It is necessary to check that the cusp forms constructed in [GHS1] are indeed cusp forms in the strong sense, i.e. that on the toroidal compactification they vanish to order at least 1 on every boundary component.

**Lemma 1.1** Let $L = 2U \oplus L_0$ be a lattice of signature $(2,n)$ containing two hyperbolic planes and let $f$ be a modular form with character det or trivial character that vanishes at every cusp. Then $f$ is a cusp form, vanishing to order at least 1 on every toroidal boundary component.

**Proof.** It is clearly sufficient to show that the order of vanishing of $f$ along any boundary component $F$ is an integer. If $f$ is of weight $k$ then near the boundary component $F$ we have

$$f(gZ) = j(g,Z)\chi(g)f(Z)$$

where $Z \in D_L(F)$ and $g \in U(F)_Z$, for some factor of automorphy $j$ and $\chi$ the character of the modular form $f$. If the factor $j(g,Z)\chi(g)$ is equal to 1 for every $g \in U(F)_Z$ then $f$ is a section of a line bundle near $F$ and its order of vanishing along $F$ is therefore an integer.

Under the hypotheses of the lemma, we do indeed have $\chi(g) = 1$ because $g$ is unipotent and therefore has trivial determinant. It therefore remains to check that the factor of automorphy $j(g,Z)$ is also trivial for $g \in U(F)_Z$.

If $F$ is of dimension 1 then according to [GHS1, Lemma 2.25] we have

$$U(F) = \left\{ \begin{pmatrix} I & 0 & \left( \begin{array}{cc} 0 & e^x \\ -x & 0 \end{array} \right) \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \right\} | x \in \mathbb{R} \right\}. \quad (1)$$

But the automorphy factor is given by the last ($(n+2)$-th) coordinate of $g(p(Z)) \in D_L$, where

$$p: \mathcal{H}_n \rightarrow D_L$$

$$Z = (z_n, \ldots, z_1) \mapsto \left( -\frac{1}{2}(Z,Z)_{L_1} : z_n : \cdots : z_1 : 1 \right)$$

is the tube domain realisation of $D_L$: see [GHS2, Section 3] or [G, Section 2]. From this description it is immediate that $j(g,Z) = 1$ for $g \in U(F)_Z$.

If $F$ is of dimension 0 then $F$ corresponds to some isotropic vector $v \in L$, and $U(F)$ is the centre of the unipotent radical of the stabiliser of $v$. With respect to a basis of $L \otimes \mathbb{Q}$ in which $v$ is the last ($(n+2)$-th) element,
the penultimate \((n + 1)\)-th element \(w\) is also isotropic and the remaining elements span the orthogonal complement \(L'\) of those two, we have

\[
U(F) = \left\{ \begin{pmatrix} I_n & b & 0 \\ 0 & 1 & 0 \\ c & x & 1 \end{pmatrix} \mid L'b + \alpha c = 0, \quad \lambda L'b + 2\alpha x = 0 \right\}. \tag{3}
\]

Here \(b\) and \(c\) are column vectors, \(x \in \mathbb{R}\) and \(\alpha = (w, v)_L\): compare [Ko, (2.7)]. In this case the tube domain is contained in \(\mathbb{C}^n\) and is identified with a subset of the locus \(z_{n+1} = 1 \subset \mathcal{D}_L^*\). The automorphy factor \(j(g, Z)\) is therefore equal to the \((n + 1)\)-th coordinate of \(g(p(Z))\), where \(p(Z)_{n+1} = 1\); but this is 1 as \(p(Z)\) is a column vector.

From the proof it follows that any cusp form \(f\) for an arithmetic subgroup \(\Gamma < O(L)\) vanishes to order at least 1 along a toroidal divisor unless the character \(\chi\) associated with \(f\) is non-trivial (and not det) on \(U(F)_Z = U(F) \cap \Gamma\). The existence of such a character appears to be a strong condition on \(\Gamma\): see [GHS3].

References


