Abelian surfaces in toric 4-folds

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There are embeddings of complex abelian surfaces in $\mathbb{P}^4$ but it was shown by Van de Ven in [17] that no abelian d-fold can be embedded in $\mathbb{P}^{2d}$ if $d \geq 3$. Hulek [9], Lange [13], Birkenhake [4] and Bauer and Szemberg [3] have all investigated the possibility of replacing $\mathbb{P}^{2d}$ with a product of projective spaces. Furthermore, Lange [14] studies the rank 2 bundle on $\mathbb{P}^1 \times \mathbb{P}^3$ that arises from the abelian surfaces in $\mathbb{P}^1 \times \mathbb{P}^3$ by Serre's construction. The analogous bundle associated with the abelian surfaces in $\mathbb{P}^4$ is, of course, the Horrocks-Mumford bundle.

In this paper we shall work over the complex numbers and consider embeddings of abelian surfaces in slightly more general ambient spaces of dimension 4, namely smooth toric varieties. The most tractable and probably the most interesting cases are when the ambient variety has small Picard number. We shall therefore consider the following question. Suppose X is a smooth complete toric variety of dimension 4 and $\rho(X) \leq 2$. Does there exist an abelian surface $A \subseteq X$ and, if so, can we describe the embedding?

In the first section we shall give some numerical conditions that such an embedding must satisfy and show that for many X of this type there can be no totally nondegenerate (see Definition 1.1, below) abelian surfaces in X. In Section 2, which is based on unpublished joint work with Professor T. Oda, we show how to describe morphisms into smooth toric varieties in a particularly simple way. The results of this section overlap with work of Cox [6], Guest [8] and Kajiwara [11] but it is useful to us to have them in the form given here. We apply this description in Section 3, in which we exhibit a new 2-dimensional family of abelian surfaces embedded in a particular toric 4-fold X of Picard number 2.

The normal bundles of the surfaces described in Section 3 give rise to rank 2 bundles on X which should be interesting to study. However, we do not attempt this here, but in Section 4 we make a few comments on this and other related matters.

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1. Numerical conditions.

If X is a smooth toric 4-fold and $\rho(X) = 1$ then $X \cong \mathbb{P}^4$ and the only possibility is that A is a Horrocks-Mumford surface. So we consider the case $\rho(X) = 2$.

Definition 1.1. *An abelian surface $A \subseteq X$, X a toric 4-fold, is totally nondegenerate if $A \cap X_\sigma$ is of dimension 1 for every torus-invariant divisor $X_\sigma \subseteq X$.*

We shall be interested only in totally nondegenerate embeddings. An example of an embedding that fails to be totally nondegenerate may be obtained by taking a Horrocks-Mumford surface $A \subseteq \mathbb{P}^4$ and taking X to be the blow-up of $\mathbb{P}^4$ in a point outside A.

Smooth toric 4-folds of Picard number 2 are well understood. In fact, smooth complete toric varieties of Picard number 2 in any dimension have been classified by Kleinschmidt [12]. Such a toric variety is a projectivisation of a decomposible vector bundle over a projective space of smaller dimension. So in our case X is a $\mathbb{P}^3$-bundle over $\mathbb{P}^1$, a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$, or a $\mathbb{P}^1$-bundle over $\mathbb{P}^3$.

Theorem 1.2. *If X is the projectivisation of a decomposible $\mathbb{P}^3$-bundle over $\mathbb{P}^1$ then X contains no totally nondegenerate abelian surfaces unless $X = \mathbb{P}^1 \times \mathbb{P}^3$.*

The proof will be given as part of the analysis below. First, we want a toric description of X (this is a convenient way to do the calculations). We can write

$$X = \mathbb{P}\big(\mathcal{O} \oplus \mathcal{O}(\kappa_1) \oplus \mathcal{O}(\kappa_2) \oplus \mathcal{O}(\kappa_3)\big)$$

and without loss of generality we may suppose that $\kappa_1 \geq \kappa_2 \geq \kappa_3 \geq 0$. We put $\kappa = \kappa_1 + \kappa_2 + \kappa_3$. Then $X = X_\Sigma$, where Σ is the fan in $\mathbb{R}^4$ whose 1-skeleton consists of $\sigma_1 = (1,0,0,0)$, $\sigma_2 = (0,1,0,0)$ $\sigma_3 = (0,0,1,0)$, $\sigma_4 = (-1,-1,-1,0)$ (these four form a primitive collection in the sense of Batyrev [1]), $\tau_1 = (0,0,0,1)$ and $\tau_2 = (\kappa_1, \kappa_2, \kappa_3, -1)$, and whose top-dimensional cones are spanned by a τ and three of the σs.

Let $D_i = \overline{\operatorname{orb}\sigma_i}$ and $E_j = \overline{\operatorname{orb}\tau_j}$. Then $\operatorname{Pic} X = \mathbb{Z}\mathbf{a} \oplus \mathbb{Z}\mathbf{b}$, where $\mathbf{a} = [E_1]$ and $\mathbf{b} = [D_4]$. Note that $\mathbf{a} = [E_1] = [E_2]$ is the class of a fibre of the projection $p : X \to \mathbb{P}^1$ and that the restriction of $\mathbf{b}$ to a fibre is $\mathcal{O}_{\mathbb{P}^3}(1)$. Also $[D_i] - [D_4] = -\kappa_i[E_2]$ so $[D_i] = \mathbf{b} - \kappa_i\mathbf{a}$ for $i = 1,\ 2,\ 3$. The intersection numbers are $\mathbf{a}^4 = \mathbf{a}^3\mathbf{b} = \mathbf{a}^2\mathbf{b}^2 = 0$ (in fact $\mathbf{a}^2 = 0$ in $H^4(X;\mathbb{Z})$ since $\mathbf{a}$ is the class of a fibre), $\mathbf{a}\mathbf{b}^3 = 1$ and $\mathbf{b}^4 = \kappa$.

Now suppose $A \subseteq X$ is an abelian surface. The class of A in $H^4(X;\mathbb{Z})$ (or in $A^2(X)$, which is the same thing in this case) is $\lambda\mathbf{a}\mathbf{b} + \mu\mathbf{b}^2$ for some $\lambda, \mu \in \mathbb{Z}$. By Proposition 3 of [17], which is a version of the self-intersection formula in [7]

$$c_2(\mathcal{N}_{A/X}) \cdot [A] = [A]^2.$$

We have $c(\mathcal{N}_{A/X}) = c(X)/c(A)$ and $c(A) = 1$ so $c_2(\mathcal{N}_{A/X}) = c_2(X)$. The total Chern class of a smooth complete toric variety is well known ([16], Theorem 3.12). Here

$$c(X) = \prod_{i=1}^{4} \Big(1 + [D_i]\Big) \prod_{j=1}^{2} \Big(1 + [E_j]\Big)$$

and in degree 2

$$\begin{aligned} c_2(X) &= \sum_{i<j} [D_i][D_j] + \sum_{i,j} [D_i][E_j] \\ &= \sum_{i<j} (\mathbf{b} - \kappa_i \mathbf{a})(\mathbf{b} - \kappa_j \mathbf{a}) + 8\mathbf{ab} \\ &= (8 - 3\kappa)\mathbf{ab} + 6\mathbf{b}^2. \end{aligned}$$

So, by the self-intersection formula,

$$\big((8 - 3\kappa)\mathbf{ab} + 6\mathbf{b}^2\big)\big(\lambda\mathbf{ab} + \mu\mathbf{b}^2\big) = \big(\lambda\mathbf{ab} + \mu\mathbf{b}^2\big)^2$$

which simplifies to

$$\lambda(2\mu - 6) = \kappa(3\mu - \mu^2) + 8\mu. \qquad (\dagger)$$

Next, note that $\mu = [A]\mathbf{ab}$ which is the degree of the space curve obtained by intersecting A with a general fibre of $p : X \to \mathbb{P}^1$. As this curve is contained in an abelian variety it cannot be rational, so $\mu \geq 3$. Put $\nu = \mu - 3$. From ($\dagger$) we know that $\nu \neq 0$ and

$$\begin{aligned} 2\lambda\nu &= \kappa\big(3\nu + 9 - (\nu + 3)^2\big) + 8\nu + 24 \\ &= -\kappa(\nu^2 + 3\nu) + 8\nu + 24 \end{aligned}$$

so $2\nu | -\kappa(\nu^2 + 3\nu) + 24$.

