Algebraic construction of normalized coprime factors for delay systems

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January 10, 2001

Abstract

We introduce an algebraic approach to the problem of constructing explicit normalized coprime factorizations for retarded delay systems. A parametrization is given of all the possible factorizations that can be obtained by solving algebraic equations over the field generated by s and e^{-s}. This enables us to provide a means of determining when such factors can be calculated explicitly, and to show that in general they cannot. Some illustrative examples are given.

Keywords: Delay system, normalized coprime factorization, spectral factorization, commutative algebra, Galois theory.

Notation

$\mathbb{C}_+$ denotes the right half-plane $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$.

$H_\infty(C_+)$ or H_∞ denotes the Hardy space of bounded analytic functions on $\mathbb{C}_+$, with $\|f\|_\infty = \sup_{s\in\mathbb{C}_+} |f(s)|$.

$A(\mathbb{C}_+)$ (the half-plane algebra) denotes the subalgebra consisting of all continuous functions on $\overline{\mathbb{C}}_+$ that tend to a unique limit at $\pm i\infty$.

The Laplace transform $\mathcal{L}$ is defined formally by

$$\mathcal{L}g(s) = \int_0^\infty e^{-st} g(t)\, dt \qquad s \in \mathbb{C}_+.$$

The notation $\hat{g}$ is also used for $\mathcal{L}g$.

$W(\mathbb{C}_+)$ denotes the Wiener algebra on $\mathbb{C}_+$, namely the space of functions of the form $a + \mathcal{L}g$, where $a \in \mathbb{C}_+$ and $g \in L_1(0,\infty)$.

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1 Introduction

Let $G(s)$ be a meromorphic function in the right half-plane $\mathbb{C}_+$, and let $\mathcal{A}$ denote one of the algebras $H_\infty(C_+)$, $A(\mathbb{C}_+)$ or $W(\mathbb{C}_+)$ defined above. Often we regard G as the transfer function of a linear time-invariant system, with the input–output relation written $y = Gu$ when $\hat{y}(s) = G(s)\hat{u}(s)$, for $s \in \mathbb{C}_+$. In this paper we shall restrict ourselves entirely to functions G that satisfy $G(\bar{s}) = \overline{G(s)}$, so that in the time-domain real functions u are mapped to real functions y.

We say G has a *coprime factorization* over $\mathcal{A}$ when $G = N/D$ with N, D coprime functions in $\mathcal{A}$, in the sense that we can solve the Bézout identity $XN + YD = 1$ over $\mathcal{A}$. In particular this means that

$$\inf_{s \in \mathbb{C}_+} |N(s)| + |D(s)| > 0.$$

Further, a coprime factorization $G = N'/D'$ is said to be *normalized*, if $|N'(s)|^2 + |D'(s)|^2 = 1$ for $s \in i\mathbb{R}$. It is clear that to normalize a coprime factorization requires us to find a function $F \in \mathcal{A}$ solving the equation

$$|F(s)|^2 = |N(s)|^2 + |D(s)|^2, \qquad s \in i\mathbb{R},$$

after which the factorization $N' = N/F$, $D' = D/F$ will be normalized. The process of obtaining the function F is usually referred to as *spectral factorization*.

Coprime factorizations in general, and normalized coprime factorizations in particular are of great importance in robust and optimal control. See [CZ, GS, MG, ZDG].

For proper rational functions $G(s) = p(s)/q(s)$ with $\deg q \geq \deg p$, and p, q having no common factors, the problem of constructing explicit normalized coprime factorizations is well-understood, and it may be regarded as a very special case of the results we present below (see Example 4.1).

For transfer functions in the more general classes defined above, analytic expressions are frequently available for normalized coprime factors, but they do not in general lead to closed-form expressions.

For example, a construction was given in [MP], which is valid in the algebra $W(\mathbb{C}_+)$. This requires writing

$$\log(|N(s)|^2 + |D(s)|^2) = V(s) + \overline{V(s)}, \qquad s \in i\mathbb{R},$$

for $V(s) \in W(\mathbb{C}_+)$ (this can be done with the aid of the inverse Laplace transform), after which $N \exp(-V)$ and $D \exp(-V)$ are normalized coprime factors. On the other hand, Treil [T] has shown that a similar construction does not exist in $A(\mathbb{C}_+)$. This alerts us to the fact that approximate constructions need a certain level of sophistication: rational approximation in the uniform norm is not enough, since the process of taking spectral factors is discontinuous in this topology. We refer to [JP] for a more systematic analysis of this question.

