Abelian surfaces with odd bilevel structure

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Abelian surfaces with weak bilevel structure were introduced by S. Mukai in [14]. There is a coarse moduli space, denoted $\mathcal{A}_t^{\text{bilevel}}$, for abelian surfaces of type $(1,t)$ with weak bilevel structure. $\mathcal{A}_t^{\text{bilevel}}$ is a Siegel modular threefold, and can be compactified in a standard way by Mumford’s toroidal method [1]. We denote the toroidal compactification (in this situation also known as the Igusa compactification) by $\mathcal{A}_t^{\text{bilevel}}$. It is a projective variety over $\mathbb{C}$, and it is shown in [14] that $\mathcal{A}_t^{\text{bilevel}}$ is rational for $t \leq 5$. In this paper we examine the Kodaira dimension $\kappa(\mathcal{A}_t^{\text{bilevel}})$ for larger $t$. Our main result is the following (Theorem VIII.1).

**Theorem.** $\mathcal{A}_t^{\text{bilevel}}$ is of general type for $t$ odd and $t \geq 17$.

It follows from the theorem of L. Borisov [2] that $\mathcal{A}_t^{\text{bilevel}}$ is of general type for $t$ sufficiently large. If $t = p$ is prime, then it follows from [7] and [12] that $\mathcal{A}_p^{\text{bilevel}}$ is of general type for $p \geq 37$. Our result provides an effective bound in the general case and a better bound in the case $t = p$. As far as we know, all previous explicit general type results (for instance [7, 12, 15, 8, 16]) have been for the cases $t = p$ or $t = p^2$ only.

It is for brevity that we assume $t$ is odd. If $t$ is even the combinatorial details are more complicated, especially when $t \equiv 2 \mod 4$, but the method is still applicable. In fact the method is essentially that of [12], with some modifications.

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I Background

If $A$ is an abelian surface with a polarisation $H$ of type $(1,t)$, $t > 1$, then a **canonical level structure**, or simply **level structure**, is a symplectic isomorphism

$$\alpha : \mathbb{Z}_2^2 \longrightarrow K(H) = \{ x \in A \mid t_{\mathfrak{L}}^x \mathcal{L} \cong \mathcal{L} \text{ if } c_1(\mathcal{L}) = H \}.$$ 

The moduli space $\mathcal{A}_t^{\text{level}}$ of abelian surfaces with a canonical level structure has been studied in detail in [11], chiefly in the case $t = p$. 
A *coleval structure* on $A$ is a level structure on the dual abelian surface $A^\vee$: note that $H$ induces a polarisation $H$ on $A$, also of type $(1, t)$. Alternatively, a coeval structure may be thought of as a symplectic isomorphism

$$\beta : \mathbb{Z}_t^2 \rightarrow A[t]/K(H)$$

where $A[t]$ is the group of all $t$-torsion points of $A$. Obviously the moduli space $A_t^\text{col}$ of abelian surfaces of type $(1, t)$ with a coeval structure is isomorphic to $A_t^\text{lev}$, and each of them has a forgetful morphism $\psi^\text{lev}, \psi^\text{col}$ to the moduli space $A_t$ of abelian surfaces of type $(1, t)$. We define

$$A_t^\text{biv} = A_t^\text{lev} \times A_t^\text{col}.$$ 

The forgetful map $\psi^\text{lev} : A_t^\text{lev} \rightarrow A_t$ is the quotient map under the action of $\text{SL}(2, \mathbb{Z})$ given by

$$\gamma : [(A, H, \alpha)] \mapsto [(A, H, \alpha \gamma)]$$

where $\gamma \in \text{SL}(2, \mathbb{Z})$ is viewed as a symplectic automorphism of $\mathbb{Z}_t^2$. The action is not effective, because $(A, H, \alpha)$ is isomorphic to $(A, H, -\alpha)$ via the isomorphism $x \mapsto -x$; so $-1 \in \text{SL}(2, \mathbb{Z})$ acts trivially. Thus $\psi^\text{lev}$ is a Galois morphism with Galois group $\text{PSL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})/\pm 1_2$.

A point of $A_t^\text{biv}$ thus corresponds to an equivalence class $[(A, H, \alpha, \beta)]$, where $(A, H)$ is a polarised abelian surface of type $(1, t)$, $\alpha$ and $\beta$ are level and coeval structures, and $(A, H, \alpha, \beta)$ is equivalent to $(A', H', \alpha', \beta')$ if there is an isomorphism $\rho : A \rightarrow A'$ such that $\rho^*H' = H$, $\rho \alpha = \alpha'$ and $\rho^*\beta = \beta'$.

In particular, for general $A$, we have $(A, H, \alpha, \beta) \cong (A, H, -\alpha, -\beta)$ but $(A, H, \alpha, \beta) \not\cong (A, H, -\alpha, \beta)$. Another way to express this is to say that the wreath product $\mathbb{Z}_2 \wr \text{PSL}(2, \mathbb{Z})$, acts on $A_t^\text{biv}$ with quotient $A_t$.

**Theorem I.1** (Mukai [14]) $A_t^\text{biv}$ is the quotient of the Siegel upper half-plane $\mathbb{H}_2$ by the group

$$\Gamma_t^\text{biv} = \Gamma_t^1 \cup \zeta \Gamma_t^1$$

where

$$\Gamma_t^1 = \{ \gamma \in \text{Sp}(4, \mathbb{Z}) \mid \gamma - 1_4 \in \begin{pmatrix} \mathbb{Z}^3 & \mathbb{Z}^2 & \mathbb{Z} \\ \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z} \\ \mathbb{Z}^3 & \mathbb{Z}^2 & \mathbb{Z} \\ * & * & \mathbb{Z} \end{pmatrix} \}$$

and $\zeta = \text{diag}(1, -1, 1, -1)$, acting by fractional linear transformations.

Thus $\Gamma_t^\text{biv}$ should be thought of as a subgroup of the paramodular group

$$\Gamma_t = \{ \gamma \in \text{Sp}(4, \mathbb{Q}) \mid \gamma - 1_4 \in \begin{pmatrix} \mathbb{Q}^3 & \mathbb{Q}^2 \mathbb{Q} \\ \mathbb{Q}^2 & \mathbb{Q} \mathbb{Q} & \mathbb{Q} \mathbb{Q} \\ \mathbb{Q}^3 & \mathbb{Q} \mathbb{Q} & \mathbb{Q} \mathbb{Q} \\ * & * & \mathbb{Q} \end{pmatrix} \}.$$
(The paramodular group is the group denoted $\Gamma_0^1, t$ in [11] and [5].)

For some purposes it is more convenient to work with the conjugate $\tilde{\Gamma}_1^{1,0} = R_1 \tilde{\Gamma}_1^{1,0} R_2$, of $\tilde{\Gamma}_1^{1,0}$ by $R_1 = \text{diag}(1, 1, 1, t)$, and with the corresponding conjugates $\tilde{\Gamma}_1^1, \tilde{\Gamma}_1^{1,0}$, etcetera. These groups have the advantage that they are subgroups of $\mathrm{Sp}(4, \mathbb{Z})$ rather than $\mathrm{Sp}(4, \mathbb{Q})$, and are defined by congruences mod $t$, not mod $t^2$, but their action on $\mathbb{H}_2$ is not the usual one by fractional linear transformations.

If $E_i$ are elliptic curves and $(A, H) = (E_1 \times E_2, \tau_1 (E_1(1) \otimes E_2(t)))$, we say that $(A, H)$ is a product surface. In this case $K(H) = \{0_{E_1}\} \times E_2[t]$, so a level structure on $A$ may be thought of as a full level-$t$ structure on $E_2$. The automorphism $(x, y) \mapsto (x, -y)$ of $A = E_1 \times E_2$ induces an isomorphism $(A, H, \alpha, \beta) \to (A, H, -\alpha, \beta)$ in this case, so a product surface with a weak bidegree structure still has an extra automorphism. The corresponding locus in the moduli space arises from the fixed locus of $\zeta$ in $\mathbb{H}_2$, and will be of great importance in this paper.