Put $\nu = 2^r\nu'$ with ν' odd. Then

$$2^{r+1}\nu' | -\kappa(2^{2r}\nu'^2 + 3 \cdot 2^r\nu') + 24$$

so $\nu'|24$ so $\nu' = 1$ or $\nu' = 3$. Moreover, if $\nu' = 1$ we have

$$2^{r+1} | -\kappa(2^{2r} + 3 \cdot 2^r) + 24$$

so $r \leq 2$ or $2^{r-2}| - 3 \cdot 2^{r-3}\kappa + 3$ and so $r = 3$. Thus if $\nu' = 1$ then $\nu = 1$, 2, 4 or 8 and $\mu = 4$, 5, 7 or 11. Similarly, if $\nu' = 3$, then

$$2^{r+1} | -\kappa(9 \cdot 2^{2r} + 9 \cdot 2^r) + 24$$

so $r \leq 2$ or $2^{r-2} | -9 \cdot 2^{r-3}\kappa + 3$ and again $r = 3$. So if $\nu' = 3$ then $\nu = 3$, 6, 12 or 24 and $\mu = 6$, 9, 15 or 27.

Consider the curves $B_i = A \cap D_i$ on A. We have

$$p_g(B_i) = \frac{1}{2}[A](\mathbf{b} - \kappa_i \mathbf{a})^2 + 1 = \frac{\lambda + \kappa\mu}{2} - \kappa_i \mu + 1$$

and this must of course be a positive integer. Adding together the inequalities

$$0 \leq \frac{\lambda}{2} + \big(\frac{\kappa}{2} - \kappa_i\big)\mu \qquad (*_i)$$

for $i = 1$, 2, 3 and using (†),

$$0 \leq 3\lambda + \kappa\mu = \frac{12\mu}{\mu - 3} - \frac{\kappa\mu}{2}.$$

If equality holds here then $0 = \frac{\lambda}{2} + \big(\frac{\kappa}{2} - \kappa_i\big)\mu$ for all i, so $\kappa_1 = \kappa_2 = \kappa_3$ and $3|\kappa$. So

$$\kappa \leq \frac{24}{\mu - 3} = \frac{24}{\nu}$$

with equality only if $3|\kappa$.

For $\mu = 27$ this implies $\kappa = 0$ and then $X = \mathbb{P}^3 \times \mathbb{P}^1$ which is treated in [9]. In any case it does not occur as it gives $p_g(B_4) = \frac{9}{4}$. If $\mu = 15$ then $\kappa = 0$ or 1 and in both cases $p_g(B_4)$ fails to be an integer. Similarly if $\mu = 9$ we have $\kappa \leq 3$ but $p_g(B_4) = 3 + \frac{9}{4}\kappa$ so $4|\kappa$, so $\kappa = 0$ and $X = \mathbb{P}^3 \times \mathbb{P}^1$ (and according to [9] this case does not occur either). We shall see shortly that $\mu = 11$, 7, 5 or 4 is impossible for a different reason, so we are left with $\mu = 6$.

If $\mu = 6$ then $\kappa < 8$ and $p_g(B_4) = 4 + \frac{3}{2}\kappa$ so $\kappa = 0$, 2, 4 or 6. The case $\kappa = 0$ is covered by [9] and [13] (and this case really does occur). If $\kappa = 4$ then $\kappa \geq 2$ and then $\lambda = 8 - 3\kappa = -4$ so $*_1$ fails. Similarly if $\kappa = 6$ then $\lambda = -10$ so $*_1$ fails unless $\kappa_1 \leq 2$, which implies $\kappa_1 = \kappa_2 = \kappa_3 = 2$. If $\kappa = \kappa_1 = 2$ then $*_1$ fails so the remaining case is $\kappa = 2$, $\kappa_1 = \kappa_2 = 1$, $\kappa_3 = 0$.

However, neither of these cases is possible, because in either case $h^0(\mathcal{O}_A(B_1)) = 1$ since $B_1^2 = 2$, so $|B_1|$ is a point and therefore $B_1 = B_2$. But then A is contained in the closure of a smaller torus, namely $\{(t_1, t_1, t_3, t_4)\} \cong (\mathbb{C}^*)^3$, and no abelian surface can be embedded in a smooth toric 3-fold.

It remains to eliminate the possibilities $\mu = 4, 5, 7, 11$. By a standard theorem ([15], Section 3.3, or [4]) there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & A & \longrightarrow & C' & \longrightarrow & 0 \\ & & & & & \searrow^{p} & \downarrow & & \\ & & & & & & \mathbb{P}^1 & & \end{array}$$

where the top row is an exact sequence of abelian varieties, so C and C' are elliptic curves. A general fibre F of $p : A \to \mathbb{P}^1$ is therefore a disjoint union of d translates of C, where d is the degree of $C' \to \mathbb{P}^1$ (and therefore $d \geq 2$). Now $\mu = [A]\mathbf{ab} = (F.B_4)_A = d(C.B_4)_A$, and since $C.B_4$ is also the degree of $C \subseteq \mathbb{P}^3$ we have $C.B_4 \geq 3$. This shows that μ cannot be equal to 4, 5, 7 or 11 and completes the proof of the theorem. ■

Next we consider the case where X is a $\mathbb{P}^1$-bundle over $\mathbb{P}^3$, which is easy.

Theorem 1.3. *If X is a $\mathbb{P}^1$-bundle over $\mathbb{P}^3$ then X contains no totally nondegenerate abelian surfaces unless $X = \mathbb{P}^1 \times \mathbb{P}^3$.*

Proof: Suppose $X = \mathbb{P}\big(\mathcal{O} \oplus \mathcal{O}(\kappa)\big)$, $\kappa > 0$. Let $p : X \to \mathbb{P}^3$ be the projection and put $\mathbf{a} = [p^*\mathcal{O}_{\mathbb{P}^3}(1)]$. Take coordinates (x, y) on $\mathcal{O} \oplus \mathcal{O}(\kappa)$ and put $\mathbf{b} = [(x = 0)]$. Then $\mathbf{b} - \kappa\mathbf{a} = [(y = 0)]$ is also the class of a section, and since $(y = 0)$ is disjoint from $(x = 0)$ we have $\mathbf{b}(\mathbf{b} - \kappa\mathbf{a}) = 0$ in $H^4(X; \mathbb{Z})$. So $H^4(X; \mathbb{Z})$ is generated by $\mathbf{a}^2$ and $\mathbf{ab}$. Suppose A is an abelian surface in X and that $[A] = \lambda\mathbf{a}^2 + \mu\mathbf{ab}$. Since $\mathbf{a}|_A$ and $(\mathbf{b} - \kappa\mathbf{a})|_A$ are disjoint effective curves on A neither can be ample, but any effective class with positive self-intersection on an abelian surface is ample (see [15]). So $\mathbf{a}^2[A] = \mathbf{a}(\mathbf{b} - \kappa\mathbf{a})[A] = (\mathbf{b} - \kappa\mathbf{a})^2[A] = 0$, which, combined with the intersection numbers $\mathbf{a}^4 = 0$, $\mathbf{a}^3\mathbf{b} = 1$, $\mathbf{a}^2\mathbf{b}^2 = \kappa$, $\mathbf{ab}^3 = \kappa^2$ and $\mathbf{b}^4 = \kappa^3$, gives $\lambda = \mu = 0$ if $\kappa \neq 0$. This is impossible. ■