The class of infinite-dimensional systems with which we shall work is the set of retarded delay systems [BC]. These have transfer functions of the form

$$G(s) = h_2(s)/h_1(s) \tag{1}$$

where

$$h_1(s) = \sum_{j=0}^{n_1} p_j(s)e^{-\beta_j s},$$
$$h_2(s) = \sum_{k=0}^{n_2} q_k(s)e^{-\gamma_k s},$$

and $0 \le \gamma_0 < \gamma_1 \cdots < \gamma_{n_1}$, $0 = \beta_0 < \beta_1 \cdots < \beta_{n_2}$, the p_j being polynomials of degree δ_j and $\delta_j < \delta_0$ for $j \neq 0$ and the q_k being polynomials of degree $d_k < \delta_0$ for each k. The technical conditions above guarantee that there will only be finitely many poles of G in any right half plane. We make the standing assumption that h_1 and h_2 have no common zeroes in $\mathbb{C}_+$.

Kamen, Khargonekar and Tannenbaum [KKT] and more recently Brethe and Loiseau [BL] and Glüsing-Lüerßen [G] considered the existence of coprime factorizations of time-delay systems with commensurate time-delays. In particular, in [BL] there is an algorithm to compute the coprime factorizations for such delay-systems. An explicit formula for the Bézout factors was given in [BP, P], in the case of arbitrary time-delays.

Since these functions all have coprime factorizations over the algebra $W(\mathbb{C}_+)$, the existence of *normalized* coprime factorizations over $W(\mathbb{C}_+)$ is guaranteed. It would be desirable to extend the algebraic approach to the construction of explicit normalized coprime factors, and this is what we shall do in the next section. However, in order to limit the number of algebraically independent variables to two, we shall only analyse the case when the delays are commensurate. We conclude with some examples.

2 Construction of normalized coprime factors

We begin with the observation that the transfer function $G(s) = h_2(s)/h_1(s)$ defined in (1) has many explicit coprime factorizations, such as $N(s) = h_2(s)/p(s)$ and $D(s) = h_1(s)/p(s)$, where $p(s)$ is any polynomial of degree $d = \delta_0$, all of whose zeroes lie in the open left half plane $\mathbb{C}_-$. A convenient choice is $p(s) = (s+1)^d$.

We shall assume that the delays are commensurate, so that each β_j and γ_k is an integer multiple of some $\alpha > 0$. Now we regard $z = e^{-\alpha s}$ as an independent variable. We have functions $N(s,z)$, $D(s,z) \in \mathbb{R}(s)[z]$ and want $|N|^2 + |D|^2 = |F|^2$ on the line $s \in i\mathbb{R}$. Since $\overline{N(s,z)} = N(-s, 1/z)$ when $s = iy$ and $|e^{-s}| = 1$, we try to solve

$$N(s,z)N(-s,1/z) + D(s,z)D(-s,1/z) = F(s,z)F(-s,1/z)$$

for F, given the positive definiteness condition $\inf_{y\in\mathbb{R}} |F(iy)| > 0$.

We define an involution $*$ on $\mathbb{R}(s,z)$ by the relation $f^*(s,z) = f(-s,1/z)$. Algebraically, we therefore wish to solve the equation $F^*F = N^*N + D^*D$. Accordingly we make the change of coordinates

$$t = \frac{1-z}{1+z} = -2\tanh \alpha s.$$

It is helpful to note that $t \in \mathbb{C}_+$ precisely when $s \in \mathbb{C}_+$. Now $\mathbb{R}(s,z) \cong \mathbb{R}(s,t)$ and the involution has the more convenient form $g^*(s,t) = g(-s,-t)$. We write $M = N^*N + D^*D$.