The geometry of $\mathcal{A}_{1,0}^{1,0}$ shows many similarities with that of $\mathcal{A}_{1,0}^{1,0}$, which was studied (in the case of $t$ an odd prime) in the book [11]. In many cases where the proofs of intermediate results are very similar to those of corresponding results in [11] we omit the details and simply indicate the appropriate reference.

II  Modular groups and modular forms

We first collect some facts about congruence subgroups in $\mathrm{SL}(2, \mathbb{Z})$ and some related combinatorial information. For $r \in \mathbb{N}$ we denote by $\Gamma_1(r)$ the principal congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$. We denote the modular curve $\Gamma_1(r) \backslash \mathbb{H}$ by $X^*(r)$, and the compactification obtained by adding the cusps by $X(r)$.

For $m, r \in \mathbb{N}$, define

$$\Phi_m(r) = \{ a \in \mathbb{Z}_r^* \mid a \text{ is not a multiple of a zerodivisor in } \mathbb{Z}_r \},$$

that is, $a \in \Phi_m(r)$ if and only if $a = za'$ implies $z \in \mathbb{Z}_r^*$; and put $\phi_m(r) = \# \Phi_m(r)$. We also put $\Phi_m(r) = \Phi_m(r) / \pm 1$.

**Lemma II.1** If the primes dividing $r$ are $p_1 < p_2 < \ldots < p_n$ then

$$\phi_m(r) = \sum_{i=0}^n (-1)^i \sum_{p_1, \ldots, p_i} \left( r \prod_{k=1}^i p_j^{-1} \right)^m = r^m \prod_{p | r} (1 - p^{-m}).$$

**Proof.** We first prove that $\phi_m(p)$ is a multiplicative function. First we suppose that $r = pq$, with $\gcd(p, q) = 1$. It is easy to see that $a \in \Phi_m(p)$ if and only if $a_p \in \Phi_m(p)$ and $a_q \in \Phi_m(q)$, where $a_p$ denotes the reduction of $a$ mod $p$.
We divide \( \mathbb{Z}_r^m \) into residue classes mod \( p \); that is, we write \( \mathbb{Z}_r^m \) as the disjoint union of subsets \( S_c \) for \( c \in \mathbb{Z}_r^m \), where \( S_c = \{ a | a_r = c \} \). There are \( \phi_m(p) \) subsets \( S_c \) such that \( r \in \Phi_m(p) \).

The reduction mod \( q \) map \( S_c \rightarrow \mathbb{Z}_q^m \) is bijective, since it is the inverse of the injective map \( b \mapsto c + pb \in \mathbb{Z}_r^m \). Hence in each of the \( \phi_m(p) \) subsets \( S_c, c \in \Phi_m(p) \) there are \( \phi_m(q) \) elements whose reduction mod \( q \) belongs to \( \Phi_m(q) \). It follows that \( \phi_m(r) = \phi_m(p)\phi_m(q) \).

Finally, we check that if \( r = p^k \), \( p \) prime, then \( \phi_m(r) = r^m(1 - p^{-m}) \). If \( a \notin \Phi_m(r) \), then \( a = pa' \) for a unique \( a' \in \mathbb{Z}_{r^f}^m \), so there are \( (p^k - 1)^m \) such elements \( a \).

Note that \( \phi_1 \) is the Euler \( \phi \) function, and \( \Phi_1(r) \) is the set of non-zerodivisors of \( \mathbb{Z}_r^m \).

**Corollary II.2** The order of \( \text{SL}(2, \mathbb{Z}_t) \) is given by

\[
| \text{SL}(2, \mathbb{Z}_t) | = t\phi_2(t) = t^3 \prod_{p | t} (1 - p^{-2}).
\]

**Proof.** (See also [18, §1.6].) If \( A \in \text{SL}(2, \mathbb{Z}_t) \), then \( A_1 = (a_{11}, a_{12}) \in \Phi_2(t) \).

So by Euclid’s algorithm we can find \( A'_2 = (a'_{21}, a'_{22}) \) such that \( \det \begin{pmatrix} A_1 \\ A'_2 \end{pmatrix} = \gcd(a_{11}, a_{12}) = r \). Replacing \( A'_2 \) by \( A_2 = r^{-1}A'_2 \), we get a matrix \( A \) with \( \det(A) = 1 \). Furthermore, if \( B_j = \begin{pmatrix} A_1 \\ A_2 + jA_1 \end{pmatrix} \), \( j = 0, \ldots, t - 1 \), then \( \det(B_j) = \det(A) = 1 \), and \( B_j \neq B_{j'} \) if \( j \neq j' \). So \( | \text{SL}(2, \mathbb{Z}_t) | = t\phi_2(t) \). \( \square \)

For \( r > 2 \), put \( \mu(r) = [\text{PSL}(2, \mathbb{Z}) : \Gamma_1(r)] \). By Corollary II.2 we have

\[
\mu(r) = r^3 \prod_{p | r} (1 - p^{-2}).
\]

We need the following well-known lemma.

**Lemma II.3** If \( r > 2 \) then \( X(r) \) has

\[
\nu(r) = \mu(r)/r = r^2 \prod_{p | r} (1 - p^{-2})
\]

cusps and is a smooth complete curve of genus \( g = 1 + \frac{\mu(r)}{12} - \frac{\nu(r)}{2} \).

**Proof.** See [18, pp. 23–24]. \( \square \)

We denote \( \mu(t) \) by \( \mu \) and \( \nu(t) \) by \( \nu \). Note that \( \phi_2(1) = \nu(1) = 1 \) and \( \phi_2(r) = 2\nu(r) \) for \( r > 2 \).
Now we turn to subgroups of Sp(4, \mathbb{Q}) and to modular forms. Denote by \( \mathcal{S}_n^*(\Gamma) \) the space of weight \( n \) cusp forms for \( \Gamma \subseteq \text{Sp}(4, \mathbb{Q}) \). We need the groups \( \Gamma(1) = \text{PSp}(4, \mathbb{Z}) \) and, for \( \ell \in \mathbb{N} \),

\[
\Gamma(\ell) = \{ \gamma \in \text{Sp}(4, \mathbb{Z}) \mid \gamma = 1_4 \in \text{Sp}(4, \mathbb{Z}, \ell) \}.
\]

If \( \ell^2 \mid \ell \) then \( \Gamma(\ell) \triangleleft \Gamma_\ell^{1\text{al}} \), because \( \Gamma(\ell) \subseteq \Gamma_\ell^{1\text{al}} \) and \( \Gamma(\ell) \) is normal in \( \Gamma(1) = \text{Sp}(4, \mathbb{Z}) \).

By a previous calculation [19] we know that

\[
\dim \mathcal{S}_n^*(\Gamma(\ell)) = \frac{n^3}{8640} \left[ \Gamma(1) : \Gamma(\ell) \right] + O(n^2)
\]

(as long as \( \ell > 2 \) we can consider \( \Gamma(\ell) \) as a subgroup of \( \text{PSp}(4, \mathbb{Z}) \) rather than \( \text{Sp}(4, \mathbb{Z}) \)). A standard application of the Atiyah–Bott fixed-point theorem (see [9], or in this context [12]) gives

\[
\dim \mathcal{S}_n^*(\Gamma_\ell^{1\text{al}}) = \frac{a}{[\Gamma_\ell^{1\text{al}} : \Gamma(\ell)]} \dim \mathcal{S}_n^*(\Gamma(\ell)) + O(n^2)
\]

where \( a \) is the number of elements \( \gamma \in \Gamma_\ell^{1\text{al}} \) whose fixed locus in \( \mathbb{H}_2 \) has dimension 3. Thus \( a \) is the number of elements of \( \Gamma_\ell^{1\text{al}} \) that act trivially on \( \mathbb{H}_2 \). In \( \text{Sp}(4, \mathbb{Z}) \) there are two such elements, \( \pm 1_4 \), but if \( t > 2 \) then \( -1_4 \not\in \Gamma_\ell^{1\text{al}} \). So \( a = 1 \), and hence

\[
\dim \mathcal{S}_n^*(\Gamma_\ell^{1\text{al}}) = \frac{1}{[\Gamma_\ell^{1\text{al}} : \Gamma(\ell)]]} \dim \mathcal{S}_n^*(\Gamma(\ell)) + O(n^2)
\]

\[
= \frac{n^3}{8640} \left[ \Gamma(1) : \Gamma(\ell) \right] + O(n^2)
\]

\[
= \frac{n^3}{8640} \left[ \Gamma(1) : \Gamma_\ell^{1\text{al}} \right] + O(n^2).
\]