For the remaining case, when X is a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$, the methods above do not suffice to determine a finite list of possible cases. However, we can give some quite strong necessary conditions. Suppose then that

$$X = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(\kappa_1) \oplus \mathcal{O}_{\mathbb{P}^2}(\kappa_2)\big)$$

with $\kappa_1 \geq \kappa_2 \geq 0$. Put $\kappa = \kappa_1 + \kappa_2$. Then $X = X_\Sigma$, where the 1-skeleton of the fan Σ consists of

$$\begin{array}{lll} \sigma_1 = (1,0,0,0), & \sigma_2 = (0,1,0,0), & \sigma_3 = (-1,-1,0,0), \\ \tau_1 = (0,0,1,0), & \tau_2 = (0,0,0,1), & \tau_3 = (\kappa_1, \kappa_2, -1, -1), \end{array}$$

and the top-dimensional cones are spanned by two σs and two τs. If we put $D_i = \overline{\mathrm{orb}\,\sigma_i}$ and $E_j = \overline{\mathrm{orb}\,\tau_j}$ and $\mathbf{a} = [E_1]$, $\mathbf{b} = [D_3]$, then $[E_1] = [E_2] = [E_3] = \mathbf{a} = p^*\mathcal{O}_{\mathbb{P}^2}(1)$, where $p : X \to \mathbb{P}^2$ is the projection, and $[D_i] = \mathbf{b} - \kappa_i\mathbf{a}$. The intersection numbers are $\mathbf{a}^4 = \mathbf{a}^3\mathbf{b} = 0$, $\mathbf{a}^2\mathbf{b}^2 = 1$, $\mathbf{ab}^3 = \kappa$ and $\mathbf{b}^4 = \kappa^2$.

If $A \subseteq X$ is an abelian surface we can take $[A] = \lambda'\mathbf{a}^2 + \mu'\mathbf{ab} + \nu'\mathbf{b}^2 \in H^4(X; \mathbb{Z})$. The notation is convenient because it is easier to work with $\nu = \nu'$, $\mu = \mu' + \kappa\nu$ and $\lambda = \lambda' + \kappa\mu$

than with λ', μ' and ν' directly. Then $\mathbf{a}^2[A] = \nu$, $\mathbf{ab}[A] = \mu$ and $\mathbf{b}^2[A] = \lambda$, so λ, μ and ν are all non-negative and λ and ν are even. We assume that $\kappa > 0$, since otherwise $X = \mathbb{P}^2 \times \mathbb{P}^2$ and then by [9] we know that A is the product of two plane cubics. We also make the nondegeneracy assumption that $\nu > 0$, that is, that $p : A \to \mathbb{P}^2$ is surjective. Now the Hodge index theorem on A gives

$$\lambda\nu \leq \mu^2 \tag{1}$$

and the self-intersection formula gives

$$(3 - 3\kappa + \kappa_1\kappa_2)\nu + (9 - 2\kappa)\mu + 3\lambda = 2\lambda\nu - 2\kappa\mu\nu + \mu^2 \tag{2}$$

Since $D_i|_A \geq 0$ we also have $(\mathbf{b} - \kappa_1\mathbf{a})^2[A] \geq 0$ and $\mathbf{a}(\mathbf{b} - \kappa_1\mathbf{a})[A] \geq 0$ so

$$\lambda - 2\kappa_1\mu + \kappa_1^2\nu \geq 0 \tag{3}$$

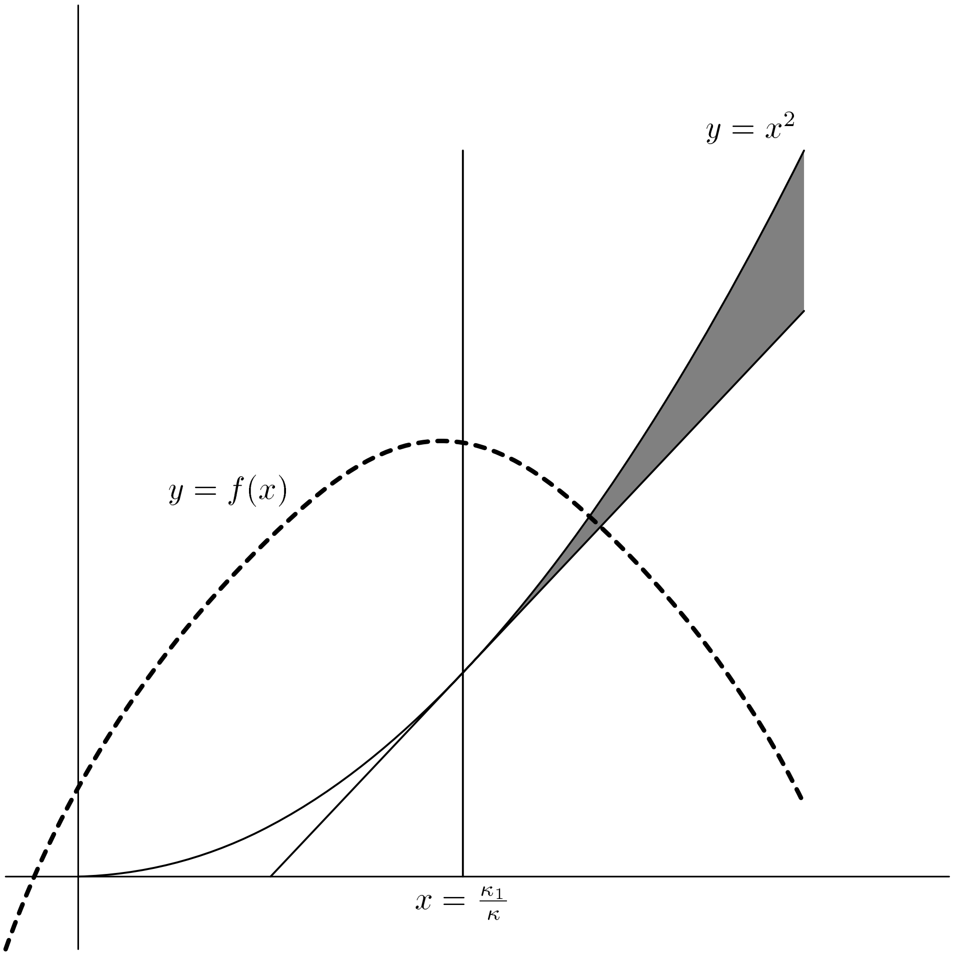
and

$$\mu - \kappa_1\nu \geq 0 \tag{4}$$

We can rewrite all of these in terms of $x = \mu/\kappa\nu$ and $y = \lambda/\kappa^2\nu$:

$$\begin{aligned}
&y \leq x^2 \\
&y = \frac{-\nu}{2\nu - 3}x^2 + \frac{2\nu - 2 + 9/\kappa}{2\nu - 3}x + \frac{3 + \kappa_1\kappa_2 - 3\kappa}{\kappa^2(2\nu - 3)} = f(x) \\
&y \geq \frac{2\kappa_1}{\kappa}x - \frac{\kappa_1^2}{\kappa^2} \\
&x \geq \frac{\kappa_1}{\kappa}
\end{aligned}$$

The three inequalities are satisfied for (x, y) in the shaded area in the diagram.



From this we can deduce the following (tidy but not very sharp) result.

Theorem 1.4. *Suppose* $X = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(\kappa_1) \oplus \mathcal{O}_{\mathbb{P}^2}(\kappa_2)\big)$ *with* $\kappa_1 \geq \kappa_2 \geq 0$ *and* $p : X \to \mathbb{P}^2$ *is the projection. If* $\kappa_1 > 2\kappa_2$ *then* X *contains no totally nondegenerate abelian surface* A *for which* $p : A \to \mathbb{P}^2$ *is surjective.*

Proof: Clearly the curve with equation $y = f(x)$ will not pass through the shaded area if we have $f(\kappa_1/\kappa) < \kappa_1^2/\kappa^2$ and $f'(\kappa_1/\kappa) \leq 2\kappa_1/\kappa$. So if these inequalities hold no such abelian surface will exist. Moreover, if such a surface does exist then $\nu \geq 6$ since by Riemann-Roch

$$h^0\big(p^*\mathcal{O}_{\mathbb{P}^2}(1)\big) = \frac{1}{2}\mathbf{a}^2[A] = \frac{1}{2}\nu$$

and $h^0\big(p^*\mathcal{O}_{\mathbb{P}^2}(1)\big) \geq h^0\big(\mathcal{O}_{\mathbb{P}^2}(1)\big) = 3$.