If $m_i = \beta_{n_i}/\alpha$ then m_i is the z-degree of $h_i(s,z) \in \mathbb{R}(s)[z])$ and we have

$$\begin{aligned} N = N(s,t) &= (1+s)^{-d}(1+t)^{-m_1}N_0(s,t) \\ D = D(s,t) &= (1+s)^{-d}(1+t)^{-m_2}D_0(s,t) \end{aligned}$$

with $N_0, D_0 \in \mathbb{R}[s,t]$. Now we define M_0 by

$$\begin{aligned} M &= NN^* + DD^* \\ &= \frac{N_0N_0^*}{(1-s^2)^d(1-t^2)^{m_2}} + \frac{D_0D_0^*}{(1-s^2)^d(1-t^2)^{m_1}} \\ &= (-1)^d\lambda^2(1-s^2)^{-d}(1-t^2)^{-m}M_0(s,t) \end{aligned}$$

where $\lambda \in \mathbb{R}$ is the leading coefficient of $p_0(s)$ and $m = \max\{m_1, m_2\}$. If we do this then $M_0 \in \mathbb{R}[s,t] \subset \mathbb{R}(t)[s]$ is a polynomial in s of degree $2d$, with coefficients in $\mathbb{R}(t)$, satisfying $M_0^* = M_0$. In fact $M_0 = (1-t^2)^n M_1$ with $M_1 \in \mathbb{R}(t)[s]$ a monic polynomial in s, for a suitable n.

The denominator $\lambda^{-2}(1-s^2)^d(1-t^2)^m$ can be factorized as EE^* with $E \in \mathbb{R}[s,t]$ simply by taking $E(s,t) = \lambda^{-1}(1+s)^d(1+t)^m$. This is holomorphic and nonzero in the right half-plane, so we want to factorize M_0 similarly. In this section, we seek to do this by regarding M_0 as a polynomial in s. More precisely, we seek a finite field extension $K : \mathbb{R}(t)$ such that $* : \mathbb{R}(t) \to \mathbb{R}(t)$ extends to a Galois involution $* : K \to K$ over $\mathbb{R}$ and a factorization

$$(-1)^d M_0 = HH^*, \qquad H \in K[s].$$

As a general reference for Galois theory we use [S].

Once we have found such a factorization we have to find out whether the resulting H determines a holomorphic function on $\mathbb{C}_+$, and whether the extension of $*$ still satisfies $\overline{H(s,z)} = H^*(-s,1/z)$ for $|e^{-s}| = 1$.

Theorem 2.1 *Suppose* $K_0 : \mathbb{R}(t)$ *is a finite field extension with an involution* $* : K_0 \to K_0$ *over* $\mathbb{R}$ *extending* $* : \mathbb{R}(t) \to \mathbb{R}(t)$*. If* $P \in K_0[s]$ *is a monic polynomial of even degree* $2d$ *or odd degree* $2d-1$ *and* $P^* = P$*, then there is a finite field extension* $K : K_0$ *and an extension of* $*$ *to* $* : K \to K$ *such that in* $K[s]$ *we can write*

$$(-1)^d P = \prod_i P_i P_i^* \prod_j L_j$$

where P_i *(which is monic) and* P_i^* *have the same splitting field over* K *and* $L_j = \lambda_j - s$*,* $\lambda_j^* = -\lambda_j$*.*

Proof: We proceed by induction on d (which explains why we introduced K_0 and allowed odd degree: in applications we shall have $K_0 = \mathbb{R}(t)$ and $P = M_1$). Over $K_0[s]$ we write P as a product of monic irreducibles, $P = \prod Q_k$. Certainly $Q_k^* = (-1)^{\deg Q_k} Q_{k'}$ for some k', since this expression for P is unique. If the only fixed points of the permutation (of order 2) $\tau : k \mapsto k'$ correspond to linear factors, then we already have a factorization of the type required, though without the condition on the splitting fields yet: if the ith transposition in τ, written as a product of disjoint cycles, is (kk') then we take $P_i = Q_k$, and if the jth fixed point is k then we take $L_j = -Q_k$.