(1)

The number \( [\tilde{\Gamma}(1) : \Gamma_\ell^{1\text{al}}] \) is equal to the degree of the map \( \mathcal{A}_\ell^{1\text{al}} \rightarrow \mathcal{A}_1 \) (actually there are two such maps of the same degree), where \( \mathcal{A}_1 \) is the moduli space of principally polarized abelian surfaces. Now

\[
[\tilde{\Gamma}(1) : \Gamma_\ell^{1\text{al}}] = \frac{1}{2} [\tilde{\Gamma}(1) : \Gamma_\ell^{1\text{lev}}]
\]

\[
= \frac{1}{2} \left[ \Gamma(1) : \Gamma_\ell^{1\text{lev}} \right] \left[ \Gamma_\ell^{1\text{lev}} : \Gamma_\ell^{1\text{al}} \right].
\]

We can see directly that \( \Gamma_\ell^{1\text{lev}} \supset \Gamma_\ell^{1} \) since

\[
\Gamma_\ell^{1\text{lev}} = \left\{ \gamma \in \text{Sp}(4, \mathbb{Z}) \mid \gamma - 1_4 \in \begin{pmatrix}
* & * & * & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}^2 \\
* & * & * & \mathbb{Z} \\
* & * & * & \mathbb{Z}
\end{pmatrix} \right\}.
\]
Lemma II.4 The map
\[ \varphi : \Gamma_{t}^{\text{lev}} \rightarrow \text{SL}(2, \mathbb{Z}_t), \quad A \mapsto \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} \]
is a surjective group homomorphism, and the kernel is \( \Gamma_{t}^{1} \).

Proof. The surjectivity follows from the well-known fact that the reduction mod \( t \) map \( \text{red}_t : \text{SL}(2, \mathbb{Z}) \rightarrow \text{SL}(2, \mathbb{Z}_t) \) is surjective, and the rest is obvious. \( \square \)

Lemma II.5 For \( t > 2 \), the index \( [\bar{\Gamma}(1) : \Gamma_{t}^{\text{lev}}] \) is equal to \( t\phi_4(t)/2 \).

Proof. The proof is almost the same as proof of [13, Lemma 0.5]. In place of the chain of groups \( \Gamma_{1,p} < \Gamma_{1,p} < \Gamma' = \Gamma(1) \), we use the chain \( \Gamma_{t}^{\text{lev}} < \Gamma_{1,t} < \Gamma(1) \). Furthermore, we use the set \( \Phi_4(t) \) where \( \text{SL}(4, \mathbb{Z}_t) \) acts. Note that \( \text{SL}(4, \mathbb{Z}) \) still acts transitively on \( \Phi_4(t) \), via
\[
\begin{pmatrix} b_{11} & 0 & 0 \\ 0 & 1 & 0 \\ b_{21} & 0 & b_{22} \\ 0 & 0 & 1 \end{pmatrix}
\quad \text{and} \quad \begin{pmatrix} B & 0 \\ 0 & tB^{-1} \end{pmatrix},
\]
for \( B \in \text{SL}(2, \mathbb{Z}) \).
Following the same steps as in [13], and substituting \( \phi_m(t) \) for \( p^m - 1 = \phi_m(p) \), we then find that \( [\Gamma_{1,t} : \Gamma_{t}^{\text{lev}}] = t\phi_1(t) \) and \( [\Gamma_{1,t} : \Gamma(1)] = \phi_4(t)/\phi_1(t) \), so \( [\bar{\Gamma}(1) : \Gamma_{t}^{\text{lev}}] = t\phi_4(t)/2 \). \( \square \)

Theorem II.6 The number of cusp forms of weight \( n \) for \( \Gamma_{t}^{\text{kl}} \) (for \( t > 2 \)) is given by
\[
\dim \mathfrak{S}_{n}(\Gamma_{t}^{\text{kl}}) = \frac{r^3}{34560} \phi_2(t)\phi_4(t) = \frac{r^3}{34560} \prod_{p|t} (1 - p^{-2})(1 - p^{-4}).
\]

Proof. Immediate from equation (1), Corollary II.2 and Lemma II.5. \( \square \)

III Torsion in the modular group

We know that \( \Gamma_{t}^{\text{kl}} \subset \text{Sp}(4, \mathbb{Z}) \), and the conjugacy classes of torsion elements in \( \text{Sp}(4, \mathbb{Z}) \) are known ([6, 20]). See [10] for a summary of the relevant information.
If $\gamma \in \Gamma_t^1$ then the reduction mod $t$ of $\gamma$ is

$$\bar{\gamma} = \begin{pmatrix} 1 & * & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & * & 1 & 0 \\ * & * & * & 1 \end{pmatrix} \in \text{Sp}(4, \mathbb{Z}_t),$$

so the characteristic polynomial $\chi(\bar{\gamma})$ is $(1-x)^4 \in \mathbb{Z}_t[x]$. On the other hand, if $\gamma \in \zeta \Gamma_t^1$ then

$$\bar{\gamma} = \zeta \begin{pmatrix} 1 & * & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & * & 1 & 0 \\ * & * & * & 1 \end{pmatrix} = \begin{pmatrix} 1 & * & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ * & * & * & -1 \end{pmatrix} \in \text{Sp}(4, \mathbb{Z}_t),$$

so $\chi(\bar{\gamma}) = (1-x)^2(1+x)^2 \in \mathbb{Z}_t[x]$.

The only classes in the list in [20], up to conjugacy, where the characteristic polynomials have this reduction mod $t$ ($t > 2$) are I(1), where $\chi(\gamma) = (1-x)^4$, $\Pi(1)a$ and $\Pi(1)b$. Class I(1) consists of the identity; class $\Pi(1)a$ includes $\zeta$ so this just gives us the conjugacy class of $\zeta$. Class $\Pi(2)b$ is the Sp($4,\mathbb{Z}$)-conjugacy class of $\xi$, where

$$\xi = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \in \Gamma_t^{1b}.$$

**Proposition III.1** Every nontrivial element of finite order in $\Gamma_t^{1b}$ (for $t > 2$) has order 2, and is conjugate to $\zeta$ or to $\xi$ in $\Gamma_t^{1b}$ if $t$ is odd.

**Proof.** It follows from the list in [20] that the only torsion for $t > 2$ is 2-torsion (this is still true if $t$ is even). The 2-torsion of the group $\Gamma_t^{1v}$ was studied by Brasch [3]. There are five types but only two of them occur for odd $t$. The representatives for these conjugacy classes given in [3] are (up to sign) $\zeta$ and $\xi$; so the assertion of the theorem is that the $\Gamma_t^{1b}$-conjugacy classes of $\zeta$ and $\xi$ coincide with the intersections of their $\Gamma_t^{1v}$-conjugacy classes with $\Gamma_t^{1b}$. This is checked in [17, Proposition 3.2] for the case $t = 6$ (the relevant cases are called $\zeta_0$ and $\zeta_3$ there), but the proof is valid for all $t > 2$. \[\Box\]

We put

$$\mathcal{H}_1 = \left\{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix} \left| \begin{array}{c} \text{Im} \tau_1 > 0, \text{Im} \tau_3 > 0 \end{array} \right. \right\} \subset \mathbb{H}_2$$

and

$$\mathcal{H}_2 = \left\{ \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \left| 2\tau_2 + \tau_3 = 0 \right. \right\} \subset \mathbb{H}_2.$$
These are the fixed loci of $\zeta$ and $\xi$ respectively. We denote by $H_1^i$ and $H_2^i$ the images of $H_1$ and $H_2$ in $\mathcal{A}_t^{4\text{cl}}$, and by $H_1$ and $H_2$ their respective closures in $\mathcal{A}_t^{4\text{cl}}$.

