Blowups with log canonical singularities

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Abstract

We show that the minimum weight of a weighted blow-up of $\mathbb{A}^d$ with $\varepsilon$-log canonical singularities is bounded by a constant depending only on $\varepsilon$ and $d$. This was conjectured by Birkar.

Using the recent classification of 4-dimensional empty simplices by Iglesias-Valiño and Santos, we work out an explicit bound for blowups of $\mathbb{A}^4$ with terminal singularities: the smallest weight is always at most 32, and at most 6 in all but finitely many cases.

1 Introduction

At a meeting of the COW seminar at City, University of London on 7th February 2018, Caucher Birkar asked the following question.

Question 1.1. Denote by $\mathbb{A}_n^4$ the weighted blowup of $\mathbb{A}^4$ at $0 \in \mathbb{A}^4$ with coprime weights $n = (n_1, n_2, n_3, n_4) \in \mathbb{N}^4$. If $\mathbb{A}_n^4$ has terminal singularities, is the smallest of the weights bounded?

By “coprime” we mean only that $n$ is primitive: we do not require the weights to be pairwise coprime.

This is a simplified version of a more ambitious conjecture.

Conjecture 1.2 (Birkar). Denote by $\mathbb{A}_n^d$ the weighted blowup of $\mathbb{A}^d$ at $0 \in \mathbb{A}^d$ with coprime weights $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$. If $\mathbb{A}_n^d$ has $\varepsilon$-log canonical singularities, then the smallest of the weights is bounded by a constant depending only on $d$ and $\varepsilon$.

Our main result, Theorem 1.3, is a proof of Conjecture 1.2.

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Theorem 1.3. In each fixed dimension $d$ and for each $\varepsilon \in (0, 1]$ there is an integer $\ell_{\varepsilon,d} \in \mathbb{N}$ such that if $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ is primitive and the weighted blowup $\mathbb{A}^d_n$ has only $\varepsilon$-log canonical singularities then $n_{\min} := \min\{n_1, \ldots, n_d\} \leq \ell_{\varepsilon,d}$.

Our proof relies on a general result about subgroups of $\mathbb{R}^n$ that miss a given open set, due to Lawrence [11], which we state here as Theorem 3.1. The connection of that result to terminal and canonical singularities, and to hollow and empty simplices, was first noticed by A. Borisov [6]. Independently of us, and by somewhat different methods, Y. Chen [7] has proved Conjecture 1.2 for the case $d = 3$.

We also give a precise answer to Question 1.1.

Theorem 1.4. If the weighted blowup $\mathbb{A}^4_n$ has terminal singularities then $n_{\min} \leq 32$. Moreover, with finitely many exceptions $n_{\min} \leq 6$.

The proof of this statement relies on the complete classification of empty simplices in dimension four due to Iglesias-Valiño and Santos [9]. The bound of 6 is attained by the infinite family of blowups with $n = (6, 10, 15, n)$, which have terminal singularities whenever $n$ is coprime with 30 (see Remark 4.10). The bound of 32 is attained only by the blowup with $n = (32, 41, 71, 102)$. There are a total of 1784 blowups of $\mathbb{A}^4$ with $n_{\min} > 6$; the number of them for each value of $n_{\min}$ is listed in Proposition 4.11.

These results extend a theorem of Kawakita [10, Theorem 3.5], which says that a weighted blowup $\mathbb{A}^3_n$ is terminal if and only if the weights are $(1, a, b)$ with $a$ and $b$ coprime. Kawakita’s result also follows from our methods: see Corollary 4.4 below.

The context of [10] is the Sarkisov program, in particular birational rigidity. To investigate Sarkisov links involving a Fano 3-fold $F$ of Picard rank 1 requires in principle an understanding of all possible divisorial contractions in the Mori program with target $F$. The main outcome of [10] is that any divisorial contraction in the Mori program with centre a smooth point is a weighted blowup, and [10, Theorem 3.5] says that the weights must then be $(1, a, b)$.

This is important because, at least in dimension 3, we understand divisorial contractions well if we know their sources, but not so well if we know their targets. So [10] provides a description of all possible baskets of singularities in a terminal 3-fold with a divisorial contraction whose centre is a smooth point. This may be thought of as a relative boundedness result, showing that exceptional divisors are weighted projective planes of the form $\mathbb{P}(1, a, b)$.

Birkar’s Conjecture 1.2 arises analogously in his work [3] on boundedness of log Calabi-Yau fibrations. One way to view it is as a local version of the BAB conjecture, in a quite special case.
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2 Singularities and simplices

Geometrically, our approach is to use toric geometry to rephrase the problem in terms of polytopes. We shall be working in $\mathbb{R}^d$ with its standard basis $e_1 = (1,0,\ldots,0),\ldots,e_d$. We shall frequently need to add up the coordinates of a vector, so we write $\sum x_i$ to abbreviate $\sum_{i=1}^d x_i$.

Definition 2.1. Let $\Lambda \subseteq \mathbb{R}^d$ be a lattice: that is, a finitely generated free abelian subgroup of rank $d$ such that $\mathbb{R}^d = \Lambda \otimes \mathbb{R}$. A polytope $\Pi$ in $\mathbb{R}^d$ is a bounded intersection of finitely many closed half-spaces. A point $v \in \Pi$ is a vertex if $\Pi \cap H = \{v\}$ for some affine hyperplane $H \subset \mathbb{R}^d$: we denote the set of vertices of $\Pi$ by $Vx(\Pi)$. The convex hull of a set $X \subset \mathbb{R}^d$ is denoted $\text{Conv}(X)$: a polytope $\Pi$ is always equal to the convex hull $\text{Conv}(Vx(\Pi))$ of its vertices. $\Pi$ is a lattice polytope if $Vx(\Pi) \subset \Lambda$.

The next definition is usually made only for the case where $\Gamma$ is a lattice and $\Pi$ is a lattice polytope, but we need it in a more general setting.

Definition 2.2. Fix a subgroup $\Gamma$ of $\mathbb{R}^d$. We say that a polytope $\Pi$ is hollow with respect to $\Gamma$ if $\Pi \cap \Gamma \subseteq \partial \Pi$ and empty with respect to $\Gamma$ if $\Pi \cap \Gamma \subseteq Vx(\Pi)$. We omit “with respect to $\Gamma$” when $\Gamma$ is understood.

Let $\sigma = \sum \mathbb{R}_{\geq 0}w_r$ be a nondegenerate closed rational polyhedral cone in $\mathbb{R}^d$, where $w_r \in \Lambda$ are primitive generators of the rays of $\sigma$. We denote by $\Delta(\sigma)$ the lattice polytope $\text{Conv}\{0\} \cup \{w_i\}$, and let $X_\sigma$ be the affine variety $\text{Spec}\mathbb{C}[\sigma^\vee \cap \Lambda]$, as usual in toric geometry. With this notation, $X_\sigma$ is $\mathbb{Q}$-Gorenstein if and only if all the $w_i$ lie in an affine hyperplane, and is $\mathbb{Q}$-factorial if and only if $\sigma$ is simplicial; that is, if $\Delta(\sigma)$ is a simplex.

The following fundamental fact is well known.

Lemma 2.3. Let $\varepsilon \in (0,1]$. Then:

(a) $X_\sigma$ is $\varepsilon$-log terminal if and only if $\varepsilon \Delta(\sigma)$ is an empty polytope.

(b) $X_\sigma$ is $\varepsilon$-log canonical if and only if $\varepsilon \Delta(\sigma)$ is hollow and all nonzero lattice points in it lie in facets not containing the origin.
Proof. $X_\sigma$ is \(\varepsilon\)-log canonical if and only if for some (hence any) birational morphism \(f: Y \to X_\sigma\) with \(Y\) smooth, the discrepancies \(e_j\) defined by \(K_Y - f^*K_X = \sum e_jE_j\) (with \(E_j\) being \(f\)-exceptional prime divisors) satisfy \(e_j \geq -1 + \varepsilon\). To check this, consider a toric resolution \(f: Y = Y_\Sigma \to X_\sigma\) obtained by subdividing \(\sigma\) into a regular fan \(\Sigma\). The exceptional divisors are given by some rays \(r_j \in \Lambda\). The \(\mathbb{Q}\)-divisors \(K_Y\) and \(f^*K_X\) are given by support functions \(h_Y\) and \(h_{X_\sigma}\) as in [15, Proposition 2.1(v)]. The function \(h_Y\) satisfies \(h_Y(r_j) = h_Y(w_i) = 1\), while \(h_{X_\sigma}\) is linear and is determined by \(h_{X_\sigma}(w_i) = 0\). Therefore \(e_j = -1 + h_{X_\sigma}(r_j)\), so in part (b) we have \(h_{X_\sigma}(r) \geq \varepsilon\), for all \(r \in \Lambda\). The result follows at once from this: part (a) is identical, replacing \(e_j \geq -1 + \varepsilon\) by \(e_j > -1 + \varepsilon\).