Suppose then that some nonlinear factor is preserved up to sign by $*$, say $Q_1 = \pm Q_1^*$. Then we define a field extension $K_1 : K_0$ by adjoining a zero of Q_1: we put

$$K_1 = K_0[x]/Q_1(x)K_0[x].$$

We extend $*$ to $K_0[x]$ by $* : x \mapsto -x$. Then the principal ideal $Q_1(x)K_0[x]$ is $*$-invariant and consequently $*$ extends to K_1. Since Q_1 has the linear factor $s - [x]$ in K_1 we can reduce to lower degree.

It remains to show that we can also arrange for the condition on the splitting fields. Suppose then that there are no more nonlinear factors preserved up to sign by $*$. Choose (if possible) a nonlinear factor Q_1 and suppose that $\tau(1) = 2$, so $Q_1^* = \pm Q_2$. Suppose that Q_1 and Q_2 have different splitting fields over K_0. As before we put

$$K_1 = K_0[x]/Q_1(x)K_0[x],$$

which need not now have an extension of $*$. Similarly we put

$$K_1' = K_0[x]/Q_2(x)K_0[x],$$

and then

$$K_2 = K_1[x']/Q_2(x')K_1[x'].$$

Generally for a commutative ring R with ideals I, J one has $(R/I)/f_I(J) \cong (R/J)/f_J(I) \cong R/(I+J)$, where $f_I : R \to R/I$ denotes the quotient map, so

$$\begin{aligned} K_1[x']/Q_2(x')K_1[x'] &\cong K_0[x,x']/Q_1(x)K_0[x,x'] + Q_2(x')K_0[x,x'] \\ &\cong K_1'[x]/Q_1(x)K_1'[x]. \end{aligned}$$

The allows us to extend $*$ to an involution on K_2 by putting $* : x \mapsto x'$, since the ideal $Q_1(x)K_0[x,x'] + Q_2(x')K_0[x,x']$ is $*$-invariant.

We claim that K_2 is a field. If not then Q_2 is reducible over K_1 and is therefore reducible over the splitting field K_{Q_1} of Q_1, which contains K_1. But then Q_2 splits completely over K_{Q_1} because K_{Q_1}, being a splitting field, is normal. Similarly Q_1 splits completely over the splitting field K_{Q_2} of Q_2: so $K_{Q_1} = K_{Q_2}$. ■

To apply this result to our present purpose we take $K_0 = \mathbb{R}(t)$ and $P = M_1$ and consider the resulting factorization. (Notice that the proof of the theorem constructs K by an explicit algorithm.)

If the L_j actually occur then the algebraic method has failed: there is no field extension in which $(-1)^d M_0 = HH^*$. This can happen: it is analogous to a real polynomial f of even degree having a pair of distinct real roots, in which case it cannot be written over $\mathbb{C}$ as $f = g\bar{g}$. Indeed one should expect such polynomials to form a set of positive measure in the real vector space of $*$-invariant polynomials of fixed even degree.

If the L_j do not occur then we can take, for instance, $H = \prod P_j$. Of course there is no preferred choice of which factor is labelled P_j and which P_j^* so there will be many ways of doing this: we would select one which gives an H which is holomorphic in the right half-plane, if such a choice exists. There may not be any such choice (see below), in which case the method fails to produce normalized factors.

The involution $*$ that we have constructed on K is not the only possible one extending $\mathbb{R}(t)$. If $\gamma : K \to K$ is an element of the Galois group $\mathrm{Gal}\big(K : \mathbb{R}(t)\big)$ then $\gamma^{-1} \circ * \circ \gamma$ would do as well. However, the correct choice of involution will be determined by the requirement that P_j^* should agree with $\overline{P}_j$ when we set $z = e^{-s}$ and s is purely imaginary.

In practice one expects that M_0 will be irreducible and will have the full symmetric group as its Galois group. In this case most of the procedure outlined above is redundant, because as soon as we adjoin a zero of M_0 it will split completely into linear factors. If this happens we at once know all possible factorizations $(-1)^d M_0 = HH^*$. Only exceptionally will we have to repeat the procedure at all, or be left with nonlinear factors.

The result as stated above is formulated so as to break down P as far as possible: if all we want is some factorization $(-1)^d P = HH^*$ we do not need the second reduction step that gives the condition on the splitting field of the factors.