I claim that in fact $f'(\kappa_1/\kappa) \leq 2\kappa_1/\kappa$ unless $\kappa = 1$, when $\kappa_1 = 1 > 2\kappa_2 = 0$: we deal with this possibility below. For if $f'(\kappa_1/\kappa) > 2\kappa_1/\kappa$ then

$$\frac{-2\nu}{2\nu - 3}\,\frac{\kappa_1}{\kappa} + \frac{2\nu - 2 + 9/\kappa}{2\nu - 3} > 2\frac{\kappa_1}{\kappa}$$

so

$$(2\nu - 2)\kappa + 9 > (6\nu - 6)\kappa_1$$
$$\geq (3\nu - 3)\kappa$$

since $\kappa_1 \geq \kappa_2 = \kappa - \kappa_1$. So $9 \geq (\nu - 1)\kappa \geq 10$ unless $\kappa = 1$.

Thus no abelian surface as in the theorem will exist if $f(\kappa_1/\kappa) < \kappa_1^2/\kappa^2$, that is, if

$$-\frac{\nu}{2\nu - 3}\frac{\kappa_1^2}{\kappa^2} + \frac{2\nu - 2 + 9/\kappa}{2\nu - 3}\frac{\kappa_1}{\kappa} + \frac{3 + \kappa_1\kappa_2 - 3\kappa}{\kappa^2(2\nu - 3)} < \frac{\kappa_1^2}{\kappa^2}$$

which simplifies to

$$\kappa_1^2 - \frac{2\nu - 1}{\nu + 1}\kappa_1\kappa_2 - \frac{6}{\nu + 1}\kappa_1 + \frac{3}{\nu + 1}(\kappa_2 - 1) > 0.$$

We first assume $\kappa_2 \geq 1$. Then this will certainly hold if

$$\kappa_1^2 - \left(2 - \frac{3}{\nu + 1}\right)\kappa_1\kappa_2 - \frac{6}{\nu + 1}\kappa_1 > 0,$$

that is, if

$$\kappa_1 > \left(2 - \frac{3}{\nu + 1}\right)\kappa_2 + \frac{6}{\nu + 1}$$

which is true if $\kappa_1 > 2\kappa_2$.

It remains to deal with the possibility that $\kappa_2 = 0$. Then there are no abelian surfaces as long as

$$(\nu + 1)\kappa_1^2 - 6\kappa_1 - 3 > 0$$

and since $\nu \geq 6$ and is even the only possibilities are $\kappa = \kappa_1 = 1$, $\nu = 6$ or $\nu = 8$. If $\nu = 6$ then by (1) and (2)

$$9\lambda = -\mu^2 + 19\mu \geq 54$$

so $\mu = 9$ or $\mu = 10$ and in both cases $\lambda = 10$ and (3) fails. If $\nu = 8$ then an identical argument shows that $\lambda = 10$ and $\mu = 10$ or $\mu = 13$, contradicting (3). ∎

2. Morphisms to smooth toric varieties

This section is based on joint work with Tadao Oda. I am grateful to Professor Oda for allowing me to use these results here.

Let Δ be a finite (but not necessarily complete) smooth fan for a free $\mathbb{Z}$-module $N \cong \mathbb{Z}^r$ of rank r, and denote by X and $T := T_N$ the corresponding toric variety and the algebraic torus. For simplicity we work over $\mathbb{C}$. $M := \mathrm{Hom}(N, \mathbb{Z})$ is the $\mathbb{Z}$-module dual to N with the duality pairing $< \, , \, >: M \times N \to \mathbb{Z}$. As a general reference for the theory of toric varieties, we use [16].

As usual, $\Delta(1)$ denotes the set of one-dimensional cones in Δ. For each $\rho \in \Delta(1)$, we denote by $V(\rho)$ the corresponding irreducible Weil divisor $\overline{\mathrm{orb}\,\rho}$ on X.

Theorem 2.1. *Let Y be a normal algebraic variety over $\mathbb{C}$. Then the set of morphisms $f : Y \to X$ such that $f(Y) \cap T \neq \emptyset$ is in one-to-one correspondence with the set of pairs $(\{D(\rho)\}_{\rho\in\Delta(1)}, \varepsilon)$ consisting of a set $\{D(\rho)\}_{\rho\in\Delta(1)}$ of effective Weil divisors $D(\rho)$ on Y for $\rho \in \Delta(1)$ such that*

$$D(\rho_1) \cap D(\rho_2) \cap \cdots \cap D(\rho_s) = \emptyset \qquad \text{whenever } \rho_1 + \rho_2 + \cdots + \rho_s \notin \Delta$$

and a group homomorphism

$$\varepsilon : M \to H^0\Big(Y \setminus \textstyle\bigcup_{\rho\in\Delta(1)} D(\rho), \mathcal{O}_Y\Big)^\times$$

to the multiplicative group of invertible regular functions on $Y \setminus \bigcup_{\rho\in\Delta(1)} D(\rho)$ such that

$$\mathrm{div}(\varepsilon(m)) = \sum_{\rho\in\Delta(1)} < m, n(\rho) > D(\rho) \qquad \text{for all} \quad m \in M.$$

Proof: Suppose a morphism $f : Y \to X$ with $f(Y) \cap T \neq \emptyset$ is given. For each $\rho \in \Delta(1)$, the pull-back Weil divisor $D(\rho) := f^{-1}(V(\rho))$ is well-defined, since Y is assumed to be normal and X smooth and $f(Y) \not\subset V(\rho)$.

If $\rho_1, \ldots, \rho_s \in \Delta(1)$ satisfy $\rho_1 + \cdots + \rho_s \notin \Delta$, then we obviously have $V(\rho_1) \cap V(\rho_2) \cap \cdots \cap V(\rho_s) = \emptyset$, hence $D(\rho_1) \cap D(\rho_2) \cap \cdots \cap D(\rho_s) = \emptyset$. By assumption, $f^{-1}(T) = Y \setminus \bigcup_{\rho\in\Delta(1)} D(\rho)$ is a nonempty open set of Y, and the restriction of f to it induces a ring homomorphism

$$f^* : \mathbb{C}[M] \to H^0(Y \setminus \textstyle\bigcup_{\rho\in\Delta(1)} D(\rho), \mathcal{O}_Y),$$

where $\mathbb{C}[M] := \bigoplus_{m\in M} \mathbb{C}\mathbf{e}(m)$ is the semigroup ring of M over $\mathbb{C}$ so that $T = \mathrm{Spec}(\mathbb{C}[M])$. The composite $\varepsilon := f^* \circ \mathbf{e}$ with $\mathbf{e} : M \to \mathbb{C}[M]$ obviously satisfies our requirements, since

$$\mathrm{div}(\mathbf{e}(m)) = \sum_{\rho\in\Delta(1)} < m, n(\rho) > V(\rho) \qquad \text{for all} \quad m \in M.$$

Conversely, suppose $(\{D(\rho)\}_{\rho\in\Delta(1)}, \varepsilon)$ satisfying the requirements are given. Put

$$\hat{\sigma} := \{\rho \in \Delta(1) \mid \rho \not\prec \sigma\} \qquad \text{for} \quad \sigma \in \Delta.$$

Then we have

$$\begin{aligned}\mathrm{Spec}(\mathbb{C}[M \cap \sigma^\vee]) &= U_\sigma \\ &= \bigcap_{\rho\in\hat{\sigma}} (X \setminus V(\rho)) \\ &= X \setminus \bigcup_{\rho\in\hat{\sigma}} V(\rho)\end{aligned}.$$

If we denote $Y_\sigma := Y \setminus \bigcup_{\rho\in\hat{\sigma}} D(\rho)$, then we have $Y = \bigcup_{\sigma\in\Delta} Y_\sigma$. Indeed, the right hand side is the complement in Y of $\bigcap_{\sigma\in\Delta}(\bigcup_{\rho\in\hat{\sigma}} D(\rho))$, which is the union of $\bigcap_{\sigma\in\Delta} D(\rho(\sigma))$ for all $\{\rho(\sigma) \in \Delta \mid \rho(\sigma) \in \hat{\sigma}, \forall\sigma \in \Delta\}$, hence is empty by assumption.