In particular, since canonical is the same as 1-log canonical, \(X_\sigma\) has \(\mathbb{Q}\)-factorial canonical singularities if and only if \(\Delta(\sigma)\) is a hollow simplex with \(\Delta(\sigma) \cap \Lambda \setminus \{0\}\) contained in the facet opposite to the origin.

Any nonnegative primitive integer vector \(n = (n_1, \ldots, n_d) \in \mathbb{N}^d\) induces a weighted blowup \(\mathbb{A}^d_n\), which is the toric variety associated with the fan in \(\mathbb{R}^d\) (and the lattice \(\mathbb{Z}^d\)) that consists of all the faces of the cones \(\sigma_n^j = \mathbb{R}_{\geq 0}n + \sum_{i \neq j} \mathbb{R}_{\geq 0}e_i\). Note that all such faces are contained in \(\mathbb{R}_{\geq 0}^d\), and that the \(\sigma_n^j\) are simplicial so \(\mathbb{A}^d_n\) always has \(\mathbb{Q}\)-factorial singularities.

The standard simplex in \(\mathbb{R}^d\) is \(\Delta := \Delta(\mathbb{R}_{\geq 0}^d) = \text{Conv}(\{0, e_1, \ldots, e_d\})\) and its interior is denoted \(\Delta^\circ\). That is,

\[
\Delta^\circ = \{x \in \mathbb{R}^d \mid \sum x_i < 1 \text{ and } \forall i \ x_i > 0\}.
\]

The facet of \(\Delta\) opposite to the origin, which is \(\text{Conv}(\{e_1, \ldots, e_d\})\), is denoted by \(\Delta_1\).

For any non-zero \(n \in \mathbb{N}^d\) we set \(\Delta_n = \text{Conv}(\{e_1, \ldots, e_d, n\})\).

**Proposition 2.4.** For \(\varepsilon \in (0, 1]\)

(a) \(\mathbb{A}_n^d\) has \(\varepsilon\)-log terminal singularities if and only if \(\varepsilon\Delta_n\) is empty.

(b) \(\mathbb{A}_n^d\) has \(\varepsilon\)-log canonical singularities if and only if \(\varepsilon\Delta_n\) is hollow.

**Proof.** (a) The singularities of \(\mathbb{A}_n^d\) are \(\varepsilon\)-log terminal if and only if all the polytopes \(\varepsilon\Delta(\sigma_n^j)\) are empty: that is, if \(\bigcup_{j=1}^d \varepsilon\Delta(\sigma_n^j)\) is empty. But

\[
\bigcup_{i=1}^n \varepsilon\Delta_{\sigma_n^i} = \varepsilon \text{Conv}(0, e_1, \ldots, e_d, n)
\]

\[
= \varepsilon \text{Conv}(0, e_1, \ldots, e_d) \cup \varepsilon \text{Conv}(e_1, \ldots, e_d, n)
\]

\[
= \varepsilon \Delta \cup \varepsilon \Delta_n
\]

and \(\varepsilon \text{Conv}(\{0, e_1, \ldots, e_d\})\) is empty anyway.
(b) All lattice points of \( \bigcup_{i=1}^{n} \varepsilon \Delta(\sigma_n^i) \) other than the origin lie in \( \varepsilon \Delta_n \) by construction. Hence they all lie in facets not containing the origin if and only if they do not lie in the interior of \( \varepsilon \Delta_n \) or in \( \varepsilon \Delta_n \cap \varepsilon \Delta = \varepsilon \text{Conv}\{e_1, \ldots, e_d\} = \varepsilon \Delta_1 \). The latter is empty, and except for the trivial case \( \varepsilon = 1 \) has no lattice points among its vertices either.

The following change of coordinates sends the simplex \( \Delta_n \) of Proposition 2.4 to the standard simplex \( \Delta \), which will be useful for us.

**Lemma 2.5.** Let \( n = (n_1, \ldots, n_d) \in \mathbb{R}_{\geq 0}^d \) be a non-negative vector with \( \Sigma n_i > 1 \). Then the unique affine-linear transformation sending \( n \) to the origin and fixing all of \( e_1, \ldots, e_d \) sends the origin to \( n/(-1 + \Sigma n_i) \).

**Proof.** The unique (modulo multiplication by a scalar) affine dependences among \( \{0, e_1, \ldots, e_d, n\} \) and among \( \{n/(-1 + \Sigma n_i), e_1, \ldots, e_d, 0\} \) are the same one: its coefficients are \((1 - \Sigma n_i, n_1, \ldots, n_d, -1)\).

**Corollary 2.6.** Let \( n \in \mathbb{N}^d \). Define \( V = -1 + \Sigma n_i \) and \( p = \frac{1}{V} n \in \mathbb{Q}^d \). Let \( \Lambda_p = \mathbb{Z}^d + \mathbb{Z} p \) be the lattice generated by \( p \) and \( \mathbb{Z}^d \). Then, for any \( \varepsilon \in (0, 1] \):

(a) \( A_n^d \) has \( \varepsilon \)-log terminal singularities if and only if \( \Delta_{p,\varepsilon} = p + \varepsilon(\Delta - p) \) is empty with respect to the lattice \( \Lambda_p \).

(b) \( A_n^d \) has \( \varepsilon \)-log canonical singularities if and only if \( \Delta_{p,\varepsilon} \) is hollow with respect to the lattice \( \Lambda_p \).

**Proof.** This is just Proposition 2.4, rephrased via the change of coordinates of Lemma 2.5. The notation here will be used more widely: see Definition 3.2 below.

### 3 \( \varepsilon \)-log canonical singularities

This section is devoted to the proof of Theorem 1.3.

#### 3.1 Lawrence’s Theorem and hollow points

Apart from the relation between \( \varepsilon \)-log canonical singularities and hollow simplices described in Corollary 2.6, our main technical tool is the following result of Jim Lawrence (see also [6]).

**Theorem 3.1** (Lawrence [11, Theorem 1]). Fix \( d \in \mathbb{N} \) and an open subset \( U \subset \mathbb{R}^d \), and let \( G \) be a closed subgroup of \( \mathbb{R}^d \) containing \( \mathbb{Z}^d \). Then there are only finitely many maximal subgroups \( G < G \) such that \( \mathbb{Z}^d \subset G \) and \( G \cap U = \emptyset \).
In other words, any subgroup of $G$ that contains $Z^d$ and misses $U$ is contained in one (at least) of finitely many such subgroups of $G$.

These maximal subgroups $G$ are automatically closed. Hence $G$ is a Lie subgroup of $\mathbb{R}^d$, and its identity component, which we call $L$, is a linear subspace of dimension equal to $\dim G$. Some of the groups containing $Z^d$ that we consider below are not closed, however.

The relation to our problem comes from the fact that the lattice $\Lambda_p$ in Corollary 2.6 is a subgroup of $\mathbb{R}^d$ containing $Z^d$. This implies, for example, that taking $U = \Delta^0$, we may interpret the case $\varepsilon = 1$ of Corollary 2.6(b) as saying that if $A_n^d$ has only canonical singularities then $p$ lies in one of finitely many subgroups of $\mathbb{R}^d$ containing $Z^d$ and not intersecting $\Delta^0$.

Our aim is to extend this approach to any value of $\varepsilon \in (0, 1]$. We first extend the notation introduced in Corollary 2.6, using Definition 2.2.

**Definition 3.2.** We define

$$
\Omega := \mathbb{R}^d_{\geq 0} \setminus \Delta = \{ x \in \mathbb{R}^d \mid \Sigma x_i > 1 \text{ and } \forall i x_i \geq 0 \}.
$$

For each point $p \in \Omega$:

(a) We call the number $V := -1 + \frac{1}{\Sigma n_i} \in \mathbb{R}_{\geq 0}$ the *index* of $p$. The entries of the vector $n := Vp \in \mathbb{R}^d_{\geq 0}$ are called the *weights* of $p$, and the smallest of them is called the *smallest weight* $n_{\min} = n_{\min}(p)$ of $p$.

