h} = Z_{\Sigma(Q)}.$$

From the group law (3) we have

$$\gamma^k = \Big(\exp\Big(u + \eta\big(\sum_{j=0}^{k-1} v^{l^j}\big)\Big), a^k, g^k, h^k\Big).$$

for an appropriate $u \in \mathrm{Lie}(U_Q)$.

The torsion element $h \in G_{Q,h}$ has a fixed point $z_1 \in D_{1,h}(Q)$. Moreover the point $o_\rho = \lim_{t\to\infty} \mathbf{e}_Q(u_t + itu_\rho) \in X_{\Sigma(Q)}$ is fixed by γ, and

$$\gamma(o_\rho, v_1, z_1) = (o_\rho, v + v_1^l, z_1) = l(o_\rho, v_1, z_1)$$

for any $v_1 \in \mathrm{Lie}(V_Q)$. So it is enough to show that there exists $v_1 \in \mathrm{Lie}(V_Q)$ with

$$v + v_1^l = v_1 \tag{11}$$

if $l^k = a^k \in A_Q \setminus \{1\}$ or $\gamma^k \in U_Q$. With a slight abuse of notation, we identify the split component A_Q of Q with $(\mathbb{R}_{>0}, .)$ and recall that it acts on $\mathrm{Lie}(V_Q)$ by scalar multiplication, $(a, v) \mapsto av$.

If γ fixes y_1 then so does γ^k, so if v_1 satisfies (11) then it also satisfies

$$\sum_{j=0}^{k-1} v^{l^j} + a^k v_1 = v_1. \tag{12}$$

In the case of $a^k \in A_Q \setminus \{1\}$ we may simply take $v_1 = \frac{1}{1-a^k} \sum_{j=0}^{k-1} v^{l^j}$: it is straightforward to verify that this does satisfy (11).

If $\gamma^k \in U_Q$ then clearly $\gamma^k(o_\rho, v_1, z_1) = (o_\rho, v_1, z_1)$ for any $v_1 \in \mathrm{Lie}(V_Q)$. To show that γ itself has a fixed point, we note that the adjoint action of L_Q on $\mathrm{Lie}(V_Q) \cong \mathbb{C}^n$, given by $l\colon v \mapsto \mathrm{Ad}(l)(x) = lvl^{-1}$, is $\mathbb{R}$-linear. Since $\mathrm{Ad}(l)^k = \mathrm{Id}_{\mathrm{Lie}(V_Q)}$, both $\mathrm{Ad}(l)$ and $\mathcal{L} = \mathrm{Id}_{\mathrm{Lie}(V_Q)} - \mathrm{Ad}(l)$ are semi-simple. Thus $\mathrm{Ker}(\mathcal{L}) \cap \mathrm{Im}(\mathcal{L}) = \{0\}$ and $\mathrm{Lie}(V_Q) = \mathrm{Ker}(\mathcal{L}) \oplus \mathrm{Im}(\mathcal{L})$. Note that (11) is equivalent to $v = \mathcal{L}(v_1)$, so it suffices to prove that $v \in \mathrm{Im}(\mathcal{L})$. Since $\gamma^k \in U_Q$, i.e. its V_Q component vanishes, we have

$$\sum_{j=0}^{k-1} \mathrm{Ad}(l^j)(v) = 0. \tag{13}$$

We decompose $v = v' + v''$ into $v' \in \mathrm{Ker}(\mathcal{L})$ and $v'' \in \mathrm{Im}(\mathcal{L})$: then $0 = \mathcal{L}(v') = v' - \mathrm{Ad}(l)(v')$ implies $\mathrm{Ad}(l^j)(v') = v'$ for all $j \geq 0$. Hence

$$\begin{aligned}\mathrm{Im}(\mathcal{L}) \ni \sum_{j=0}^{k-1} \mathrm{Ad}(l^j)(v'') &= \sum_{j=0}^{k-1} \mathrm{Ad}(l^j)(v' + v'') - \sum_{j=0}^{k-1} \mathrm{Ad}(l^j)(v') \\ &= \sum_{j=0}^{k-1} \mathrm{Ad}(l^j)(v) - kv' = -kv' \in \mathrm{Ker}(\mathcal{L}),\end{aligned}$$

making use of (13). Therefore $v' = 0$ and $v = v'' \in \mathrm{Im}(\mathcal{L})$.

This concludes the proof of Proposition 4.4. □

4.2 The first Betti number

We can use Theorem 4.3 to give bounds on the first Betti number of the toroidal compactifications.