For each $\sigma \in \Delta$, $M \cap \sigma^\vee$ is the semigroup consisting of $m \in M$ such that $\mathbf{e}(m)$ is regular on U_σ. By assumption, we thus see that $\varepsilon(M \cap \sigma^\vee)$ consists of regular functions on Y_σ. Hence we get a morphism $f_\sigma : Y_\sigma \to U_\sigma$. Clearly, we can glue $\{f_\sigma\}_{\sigma\in\Delta}$ together to get a morphism $f : Y \to X$. ■

Although we shall not need it in the rest of the paper we mention here a simple consequence of this result and an example.

Corollary 2.2. *Let $y_0 \in Y$ be a point of a normal algebraic variety Y over $\mathbb{C}$. Then the set of morphisms $f : Y \to X$ such that $f(y_0)$ coincides with the identity element $1 \in T$ is in one-to-one correspondence with the set of pairs $(\{D(\rho)\}_{\rho\in\Delta(1)}, \varepsilon)$ satisfying the same conditions as in Theorem 2.1 and such that $\varepsilon(m)$ has value 1 at y_0 for all $m \in M$.* ■

Let us consider the case (first investigated by Guest in [8]) where $Y = \mathbb{P}^1$ is the projective line with $y_0 = \infty$. The morphisms $f : Y \to X$ satisfying $f(\infty) = 1$ are in one-to-one correspondence with the pairs $(\{D(\rho)\}_{\rho\in\Delta(1)}, \varepsilon)$ satisfying the conditions of Corollary 2.2, so that $\varepsilon(m)$ has value 1 at ∞ for all $m \in M$.

In terms of an inhomogeneous coordinate z on $Y = \mathbb{P}^1$, let us identify the effective divisor $D(\rho)$ as usual with a monic polynomial $P_\rho(z) \in \mathbb{C}[z]$ for each $\rho \in \Delta(1)$. Then for each $m \in M$ we have

$$\varepsilon(m) = \prod_{\rho\in\Delta(1)} P_\rho(z)^{<m,n(\rho)>}.$$

Our requirements amount to the following: $P_{\rho_1}, P_{\rho_2}, \ldots, P_{\rho_s}$ have no common factors whenever $\rho_1 + \rho_2 + \cdots + \rho_s \notin \Delta$, and

$$\sum_{\rho \in \Delta(1)} < m, n(\rho) > \deg P_\rho = 0 \qquad \text{for all} \quad m \in M.$$

A strong result describing morphisms into toric varieties is proved by Cox in [6]. The version given here is perhaps simpler to apply but is much more limited in its scope. It has been extended by Kajiwara [11] to certain singular toric varieties X, including all projective toric varieties.

3. An example

In this section we shall show that one possibility not excluded by the results of section 1 does indeed occur.

Theorem 3.1. *There is a 2-dimensional family of abelian surfaces* $A \subseteq X = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)\big)$ *such that* $[A] = -6\mathbf{a}^2 + 2\mathbf{ab} + 6\mathbf{b}^2$.

Before starting to prove this theorem, let us compare this case with the restrictions given in Theorem 1.4. It is really the simplest case not excluded already. We have taken $\kappa_1 = \kappa_2 = 1$, thus complying with 1.4, and $\nu = \nu' = 6$, which is minimal. Then the values $\lambda = 22$, $\mu = 14$ are dictated by the equations (1–4) of Section 1. In fact the inequality (3) is in this case an equality: geometrically this means that the abelian surface A which arises turns out to be isogenous to a product of two elliptic curves.

The strategy for proving Theorem 3.1 is as follows. We first show that there exist abelian surfaces having curves which behave numerically like the intersections of a surface of class $-6\mathbf{a}^2 + 2\mathbf{ab} + 6\mathbf{b}^2$ with toric strata in X. Given such a surface A, we apply Theorem 2.1 to obtain a morphism $\phi : A \to X$. This morphism will depend on the choice of the curves. We show also, again using Theorem 2.1, that such a choice of curves also determines a morphism $\psi : A \to \mathbb{P}^2 \times \mathbb{P}^1$ and that, for a general choice of A and of the curves, ψ is birational onto its image. Furthermore, ϕ factors through ψ as a rational map and is therefore also birational onto its image. We describe the singular locus of $\psi(A)$ and show that, for general A, we can choose things so that $\phi(A)$ has isolated singularities. Then by an application of the double point formula we can deduce that $\phi(A)$ is smooth.

Proposition 3.2. *There exists a 2-dimensional family of abelian surfaces A containing curves E_1 and C such that $E_1^2 = 6$, $C^2 = 0$ and $E_1.C = 4$.*

Proof: Take $A = \mathbb{C}^2/\Lambda$, where Λ is the lattice spanned by the columns $\mathbf{f}_i$ of the period matrix

$$\Pi = \begin{pmatrix} 4\tau_1 & 3\tau_1 & 1 & 0 \\ 3\tau_1 & \tau_3 & 0 & 3 \end{pmatrix}$$

with $\begin{pmatrix} 4\tau_1 & 3\tau_1 \\ 3\tau_1 & \tau_3 \end{pmatrix}$ in the Siegel upper half-space of degree 2. The complex torus A is then an abelian surface equipped with a polarisation H of type $(1,3)$. We take E_1 to be some curve on A giving rise to this polarisation. Additionally, A contains the elliptic curve $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau_1$, embedded by $\gamma : z \mapsto \begin{pmatrix} 4z \\ 3z \end{pmatrix}$: we take C to be this image. Then $\gamma(1) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4\mathbf{f}_3 + 3\mathbf{f}_4$ and $\gamma(\tau_1) = \begin{pmatrix} 4\tau_1 \\ 3\tau_1 \end{pmatrix} = \mathbf{f}_1$, so $E_1.C = \deg_H C = H(\mathbf{f}_1, 4\mathbf{f}_3 + 3\mathbf{f}_4) = 4$ as required. ■

According to Theorem 2.1, we must now specify the curves E_1, E_2, E_3, D_1, D_2, D_3 on A and also specify trivialisations of certain line bundles on A. We choose D_1 to be some element of the linear system $|2C|$. This is of dimension 1, because in $H^2(A,\mathbb{Z}) \cong \bigwedge^2 \operatorname{Hom}(\Lambda,\mathbb{Z})$ we have

$$[C] = \mathbf{f}_1^* \wedge 4\mathbf{f}_3^* + \mathbf{f}_1^* \wedge \mathbf{f}_4^*$$

which is not divisible. Hence $\mathcal{O}_A(C)$ is of type $(0,1)$ and according to [15] it follows that $\dim H^0\big(\mathcal{O}_A(2C)\big) = 2$. A general D_1 in this system is a union of two disjoint elliptic curves, both translates of C. We choose E_1 to be an element of the polarising class. Now choose homogeneous coordinates $(x_1 : x_2)$ in $|D_1| \cong \mathbb{P}^1$ and $(y_1 : y_2 : y_3)$ in $|E_1| \cong \mathbb{P}^3$ such that $D_1 = (1:0)$ and $E_1 = (1:0:0)$ and put $D_2 = (0:1) \in |D_1|$, $E_2 = (0:1:0), E_3 = (0:0:1) \in |E_1|$.