(b) We put $\Delta_{p, \varepsilon} = p + \varepsilon(\Delta - p)$ and $\Lambda_p = Z^d + Zp$.

(c) We say that $p$ is *$\varepsilon$-hollow* if $\Delta_{p, \varepsilon}$ is hollow with respect to the group $\Lambda_p$.

The notation in Definition 3.2(a) is compatible with the notation of Corollary 2.6 because

$$
-1 + \Sigma n_i = -1 + V \Sigma p_i = -1 + V \left( \frac{1}{V} + 1 \right) = V,
$$

but at this stage we do not require the weights to be integers: $V$ and $n$ need not even be rational, so the group $\Lambda_p$ may not be a lattice.

Observe that $\Delta_{p, \varepsilon}$ is $\Delta$ shrunk towards $p$ by a factor $\varepsilon$, so it is a simplex with facets parallel to the facets of $\Delta$.

**3.2 The canonical case of Birkar’s conjecture**

We let $H_0 = \{ x \mid \Sigma x_i = 0 \}$ and $H_1 = \{ x \mid \Sigma x_i = 1 \}$. Thus $H_1$ is the affine hyperplane containing $\Delta_1$ and $H_0$ is the linear hyperplane parallel to it. Let $\Delta^0_1$ denote the relative interior of $\Delta_1$.

Fix a linear subspace $L \subset \mathbb{R}^d$, of codimension $k$. Assuming that $L \not\subset H_0$ we are going to prove a bound $\ell_L$, depending only on $L$, for the minimum weight of every point $p \in \Omega$ such that $L + p$ does not meet $\Delta^0_1$.  

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For this, let $\pi_L: \mathbb{R}^d \rightarrow \mathbb{R}^d/L \cong \mathbb{R}^k$ be the canonical projection along $L$, let $s_i = \pi_L(e_i)$, and let $S = \{0, s_1, \ldots, s_d\}$, so that $\text{Conv}(S) = \pi_L(\Delta)$. The condition $L \not\subseteq H_1$ implies that no affine hyperplane in $\mathbb{R}^d/L$, in particular no facet of $\text{Conv}(S)$, contains $\{s_1, \ldots, s_d\}$. This makes the minimum in the following statement well-defined.

**Proposition 3.3.** Suppose that $L \subseteq \mathbb{R}^d$ is a linear subspace not contained in $H_1$. For each facet-supporting hyperplane $H$ of $\pi_L(\Delta)$ let

$$\ell_H := \min_{s_i \not\in H} \frac{\text{dist}(H, 0)}{\text{dist}(H, s_i)},$$

and let $\ell_L = \max_H \ell_H$. Then every point $p \in \Omega$ such that $p + L$ does not meet $\Delta^\circ$ has $\n_{\text{min}}(p) \leq \ell_L$.

**Remark 3.4.** Let $k = d - \dim L$. In $\mathbb{R}^d/L \cong \mathbb{R}^k$, an affine hyperplane $H$ is expressed as $H = \{x \in \mathbb{R}^k \mid f(x) = c\}$, where $f: \mathbb{R}^k \rightarrow \mathbb{R}$ is a linear functional. For $y \in \mathbb{R}^k$, we define the distance $\text{dist}(H, y) = |f(y) - c|$. This depends on the choice of $f$, which is only unique up to a scalar and, implicitly, on the choice of isomorphism $\mathbb{R}^d/L \cong \mathbb{R}^k$. But in the statement of Proposition 3.3 and the rest of this section we only consider ratios of two distances, which do not depend on choice. In Section 4 we shall need to be more definite.

**Proof.** Since $(p + L) \cap \Delta^\circ = \emptyset$ and $p \in \Omega$, we also have $(p + L) \cap \Delta = \emptyset$, and the point $\pi_L(p)$ is not in the interior of $\text{Conv}(S)$. Hence there is a facet-supporting hyperplane $H$ of $\text{Conv}(S)$ that weakly separates $\pi_L(p)$ from $\text{Conv}(S)$. Let $H = \pi_L^{-1}(H)$, which is a hyperplane weakly separating $L + p$ from $\Delta$ (but is not necessarily facet-supporting for $\Delta$).

If $0 \in H$ then, in order for $p$ to be in $\Omega$, one of the coordinates of $p$, hence one of the weights of $p$, must be zero. Thus we assume $0 \not\in H$ and we can find an $a \in \mathbb{R}^d$ such that $H = \{x \in \mathbb{R}^d \mid a.x = 1\}$, where $a.x := \sum_{i=1}^d a_ix_i$ is the usual Euclidean inner product.

Since $H$ weakly separates $\Delta$ from $p$ we have $\sum_i a_ip_i = a.p \geq 1$ but $a.x \leq 1$ for every $x \in \Delta$; in particular, $a_i = a.e_i \leq 1$ for every $i$. Thus

$$\sum_{i=1}^d (1-a_i)n_i = \sum_{i=1}^d n_i - V \sum_{i=1}^d a_ip_i \leq (V + 1) - V = 1.$$

Since the terms in the first sum are non-negative, $(1-a_i)n_i \leq 1$ for every $i$.

Observe that $\text{dist}(H, 0) = 1$ and $\text{dist}(H, e_i) = (1 - a.e_i)$ so

$$\frac{\text{dist}(H, s_i)}{\text{dist}(H, 0)} = \frac{\text{dist}(H, e_i)}{\text{dist}(H, 0)} = 1 - a_i.$$
Hence, for any $i$ with $s_i \not\in H$ (which exists, because otherwise we would have $\tilde{H} = \{\Sigma x_i = 1\} = H_1$ and that would imply $L \subset H_0$) we have

$$n_i \leq \frac{1}{1 - a_i} \frac{\text{dist}(H, 0)}{\text{dist}(H, s_i)}.$$ 

Thus $n_{\min}(p) \leq \ell_H$. This does not yet give a bound for $n_{\min}(p)$ because $H$ depends on $p$, but $H$ is one of the finitely many facet-supporting hyperplanes of $\pi_L(\Delta)$, so $n_{\min}(p) \leq \max_H \ell_H = \ell_L$ as claimed. 

Although we give below a separate proof of the general case, it is interesting to observe that Proposition 3.3 leads to the following easy proof of the canonical case of Theorem 1.3.

**Proof of Theorem 1.3 for $\varepsilon = 1$.** By Theorem 3.1 there is a finite collection \{\(G_1, \ldots, G_t\)\} of closed subgroups of \(\mathbb{R}^d\) containing \(\mathbb{Z}^d\) and not meeting \(\Delta^\circ\), such that any subgroup of \(\mathbb{R}^d\) containing \(\mathbb{Z}^d\) and not meeting \(\Delta^\circ\) is contained in one of them. We denote \(L_j\) the identity component of \(G_j\).

If \(L_j \subset H_0\), then the quotient \(G_j / (G_j \cap H_0) \cong \pi_{H_0}(G_j)\) is a discrete subgroup of \(\mathbb{R}^d / H_0 \cong \mathbb{R}\). Let \(y\) be the minimum of \(\pi_{H_0}(G_j)\) in the interval \((1, \infty)\) and define \(\ell_{G_j} = 1 / (-1 + y)\). Then the index (and hence each weight) of every \(p \in G_j \cap \Omega\) is bounded by \(\ell_{G_j}\).

If \(L_j \not\subset H_0\), then Proposition 3.3 applies, since \(L_j + p \subset G_j\) does not meet \(\Delta^\circ\). The proposition gives us an \(\ell_{G_j} = \ell_{L_j}\) (depending only on \(L_j\)) with \(n_{\min}(p) \leq \ell_{G_j}\) for every \(p \in G_j \cap \Omega\).

We can then take \(\ell_{1,d} = \max_{j = 1, \ldots, t} \ell_{G_j}\). Indeed, let \(n \in \mathbb{N}^d\) be such that \(A_n^d\) has only canonical singularities. As above, let \(V = -1 + \Sigma n_i\) and let \(p = \frac{1}{V} n\), which lies in \(\Omega\). By Corollary 2.6 the lattice \(\Lambda_p = \mathbb{Z}^d + \mathbb{Z}p\) does not meet \(\Delta^\circ\) and is thus contained in some \(G_j\) from our list. Thus, \(n_{\min} = n_{\min}(p) \leq \ell_{G_j} \leq \ell_{1,d}\). 