**Lemma III.2** $H_i^i$ is irreducible for $i = 1, 2$.

**Proof.** This follows at once from Proposition III.1 together with equations (2) and (3). \hfill \Box

The abelian surfaces corresponding to points in $H_1^i$ and $H_2^i$ are, respectively, product surfaces and bielliptic abelian surfaces, as described in [13] for the case $t$ prime.

We define the subgroup $\Gamma(2t, 2t)$ of $\Gamma(t) \times \Gamma(t)$ by

$$\Gamma(2t, 2t) = \{(M, N) \in \Gamma(t) \times \Gamma(t) \mid M \equiv N^{-1} \mod 2\}$$

**Lemma III.3** $H_1^i$ is isomorphic to $X^0(t) \times X^0(t)$, and $H_2^i$ is isomorphic to $\Gamma(2t, 2t) \backslash \mathbb{H} \times \mathbb{H}$.

**Proof.** Identical to the proofs of the corresponding results [11, Lemma I.5.43] and [11, Lemma I.5.45]. The level-$t$ structure now occurs in both factors, whereas in [11] there is level-1 structure in the first factor and level-$p$ structure in the second. In [11] the level $p$ is assumed to be an odd prime but this fact is not used at that stage: $p$ odd suffices, so we may replace $p$ by $t$. Thereafter one simply replaces all the groups with their intersection with $\Gamma_t^{4\text{cl}}$, which imposes a level-$t$ structure in the first factor and causes it to behave exactly like the second factor. \hfill \Box

**Lemma III.4** $H_1^i$ and $H_2^i$ are disjoint.

**Proof.** The stabiliser of any point of $\mathbb{H}_2$ in $\Gamma_t^{4\text{cl}}$ is cyclic (of order 2), since $\Gamma_t^{4\text{cl}}$ is torsion-free and therefore has no fixed points. A point of $H_1 \cap H_2$ would be the image of a point of $\mathbb{H}_2$ stabilised by the subgroup generated by $\zeta$ and $\xi$, which is not cyclic. \hfill \Box

## IV Boundary divisors

We begin by counting the boundary divisors. These correspond to $\tilde{\Gamma}_t^{4\text{cl}}$-orbits of lines in $\mathbb{Q}^4$: we identify a line by its unique (up to sign) primitive generator $\nu = (v_1, v_2, v_3, v_4) \in \mathbb{Z}^4$ with $\text{hcf}(v_1, v_2, v_3, v_4) = 1$. We denote the reduction of $\nu$ mod $t$ by $\tilde{\nu} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \in \mathbb{Z}_t^4$. To fix things we shall say, arbitrarily, that $\nu$ is positive if the first non-zero entry $\tilde{v}_1$ of $\tilde{\nu}$ satisfies $\tilde{v}_1 \in \{1, \ldots, (t-1)/2\}$ (remember that we have assumed that $t$ is odd). Then each line has a unique positive primitive generator.

If $\nu = (v_1, v_2, v_3, v_4) \in \mathbb{Z}^4$, we define the $t$-divisor to be $r = \text{hcf}(t, v_1, v_3)$.
**Proposition IV.1** The lines \( \mathbb{Q}v \) and \( \mathbb{Q}w \) spanned by positive primitive vectors \( v, w \in \mathbb{Z}^4 \) are in the same \( \Gamma_l^{\text{pl}} \)-orbit if and only if \((\bar{v}_1, \bar{v}_3) = (\bar{w}_1, \bar{w}_3)\) (in particular \( v \) and \( w \) have the same \( t \)-divisor, \( r \)), and \((v_2, v_4) \equiv \pm (w_2, w_4) \mod r \).

**Proof.** Note that if \( \Gamma(t) \) is the principal congruence subgroup of level \( t \) in \( \text{Sp}(4, \mathbb{Z}) \) then \( \Gamma(t) < \Gamma_t^1 \) and the quotient is

\[
\Gamma^1_t(t) = \left\{ \begin{pmatrix} 1 & k & 0 & k' \\ 0 & 1 & 0 & 0 \\ 0 & l & 1 & l' \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \text{Sp}(4, \mathbb{Z}_t) \right\} \cong \mathbb{Z}_t^4.
\]

We claim that two primitive vectors \( v \) and \( w \) are equivalent modulo \( \Gamma(t) \) if and only if \( \vec{v} = \vec{w} \). It is obvious that \( \Gamma(t) \) preserves the residue classes \( \mod t \). Conversely, suppose that \( \vec{v} = \vec{w} \). Then we can find \( \gamma \in \text{Sp}(4, \mathbb{Z}) \) such that \( \gamma v = (1, 0, 0, 0) \) (the corresponding geometric fact is that the moduli space \( \mathcal{A}_2 \) of principally polarised abelian surfaces has only one rank 1 cusp). Since \( \Gamma(t) < \text{Sp}(4, \mathbb{Z}) \) this means that in order to prove the claim we may assume \( v = (1, 0, 0, 0) \). Then we proceed exactly as in the proof of [5, Lemma 3.3], taking \( p = 1 \) and \( q = t \) (the assumptions that \( p \) and \( q \) are prime are not used at that point).

The group \( \Gamma^1_t(t) \) acts on the set \((\mathbb{Z}_t^4)^\times \) of non-zero elements of \( \mathbb{Z}_t^4 \) by \( \vec{v}_2 \mapsto \vec{v}_2 + k\vec{v}_1 + l\vec{v}_3 \) and \( \vec{v}_4 \mapsto \vec{v}_4 + k'\vec{v}_1 + l'\vec{v}_3 \); so \( \vec{v} \) is equivalent to \( \vec{w} \) if and only if \((\vec{v}_1, \vec{v}_3) = (\vec{w}_1, \vec{w}_3) \), so they have the same \( t \)-divisor, and \( \vec{v}_2 \in \vec{w}_2 + \mathbb{Z}_t r \) and \( \vec{v}_4 \in \vec{w}_4 + \mathbb{Z}_t r \). These are therefore the conditions for primitive vectors \( v \) and \( w \) to be equivalent under \( \Gamma^1_t \). For equivalence under \( \Gamma^{\text{pl}}_l \), we get the extra element \( \zeta \) which makes \((v_1, v_2, v_3, v_4) \equiv (v_1, -v_2, v_3, -v_4) \) equivalent to \((v_1, v_2, v_3, v_4) \).

Since we are interested in orbits of lines, not primitive generators, we may restrict ourselves to positive generators \( v \).

The irreducible components of the boundary divisor of \( \mathcal{A}^{\text{pl}}_l \) correspond to the \( \Gamma^{\text{pl}}_l \)-orbits (or equivalently to \( \Gamma^1_l \)-orbits) of lines in \( \mathbb{Q}^4 \). We denote the boundary component corresponding to \( \mathbb{Q}v \) by \( D_v \). We shall be chiefly interested in the cases \( r = t \) and \( r = 1 \). We refer to these as the standard components. They are represented by vectors \((0, a, 0, b)\) and \((a, 0, b, 0)\) respectively, in both cases with \( \gcd(a, b) = 1 \), \( 0 \leq a \leq (t-1)/2 \) and \( 0 \leq b < t \). Note that there are \( \nu \) of each of these.

**Corollary IV.2** If \( t \) is odd then the number of irreducible boundary divisors of \( \mathcal{A}^{\text{pl}}_l \) with \( t \)-divisor \( r \) is \( \#\Phi_2(h) \#\overline{\Phi}_2(r) \), where \( h = t/r \). For \( r \neq 1, t \), this is equal to \( \frac{1}{4} \phi_2(h) \phi_2(r) \).

**Proof.** See above for the standard cases. In general, the \( \Gamma^1_t \)-orbit of a primitive vector \( v \) is determined by the classes of \((v_1/r, v_3/r) \) in \( \Phi_2(h) \) and of
$(\bar{v}_2, \bar{v}_4) \in \Phi_2(r)$. The extra element $\zeta$ and the freedom to multiply $v$ by $-1 \in \mathbb{Q}$ allow us to multiply either of these classes by $-1$ and the choices therefore lie in $\Phi_2(h)$ and $\Phi_2(r)$.