Theorem 3.3. *The complete linear systems $|D_1|$ and $|E_1|$ determine a morphism $\psi = (\phi_{|E_1|}, \phi_{|D_1|}) : A \to \mathbb{P}^2 \times \mathbb{P}^1$ which for general τ_1, τ_3 is birational onto its image.*

Proof: The only nontrivial assertion is the last one. We shall show, laboriously, that $\phi_{|E_1|}$ is birational on C and hence on every general translate of C. Given this, ψ must either be itself birational or be 2-to-1, identifying the two components of a general fibre of $\phi_{|D_1|}$. In that case we consider the corresponding birational involution $\iota : A \to A$, which is biregular because A is minimal. It preserves the fibres of $\phi_{|D_1|}$ and in particular it preserves the four double fibres $C_1, \ldots, C_4$ which correspond to the branch points of $C' \to \mathbb{P}^1$. We know that ι is not the Kummer involution of A (with some choice of origin) because in that case the ± 1-eigenspaces of ι in $H^0(\mathcal{O}_A(E_1))$ both have positive dimension, so $\phi_{|E_1|}$ does not factor through ι. We also know that ι is not translation by a 2-torsion point of A, because in order to preserve the fibres it would have to be a 2-torsion point of C and then ι would preserve every translate of C instead of interchanging different components of the general fibre. So ι has fixed points, and they all lie on the double fibres.

If these fixed points are isolated then ι is after all conjugate to the Kummer involution. The alternative is that ι fixes each C_i pointwise. If we assume, as we may do, that $\mathcal{O}_A(E_1)$ is a symmetric line bundle then $\phi_{|E_1|}$ becomes equivariant for the action of the extended Heisenberg group $H(3)^e$, as described in [5], and in particular the ramification curve $R \subseteq \mathbb{P}^2$ is $H(3)^e$-invariant. But R certainly includes the image of the branch locus of ψ which in this case is the image of $\sum C_i$. Each of these curves is of degree 4, so $R = R' + \sum \phi_{|E_1|}(C_i)$, and $\deg R = 18$ so R' must be of degree 2. It is the only reduced degree 2 component of R, so it must be $H(3)^e$-invariant. But it is easy to see, using the generators of $H(3)^e$ given in [5] that no such conic exists. (See the remark below for an alternative argument.)

It remains to show that for general τ_1, τ_3, the map $\phi_{|E_1|}|_C : C \to \mathbb{P}^2$ is birational.

The linear system $|\mathcal{O}_C(E_1)|$ embeds C as the intersection of two quadrics in $\mathbb{P}^3 = \mathbb{P}H^0\big(\mathcal{O}_C(E_1)\big)^*$. The restriction map $\mathcal{O}_A(E_1) \to \mathcal{O}_C(E_1)$ induces

$$0 \longrightarrow H^0\big(\mathcal{O}_A(E_1 - C)\big) \longrightarrow H^0\big(\mathcal{O}_A(E_1)\big) \longrightarrow H^0\big(\mathcal{O}_C(E_1)\big)$$

and $(E_1 - C)^2 = -2 < 0$ so the right-hand map is injective. So the image of C under $\phi_{|E_1|}$ is the projection of $C \subseteq \mathbb{P}^3$ to some $\mathbb{P}^2$ which is determined by the 3-dimensional subspace $H^0\big(\mathcal{O}_A(E_1)\big)$.

Projection from a point $P \in \mathbb{P}^3$ will map C onto a double conic if and only if P is the vertex of a quadric cone containing C. Since C is the intersection of two quadrics, $h^0\big(\mathcal{I}_{C/\mathbb{P}^3}(2)\big) = 2$, so there are only finitely many (actually four) quadric cones containing C and therefore only finitely many projections that fail to be birational on C. Fixing τ_1 and letting τ_3 vary we get a family of projections: if we can show that this family is nonconstant (for some choice of τ_1) we shall have finished.

$H^0\big(\mathcal{O}_A(E_1)\big)$ is spanned by the classical theta functions

$$\theta\begin{bmatrix} 0 & \frac{j}{3} \\ 0 & 0 \end{bmatrix}\left(z_1, z_2, \begin{pmatrix} 4\tau_1 & 3\tau_1 \\ 3\tau_1 & \tau_3 \end{pmatrix}\right) = \sum_{m,n\in\mathbb{Z}} e^{\left\{\pi\sqrt{-1}(m,n+\frac{j}{3})\begin{pmatrix} 4\tau_1 & 3\tau_1 \\ 3\tau_1 & \tau_3 \end{pmatrix}\begin{pmatrix} m \\ n+\frac{j}{3} \end{pmatrix}+2\pi\sqrt{-1}(mz_1+(n+\frac{j}{3})z_2)\right\}}$$

where we have chosen E_1 so that $\mathcal{O}_A(E_1)$ has characteristic zero with respect to the decomposition determined by the period matrix Π. We use [15] as our general reference for this theory.

The restriction of this bundle to C is of characteristic zero with respect to the decomposition $\mathbb{Z}\oplus\mathbb{Z}\tau_1$ since $\gamma(1)$ and $\gamma(\tau_1)$ are in the sublattices $\mathbb{Z}\mathbf{f}_3+\mathbb{Z}\mathbf{f}_4$ and $\mathbb{Z}\mathbf{f}_1+\mathbb{Z}\mathbf{f}_2$ respectively. In particular if we fix τ_1 the bundle $\mathcal{O}_C(E_1)$ does not depend on τ_3. If we restrict these theta functions to C we shall get (non-classical) theta functions determining a 3-dimensional subspace of $H^0\big(\mathcal{O}_C(E_1)\big)$ which we must show really does vary with τ_3.

We denote by $\vartheta_j(z,\tau_1,\tau_3)$ the restriction of $\theta\left[\begin{smallmatrix}0 & \frac{j}{3}\\ 0 & 0\end{smallmatrix}\right]$ to $\tilde{C}=\{(4z,3z)\mid z\in\mathbb{C}\}$.

$$\begin{aligned}\vartheta_j(z,\tau_1,\tau_3) &= \sum_{m,n\in\mathbb{Z}} e^{\pi\sqrt{-1}[(4m^2+6mn+2mj)\tau_1+(n+j/3)^2\tau_3]}e^{2\pi\sqrt{-1}(4m+3n+j)z}\\ &= \sum_{n\in\mathbb{Z}} s^{(3n+j)^2}\sum_{m\in\mathbb{Z}} t^{4m^2+6mn+2mj}e^{2\pi\sqrt{-1}(4m+3n+j)z}\end{aligned}$$

where we have set $s=e^{\pi\sqrt{-1}\tau_3/3}$ and $t=e^{\pi\sqrt{-1}\tau_1}$.

We now need some coordinates in $\mathbb{P}H^0\big(\mathcal{O}_C(E_1)\big)^*$. This we do by selecting four arbitrary fixed points z_0, z_1, z_2, z_3 on C and taking the evaluation maps at those points as a basis. We cannot take z_i to be the 2-torsion points, however, as that does not give a basis, since the 2-torsion points are coplanar in $\mathbb{P}^3$ in this embedding. Instead we pick $z_0=0$, $z_1=1/2$, $z_2=\tau_1/2$ and $z_3=1/3$ in $\tilde{C}$. Then we consider the matrix $\Theta=\Big(\vartheta_j(z_i)\Big)$, $0\le j\le 2$, $0\le i\le 3$, and its four 3×3 minors $\hat{\Theta}_k=\det\Big(\big(\vartheta_j(z_i)\big)_{i\neq k}\Big)$. The point $(\hat{\Theta}_0:\hat{\Theta}_1:\hat{\Theta}_2:\hat{\Theta}_3)\in\mathbb{P}^3$ is the point of $\mathbb{P}\big(\bigwedge^3 H^0\big(\mathcal{O}_C(E_1)\big)\big)=\mathbb{P}H^0\big(\mathcal{O}_C(E_1)\big)^*$ which is the vertex of the projection induced by $\mathcal{O}_A(E_1)\to\mathcal{O}_C(E_1)$.

Now one calculates directly, writing out the first few terms of each $\vartheta_j(z_i)$ as a power series in s, whose coefficients are Laurent series (with bounded negative degree) in t. From this one can calculate

$$\hat{\Theta}_k=s^2g_{k2}(t)+s^5g_{k5}(t)+O(s^8)$$

and then the point will depend on t unless (inter alia) $g_{02}g_{15}-g_{12}g_{05}\equiv 0$. But this can be calculated from the Laurent expansions of $g_{kl}(t)$. I did this using MAPLE (it is not beyond the capacity of a determined human) and found that this expression has the constant term 36. As this is not zero, we are done. ■

Remark. In fact a general abelian surface A in this family has no order 2 automorphisms apart from -1, because the family corresponds to an Humbert surface of discriminant 16 in the moduli space of $(1,3)$-polarised abelian surfaces. The abelian surfaces that do have extra automorphisms of order 2 are the product surfaces and the bielliptic abelian surfaces, and those are parametrised by Humbert surfaces of discriminants 1 and 4 respectively, as is shown in [10]. Since it is easy to see that ψ has degree at most 2 we could use this fact to replace the argument above if we knew that ι could not be translation by a 2-torsion point in C.