**3.3 Local weight bound**

In this section we examine the situation near a given point \(x\) of \(\Delta_1\) and show the following.

**Proposition 3.5.** Let \(\varepsilon \in (0, 1]\) and \(d \in \mathbb{N}\) be fixed. Then, for each point \(x \in \Delta_1\), there is a non-negative integer \(\ell_x \in \mathbb{N}\) and an open neighbourhood \(W_x\) of \(x\) in \(\mathbb{R}^d\), such that if \(p \in \Omega \cap W_x\) is \(\varepsilon\)-hollow then its smallest weight \(n_{\min}(p)\) satisfies \(n_{\min}(p) \leq \ell_x\).

To prove this we introduce the following notation. For each set \(U\) with \(x \in U \subseteq \mathbb{R}^d\) we define \(\Delta_{U,\varepsilon} = \bigcap_{q \in U} \Delta_{q,\varepsilon}\), and we let \(\mathcal{G}_{U,\varepsilon}\) be the family of all subgroups of \(\mathbb{R}^d\) containing \(\mathbb{Z}^d\) and not meeting \(\Delta_{U,\varepsilon}\). Observe that

\[
U \supseteq U' \implies \Delta_{U,\varepsilon} \subseteq \Delta_{U',\varepsilon} \implies \mathcal{G}_{U,\varepsilon} \supseteq \mathcal{G}_{U',\varepsilon}.
\]

We are interested in the case where \(U\) is a neighbourhood of \(x\).
Lemma 3.6. Let \( B_1 \supset B_2 \supset \ldots \) be a countable base of neighbourhoods of \( x \), so that \( \bigcap_{r \in \mathbb{N}} B_r = \{x\} \). Then \( \bigcup_{r \in \mathbb{N}} \Delta_{B_r, \varepsilon}^o = \Delta_{x, \varepsilon}^o \).

Proof. The inclusion \( \bigcup_{r \in \mathbb{N}} \Delta_{B_r, \varepsilon}^o \subseteq \Delta_{x, \varepsilon}^o \) is immediate. For the other direction, if \( y \in \Delta_{x, \varepsilon}^o \) then

\[
x \in \{z \mid y \in \Delta_{z, \varepsilon}^o\} = \{z \mid \exists w \in \varepsilon \Delta^o \text{ such that } y = z(1 - \varepsilon) + w\} = \{z \mid y - z(1 - \varepsilon) \in \varepsilon \Delta^o\},
\]

which is open because \( \varepsilon \Delta^o \) is open and \( z \mapsto y - z(1 - \varepsilon) \) is continuous.

Hence \( y \in \Delta_{x, \varepsilon}^o \) for all \( z \) in some neighbourhood of \( x \), and in particular for all \( z \in B_r \) for some sufficiently large \( r \). Hence \( y \in \bigcup_{r \in \mathbb{N}} \Delta_{B_r, \varepsilon}^o \).

By analogy with Definition 3.2 we say that a closed group \( G \) with identity component \( L \) is \( \varepsilon \)-hollow at \( x \) if \( G \cap (x + L) \cap \Delta_{x, \varepsilon}^o = \emptyset \).

Observe that this includes all closed groups with \( x \notin G \), since in this case \( G \cap (x + L) \) is already empty. Our next two lemmas prepare the proof of Proposition 3.5, dealing separately with groups that are and are not \( \varepsilon \)-hollow at \( x \).

Lemma 3.7. Every \( x \in \Delta_1 \) has an open neighbourhood \( U_x \) such that every closed group in \( G_{U_x, \varepsilon} \) is \( \varepsilon \)-hollow at \( x \).

Proof. Let \( B_1 \supset B_2 \supset \ldots \) be a countable base of neighbourhoods of \( x \). We will prove the following, which has Lemma 3.7 as the case \( k = 0 \):

For every \( k \in \{0, \ldots, d\} \) there is an \( r \) such that every closed group of dimension \( \geq k \) in \( G_{B_r, \varepsilon} \) is \( \varepsilon \)-hollow at \( x \).

The proof of this is by induction on \( d - k \). The base case \( k = d \) is trivial since the only group of dimension \( d \) is the whole space \( \mathbb{R}^d \), and this group does not lie in \( G_{B_1, \varepsilon} \). (We assume that \( \Delta_{B_1, \varepsilon} \) has non-empty interior: Lemma 3.6 allows us to do this.)

Now, for a fixed \( k \), our induction hypothesis is that there is an \( r \) such that every closed group of dimension greater than \( k \) in \( G_{B_r, \varepsilon} \) is \( \varepsilon \)-hollow at \( x \). That is, every closed group in \( G_{B_r, \varepsilon} \) that is not \( \varepsilon \)-hollow at \( x \) has dimension at most \( k \). By Theorem 3.1, \( G_{B_r, \varepsilon} \) contains finitely many maximal groups, all closed. Let us denote \( G_1, \ldots, G_t \) the ones of dimension \( k \) that are not \( \varepsilon \)-hollow (if any), and let \( L_1, \ldots, L_t \) be their corresponding identity components. Observe that, although \( G_{B_r, \varepsilon} \) may contain additional non-\( \varepsilon \)-hollow groups of dimension \( k \), apart from the \( G_i \)'s, any such group must be contained in one of the \( G_i \)'s and, in particular, its identity component must equal the corresponding \( L_i \).

For each \( i \in \{1, \ldots, t\} \), since \( G_i \) is non-\( \varepsilon \)-hollow, \( x + L_i \) meets \( \Delta_{x, \varepsilon}^o \); by Lemma 3.6, \( x + L_i \) meets \( \Delta_{B_r, \varepsilon}^o \) for some \( r_i \). In particular, \( G_{B_{r_i}, \varepsilon} \) contains
neither $G_i$ nor any other group whose identity component equals $L_i$. Obviously, the same holds for any $r \geq r_i$.  

Hence, taking $r' = \max\{r_1, \ldots, r_t\}$ we have that $G_{B_{r'},\varepsilon}$ does not contain any group with identity component equal to any of the $L_i$’s. Since $B_{r'} \supset B_r$ we have $G_{B_{r'},\varepsilon} \subset G_{B_r,\varepsilon}$, and hence all the non-$\varepsilon$-hollow groups in $G_{B_r,\varepsilon}$ are non-$\varepsilon$-hollow groups in $G_{B_r,\varepsilon}$ too, but necessarily of smaller dimension. □

**Lemma 3.8.** Let $x \in \Delta_1$ and let $G$ be a closed group containing $\mathbb{Z}^d$ and $\varepsilon$-hollow at $x$. Then there is a neighbourhood $W_G$ of $x$ and a natural number $\ell_G$ such that every $p \in \Omega \cap G \cap W_G$ has $n_{\min}(p) \leq \ell_G$.

**Proof.** Let $L$ be the identity component of $G$. There are three possibilities:

- If $x \not\in G$, simply take $W_G = \mathbb{R}^d \setminus G$ and $\ell_G = 0$.
- If $L \subset H_0$, then $\pi_{H_0}(G) = G/(G \cap H_0) \subset \mathbb{R}$ is discrete. Let $s$ be its minimum in $(1, \infty)$. We can take $W_G = \{p \mid \Sigma p_i < s\}$ and $\ell_G = 0$, since $\Omega \cap G \cap W_G = \emptyset$.
- If $x \in G$ and $L \not\subset H_0$, then $x + L \subset G$ but $(x + L) \cap \Delta^0_\varepsilon = \emptyset$, because $G$ is $\varepsilon$-hollow. But then $L + x$ does not meet $\Delta^0_\varepsilon$, so we may apply Proposition 3.3 to $L$. We then get an $\ell_L$ such that for every $p \in \Omega \cap (x + L)$ we have that the minimum weight of $p$ is bounded by $\ell_L$. We can then take $W_G = \mathbb{R}^d \setminus (G \setminus (x + L))$, so that $G \cap W_G = x + L$ and $\Omega \cap G \cap W_G = \Omega \cap (x + L)$. □

We can now prove Proposition 3.5.