\section{Jacobi forms}

In this section we shall describe the behaviour of a modular form $F \in S_k^+(\Gamma_{h}^{\text{hol}})$ near a boundary divisor $D_v$. The standard boundary divisors are best treated separately, since it is in those cases only that the torsion plays a role: on the other hand, the standard boundary divisors occur for all $t$ and their behaviour is not much dependent on the factorisation of $t$. We assume at first, then, that $D_v$ is a nonstandard boundary divisor. Since all the divisors of given $t$-divisor are equivalent under the action of $\mathbb{Z}_2 \wr \text{SL}(2, \mathbb{Z}_2)$, (because the $t$-divisor is the only invariant of a boundary divisor of $\mathcal{A}_t$: see [5]) it will be enough to calculate the number of conditions imposed by one divisor of each type. That is to say, we only need consider boundary components in $A_t^+$. In view of this we may take $v = (0, 0, r, 1)$ for some $r \mid t$ with $1 < r < t$. We write $(0, 0, 0, 1) = v_{(0, 1)}$ (for consistency with [11]) and we put $h = t/r$. Since we want to work with $\Gamma_h^{\text{hol}}$ rather than $\Gamma_h^1$ (so as to use fractional linear transformations) we must consider the lines $\mathbb{Q}vR_t = \mathbb{Q}v'$, where $v' = (0, 0, 1, h)$, and $\mathbb{Q}v_{(0, 1)}R_t = \mathbb{Q}v_{(0, 1)}$. Note that $v'Q_r = v_{(0, 1)}$, where

$$Q_r = \begin{pmatrix} 1 & 1 & 0 & 0 \\ h - 1 & h & 0 & 0 \\ 0 & 0 & h & 1 - h \\ 0 & 0 & -1 & 1 \end{pmatrix} \in \text{Sp}(4, \mathbb{Z}).$$

**Proposition V.1** If $v$ has $t$-divisor $r \neq t$, 1, and $F \in S_k^+(\Gamma_{h}^{\text{hol}})$ is a cusp form of weight $k$, then there are coordinates $\tau_1^v, \tau_2^v$ such that $F$ has a Fourier expansion near $D_v$ as

$$F = \sum_{u \geq 0} \theta_v^u(\tau_1^v, \tau_2^v) \exp 2\pi i uw\tau_2^v/r.$$

**Proof.** As usual (cf. [11]) we write $P_v'$ for the stabiliser of $v'$ in Sp(4, $\mathbb{R}$), so $P_v = Q^{-1}P_{v_{(0, 1)}}Q$. We take $P_v^t = P_v' \cap \Gamma_h^{\text{hol}}$: this group determines the structure of $A_t^+$. It is shown in [11, Proposition I.3.87] that $P_{v_{(0, 1)}}$ is generated by $g_1(\gamma)$ for $\gamma \in \text{SL}(2, \mathbb{R})$, $g_2 = \zeta$, $g_3(m, n)$ and $g_4(s)$ for $m, n, s \in \mathbb{R}$, where

$$g_1(\gamma) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
and $g_3$ and $g_4$ are given by

$$g_3(m, n) = \begin{pmatrix} 1 & 0 & 0 & n \\ m & 1 & n & 0 \\ 0 & 1 & 0 & -m \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_4(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$ 

So $P_v'$ includes the subgroup generated by all elements of the form $Q_{r}^{-1}g_iQ_r$ with $a, b, c, d, m, n, s \in \mathbb{Z}$ which lie in $\Gamma_{t}^{\text{isl}}$. In particular it includes the lattice

$$\{Q_{r}^{-1}g_{i}(rts)Q_r \mid s \in \mathbb{Z}\}.$$ 

If we take $Z^v = Q_{r}^{-1}(Z)$ for $Z = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}$ then we obtain

$$Z^v = \begin{pmatrix} h^2\tau_1 - 2h\tau_2 + \tau_3 & -h(h - 1)\tau_1 + (2h - 1)\tau_2 - \tau_3 \\ -(h - 1)^2\tau_1 - 2(h - 1)\tau_2 + \tau_3 \end{pmatrix}.$$

One easily checks that

$$Q_{r}^{-1}g_i(rt)Q_r : Z^v \rightarrow \begin{pmatrix} \tau_1^v & \tau_2^v \\ \tau_2^v & \tau_3^v + rt \end{pmatrix}$$

and this proves the result. \hfill \Box

We define a subgroup $\Gamma(t, r)$ of $\text{SL}(2, \mathbb{Z})$ by

$$\Gamma(t, r) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \equiv d \equiv 1 \text{ mod } t, \ b \equiv 0 \text{ mod } r^2, \ c \equiv 0 \text{ mod } r \right\}.$$ 

**Lemma V.2** If $D_v$ is nonstandard then $P_v'$ is torsion-free.

**Proof.** The only torsion in $\Gamma_{t}^{\text{isl}}$ is 2-torsion and a simple calculation shows that if $1_4 \neq g \in P_v(0, 1)$ and $g^2 = 1_4$, then $Q_{r}^{-1}gQ_r \in \Gamma_{t}^{\text{isl}}$ for $r \neq 1, t$. \hfill \Box

**Proposition V.3** If $D_v$ is nonstandard and $F \in \mathcal{H}_v^*(\Gamma_{t}^{\text{isl}})$ then $\theta_v^*(r\tau_1^v, t\tau_2^v)$ is a Jacobi form of weight $k$ and index $w$ for $\Gamma(t, r)$.

**Proof.** By direct calculation we find that $Q_{r}^{-1}g_1(\gamma)Q_r \in \Gamma_{t}^{\text{isl}}$ if $\gamma \in \Gamma(t, r)$ and $Q_{r}^{-1}g_3(rm, tn)Q_r \in \Gamma_{t}^{\text{isl}}$ for $m, n \in \mathbb{Z}$. Using these two elements, another elementary calculation verifies that the transformation laws for Jacobi forms given in [4] are satisfied, since

$$Q_{r}^{-1}g_3(rm, tn)Q_r : Z^v \rightarrow \begin{pmatrix} \tau_1^v & \tau_2^v + rm\tau_1^v + tn \\ \tau_2^v + rm\tau_1^v + tn & \tau_3^v + 2rm\tau_2^v + r^2m^2\tau_1^v \end{pmatrix}$$

and

$$Q_{r}^{-1}g_1(\gamma)Q_r : Z^v \rightarrow \begin{pmatrix} \gamma(\tau_1^v) & \tau_2^v/(c\tau_1^v + d) \\ \tau_2^v/(c\tau_1^v + d) & \tau_3^v - c\tau_2^v/(c\tau_1^v + d) \end{pmatrix}.$$ 

\hfill \Box
Lemma V.4 The index of $\Gamma(t,r)$ in $\Gamma(1)$ is equal to $rt\phi(t)$ for $r \neq 1, t$.

Proof. Consider the chain of groups

$$
\Gamma(1) = \text{SL}(2, \mathbb{Z}) > \Gamma_0(t) > \Gamma_0(t)(r) > \Gamma(t,r)
$$

and the normal subgroup $\Gamma_1(t) \triangleleft \Gamma_0(t)$, where

$$
\Gamma_0(t) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \mid \begin{array}{l}
    a \equiv d \equiv 1 \pmod{t}, \\
    b \equiv 0 \pmod{t}
\end{array} \right\},
$$

$$
\Gamma_1(t) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \mid \begin{array}{l}
    a \equiv d \equiv 1 \pmod{t}, \\
    b \equiv c \equiv 0 \pmod{t}
\end{array} \right\},
$$

$$
\Gamma_0(t)(h) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \mid \begin{array}{l}
    a \equiv d \equiv 1 \pmod{t}, \\
    b \equiv 0 \pmod{t},
    c \equiv 0 \pmod{h}
\end{array} \right\}.
$$

Thus $\Gamma_0(t)(r)$ is the kernel of reduction mod $r$ in $\Gamma_0(t)$. By Corollary II.2, $[\Gamma(1) : \Gamma_1(t)] = t\phi(t)$. By the exact sequence

$$
0 \longrightarrow \Gamma_1(t) \longrightarrow \Gamma_0(t) \longrightarrow \left\{ \begin{pmatrix} 1 & 0 \\ \bar{c} & 1 \end{pmatrix} \mid \bar{c} \in \mathbb{Z}_t \right\} \cong \mathbb{Z}_t \longrightarrow 0
$$

we have $[\Gamma_0(t) : \Gamma_1(t)] = t$, and similarly

$$
0 \longrightarrow \Gamma_0(t)(r) \longrightarrow \Gamma_0(t) \longrightarrow \left\{ \begin{pmatrix} 1 & 0 \\ \bar{c} & 1 \end{pmatrix} \mid \bar{c} \in \mathbb{Z}_r \right\} \cong \mathbb{Z}_r \longrightarrow 0
$$

gives $[\Gamma_0(t) : \Gamma_0(t)(r)] = r$.