3 More general extensions

We sacrificed some generality in the last section, in the intersts of easy calculation, by regarding M_0 as a polynomial in s and allowing only field extensions of $\mathbb{R}(t)$. For instance, we could just as well have worked in $\mathbb{R}(s)[t]$. In this section we will allow finite extensions of $\mathbb{R}(s,t)$.

Take M_0 and $M_1 = (1-t^2)^{-n}M_0$ as before, so that $M_0 \in \mathbb{R}[s,t] \subset \mathbb{R}(s,t) = K$ satisfies $M_0^* = M_0$ and $(-1)^d M_1$ is monic in s of degree $2d$. We seek, first of all, a finite extension $\tilde{K} : K$ with an extension of $* : K \to K$ to $* : \tilde{K} \to \tilde{K}$ (so $* \in \mathrm{Gal}(\tilde{K} : \mathbb{R})$, and a factorization $M_0 = HH^*$ with $H \in \tilde{K}$. We put $L = \mathbb{R}(s^2, t^2, st)$, which is the fixed field of the automorphism $*$ of K. Notice that L is the field of fractions of $\mathbb{R}[s^2,t^2,st] = \mathbb{R}[s,t] \cap L$ and that $M_0 \in \mathbb{R}[s^2,t^2,st]$. We can always make $\tilde{K}$ bigger if we wish, so let us assume that $\tilde{K} : L$ is Galois. We let $\tilde{L}$ be the fixed field of $*$ in $\tilde{K}$: then $[\tilde{K} : \tilde{L}] = 2$ so $\tilde{K} = \tilde{L}(s) = \tilde{L}(t)$.

Since we are dealing with the general case, we may as well assume that M_0 is absolutely irreducible; that is, it is irreducible in $\mathbb{C}[s,t]$. Then $K(\sqrt{M_0})$ is a field with an extension of $*$, namely

$$K(\sqrt{M_0}) = K[x]/(x^2 - M_0)$$

and $x^* = x$. We may suppose that $\tilde{K} \supseteq K(\sqrt{M_0})$, and then $\tilde{L} \supseteq L(\sqrt{M_0})$: in particular, $(\sqrt{M_0})^* = \sqrt{M_0^*} = \sqrt{M_0}$.

Theorem 3.1 *Any factorization $M_0 = HH^*$ is of the form $H = X + sY$, $H^* = X - sY$, where $X, Y \in \tilde{L}$ and there is a $Z \in \tilde{L}$ such that*

$$X = \sqrt{M_0}\left(\frac{s^2Z^2+1}{s^2Z^2-1}\right), \qquad Y = 2\sqrt{M_0}\left(\frac{Z}{s^2Z^2-1}\right).$$

Thus

$$H = \sqrt{M_0}\left(\frac{sZ+1}{sZ-1}\right).$$

Proof. The map $H \mapsto HH^* = Q(H)$ is a quadratic form on the vector space $\tilde{K} = \tilde{L}.1 + \tilde{L}.s$ over $\tilde{L}$. If we write $H = X + sY$, with $X, Y \in \tilde{L}$, then $Q(H) = (X+sY)(X-sY) = X^2 - s^2Y^2$. We want to know whether Q represents $M_0 \in L \subset \tilde{L}$ and if so to find all solutions in $\tilde{L}$ to the equation $Q(H) = M_0$. One such solution is immediately to hand: by our choice of $\tilde{K}$, $Q(\sqrt{M_0}) = M_0$. From this we can find all the solutions by a simple and well-known procedure. If we put $X_0 = \sqrt{M_0}$ and $Y_0 = 0$ then the point (X_0, Y_0) lies on the quadric (the hyperbola) $X^2 - s^2Y^2 = M_0$ in the X–Y-plane $\mathbb{A}^2(\tilde{L})$ over $\tilde{L}$. Suppose (X_1, Y_1) is another solution and $X_0 \neq X_1$: then the line joining (X_0, Y_0) and (X_1, Y_1) has slope $(Y_1 - Y_0)/(X_1 - X_0) \in \tilde{L}$.