**Proof of Proposition 3.5.** By Lemma 3.7, $x$ has an open neighbourhood $U_x$ such that every group in $G_{U_x,\varepsilon}$ contains $x$ is $\varepsilon$-hollow. By Theorem 3.1, $G_{U_x,\varepsilon}$ has a finite number of maximal elements, all closed and $\varepsilon$-hollow at $x$, which we denote $G_1, \ldots, G_t$. By Lemma 3.8, each $G_i$ gives a neighbourhood $W_i$ of $x$ and a natural number $\ell_i$ such that every $p \in \Omega \cap G_i \cap W_i$ has $n_{\min}(p) \leq \ell_i$.

Now it is enough to take $W_x = U_x \cap (\bigcap_t W_i)$ and $\ell_x = \max \ell_i$. Indeed, let $p \in W_x \cap \Omega$ be $\varepsilon$-hollow, so that $\Delta_{p,\varepsilon} \cap \Lambda_p = \emptyset$. Since $p \in W_x$, we have $\Delta_{p,\varepsilon} \supset \Delta_{U_x,\varepsilon} \supset \Delta_{U_x,\varepsilon}$. In particular, the group $\Lambda_p$ is in $G_{U_x,\varepsilon}$, and hence is contained in one of the $G_i$’s. Thus $p \in \Omega \cap G_i \cap W_i$. □

### 3.4 The general case of Birkar’s conjecture

We are now in a position to give the proof of Theorem 1.3, settling Conjecture 1.2 completely.

**Proof of Theorem 1.3.** Fix $\varepsilon \in (0, 1]$. For each $x \in \Delta_1$, choose $\ell_x$ and $W_x$ as in Proposition 3.5, with $\ell_x$ as small as possible. For a non-negative integer $\ell$, define $\Delta_1(\ell) := \{x \in \Delta_1 \mid \ell_x \leq \ell\}$. Then $\Delta_1(\ell)$ is relatively open in $\Delta_1$,
because if \( y \in W_x \cap \Delta_1 \) then \( \ell_y < \ell_x \). Moreover, the \((\Delta_1(\ell))_{\ell \in \mathbb{N}}\) obviously form an increasing sequence and they cover \( \Delta_1 \). Observe, for example, that \( \Delta_1^0 \subseteq \Delta_1(0) \), because if \( x \in \Delta_1^0 \) and \( G \cap (x + L) \) meets \( \Delta_1^0 \) then \( L \subset H_0 \).

Put differently, Proposition 3.3 is not needed on \( \Delta_1^0 \). By compactness, there is an open subset \( W = \bigcup_{x \in \Delta_1^0} W_x \) and an integer \( \ell_W \) such that \( \Delta_1 \subset W \) and every \( \varepsilon \)-hollow \( p \in \Omega \cap W \) has \( n_{\min}(p) \leq \ell_W \). On the other hand, if \( p \in 2\Omega \) then \( V < 1 \), and since \( \Omega \setminus (2\Omega \cup W) \) is compact, the index (hence the minimum weight) of all \( p \in \Omega \setminus U \) has a global upper bound. \( \square \)

4 Terminal and canonical bounds

Throughout this section we take \( \varepsilon = 1 \), so that we are considering only canonical and terminal singularities. In these cases we compute more explicit bounds, assuming that \( \dim L \) or \( \mathrm{codim} L \) is small. Combining these bounds with the classification of empty 4-simplices in [9] we give precise bounds in the terminal 4-fold case: that is, a precise answer to Question 1.1.

4.1 Bounds in terms of width

We first rework the bound of Proposition 3.3 in terms of the lattice width of \( \text{Conv}(S) = \pi_L(\Delta) \).

**Definition 4.1.** A linear functional \( f : \mathbb{R}^d \to \mathbb{R} \) is called *primitive* with respect to a lattice \( \Lambda \) if \( f(\Lambda) = \mathbb{Z} \).

The *width* of a lattice polytope \( \Pi \) in the direction of \( f \) is the length of the interval \( f(\Pi) \). Its *facet width* with respect to a facet \( F \) is the width in the direction of the unique (up to a sign) primitive linear functional that is constant on \( F \).

Let \( G \subseteq \mathbb{R}^d \) be a closed group containing \( \mathbb{Z}^d \) and not meeting \( \Delta^0 \), with identity component \( L \). We keep the notation from Subsection 3.2, and we let \( \Lambda_G = \pi_L(G) \), which is a lattice in \( \mathbb{R}^d / L \), and put

\[
\ell_G = \max\{n_{\min}(p) \mid p \in \Omega \cap G\},
\]

i.e. the best possible bound for the smallest weight in \( G \).

**Proposition 4.2.** \( \ell_G \) is bounded by the maximum facet width of \( \pi_L(\Delta) \) with respect to \( \Lambda_G \).

**Proof.** Suppose first that \( L \not\subset H_0 \) and let \( H \) be a facet-supporting hyperplane of \( \pi_L(\Delta) = \text{Conv}(S) \). We normalise the distance to \( H \) by taking \( f \) to be the primitive linear functional constant on \( H \) and \( \text{dist}(H, x) = |f(x) - f(H)| \). Then \( 1 \leq \text{dist}(H, s_i) \in \mathbb{N} \) for every \( s_i \not\in H \) and \( \text{dist}(H, 0) \) is...
bounded above by the facet width with respect to the facet contained in $H$. Hence the statement follows from Proposition 3.3.

If $L \subset H_0$ then $\pi_L(H_1)$ is a facet-supporting hyperplane of $\pi_L(\Delta)$. If $p \in \Omega \cap G$ then $\pi_L(p) \in \Lambda_G$ and is strictly separated from $\pi_L(\Delta)$ by $\pi_L(H_1)$. So if $f$ is the primitive linear functional constant on $\pi_L(H_1)$, then $f_1 := f(\pi_L(H_1))$ is the facet width of $\pi_L(\Delta)$ with respect to $\pi_L(H_1)$, and $f(p) \geq f_1 + 1$. Hence $\Sigma p_i \geq \frac{f_1+1}{f_1}$, so $V \leq f_1$ and therefore $n_{\min}(p) \leq f_1$. \hfill $\Box$

**Corollary 4.3.** With the notation of Proposition 4.2,

(a) If $\pi_L(\Delta)$ has width equal to 1 in some lattice direction then $\ell_G \in \{0, 1\}$.
This is always the case if $\dim L = d - 1$.

(b) If $\dim L = d - 2$, then $\ell_G \in \{0, 1, 2\}$.

**Proof.** (a) Let $f$ be a primitive functional giving width 1 to $\Delta/L$, and $\tilde{f}$ its pull-back to $\mathbb{R}^d$. Then $G' := G + \text{Ker}(\tilde{f})$ is a closed group containing $G$ and not intersecting $\Delta^\circ$, which implies $\ell_G \leq \ell_{G'}$.

Thus there is no loss of generality in assuming $\dim L = d - 1$. In this case $L = \text{Ker}(\tilde{f})$, so $\pi_L(\Delta) = f(\Delta)$ is a hollow lattice polytope of dimension 1, that is, a unit segment. This has facet width 1 with respect to every facet, so Proposition 4.2 gives the statement.

(b) Here $\pi_L(\Delta)$ is a hollow lattice polytope of dimension 2. This implies $\pi_L(\Delta)$ either has width 1 or equals (modulo an affine isomorphism of the lattice) the triangle $\text{Conv}((0, 0), (2, 0), (0, 2))$ (see, e.g., [8]). This triangle has width 2 with respect its to all its three facets. \hfill $\Box$

We can now recover Kawakita’s result on the terminal weighted blowups in dimension 3.

**Corollary 4.4 ([10, Theorem 3.5]).** The weighted blowup $\mathbb{A}_n^3$ has terminal singularities if and only if the weights are $(1, a, b)$, with $a$ and $b$ coprime.

**Proof.** This follows immediately from Corollary 4.3(a) and the theorem of White [16] that all empty 3-simplices have width 1. \hfill $\Box$

### 4.2 Groups of dimension 1

For our application to $d = 4$ in Subsection 4.3 below, we want to consider the case $\dim L = 1$ more carefully. In this case let $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ be a primitive integer vector in $L$, which is unique up to sign, and let $a_0 := \sum_{i=1}^d a_i$. The vector $a := (a_0, \ldots, a_d) \in \mathbb{Z}^{d+1}$ is called the $(d + 1)$-tuple of $L$. We assume $L \not\subset H_0$, which is equivalent to $a_0 \neq 0$.

**Lemma 4.5.** Suppose $p \in \Omega$ and that $\dim L = 1$, and that $(p + L) \cap \Delta^\circ = \emptyset$. Then $n_{\min}(p) \leq \max_{i=1,\ldots,d}(-a_i/a_0)$. 