To calculate $[\Gamma(t)(r) : \Gamma(t,r)]$ we let $\Gamma_0(t)(r)$ act on $\mathbb{Z}_t \times \mathbb{Z}_{t^2}$ by multiplication on the right, i.e. by $\gamma : (x, y) \mapsto (ax + cy, bx + dy)$. The stabilizer of $(1, 0) \in \mathbb{Z}_t \times \mathbb{Z}_{t^2}$ is then $\{ \gamma \in \Gamma_0(t)(r) \mid a \equiv 1 \pmod{t}, b \equiv 0 \pmod{t^2}, \}$, which is $\Gamma(t,r)$. On the other hand the orbit of $(1,0) \in \mathbb{Z}_t \times \mathbb{Z}_{t^2}$ is $\left\{ (a, b) \in \mathbb{Z}_t \times \mathbb{Z}_{t^2} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(t)(r) \right\}$: that is, the set of possible first rows of a matrix in $\Gamma_0(t)(r)$ taken mod $t$ in the first column and mod $t^2$ in the second. This is evidently equal to $\{(1, t'b') \mid b' \in \mathbb{Z}_t\}$, and hence of size $t$. Thus $[\Gamma(t)(r) : \Gamma(t,r)] = t$, which completes the proof. \hfill $\Box$

The standard case is only slightly different, but now there is torsion.

Proposition V.5 If $D_v$ is standard and $F \in \mathfrak{S}_k^1(\Gamma^k_{11})$ then $\theta^+_w(r^2\gamma, t^2\gamma^2)$ is a Jacobi form of weight $k$ and index $w$ for a group $\Gamma^k(t,r)$, which contains $\Gamma(t,r)$ as a subgroup of index 2.

Proof. Although the standard boundary components are most obviously given by $(0,0,0,1)$ for $r = t$ and $(0,0,1,0)$ for $r = 1$, we choose to take advantage of the calculations that we have already performed by working instead with $(0,0,t,1)$ and $(0,0,1,1)$. Lemma V.3 is still true, but we also have $Q^{-1}_r(1)Q_t \in \Gamma^k_{11}$ and $Q^{-1}_t(-z)Q_1 \in \Gamma^k_{11}$. These give rise to the stated extra invariance. \hfill $\Box$
Lemma V.6 The dimension of the space $J_{3k, w}(\Gamma'(t, r))$ of Jacobi forms of
weight $3k$ and index $w$ for $\Gamma'(t, r)$ is given as a polynomial in $k$ and $w$ by
$$\dim J_{3k, w}(\Gamma'(t, r)) = \delta rtv \left( \frac{kw}{2} + \frac{w^2}{6} \right) + \text{linear terms}$$
where $\delta = \frac{1}{2}$ if $r = 1$ or $r = t$ and $\delta = 1$ otherwise.

Proof. By [4, Theorem 3.4] we have
$$\dim J_{3k, w}(\Gamma'(t, r)) \leq \sum_{i=0}^{2w} \dim \mathcal{S}_{3k+i}(\Gamma'(t, r)). \tag{4}$$
Since $\Gamma'(t, r)$ is torsion-free, the corresponding modular curve has genus
$$1 + \frac{\mu(t, r)}{12} - \frac{\nu(t, r)}{2},$$
where $\mu(t, r)$ is the index of $\Gamma'(t, r)$ in $\text{PSL}(2, \mathbb{Z})$ and $\nu(t, r)$
is the number of cusps (see [18, Proposition 1.40]). Hence by [18, Theorem 2.23] the space of modular forms satisfies
$$\dim \mathcal{S}_k(\Gamma'(t, r)) = k \left( \frac{\mu(t, r)}{12} - \frac{\nu(t, r)}{2} \right) + \frac{k^2}{2} \nu(t, r) + O(1)$$
$$= \frac{k\mu(t, r)}{12} + O(1) \tag{5}$$
as a polynomial in $k$. By Lemma V.4 we have $\mu(t, r) = \frac{1}{2}rt\phi_2(t) = rtv$ for the nonstandard cases, $\mu(t, 1) = \frac{1}{2}tv$ and $\mu(t, t) = \frac{1}{2}t^2v$. Now the result follows from equations (5) and (4).

If $F \in \mathcal{S}_{3k}^{\infty}(\Gamma'_t)$ then $F,(d\tau_1 \wedge d\tau_2 \wedge d\tau_3)^{\otimes k}$ extends over the component $D_F$ if and only if $\theta_{w, k}^t = 0$ for all $w < k$; see [1, Chapter IV, Theorem 1]. Hence the obstruction $\Omega_F$ coming from the boundary component $D_F$ is
$$\Omega_F = \sum_{w=0}^{k-1} \dim J_{3k, w}(\Gamma'(t, r)) \tag{6}$$
where $\Gamma'(t, r) = \Gamma(t, r)$ if $D_F$ is nonstandard.

By Corollary IV.2 the total obstruction from the boundary is
$$\Omega_\infty = \sum_{r \mid t} \#\Phi(h) \#\Phi(r) \sum_{w=0}^{k-1} \dim J_{3k, w}(\Gamma'(t, r)),$$
and we may assume that $k$ is even.

Corollary V.7 The obstruction coming from the boundary is
$$\Omega_\infty \leq \left( \sum_{r \mid t} \delta rtv \#\Phi(h) \#\Phi(r) \right) \frac{11k^3}{30} + O(k^2).$$

Proof. Summing the expression in Lemma V.6 for $0 \leq w < k$, as required by equation (6) gives the coefficient of $\frac{11k^3}{30}$ and the rest comes directly from Lemma V.6 and Corollary IV.2. \qed
VI Intersection numbers

We need to know the degrees of the normal bundles of the curves that generate $\text{Pic} \ H_1$ and $\text{Pic} \ H_2$. For this we first need to describe the surfaces $H_1$ and $H_2$. The statements and the proofs are very similar to the corresponding results for the case of $\mathcal{A}_p^{\text{lev}}$, given in [11] and [12]. Therefore we simply refer to those sources for proofs, pointing out such differences as there are.

**Proposition VI.1** $H_1$ is isomorphic to $X(t) \times X(t)$.

*Proof.* Identical to [11, I.5.53].

**Proposition VI.2** $H_2$ is the minimal resolution of a surface $\tilde{H}_2$ which is given by two $\text{SL}(2,\mathbb{Z}_2)$-covering maps

$$X(2t) \times X(2t) \twoheadrightarrow \tilde{H}_2 \twoheadrightarrow X(t) \times X(t).$$

The singularities that are resolved are $v^2$ ordinary double points, one over each point $(\alpha, \beta) \in X(t) \times X(t)$ for which $\alpha$ and $\beta$ are cusps.

*Proof.* Similar to [11, Proposition I.5.55] and the discussion before [12, Proposition 4.21]. $X(2)$ and $X(2p)$ are both replaced by $X(2t)$ and $X(1)$ and $X(p)$ by $X(t)$. Since $t > 3$ there are no elliptic fixed points and hence no other singularities in this case.