Let ℓ be a line in $\mathbb{A}^2(\tilde{L})$ of slope $Z \in \tilde{L}$, passing through (X_0, Y_0). The equation of ℓ is $Y = Z(X - X_0)$ and it meets the hyperbola where

$$X^2 - s^2Z^2(X - X_0)^2 = M_0.$$

Since $X_0 = \sqrt{M_0}$ this gives

$$(X - X_0)(X + X_0) - s^2 Z^2 (X - X_0)^2 = 0$$

and since we are looking for solutions with $X \neq X_0$ we obtain

$$X + X_0 - s^2 Z^2 (X - X_0) = 0$$

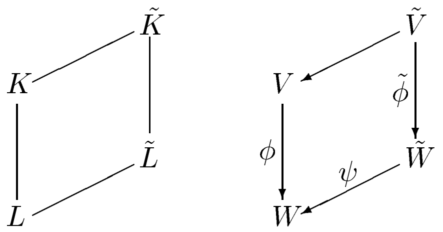
and hence $X = \sqrt{M_0}\left(\frac{s^2Z^2+1}{s^2Z^2-1}\right)$ and $Y = 2\sqrt{M_0}\left(\frac{Z}{s^2Z^2-1}\right)$. If instead $X = X_0$ then we have $s^2Y^2 = 0$ so $Y = 0$ and we do not find any more solutions (this corresponds to case where ℓ is the line $X = X_0$, which is tangent to the hyperbola and therefore does not meet it again).

For our purpose we want to know whether we can choose $\tilde{K}$ in such a way that some choice of $Z \in \tilde{L}$ will yield an H as above which defines a holomorphic function on $\mathbb{C}_+$ if we put $t = -2\tanh \alpha s$.

Theorem 3.2 *For general M_0 no such choice is possible.*

Proof: The idea is this: we consider H^2. It is a multiple of M_0 but not, on the face of it, of M_0^2. So if we take its square root we shall be left with a $\sqrt{M_0}$ and this will not give a holomorphic function in the case that M_0 has simple zeros $s \in \mathbb{C}_+$. To make this argument precise, we use the machinery of discrete valuation rings (DVRs): our reference for this is [AM].

Consider the real affine algebraic surfaces $V = \mathbb{A}^2(\mathbb{R})$ and $W = \{xy = z^2\} \subset \mathbb{A}^3(\mathbb{R})$. Their rings of algebraic functions are $\mathcal{O}_V = \mathbb{R}[s,t]$ and $\mathcal{O}_W = \mathbb{R}[s^2, t^2, st]$ respectively, and their function fields are $k(V) = K$ and $k(W) = L$. Choose integrally closed subrings $\mathcal{O}_{\tilde{V}} \subset \tilde{K}$ and $\mathcal{O}_{\tilde{W}} \subset \tilde{L}$ such that $\tilde{K}$ and $\tilde{L}$ are the fields of fractions of $\mathcal{O}_{\tilde{V}}$ and $\mathcal{O}_{\tilde{W}}$, and put $\tilde{V} = \operatorname{Spec} \mathcal{O}_{\tilde{V}}$, $\tilde{W} = \operatorname{Spec} \mathcal{O}_{\tilde{W}}$. We may assume that the rings are included in one another in the same way as the fields, so that there are diagrams of fields and of normal surfaces



Let $p \in W$ be the generic point over $\mathbb{R}$ of the subvariety $W_0 \subset W$ given by the equation $M_0 = 0$. This exists because M_0 is absolutely irreducible: irreducibility in $\mathbb{R}[s,t]$ is not enough. For the necessary details we refer to [W], especially the Introduction and Chapters I.1 and IV.1. Put $V_0 = \phi^*(W_0) \subseteq V$, $\tilde{W}_0 = \psi^*(W_0) \subseteq \tilde{W}$, and $\tilde{V}_0 = \tilde{\phi}^*\psi^*(W_0) \subseteq \tilde{V}$, and let the generic points be q, $\tilde{p}$ and $\tilde{q}$. (Strictly we should just take an irreducible component in each case, but in the most general case the varieties will all be absolutely irreducible anyway.)

It can even be shown that, for each fixed positive degree in s and t, the set of those M_0 which are absolutely irreducible is dense in the set of all possible M_0 of that degree. However, we shall not discuss this point further.