Corollary 4.5 *Suppose that $D = G/K$ is an irreducible Hermitian symmetric space of non-compact type with $\dim_{\mathbb{C}}(D) > 1$ and Γ is a non-uniform lattice of G. Let $Q_1, \ldots, Q_h$ be a complete set of representatives of the Γ-conjugacy classes of Γ-rational maximal parabolic subgroups of G, with solvable radicals R_{Q_j}. Then*

$$\mathrm{rk}_{\mathbb{Z}}\, H_1(D/\Gamma, \mathbb{Z}) - \sum_{j=1}^{h} \dim_{\mathbb{R}}(R_{Q_j}) \leq \mathrm{rk}_{\mathbb{Z}}\, H_1((D/\Gamma)'_{\Sigma}, \mathbb{Z}) \leq \mathrm{rk}_{\mathbb{Z}}\, H_1(D/\Gamma, \mathbb{Z}). \tag{14}$$

If Γ is neat then

$$\mathrm{rk}_{\mathbb{Z}}\, H_1((D/\Gamma)'_{\Sigma}, \mathbb{Z}) = \mathrm{rk}_{\mathbb{Z}}\, H_1(D/\Gamma, \mathbb{Z}).$$

Proof. For an arbitrary group $\mathbf{G}$ we denote by $\mathrm{ab}\,\mathbf{G}$ its abelianisation $\mathrm{ab}(\mathbf{G}) = \mathbf{G}/[\mathbf{G}, \mathbf{G}]$. If S is a complex analytic space then $H_1(S, \mathbb{Z}) = \mathrm{ab}\,\pi_1(S)$.

If $\mathbf{H}$ is a normal subgroup of $\mathbf{G}$ then $[\mathbf{G}/\mathbf{H}, \mathbf{G}/\mathbf{H}] = [\mathbf{G}, \mathbf{G}]\mathbf{H}/\mathbf{H}$ and

$$\mathrm{ab}(\mathbf{G}/\mathbf{H}) = (\mathbf{G}/\mathbf{H})/([\mathbf{G}, \mathbf{G}]\mathbf{H}/\mathbf{H}) \cong \mathbf{G}/[\mathbf{G}, \mathbf{G}]\mathbf{H} = \mathbf{G}/\mathbf{H}[\mathbf{G}, \mathbf{G}].$$

Therefore by Theorem 4.3

$$H_1((D/\Gamma)'_{\Sigma}, \mathbb{Z}) \cong \mathrm{ab}(\Gamma/\Upsilon\Lambda) \cong \Gamma/\Lambda\Upsilon[\Gamma, \Gamma].$$

On the other hand, D is a path connected, simply connected locally compact space with a properly discontinuous action of Γ by homeomorphisms. Let Φ be the subgroup of Γ generated by the elements $\gamma \in \Gamma$ with a fixed point on D. By [Ar], the fundamental group of D/Γ is $\pi_1(D/\Gamma) = \Gamma/\Phi$. Therefore $H_1(D/\Gamma, \mathbb{Z}) \cong \mathrm{ab}(\Gamma/\Phi) \cong \Gamma/\Phi[\Gamma,\Gamma]$ and

$$H_1((D/\Gamma)'_\Sigma, \mathbb{Z}) \cong (\Gamma/\Phi[\Gamma,\Gamma])/(\Lambda\Upsilon[\Gamma,\Gamma]/\Phi[\Gamma,\Gamma]) \cong H_1(D/\Gamma,\mathbb{Z})/F$$

for the abelian group

$$F = \Lambda\Upsilon[\Gamma,\Gamma]/\Phi[\Gamma,\Gamma] < \Gamma/\Phi[\Gamma,\Gamma] \cong H_1(D/\Gamma,\mathbb{Z}).$$

In particular,

$$\mathrm{rk}_{\mathbb{Z}}\, H_1(D/\Gamma,\mathbb{Z}) = \mathrm{rk}_{\mathbb{Z}}\, H_1((D/\Gamma)'_\Sigma,\mathbb{Z}) + \mathrm{rk}_{\mathbb{Z}}\, F.$$

To verify (14), it suffices to show that $F_o = \Phi\Upsilon[\Gamma,\Gamma]/\Phi[\Gamma,\Gamma]$ is a finite subgroup of F and that $\mathrm{rk}_{\mathbb{Z}}(F) = \mathrm{rk}_{\mathbb{Z}}(F/F_o) \leq \sum_{j=1}^{h} \dim_{\mathbb{R}}(R_{Q_j})$.

We check the rank condition first. For any $Q \in \mathrm{MPar}_\Gamma$ we define Λ_Q to be the subgroup of Γ generated by all $\gamma \in \Gamma \cap Q$ such that $\gamma^k \in R_Q$ and $\gamma^k \in U_Q$ if $\gamma^k \in N_Q$ for some $k \in \mathbb{N}$. We define Λ_Q^R to be the subgroup of $\Gamma \cap R_Q$ generated by such k-th powers.

Consider the finitely generated abelian group $\mathbf{L}_Q = \Lambda_Q\Upsilon[\Gamma,\Gamma]/\Phi\Upsilon[\Gamma,\Gamma]$ and the subgroup $\mathbf{R}_Q = \Lambda_Q^R\Upsilon[\Gamma,\Gamma]/\Phi\Upsilon[\Gamma,\Gamma]$. These both have the same rank, because the quotient $\mathbf{L}_Q/\mathbf{R}_Q$ is abelian and generated by torsion elements so it is finite.