Proposition 3.4. *Given A, E_1 and D_1 and homogeneous coordinates $(x_1 : x_2)$ in $|D_1|$ and $(y_1 : y_2 : y_3)$ in $|E_1|$, we can specify a morphism $\phi : A \to X$ by choosing a curve $D_3 \in |E_1 + D_1|$ and a trivialisation $\mathcal{O} \xrightarrow{\sim} \mathcal{O}(E_1 + D_1 - D_3)$. There is a rational map $\pi : X \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^1$ such that $\pi\phi = \psi$, and in particular $\pi|_{\phi(A)}$ is a morphism.*

Proof: According to Theorem 2.1 we need to specify D_3 as in the statement of the proposition and also a homomorphism

$$\varepsilon : M \longrightarrow H^0\big(A \setminus \big(\bigcup D_i \cup \bigcup E_j\big), \mathcal{O}_A\big)^\times \subseteq K(A)^\times.$$

Obviously it is enough to specify ε on a basis of $M \cong \mathbb{Z}^4$. In X we have $D_1 - D_2 = \operatorname{div}\big(\mathbf{e}(1,-1,0,0)\big)$, $E_1 - E_2 = \operatorname{div}\big(\mathbf{e}(0,0,1,-1)\big)$ and $E_1 - E_3 = \operatorname{div}\big(\mathbf{e}(0,0,1,0)\big)$, so we should define ε on the space $m_1 + m_2 = 0$ spanned by these three by putting $\varepsilon(1,-1,0,0) = x_1/x_2$, $\varepsilon(0,0,1,-1) = y_1/y_2$ and $\varepsilon(0,0,1,0) = y_1/y_3$. We can think of these as functions on A by composing with ϕ_{D_1} or ϕ_{E_1}. The trivialisation $\mathcal{O} \xrightarrow{\sim} \mathcal{O}(E_1 + D_1 - D_3)$ then determines $\varepsilon(1,0,0,0)$, since $\operatorname{div}\big(\mathbf{e}(1,0,0,0)\big) = E_1 + D_1 - D_3$ in X.

The rational map π is given by the projection $\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathcal{O}(1) \oplus \mathcal{O}(1)$, which is evidently equal to $\psi\phi^{-1}$ on $\phi(A)$. The data that determine ϕ include data that determine ψ, namely E_1, E_2, E_3, D_1, D_2 and $\varepsilon|_{\{m_1+m_2=0\}}$, so $\pi|_{\phi(A)}$ is well-defined and therefore a morphism. ∎

Corollary 3.5. *$\phi : A \to X$ is birational onto its image.* ∎

Next we collect some information about the singularities of $\bar{A} := \psi(A) \subseteq \mathbb{P}^2 \times \mathbb{P}^1$, for a general choice of ψ. We do not need all of this information but it also clarifies the geometric picture.

The generic $D_\eta \in |D_1|$ is a union of two smooth curves of genus 1 in A, say $D_\eta = D^+ \coprod D^-$. The linear system E_1 has degree 4 on each of these, so the fibre $\bar{A} \cap \operatorname{pr}_1^{-1}(\eta)$ consists, for generic $\eta \in \mathbb{P}^1$, of two plane quartic curves $\bar{D}^+$, $\bar{D}^-$ with $p_g = 1$. The curve $\bar{D}_\eta = \bar{D}^+ \cup \bar{D}^-$ has only ordinary double point singularities: there are 20 of these, of which 16 are the points of $\bar{D}^+ \cap \bar{D}^-$ and 4 are singularities of $\bar{D}^\pm$ (two on each curve). At all of these points, $\bar{A}$ also has a (non-isolated) singularity. Taking the closure we get a curve $\bar{\Gamma}_{\text{int}} \cup \bar{\Gamma}_{\text{node}} \subseteq \operatorname{Sing}\bar{A}$, where $\bar{\Gamma}_{\text{int}}$ corresponds to the 16 intersection points and $\bar{\Gamma}_{\text{node}}$ to the 4 other nodes. Take $\bar{\Gamma}$ to be the union of all dimension 1 components of $\operatorname{Sing}\bar{A}$. In fact $\bar{\Gamma} = \bar{\Gamma}_{\text{int}} \cup \bar{\Gamma}_{\text{node}}$ but we shall not need this fact.

As a scheme Sing $\bar{A}$ consists of $\bar{\Gamma}$ and perhaps some points (possibly infinitely near to one another, possibly infinitely near to points of $\bar{\Gamma}$). We shall see shortly that such points may in practice be ignored. Denote by Γ_i the reduced curve in A whose image in $\bar{A}$ is an irreducible component $\bar{\Gamma}_i$ of $\bar{\Gamma}$. The map $\psi : A \to \bar{A}$ fails to be an embedding along Γ_i; in fact it maps Γ_i 2-to-1 onto $\bar{\Gamma}_i$. We need to check that the additional information carried by ϕ is sufficient to separate a general pair of points of this kind, in other words, that $\phi|_{\Gamma_i}$ is birational. Then we shall have to deal with the 0-dimensional part of the singular locus that remains, but it will turn out that this is empty.

For all of this the essential observation is the following.

Proposition 3.6. *For a general A with period matrix as above, the line bundle $\mathcal{O}_A(E_1 + D_1)$ is very ample.*

Proof: We have $[E_1+D_1] = \mathbf{a}$ and calculating intersection numbers on A gives $(E_1+D_1)^2 = \lambda' = 22$; so $E_1 + D_1$ determines a polarisation of type $(1, 11)$. According to Reider's theorem, in the form of [15], 10.4.1, such a polarisation is very ample unless either $(A, \mathbf{a})$ is a product of elliptic curves with a product polarisation or A contains an elliptic curve J such that $J.(E_1 + D_1) = 2$.

Suppose first that a general A is a product. Then there are elliptic curves $J, J' \subseteq A$ such that $A \cong J \times J'$ and $\mathbf{a} = [\mathcal{L} \boxtimes \mathcal{L}']$, where $\mathcal{L}$ and $\mathcal{L}'$ are line bundles on J and J' of degrees 1 and 11 respectively. We have the intersection numbers $J.\mathbf{a} = 1$, $J'.\mathbf{a} = 11$, $J.J' = 1$. Since we are considering a general A in the surface in the moduli space given by the condition of 3.2, we may assume that $\rho(A) = 2$, so that $\mathrm{NS}(A) \otimes \mathbb{Q} = \mathbb{Q}\mathbf{a} \oplus \mathbb{Q}\mathbf{b}$. Suppose $[J] = \xi\mathbf{a} + \zeta\mathbf{b}$: then

$$0 = (\xi\mathbf{a} + \zeta\mathbf{b})_A^2 = 2(\xi + \zeta)(11\xi + 3\zeta),$$

so $[J] = \xi(\mathbf{a} - \mathbf{b})$ or $[J] = \xi(11\mathbf{a} - 3\mathbf{b})$; similarly $[J'] = \xi'(\mathbf{a} - \mathbf{b})$ or $[J'] = \xi'(11\mathbf{a} - 3\mathbf{b})$, with $\xi, \xi' \in \mathbb{Q}$. If $[J] = \xi(\mathbf{a} - \mathbf{b})$ then $1 = J.\mathbf{a} = \xi(\mathbf{a} - \mathbf{b}).\mathbf{a} = 8\xi$, so $\xi = 1/8$; $1 = J.J' = \xi\xi'(\mathbf{a} - \mathbf{b})(3\mathbf{a} - 11\mathbf{b}) = 49\xi'/2$ (we cannot have $[J'] = \xi'(\mathbf{a} - \mathbf{b})$ in this case as then $J.J' = 0$); and finally $11 = J'.\mathbf{a} = \xi'(3\mathbf{a} - 11\mathbf{b}).\mathbf{a} = \frac{2}{49}(3\mathbf{a}^2 - 11\mathbf{ab}) = -\frac{176}{49}$, which is absurd. If $[J] = \xi(11\mathbf{a} - 3\mathbf{b})$ a similar calculation leads to the same result.