Since $\sqrt{M_0} \notin K$ (otherwise M_0 would be reducible) ϕ is unbranched along V_0. ψ is branched along V_0 of even order $2d$ ($d = 1$ if $\tilde{L}$ contains no other roots of M_0). Denote by $\mathcal{O}_p$, etc., the local ring at $p \in V$ and by $\hat{\mathcal{O}}_p$ its completion with respect to the maximal ideal $m_p \subset \mathcal{O}_p$. These are DVRs (with valuations v_p, etc.): by the branching conditions, $v_q(a) = v_p(a)$ and $v_{\tilde{q}}(a) = v_{\tilde{p}}(a)$ for $a \in \hat{\mathcal{O}}_p, \hat{\mathcal{O}}_{\tilde{p}}$, but $v_{\tilde{p}}(a) = 2dv_p(a)$ and similarly for q. Let $\hat{k}_p$ be the field of fractions of $\hat{\mathcal{O}}_p$, so that v_p extends to a valuation on $\hat{k}_p^\times$, and similarly for q, $\tilde{p}$ and $\tilde{q}$.

We may regard H^2 as an element of $\hat{k}_{\tilde{q}}$. We claim that it is not the square of any element of $\hat{\mathcal{O}}_p$. We have

$$H^2 = M_0 \left(\frac{sZ+1}{sZ-1} \right)^2,$$

so that

$$v_{\tilde{q}}(H^2) = v_{\tilde{q}}(M_0) + 2v_{\tilde{q}}(sZ+1) - 2v_{\tilde{q}}(sZ-1) = v_{\tilde{q}}(M_0),$$

since $v_{\tilde{q}}(a) = v_{\tilde{q}}(a^*)$ for all $a \in \hat{k}_{\tilde{q}}^\times$, and $(sZ+1)^* = (-sZ+1)$. Hence

$$v_{\tilde{q}}(H^2) = v_{\tilde{q}}(M_0) = 2dv_p(M_0) = 2d.$$

However, if $H \in \hat{\mathcal{O}}_p$ then $v_{\tilde{q}}(H^2) = 2v_{\tilde{p}}(H) = 4dv_p(H)$, which is congruent to zero mod $4d$. Hence $H \notin \hat{\mathcal{O}}_p$.

This proves that there is no global analytic function $H = \sqrt{M_0}\left(\frac{sZ+1}{sZ-1}\right) \in \mathbb{R}\{s,t\}$, because no such function exists even in $\hat{\mathcal{O}}_q$, that is, as a formal power series near the generic point of V_0.

Now, if we set $t = -2\tanh \alpha s$, then $H^2(s,t) = 0$ infinitely often because there is an essential singularity at $s = \infty$; and in the general situation H^2 will have zeros in $\mathbb{C}_+$. If we could find a local analytic function H in s near one of these zeros then by analytic continuation we would be able to find a local analytic H in s and t in a neighbourhood of the zero, and hence on an open set (in the Hausdorff topology) in V_0. But then we could choose p to be in this open set, contradicting the argument above.

4 Examples

We give three examples to illustrate the arguments above. The first is classical, the second an example when algebraic spectral factorization is possible, the third an example when it is not.

Example 4.1 *Let $G(s) = e^{-s}R(s)$, where R is rational and proper with denominator degree d.*

This case reduces to the classical Fejér–Riesz theorem [RS]. Because e^{-s} is an inner function, it plays no role in the normalization of the coprime factors. There is a coprime factorization of the form

$$N(s) = \frac{p(s)e^{-s}}{(s+1)^d}, \qquad D(s) = \frac{q(s)}{(s+1)^d},$$

where p and q are real polynomials, and we arrive at $M = NN^* + DD^*$, where

$$M(s,t) = \frac{p(s)p(-s) + q(s)q(-s)}{(1-s^2)^d} = c \prod_{k=1}^{d} \frac{(s-a_k)(-s-a_k)}{(1-s)(1+s)},$$

with $c > 0$ and $a_1, \ldots, a_d \in \mathbb{C}_+$. It is clear now that a normalized coprime factorization is

$$N(s) = \frac{c^{-1/2}p(s)e^{-s}}{r(s)}, \qquad D(s) = \frac{c^{-1/2}q(s)}{r(s)},$$

where $r(s) = \prod_{k=1}^{d}(s + a_k)$.