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Proof. The set \( S = \{0, s_1, \ldots, s_d\} \) affinely spans \( \mathbb{R}^d / L \cong \mathbb{R}^{d-1} \) and has \( d+1 \) points, so it has a unique (modulo a scalar factor) affine dependence. Since \( \sum_{i=1}^d a_i e_i \in L \), the coefficient vector of that dependence is precisely \( a \).

To bound the minimum weight we use Proposition 3.3. Let \( H \) be a facet-supporting hyperplane of \( \text{Conv}(S) \). If \( 0 \in H \) then \( \ell_H = 0 \) in Proposition 3.3.

If \( 0 \not\in H \) then, since \( L \not⊂ H \), there must be an \( i \) with \( s_i \not\in H \). Thus \( H \) contains all of \( S \) except for \( 0 \) and a single \( s_i \). Applying the affine dependence \( a \) to the affine functional vanishing on \( H \) gives \( \text{dist}(H, 0) a_0 + \text{dist}(H, s_i) a_i = 0 \), which finishes the proof since \( \min_{s_j \not\in H} \text{dist}(H, 0) \text{dist}(H, s_j) = -a_i a_0 \).

We also have the following alternative bound, which is better than the previous one in a few critical cases.

**Lemma 4.6.** Let \( p \in \Omega \) be such that \( n = V p \in \mathbb{N}^d \), where \( V = \frac{1}{-1 + \Sigma p_i} \) as usual. Suppose that there is a proper subset \( J \subset \{1, \ldots, d\} \) such that

\[
\sum_{i \in J} p_i - s \sum_{i=1}^d p_i \in \mathbb{Z}
\]

for a positive integer \( s \). Then either \( \sum_{i \in J} n_j \leq s \) or else \( n_i = 0 \) for all \( i \not\in J \).

**Proof.** Multiplying the equation in the statement by \( V \) we obtain that

\[
\sum_{i \in J} n_i - s(V + 1) \in V \mathbb{Z},
\]

so \( \sum_{i \in J} n_i \equiv s \pmod{V} \). Since \( \Sigma n_i = V + 1 \), either \( n_i = 0 \) for every \( i \not\in J \), or \( \sum_{i \in J} n_i \leq V \). The latter, together with \( \sum_{i \in J} n_i \equiv s \pmod{V} \), implies \( \sum_{i \in J} n_i \leq s \). \( \square \)

### 4.3 Terminal 4-fold case

Now we consider the case \( d = 4 \), where there is an extensive history. Notice that another interpretation of Corollary 2.6 is that \( \mathbb{A}^d_n \) has terminal (or canonical) singularities if and only if the cyclic quotient singularity \( \frac{1}{V} n \) is terminal (or canonical), where \( V = -1 + \Sigma n_i \).

In fact any non-Gorenstein terminal quotient singularity in dimension 4 is cyclic, but this fails in higher dimension: see [2] for both of these facts. The singularity \( \frac{1}{V} n \) is never Gorenstein, but we note for completeness that Gorenstein cyclic terminal 4-fold singularities were classified in [13], and Gorenstein non-cyclic terminal 4-fold singularities in [1].

In dimension 4, a classification of non-Gorenstein terminal quotient singularities was begun experimentally in [12]. The first definite result was
proved in [14] (another proof of the same result may be found in [5]): together with the results of [6] and [2], it implies that the list in [12] of such singularities of prime index is complete with possibly finitely many exceptions. Note, however, that the claim made in [2] that the results of [14] and [5] are valid for composite index is incorrect, as was pointed out in [4].

The complete classification of non-Gorenstein terminal quotient singularities in dimension 4 was recently given in [9], and we use it to prove Theorem 1.4.

In [9, Section 2] hollow simplices are divided into *fine families*. Two hollow lattice simplices $\Delta_1$ and $\Delta_2$ in $\mathbb{R}^d$, with $Vx(\Delta_i) = \{v_{ij}\} \subset \mathbb{Z}^d$, lie in the same fine family if there is an integer $k \leq d$ and integer affine maps $\pi_i : \mathbb{Z}^d \to \mathbb{Z}^k$ such that $\pi_1(Vx(\Delta_1)) = \pi_2(Vx(\Delta_2)) = S$ and $\text{Conv}(S)$ is hollow. Here $S = \{s_0, \ldots, s_d\}$ is to be thought of as a multiset: that is, there is a permutation $\sigma$ of $\{0, \ldots, d\}$ such that $\pi_1(v_1\sigma(j)) = \pi_2(v_2j)$ for all $j$.

As before, if $G$ is a closed group containing $\mathbb{Z}^d$ and with $G \cap \Delta^0 = \emptyset$ then $\pi_L(\Delta)$ is a hollow lattice polytope with respect to the lattice $\Lambda_G = \pi_L(G)$. Thus the rational points in $G$ parametrise (perhaps part of) a fine family of hollow simplices: each point $p \in G \cap \mathbb{Q}^d$ corresponds, as in Corollary 2.6, to the standard simplex $\Delta \subset \mathbb{R}^d$ considered with respect to $\Lambda_p$. In this situation we say $p$ is a *generating point* of that hollow simplex. This relation makes Theorem 3.1 equivalent to [9, Corollary 2.7].

The case $L = \{0\}$ corresponds to the *sporadic hollow simplices* that do not project to hollow polytopes of lower dimension: more generally, the codimension of $L$, which we have called $k$ here, is the same as the parameter $k$ in [9, Theorem 1.6]. In particular, cases $k = 1, 2, 3, 4$ of [9, Theorem 1.6] correspond exactly to the cases $\dim L = 3, 2, 1, 0$ in our setting. We prove Theorem 1.4 separately for each value of $k$. We have already done $k = 1$ and $k = 2$.

**Proposition 4.7.** If a blowup $\mathbb{A}_n^4$ of $\mathbb{A}^4$ belongs to the case $k = 1$ then $n_{\text{min}} \leq 1$, and if $k = 2$ then $n_{\text{min}} \leq 2$.

**Proof.** These are just parts (a) and (b) of Corollary 4.3. \qed

For the case $k = 3$, the most interesting one, we analyse the bounds from Subsection 4.2. The *index* of a family parametrised by a group $G$ as above is defined to be the index $|G : L + \mathbb{Z}^d|$. A family is called *primitive* if its index is 1, and *non-primitive* otherwise.

The classification in [9] for $k = 3$ consists of two lists: one of 29 primitive quintuples $Q_1$–$Q_{29}$ (the same as the list of quintuples that appears in [12]), and one of 17 non-primitive quintuples $N_1$–$N_{17}$.

A primitive family is fully determined by $L$. In the case $\dim L = 1$ and $d = 4$ we specify $L$ via a quintuple $q = (q_1, \ldots, q_5)$ with $\sum q_i = 0$, defined by the property that $\mathbb{R}q$ parametrises $(L + \mathbb{Z}^4)/\mathbb{Z}^4$ in barycentric coordinates.
with respect to the standard simplex. As shown in [9], the quintuple $q$ can also be interpreted as the affine dependence among the points in $S = \pi_L(\{0,e_1,\ldots,e_n\})$. Thus, modulo a permutation of the entries, $q$ is the same as the vector $a = (a_0,\ldots,a_d)$ that we used in Lemma 4.5. However, in order to apply Lemma 4.5 we need to specify which of the entries $q_l$ will be considered the distinguished entry $a_0$.

A more concrete interpretation of the quintuple is as follows: for each $V \in \mathbb{N}$, the family corresponding to $q$ contains a unique (modulo affine-integer isomorphism) hollow simplex of index $V$; the generating point $p$ of this simplex can be chosen to be $p = \frac{1}{p}(a_1,\ldots,a_d)$, where $(a_1,\ldots,a_d)$ is obtained from $q$ by deleting the entry $q_l = a_0$ corresponding to the origin and permuting the rest. The generating point is only important modulo $\mathbb{Z}^4$.

In the non-primitive case a family is determined by not only $L$ or $q$, but also by information on the group $G/(L + \mathbb{Z}^4)$. In [9] and in the table below this is expressed by adding to $q$ a vector of the form $V\mathbb{r}$ (or of the form $\pm V\mathbb{r}$, for the non-primitive quintuples of index greater than 2, which are N7–N17). Observe, however, that the statement of Lemma 4.5 depends only on $L$, namely N5, we defer the details on how to interpret $V\mathbb{r}$ to when we need it.