Suppose then that a general A contains an elliptic curve J with $J.(E_1 + D_1) = 2$. Since $\mathcal{O}_A(E_1)$ is ample this implies either $J.E_1 = J.D_1 = 1$ or $J.E_1 = 2$, $J.D_1 = 0$. Again we may suppose $\rho(A) = 2$ and $[J] = \xi\mathbf{a} + \zeta\mathbf{b}$; as above, this implies $[J] = \xi(\mathbf{a} - \mathbf{b})$ or $[J] = \xi(11\mathbf{a} - 3\mathbf{b})$. If $J.D_1 = 1$ then $[J] = \xi(11\mathbf{a} - 3\mathbf{b})$ and $1 = 2\xi(11\mathbf{a} - 3\mathbf{b})(\mathbf{a} - \mathbf{b}) = 392\xi$ so $\xi = \frac{1}{392}$. But then $2 = J.E_1 = \frac{1}{392}(3\mathbf{a} - 11\mathbf{b})\mathbf{b} = -\frac{9}{196}$ so this is impossible.

It remains to exclude the possibility that $J.E_1 = 2$. If this happens then $2[J] = [C] \in H^2(X;\mathbb{Z})$. But we saw earlier that

$$[C] = \mathbf{f}_1^* \wedge 4\mathbf{f}_3^* + \mathbf{f}_1^* \wedge \mathbf{f}_4^* \in \bigwedge^2 \mathrm{Hom}(\Lambda, \mathbb{Z}) \cong H^2(A, \mathbb{Z})$$

and this is not divisible. ■

Remark. Again one could argue, less directly, that the surfaces for which a $(1, 11)$-polarisation is not very ample are the product surfaces and the bielliptic abelian surfaces, and that those are parametrised by Humbert surfaces different from the one that occurs here.

Corollary 3.7. *For a general choice of $D_3 \in |E_1 + D_1|$ and a trivialisation of $\mathcal{O}_A(E_1 + D_1 - D_3)$, the image of the associated map $\phi : A \to X$ has only isolated singularities.*

Proof: Choose an irreducible component $\bar{\Gamma}_i$ of $\bar{\Gamma}$ and a point $P \in \bar{\Gamma}_i$. For general P, there are precisely two points P_1, $P_2 \in A$ such that $\psi(P_1) = \psi(P_2)$. By 3.6, the subspace of $H^0(\mathcal{O}_A(E_1+D_1))$ given by the condition $\sigma(P_1) = \sigma(P_2)$ is proper, so there is a non-empty Zariski-open subset $U_P \subseteq H^0(\mathcal{O}_A(E_1 + D_1))$ for which $\sigma(P_1) \neq \sigma(P_2)$. Furthermore, given $\sigma \in H^0(\mathcal{O}_A(E_1 + D_1))$, the set of points of $\bar{\Gamma}_i$ whose two preimages under ψ are separated by σ is Zariski-open. If $\sigma \in U_P$ then this open set is non-empty, and doing this for each component and taking σ to be in the intersection of the U_Ps we can find a σ which separates the preimages of all but finitely many points of $\bar{\Gamma}$. (In principle ψ might kill a tangent direction at a general point of Γ_i instead of identifying two distinct points. If so, the points P_1 and P_2 will be infinitely near but this makes no difference.)

Now take D_3 to be the set $\{\sigma = 0\}$, which we may assume to be reduced and irreducible if we like, and take $\tau \in H^0(\mathcal{O}_A(E_1 + D_1))$ such that $\{\tau = 0\} = D_1 \cup E_1$, so that $\tau \in H^0(\mathcal{O}_A(D_1)) \otimes H^0(\mathcal{O}_A(E_1))$. Then we take the trivialisation of $\mathcal{O}_A(E_1 + D_1 - D_3)$ given by τ/σ. Now we have enough data to separate P_1 and P_2, in other words $\phi|_\Gamma : \Gamma \to X$ is birational onto its image. As ϕ is birational outside Γ, except perhaps at finitely many points, we are done. ■

So the failure of ϕ to be birational can only be caused by its identifying finitely many pairs of points of A, or killing a tangent direction at finitely many points. The set of such pairs, respectively tangent directions, is the set of closed, respectively embedded, points of the double-point scheme $\tilde{D}(\phi)$. So it is enough to show that $\tilde{D}(\phi)$ is empty; then ϕ will be an embedding.

Proposition 3.8. *For general $D_3 \in |E_1 + D_1|$ and trivialisation of $\mathcal{O}_A(E_1 + D_1)$, the double point scheme $\tilde{D}(\phi)$ is empty.*

Proof: If $\tilde{D}(\phi) \neq \emptyset$, then $\operatorname{codim} \tilde{D}(\phi) = 2$ by 3.7, so (see [7], p.166 for the notation and general facts) $\tilde{\mathbb{D}}(\phi) = [\tilde{D}(\phi)]$ and is a nonzero element of $A_0(\tilde{D}(\phi))$, so $\mathbb{D}(\phi) \in A_0(A) \cong \mathbb{Z}$ is also nonzero. So we want to show that in fact $\mathbb{D}(\phi) = 0$. By the double point formula ([7], Theorem 9.3)

$$\begin{aligned}
\mathbb{D}(\phi) &= \phi^*\phi_*[A] - \big(c(\phi^*T_X)c(T_A)^{-1}\big)_2 \cap [A] \\
&= \phi^*\phi_*[A] - c_2(\phi^*T_X) \cap [A] \\
&= \big([\phi(A)] - c_2(T_X)\big) \cap [A] \\
&= 0
\end{aligned}$$

since $\big([\phi(A)] - c_2(T_X)\big).[A] = 0$ in $A_0(X)$, by the choice of the class of $\phi(A)$. ■

This concludes the proof of Theorem 3.1.

4. Further remarks

We can use the abelian surfaces constructed in the previous section to give some rank 2 vector bundles on $X = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)\big)$ via Serre's construction, extending the normal bundle $\mathcal{N}_{A/X}$ to the whole of X. One needs to check that $\det \mathcal{N}$ is the restriction of a line bundle $\mathcal{E}$ on X with $H^1(\mathcal{E}) = H^2(\mathcal{E}) = 0$, but this is immediate as $\mathcal{E} = K_X$. In fact each $A \subseteq X$ produces a rank 2 bundle in this way and there are many questions that might be asked about them. For instance, are they all isomorphic? Are they indecomposible (presumably yes)? Can one calculate their cohomology? Some of these questions are answered in [14] in the case of $X = \mathbb{P}^1 \times \mathbb{P}^3$, where an extension of the normal bundle exists for the same reasons.

Another series of questions raised by this example is the possibility of extending the procedure to other X. The results of Section 1 allow one to generate other possibilities among the smooth toric 4-folds with $\rho = 2$, but the proofs in Section 3 used some special geometry and in particular the fact that the linear system spanned by E_1, E_2 and E_3 is complete. In other cases one would presumably have to work with very far from complete linear systems and the methods of this paper might not be adequate. In any case a more interesting problem might be to revert to $\rho(X) = 1$ but allow singular toric varieties, and try to apply the results of Kajiwara from [11]. The case of weighted projective spaces is a natural starting point. Another possibility would be to work with Batyrev's list [2] of toric Fano 4-folds.

The method used to prove that the morphism $A \to X$ we produce is an embedding is very clumsy. In the case of $X = \mathbb{P}^n$ one has an elegant criterion in the form of Reider's theorem. It would be interesting to have a way of distinguishing the embeddings (or even the birational morphisms) among the morphisms into toric varieties, say in terms of Cox's description in [6].

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