In this case we have made a field extension $K : \mathbb{R}(t)$ as in section 2, namely $K = \mathbb{C}(t) = \mathbb{R}(t)(\sqrt{-1})$. Then M_0 factorizes immediately as $M_0 = \big(\prod_{k=1}^{d}(s - a_k)\big)\big(\prod_{k=1}^{d}(s - a_k)\big)^*$ and this gives us a normalized coprime factorization. The argument in section 3 does not apply here because it is not true that M_0 is absolutely irreducible: over $\mathbb{C}$ it splits into linear factors. The generic points used in section 3 do not exist.

In this situation the covers $\tilde{V} \to V$ and $\tilde{W} \to W$ are not branched: $\tilde{V}$ and $\tilde{W}$ are simply two isomorphic (over $\mathbb{C}$) copies of V and W respectively, and all we have to do is to make a field extension big enough to enable us to see the two components separately.

Example 4.2 *Let*

$$G(s) = \frac{1 + e^{-s}/2}{s + 1 + e^{-s}} = \frac{1 + z/2}{1 + s + z}.$$

Using the coprime factorization

$$N = \frac{1 + z/2}{1 + s}, \qquad D = \frac{1 + s + z}{1 + s},$$

and the substitution $t = (1 - z)/(1 + z)$, we arrive at

$$M(s,t) = -\left[\frac{s^2(t^2-1) + 4st + (25 - t^2)/4}{(1-s^2)(1-t^2)}\right]. \tag{2}$$

Factorizing the numerator and denominator as the product of polynomials we see that

$$M(s,t) = \frac{[s + (4t + t^2 - 5)/(2(t^2 - 1))]\,[-s + (-4t + t^2 + 5)/(2(t^2 - 1))]}{(1+s)(1-s)},$$

so that $M = F^*F$, where

$$F(s,t) = \frac{s + (t^2 + 4t - 5)/(2(t^2 - 1))}{1+s} = \frac{s + 3/2 + z}{1+s},$$

which is analytic and nonzero in $\mathbb{C}_+$. This gives the normalized coprime factorization

$$N = \frac{1 + z/2}{s + 3/2 + z}, \qquad D = \frac{1 + s + z}{s + 3/2 + z}.$$

In this case the argument of section 3 fails at the last step: it just happens that the polynomial M_0, the numerator of (2), has no zeros in the right half-plane; thus we can take its square root as an analytic function. Now M_0 factorizes as HH^*, where $H(s,t) = (t^2 - 1)s + (t^2 + 4t - 5)/2$. Moreover, we can write $H = \sqrt{M_0}(sZ + 1)/(sZ - 1)$, where

$$Z = \frac{H + \sqrt{M_0}}{s(H - \sqrt{M_0})} \in \tilde{L}.$$

Example 4.3 *Let*

$$G(s) = 1/(s - e^{-s}) = 1/(s - z).$$

Using the coprime factorization

$$N = \frac{1}{1+s}, \qquad D = \frac{s - z}{1+s},$$

we now evaluate $M = NN^* + DD^*$ to obtain

$$M = -\left[\frac{s^2(t^2 - 1) - 4st + 2(1 - t^2)}{(1 - s^2)(1 - t^2)}\right].$$

Clearly the only problem is how to factorize the numerator. In this case M_0 has many simple zeros in $\mathbb{C}_+$ so according to section 3 we should be unable to do this satisfactorily.

We illustrate this failure by attempting to apply the simple method of section 2. Solving for s we obtain the factors

$$\prod \left[s - \frac{2t \pm \sqrt{2t^4 + 2}}{t^2 - 1}\right].$$

The next thing to do is to examine whether either factor is an analytic function in the right half plane. Now we obtain a branch point of either factor if t is a 4th root of -1, of which there are two in $\mathbb{C}_+$, corresponding to $z = i(\pm\sqrt{2}\pm 1)$. Thus the only possible algebraic factorization of M gives functions which are inadmissible as spectral factors, and we conclude that there is no algebraic expression for the normalized coprime factorization.

One arrives at a similar conclusion on solving for t in terms of s, when the factors become

$$\prod \left[t - \frac{2s \pm \sqrt{s^4+4}}{s^2-2} \right].$$

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