We now list the quintuples, with the conventional labels Q1–Q29 and N1–N17.

<table>
<thead>
<tr>
<th>Case</th>
<th>Quintuple</th>
<th>Case</th>
<th>Quintuple</th>
<th>Case</th>
<th>Quintuple</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1</td>
<td>9,1,−2,−3,−5</td>
<td>Q18</td>
<td>15,1,−3,−5,−8</td>
<td>N1</td>
<td>6+\frac{\mathbb{r}}{2},1,−2,−2+\frac{\mathbb{r}}{2},−3</td>
</tr>
<tr>
<td>Q2</td>
<td>9,2,−1,−4,−6</td>
<td>Q19</td>
<td>15,2,−1,−6,−10</td>
<td>N2</td>
<td>4,−1,−2+\frac{\mathbb{r}}{2},−4+\frac{\mathbb{r}}{2}</td>
</tr>
<tr>
<td>Q3</td>
<td>12,3,−4,−5,−6</td>
<td>Q20</td>
<td>15,4,−2,−5,−12</td>
<td>N3</td>
<td>8,1,−2+\frac{\mathbb{r}}{2},−3,−4+\frac{\mathbb{r}}{2}</td>
</tr>
<tr>
<td>Q4</td>
<td>12,2,−3,−4,−7</td>
<td>Q21</td>
<td>18,1,−4,−6,−9</td>
<td>N4</td>
<td>6+\frac{\mathbb{r}}{3},3,−1,−2+\frac{\mathbb{r}}{3},−6</td>
</tr>
<tr>
<td>Q5</td>
<td>9,4,−2,−3,−8</td>
<td>Q22</td>
<td>18,2,−5,−6,−9</td>
<td>N5</td>
<td>8,3,−1,−4+\frac{\mathbb{r}}{2},−6+\frac{\mathbb{r}}{2}</td>
</tr>
<tr>
<td>Q6</td>
<td>12,1,−2,−3,−8</td>
<td>Q23</td>
<td>18,4,−1,−9,−12</td>
<td>N6</td>
<td>12,1,−3,−4+\frac{\mathbb{r}}{2},−6+\frac{\mathbb{r}}{2}</td>
</tr>
<tr>
<td>Q7</td>
<td>12,3,−1,−6,−8</td>
<td>Q24</td>
<td>20,1,−4,−7,−10</td>
<td>N7</td>
<td>3,1,−1±\frac{\mathbb{r}}{2},−1±\frac{\mathbb{r}}{2},−2</td>
</tr>
<tr>
<td>Q8</td>
<td>15,4,−5,−6,−8</td>
<td>Q25</td>
<td>20,1,−3,−8,−10</td>
<td>N8</td>
<td>3,2,−1,−1±\frac{\mathbb{r}}{2},−3±\frac{\mathbb{r}}{2}</td>
</tr>
<tr>
<td>Q9</td>
<td>12,2,−1,−4,−9</td>
<td>Q26</td>
<td>20,3,−4,−9,−10</td>
<td>N9</td>
<td>3,2,−1,−2±\frac{\mathbb{r}}{2},−2±\frac{\mathbb{r}}{2}</td>
</tr>
<tr>
<td>Q10</td>
<td>10,6,−2,−5,−9</td>
<td>Q27</td>
<td>20,3,−1,−10,−12</td>
<td>N10</td>
<td>4±\frac{\mathbb{r}}{3},2,−1,−1±\frac{\mathbb{r}}{3},−4</td>
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<tr>
<td>Q11</td>
<td>15,1,−2,−5,−9</td>
<td>Q28</td>
<td>24,1,−5,−8,−12</td>
<td>N11</td>
<td>6,1,−2,−2±\frac{\mathbb{r}}{2},−3±\frac{\mathbb{r}}{2}</td>
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<tr>
<td>Q12</td>
<td>12,5,−3,−4,−10</td>
<td>Q29</td>
<td>30,1,−6,−10,−15</td>
<td>N12</td>
<td>6,1,−1±\frac{2\mathbb{r}}{3},−2,−4±\frac{\mathbb{r}}{3}</td>
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<tr>
<td>Q13</td>
<td>15,2,−3,−4,−10</td>
<td>Q30</td>
<td>30,1,−6,−10,−15</td>
<td>N13</td>
<td>4,3,−1±\frac{2\mathbb{r}}{3},−2,−4±\frac{\mathbb{r}}{3}</td>
</tr>
<tr>
<td>Q14</td>
<td>12,1,−3,−4,−6</td>
<td>Q44</td>
<td>30,1,−6,−10,−15</td>
<td>N14</td>
<td>6,3±\frac{\mathbb{r}}{3},1,−2±\frac{\mathbb{r}}{3},−6±\frac{\mathbb{r}}{3}</td>
</tr>
<tr>
<td>Q15</td>
<td>14,1,−3,−5,−7</td>
<td>Q55</td>
<td>30,1,−6,−10,−15</td>
<td>N15</td>
<td>3±\frac{\mathbb{r}}{3},2,−1,−1±\frac{\mathbb{r}}{3},−3±\frac{\mathbb{r}}{3}</td>
</tr>
<tr>
<td>Q16</td>
<td>14,3,−1,−7,−9</td>
<td>Q66</td>
<td>30,1,−6,−10,−15</td>
<td>N16</td>
<td>6,1±\frac{\mathbb{r}}{3},−1,−3±\frac{\mathbb{r}}{3},−3±\frac{\mathbb{r}}{3}</td>
</tr>
<tr>
<td>Q17</td>
<td>15,7,−3,−5,−14</td>
<td>Q77</td>
<td>30,1,−6,−10,−15</td>
<td>N17</td>
<td>3,1±\frac{\mathbb{r}}{3},−1,−1±\frac{\mathbb{r}}{3},−2±\frac{2\mathbb{r}}{3}</td>
</tr>
</tbody>
</table>
In every case the entries are arranged so that 

\[ q_1 > q_2 > 0 > q_3 \geq q_4 \geq q_5. \]

With this convention, we have \( \max\{-a_j/a_0\} \leq -q_1/q_3 \) if \( a_0 \in \{q_1, q_2\} \) and \( \max\{-a_j/a_0\} \leq -q_5/q_2 \) if \( a_0 \in \{q_3, q_4, q_5\} \). Thus Lemma 4.5 implies the following. Observe that in the hypotheses of this statement we can write \( <7 \) instead of \( \leq 6 \) since all weights are integers.

**Lemma 4.8.** If a quintuple \( q \) (primitive or not) written as above satisfies 

\[ \max\{-q_1/q_3, -q_5/q_2\} < 7 \]

then every blowup coming from that quintuple has \( n_{\text{max}} \leq 6 \).

With this, we are now ready to prove the main result in this section, which gives Theorem 1.4 for the families with \( \dim L = 1 \), that is, \( k = 3 \).

**Proposition 4.9.** If a blowup \( \mathbb{A}^4 \) of \( \mathbb{A}^4 \) belongs to the case \( k = 3 \) (equivalently, \( \dim L = 1 \)) then \( n_{\text{min}} \leq 6 \).

**Proof.** The reader may easily check that the only cases where Lemma 4.8 is not sufficient to prove a bound of 6 are the ones shown (with the ratio \( q_1/-q_3 \) or \(-q_5/q_2 \) that we do get) in the table below. In all the other cases, including the ones marked “—” in the table, the ratios \( q_1/-q_3 \) and \(-q_5/q_2 \) are strictly less than 7. In the non-primitive quintuples this check is especially easy, since none of them has \(-q_5 > 6 \) and the only ones with \( q_1 > 6 \) are N3, N5, and N6.