Now F/F_o is generated by the $\mathbf{L}_{Q_i}$, and therefore

$$\mathrm{rk}_{\mathbb{Z}}(F) = \mathrm{rk}_{\mathbb{Z}}(F/F_o) \leq \sum_{i=1}^{h} \mathrm{rk}_{\mathbb{Z}}(\mathbf{L}_{Q_i}) = \sum_{i=1}^{h} \mathrm{rk}_{\mathbb{Z}}(\mathbf{R}_{Q_i}).$$

However, $\mathbf{R}_Q$ is a discrete subgroup of R_Q, so $\mathrm{rk}_{\mathbb{Z}}\,\mathbf{R}_Q < \mathrm{rk}_{\mathbb{R}}\, R_Q$, and this gives the bound on $\mathrm{rk}_{\mathbb{Z}}(F)$.

It remains to show that F_o is finite. It is certainly finitely generated, because it is generated by all $\Upsilon_Q[\Gamma,\Gamma]\Phi/[\Gamma,\Gamma]\Phi$; but each Υ_Q is finitely generated and $\Upsilon_{Q^\gamma} = \Upsilon_Q$ so we need only the Υ_{Q_i}.

Since F_o is abelian, it is now enough to show that any element of F_o is of finite order. For $Q \in \mathrm{MPar}_\Gamma$, consider the group $\mathbf{N}_Q = \Gamma \cap N_Q$: we have

$$[\mathbf{N}_Q,\mathbf{N}_Q] \leq \Gamma \cap [N_Q,N_Q] = \Gamma \cap U_Q = \Upsilon_Q.$$

It suffices to prove that $\mathrm{Span}_{\mathbb{R}}[\mathbf{N}_Q,\mathbf{N}_Q] = U_Q$, because then $[\mathbf{N}_Q,\mathbf{N}_Q]$ is of finite index in Υ_Q so $\Upsilon_Q/[\mathbf{N}_Q,\mathbf{N}_Q]$ is finite, of exponent k_Q say. Then $(\Upsilon_Q)^{k_Q} \leq [\mathbf{N}_Q,\mathbf{N}_Q] \leq [\Gamma,\Gamma]$ and any element of $\Upsilon_Q\Phi[\Gamma,\Gamma]/\Phi[\Gamma,\Gamma]$ is of order dividing k_Q.

To prove $\mathrm{Span}_{\mathbb{R}}[\mathbf{N}_Q, \mathbf{N}_Q] = U_Q$, note that the group N_Q is 2-step nilpotent, so $[\mathbf{N}_Q, \mathbf{N}_Q] \subset U_Q$ and hence $\mathrm{Span}_{\mathbb{R}}[\mathbf{N}_Q, \mathbf{N}_Q] \subseteq U_Q$. For the other inclusion, let $\beta_1, \ldots, \beta_{m+2n} \in \mathrm{Lie}(N_Q)$ be such that $b_j = \exp(\beta_j) \in N_Q$ generate the lattice $\mathbf{N}_Q$. Then $\mathrm{Lie}(N_Q) = \mathrm{Span}_{\mathbb{R}}(\beta_1, \ldots, \beta_{m+2n})$ and

$$\mathrm{Lie}(U_Q) = [\mathrm{Lie}(N_Q), \mathrm{Lie}(N_Q)] = \mathrm{Span}_{\mathbb{R}}(\{[\beta_i, \beta_j]\}).$$

Here $[\beta_i, \beta_j]$ is the Lie bracket, but U_Q is isomorphic to $\mathrm{Lie}(U_Q)$ via exp, so that $\exp[\beta_i, \beta_j] = [\exp(\beta_i), \exp(\beta_j)] = [b_i, b_j]$. Thus $U_Q = \mathrm{Span}_{\mathbb{R}}(\{[b_i, b_j]\}) \subseteq \mathrm{Span}_{\mathbb{R}}[\mathbf{N}_Q, \mathbf{N}_Q]$, as required.

For neat Γ, it is shown in [Sa] that $(\Gamma/\Upsilon)_o^{\mathrm{Fix}}$ is trivial. Combining this with $(\Gamma/\Upsilon)_o^{\mathrm{Fix}} = \Lambda\Upsilon/\Upsilon$ from Lemma 4.4, one concludes that $\Lambda \subseteq \Upsilon$. Therefore $F = F_o$ and $\mathrm{rk}_{\mathbb{Z}}(F) = 0$. □

In the case of a classical irreducible Hermitian symmetric space $D = G/K$ of non-compact type, Wolf's book [Wo] provides matrix realisations of the maximal parabolic subgroups Q of G and their solvable radicals R_Q. These allow an explicit calculation of $\dim_{\mathbb{R}}(R_Q)$, depending on the parameters of G and the parameters of the associated complex analytic boundary component $F(Q)$ of Q.

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