**Proposition VI.3** $H_1^\circ$ and $H_2^\circ$ meet the standard boundary components $D_v$ transversally in irreducible curves $C_v \cong X^\circ(t)$ and $C'_v \cong X^\circ(2t)$ respectively. $D_v$ is isomorphic to the (open) Kummer modular surface $K^\circ(t)$, $C_v$ is the zero section and $C'_v$ is the 3-section given by the 2-torsion points of the universal elliptic curve over $X(t)$.

*Proof.* This is essentially the same as [11, Proposition I.5.49], slightly simpler in fact. We may work with $v = (0,0,1,0)$ and copy the proof for the central boundary component in $\mathcal{A}_p^{\text{lev}}$, replacing $p$ by $t$ (again the fact that $p$ is prime is not used).

We do not claim that the closure of $D_v$ is the Kummer modular surface $K(t)$. They are, however, isomorphic near $H_1$ and $H_2$. We remark that $H_1$ and $H_2$ do not meet the nonstandard boundary divisors, because of Lemma V.2.

**Proposition VI.4** $\mathcal{A}_t^{\text{gal}}$ is smooth near $H_1$ and $H_2$.

*Proof.* Certainly $\mathcal{A}_t^{\text{gal}}$ is smooth since the only torsion in $\Gamma_t^{\text{gal}}$ is 2-torsion fixing a divisor in $\mathcal{H}_2$. There can in principle be singularities at infinity, but such singularities must lie on corank 2 boundary components not meeting $H_1$ nor $H_2$ (again this follows from Lemma V.2).
Corollary VI.5 $H_1$ does not meet $H_2$.

Proof. Since $A^{\text{bl}}_t$ and the divisors $H_1$ and $H_2$ are smooth at the relevant points, the intersection must either be empty or contain a curve. However, the intersection also lies in the corank 2 boundary components. These components consist entirely of rational curves, and if $t > 5$ then $H_1 \cong X(t) \times X(t)$ contains no rational curves. Hence $H_1 \cap H_2 = \emptyset$.

With a little more work one can check that this is still true for $t \leq 5$, but we are in any case not concerned with that. □

Proposition VI.6 The Picard group $\text{Pic} H_1$ is generated by the classes of $\Sigma_1 = \mathcal{C}_{0010}$ and $\Psi_1 = \mathcal{C}_{0001}$. The intersection numbers are $\Sigma_1^2 = \Psi_1^2 = 0$, $\Sigma_1 . \Psi_1 = 1$ and $\Sigma_1 . H_1 = \Psi_1 . H_1 = -\mu / 6$.

Proof. As in [12, Proposition 4.18] (but one has to use the alternative indicated in the remark that follows). □

Proposition VI.7 The Picard group $\text{Pic} H_2$ is generated by the classes of $\Sigma_2$ and $\Psi_2$, which are the inverse images of general fibres of the two projections in $X(t) \times X(t)$, and of the exceptional curves $R_{\alpha \beta}$ of the resolution $H_2 \rightarrow \hat{H}_2$. The intersection numbers in $H_2$ are $\Sigma_2^2 = \Psi_2^2 = \Sigma_2 . R_{\alpha \beta} = \Psi_2 . R_{\alpha \beta} = 0$, $R_{\alpha \beta} . R_{\alpha' \beta'} = -2 \delta_{\alpha \alpha'} \delta_{\beta \beta'}$ and $\Sigma_2 . \Psi_2 = 6$. In $A^{\text{bl}}_t$ we have $\Sigma_2 . H_2 = \Psi_2 . H_2 = -\mu$ and $R_{\alpha \beta} . H_2 = -4$.

Proof. The same as the proofs of [12, Proposition 4.21] and [12, Lemma 4.24]. The curves $R'_{(a, b)}$ from [12] arise from elliptic fixed points so they are absent here. □

Notice that $\Sigma_2$ and $\Psi_2$ are also images of the general fibres in $X(2t) \times X(2t)$ and are themselves isomorphic to $X(2t)$.

VII Branch locus

The closure of the branch locus of the map $\mathbb{H}_2 \rightarrow A^{\text{bl}}_t$ is $H_1 \cup H_2$ and modular forms of weight $3k$ (for $k$ even) give rise to $k$-fold differential forms with poles of order $k/2$ along $H_1$ and $H_2$. We have to calculate the number of conditions imposed by these poles.

Proposition VII.1 The obstruction from $H_1$ to extending modular forms of weight $3k$ to $k$-fold holomorphic differential forms is

$$\Omega_1 \leq \nu^2 \left( \frac{1}{2} - \frac{2t}{21} + \ell^2 \left( \frac{1}{21} + \frac{1}{80t} \right) \right) k^3 + O(k^2).$$
Proof. If $F$ is a modular form of weight $3k$ for $k$ even, vanishing to sufficiently high order at infinity, and $\omega = d\tau_1 \wedge d\tau_2 \wedge d\tau_3$, then $F \omega^k$ determines a section of $kK + \frac{b}{2}H_1 + \frac{b}{2}H_2$, where $K$ denotes the canonical sheaf of $\mathcal{A}_{12}^k$. From

$$0 \rightarrow \mathcal{O}(-H_1) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{H_1} \rightarrow 0$$

we get, for $0 \leq j < k/2$

$$0 \rightarrow H^0(kK + (\frac{b}{2} - j - 1)H_1 + \frac{b}{2}H_2) \rightarrow H^0(kK + (\frac{b}{2} - j)H_1 + \frac{b}{2}H_2)$$

$$\rightarrow H^0((kK + (\frac{b}{2} - j)H_1 + \frac{b}{2}H_2)|_{H_1})$$

so

$$h^0(kK + (\frac{b}{2} - j)H_1 + \frac{b}{2}H_2) \leq h^0(kK + (\frac{b}{2} - j - 1)H_1 + \frac{b}{2}H_2)$$

$$+ h^0((kK + (\frac{b}{2} - j)H_1 + \frac{b}{2}H_2)|_{H_1}).$$

Note that, by Lemma VI.5, $H_2|_{H_1} = 0$. Therefore

$$h^0(kK + \frac{b}{2}H_2) \geq h^0(kK + \frac{b}{2}H_1 + \frac{b}{2}H_2) + \sum_{j=0}^{k/2-1} h^0((kK + (\frac{b}{2} - j)H_1)|_{H_1}),$$

so

$$\Omega_1 \leq \sum_{j=0}^{k/2-1} h^0((kK + (\frac{b}{2} - j)H_1)|_{H_1}) = \sum_{j=0}^{k/2-1} h^0(kK_{H_1} - (\frac{b}{2} + j)H_1|_{H_1}). \quad (7)$$

By Lemma VI.6, $K_{H_1}$ and $H_1|_{H_1}$ are both multiples of $\Sigma_1 + \Phi_1$, and any positive multiple of $\Sigma_1 + \Psi_1$ is ample. Suppose $H_1|_{H_1} = a_1(\Sigma_1 + \Psi_1)$ and $K_{H_1} = b_1(\Sigma_1 + \Psi_1)$. Then

$$-\frac{\mu}{6} = \Sigma_1.H_1 = a\Sigma_1.(\Sigma_1 + \Psi_1) = a_1$$

and

$$\frac{\mu}{6} - \nu = 2g(\Sigma_1) - 2 = (K_{H_1} + \Sigma_1).\Sigma_1 = K_{H_1}.\Sigma_1 = b_1$$

Hence, using equation (7)

$$\Omega_1 \leq \sum_{j=0}^{k/2-1} h^0((\frac{b\nu}{6} - k\nu + \frac{b\mu}{6} + \frac{b\mu}{6})(\Sigma_1 + \Psi_1))$$

$$= \sum_{j=0}^{k/2-1} h^0((\frac{b\nu}{6} - k\nu + \frac{\mu\nu}{6})(\Sigma_1 + \Psi_1)).$$
Since $t \geq 7$ (we know from [14] that $A_{i}^{13}$ is rational for $t \leq 5$), we have $\frac{3}{4}k\nu - k\nu + \frac{i}{6}\nu - \frac{\nu}{6} + \nu > 0$ for all $j$ and hence $(\frac{3}{4}k\nu - k\nu + \frac{i}{6}\nu)(\Sigma + \Psi) - K_{H_{1}}$ is ample. So by vanishing we have

$$\Omega_{1} \leq \sum_{j=0}^{k/2-1} \left( \frac{k}{4} - k\nu + \frac{i}{6}\nu \right)^{2} (\Sigma + \Psi)^{2} + O(k^{2})$$

$$= \sum_{j=0}^{k/2-1} \left( \frac{k}{4} - k\nu + \frac{i}{6}\nu \right)^{2} + O(k^{2})$$

$$= \nu^{2} \left( \frac{1}{2} - \frac{7}{3}t + \frac{1}{2} \right) k^{3} + O(k^{2}).$$

\[\square\]

Next we carry out the same calculation for $H_{2}$.