<table>
<thead>
<tr>
<th>Quintuple</th>
<th>( q_1 : -q_3 )</th>
<th>(-q_5 : q_2 )</th>
<th>Quintuple</th>
<th>( q_1 : -q_3 )</th>
<th>(-q_5 : q_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q2</td>
<td>9 : 1</td>
<td>—</td>
<td>Q20</td>
<td>15 : 2</td>
<td>—</td>
</tr>
<tr>
<td>Q6</td>
<td>—</td>
<td>8 : 1</td>
<td>Q21</td>
<td>—</td>
<td>9 : 1</td>
</tr>
<tr>
<td>Q7</td>
<td>12 : 1</td>
<td>—</td>
<td>Q23</td>
<td>18 : 1</td>
<td>—</td>
</tr>
<tr>
<td>Q9</td>
<td>12 : 1</td>
<td>—</td>
<td>Q24</td>
<td>—</td>
<td>10 : 1</td>
</tr>
<tr>
<td>Q11</td>
<td>15 : 2</td>
<td>9 : 1</td>
<td>Q25</td>
<td>—</td>
<td>10 : 1</td>
</tr>
<tr>
<td>Q15</td>
<td>—</td>
<td>7 : 1</td>
<td>Q27</td>
<td>20 : 1</td>
<td>—</td>
</tr>
<tr>
<td>Q16</td>
<td>14 : 1</td>
<td>—</td>
<td>Q28</td>
<td>—</td>
<td>12 : 1</td>
</tr>
<tr>
<td>Q18</td>
<td>—</td>
<td>8 : 1</td>
<td>Q29</td>
<td>—</td>
<td>15 : 1</td>
</tr>
<tr>
<td>Q19</td>
<td>15 : 1</td>
<td>—</td>
<td>N5</td>
<td>8 : 1</td>
<td>—</td>
</tr>
</tbody>
</table>

Even where the bound exceeds 7, the ratios \(-q_5/q_1 \) and \(-q_1/q_4 \) (hence also \(-q_1/q_5 \)) are less than 7, which implies that for the cases with \( l = 1, 4, 5 \) the bound of Lemma 4.5 is at most 6 in every quintuple. Thus the eighteen quintuples in the table correspond to nineteen pairs (quintuple, \( l \)) that need to be checked: one of \( l = 2 \) or \( l = 3 \) for each of the quintuples, except for the quintuple Q11 where we have to check both.

Sixteen of the nineteen cases are primitive quintuples in which \( q_2 = 1 \) (if \( l = 2 \)) or \( q_3 = -1 \) (if \( l = 3 \)). This is fortunate since in these cases it is particularly simple to apply Lemma 4.6. Indeed:
• If \( a_0 = q_2 = 1 \) then we can use \( s = -q_3 \) in the lemma, by letting \( J \) be just one coordinate, the one corresponding to \( q_3 \).

• If \( a_0 = q_3 = -1 \) then we can use \( s = q_2 \) in the lemma, by letting \( J \) be just one coordinate, the one corresponding to \( q_2 \).

That is, in these sixteen cases we can use \(-q_3\) and \(q_2\) as bounds instead of the bigger \(-q_5\) and \(q_1\), respectively. The worst value obtained is 6, for \( Q_{29} \) with \( l = 2 \).

For the last three remaining cases we also apply Lemma 4.6 as follows:

• For \( Q_{11} = (15, 1, -2, -5, -9) \) with \( a_0 = q_3 = -2 \), our generating point is \( p = \frac{1}{V}(15, 1, -5, -9) \). Taking \( J \) to be the first and fourth coordinates and \( s = 3 \) we have \( \sum_{i \in J} p_i - s \sum_{i=1}^{d} p_i = \frac{1}{V}((15 - 9) - 3 \cdot 2) = 0 \). Thus, Lemma 4.6 gives \( n_1 + n_4 \leq 3 \).

• For \( Q_{20} = (15, 4, -2, -5, -12) \) with \( a_0 = q_3 = -2 \), our generating point is \( p = \frac{1}{V}(15, 4, -5, -12) \). Taking \( J \) to be the first and third coordinates and \( s = 5 \) we have \( \sum_{i \in J} p_i - s \sum_{i=1}^{d} p_i = \frac{1}{V}((15 - 5) - 5 \cdot 2) = 0 \). Thus, Lemma 4.6 gives \( n_1 + n_3 \leq 5 \).

• For \( N_5 \) the quintuple is expressed as \( (8, 3, -1, -4 + \frac{V}{2}, -6 + \frac{V}{2}) \), that is, as \( q + Vr \) with \( q = (8, 3, -1, -4, -6) \) and \( r = \frac{1}{2}(0, 0, 0, 1, 1) \). The interpretation of this is that hollow simplices in this family are those with generating point (in barycentric coordinates) equal to

\[
\frac{1}{V}(8, 3, -1, -4, -6) + \frac{1}{2}(0, 0, 0, 1, 1).
\]

See [9] for more details.

Since \( l = 3 \), we have to omit the third coordinate and get

\[
p = \frac{1}{V} \left( 8, 3, -4 + \frac{V}{2}, -6 + \frac{V}{2} \right),
\]

whose sum of coordinates is equal to \( 1 + \frac{1}{V} \).

Taking \( J \) to be just the second coordinate and \( s = 3 \) we have

\[
\sum_{i \in J} p_i - s \sum_{i=1}^{d} p_i = \frac{3}{V} - 3 \left( 1 + \frac{1}{V} \right) = 3 \in \mathbb{Z},
\]

so Lemma 4.6 gives \( n_2 \leq 2 \).

Thus, in all cases we get a bound of at most 6 for the smallest weight.
Remark 4.10. The bounds obtained by these methods are not sharp for each individual quintuple and choice of $l$, but the overall bound in Proposition 4.9 is sharp. For example, the blowup $A_4^{(V=30,6,10,15)}$ arising from Q29 with $l = 2$, has terminal singularities whenever $V$ is coprime with 30, and has minimum weight equal to 6 for every $V \geq 37$. This gives an infinite family of blowups of $A^4$ with terminal singularities and $n_{\min} = 6$.

To finish the proof of Theorem 1.4 we need to look at the case $k = 4$, that is, at the 2641 sporadic terminal 4-simplices enumerated in [9]. The full list is publicly available, and each simplex is expressed as a pair $(V, b)$ with $V \in \mathbb{N}$ and $b \in (\mathbb{Z}_V)^5$ where, as before, $V$ equals the (normalised) volume and $\frac{1}{V}b$ are the barycentric coordinates (modulo an integer vector, which does not affect the lattice) for a generator of $\Lambda/\mathbb{Z}^d$.

Each such simplex corresponds to five terminal quotient singularities (perhaps not distinct, if the simplex has symmetries) but not all such singularities correspond to blowups of $A^4$. The conditions for that are that:

- the corresponding entry $b_l$ of $b$ is coprime to $V$, so that by multiplying by a unit in $\mathbb{Z}_V$ we can assume that entry to be $-1$, and

- after this multiplication, the representatives in $\{0, \ldots, V-1\}$ of the other four entries (remember that they are only important modulo $V$) add up to $V + 1$.

When these conditions hold, the other four entries are the weights of a blowup of $A^4$.

We have computationally checked the $2641 \times 5$ possibilities, obtaining the results summarised in the following statement.

**Proposition 4.11.** Among the $2641 \times 5$ sporadic terminal quotient singularities of dimension 4 there are 4620 blowups, all with $n_{\min} \leq 32$. The number $B$ of sporadic blowups with each possible value of $n_{\min}$ is as follows.

<table>
<thead>
<tr>
<th>$n_{\min}$</th>
<th>$B$</th>
<th>$n_{\min}$</th>
<th>$B$</th>
<th>$n_{\min}$</th>
<th>$B$</th>
<th>$n_{\min}$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>9</td>
<td>194</td>
<td>17</td>
<td>65</td>
<td>25</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>964</td>
<td>10</td>
<td>130</td>
<td>18</td>
<td>34</td>
<td>26</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>804</td>
<td>11</td>
<td>178</td>
<td>19</td>
<td>57</td>
<td>27</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>413</td>
<td>12</td>
<td>81</td>
<td>20</td>
<td>26</td>
<td>28</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>468</td>
<td>13</td>
<td>137</td>
<td>21</td>
<td>16</td>
<td>29</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>187</td>
<td>14</td>
<td>63</td>
<td>22</td>
<td>11</td>
<td>30</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>408</td>
<td>15</td>
<td>63</td>
<td>23</td>
<td>23</td>
<td>31</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>212</td>
<td>16</td>
<td>48</td>
<td>24</td>
<td>7</td>
<td>32</td>
<td>1</td>
</tr>
</tbody>
</table>

The unique blowup with $n_{\min} = 32$ has $V = 245$ and $n = (32, 41, 71, 102)$. The unique sporadic simplex of maximum volume $V = 419$ produces two blowups with terminal singularities, with weight vectors

$(20, 57, 133, 210)$ and $(21, 60, 140, 199)$.  

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Theorem 1.4 now simply summarises Propositions 4.7, 4.9 and 4.11.

References