**Proposition VII.2** The obstruction from $H_{2}$ is

$$\Omega_{2} \leq \nu^{2} \left( \frac{1}{2} + \frac{1}{6} \right) t^{2} - \left( \frac{1}{4} + \frac{1}{3} \right) t - \frac{7}{8} + \frac{1}{3} t^{3} + O(k^{2}).$$

**Proof.** By the same argument as above (equation (7)) the obstruction is

$$\Omega_{2} \leq \sum_{j=0}^{k/2-1} k^{0} (kK_{H_{2}} - \frac{k}{4} + j) \Omega_{2}|_{H_{2}}.$$ 

In this case $H_{2}|_{H_{2}} = a_{2}(\Sigma + \Psi) + c_{2}R$, where $R = \sum_{\alpha,\beta} R_{\alpha,\beta}$ is the sum of all the exceptional curves of $H_{2} \to \tilde{H}_{2}$, and $K_{H_{2}} = b_{2}(\Sigma + \Psi) + d_{2}R$. Since $\Sigma \cong X(2t)$ we have by [18, 1.6.4]

$$2g(\Sigma) - 2 = \frac{1}{3}(t - 3)\nu(2t) = \mu - \frac{\nu}{7}.$$ 

Hence

$$-\mu = \Sigma_{2}.H_{2} = a_{2}\Sigma_{2} + a_{2}\Sigma_{2}.\Psi_{2} + c_{2}\Sigma_{2}.R = 6a_{2}$$

so $a_{2} = -\mu/6$, and

$$-4\nu^{2} = R.H_{2} = a_{2}\Sigma_{2}.R + a_{2}\Psi_{2}.R + c_{2}R^{2} = -2\nu^{2}c_{2}$$

so $c_{2} = 2$. Therefore

$$H_{2}|_{H_{2}} = -\frac{\nu}{6}(\Sigma + \Psi) + 2R.$$ 

Similarly

$$\mu - \frac{\nu}{7} = (K_{H_{2}} + \Sigma_{2}) = 6b_{2}$$

so $b_{2} = \mu/6 - \nu/12$, and $0 = R.K_{H_{2}} = d_{2}R^{2}$ so $d_{2} = 0$. Hence

$$K_{H_{2}} = \frac{1}{6}(\mu - \frac{\nu}{7})(\Sigma + \Psi).$$

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Moreover \( L_j = (k - 1)K_{H_2} - (\frac{k}{2} + j)H_2 \mid_{H_2} \) is ample, as is easily checked using the Nakai criterion and the fact that the cone of effective curves on \( H_2 \) is spanned by \( R_{\alpha\beta} \) and by the non-exceptional components of the fibres of the two maps \( H_2 \to X(t) \). These components are \( \Sigma_{\alpha} \equiv \Sigma_{2} - \sum_{\beta} R_{\alpha\beta} \) and \( \Psi_{\beta} \equiv \Psi_{2} - \sum_{\alpha} R_{\alpha\beta} \), and it is simple to check that \( L_j^2, L_j, \Sigma_{\alpha} = L_j, \Psi_{\beta} \) and \( L_j R_{\alpha\beta} \) are all positive for the relevant values of \( j, k \) and \( t \). Therefore

\[
\Omega_2 \leq \sum_{j=0}^{k/2-1} \frac{1}{2} (kK_{H_2} - \frac{k}{2} + j)H_2 \mid_{H_2}^2
\]

\[
= \sum_{j=0}^{k/2-1} \frac{1}{2} \left( \nu \left( \frac{k}{2} - \frac{k}{2} + j \right) \left( \Sigma_{2} + \Psi_{2} \right) + (k + 2j)R \right)^2
\]

\[
= \nu^2 k^2 \left( \frac{2}{3} + \frac{1}{3} + \frac{1}{3} - t \left( \frac{1}{3} + \frac{1}{3} + \frac{1}{3} - 2 - \frac{1}{3} \right) + O(k^2) \right)
\]

since \( \left( \Sigma_{2} + \Psi_{2} \right)^2 = 12 \). \( \square \)

**VIII Final calculation**

In this section we assemble the results of the previous sections into a proof of the main theorem.

**Theorem VIII.1** \( \mathcal{A}_t^{lab} \) is of general type for \( t \) odd and \( t \geq 17 \).

**Proof.** We put \( n = 3k \) in Theorem II.6, and use \( \phi_2(t) = 2\nu \) and the fact that

\[
\phi_4(t) = \prod_{p \mid t} (1 - p^{-1}) = t^2 \phi_2(t) \prod_{p \mid t} (1 + p^{-2}).
\]

This gives the expression

\[
\dim \mathcal{G}_n^s(\Gamma_t^{lab}) = \frac{k^3 \nu^2}{320} t^4 \prod_{p \mid t} (1 + p^{-2}) + O(k^2).
\]

From Proposition VII.1 and Proposition VII.2 we have

\[
\Omega_1 = k^3 \nu^2 \left( \frac{37}{360} t^2 - \frac{7}{24} t + \frac{1}{2} \right) + O(k^2),
\]

\[
\Omega_2 = k^3 \nu^2 \left( \frac{37}{360} t^2 - \frac{7}{24} t - \frac{5}{24} \right) + O(k^2)
\]

and from Corollary V.7 and Corollary IV.2

\[
\Omega_\infty = k^3 \nu^2 \sum_{p \mid t} \frac{11}{360} t^2 \prod_{p \mid r \mid h} (1 - p^{-2}) + O(k^2),
\]

since \( \phi_2(r) \phi_2(h) = t^2 \prod_{p \mid r \mid h} (1 - p^{-2}) \).
It follows that \( \mathcal{A}_t^{\text{bl}} \) is of general type, for odd \( t \), provided

\[
\frac{1}{320} \prod_{p|t} (1 + p^{-2}) t^4 - \frac{481}{864} t^2 + \frac{7}{12} t + \frac{43}{24} - \sum_{r|t} \frac{11}{36r^2} \prod_{p|(r,h)} (1 - p^{-2}) > 0. \tag{8}
\]

This is simple to check: since either \( r = 1 \) or \( r \geq 3 \), and since the sum of the divisors of \( t \) is less than \( t/2 \), the last term can be replaced by \(-\frac{11}{36} t^2 - \frac{11}{108} t^3\) and the \( t \) and constant terms, and the the \( p^{-2} t^4 \) term, can be discarded as they are positive. The resulting expression is a quadratic in \( t \) whose larger root is less than 40, so we need only consider odd \( t \leq 39 \). We deal with primes, products of two primes and prime powers separately. In the case of primes, the expression on the left-hand side of the inequality (8) becomes \( \frac{1}{320} t^4 - \frac{7433}{8640} t^2 + \frac{5}{18} t + \frac{43}{24} \), which is positive for \( t \geq 17 \). The expression in the case of \( t = pq \) is positive if \( t \geq 21 \). For \( t = p^2 \) we get an expression which is negative for \( t = 9 \) but positive for \( t = 25 \), and for \( t = p^3 \) the expression is positive.

One can say something even for \( t \) even, though not if \( t \) is a power of 2.

**Corollary VIII.2** \( \mathcal{A}_t^{\text{bl}} \) is of general type unless \( t = 2^a b \) with \( b \) odd and \( b < 17 \).

**Proof.** \( \mathcal{A}_n^{\text{bl}} \) covers \( \mathcal{A}_t^{\text{bl}} \) for any \( n \), and therefore \( \mathcal{A}_n^{\text{bl}} \) is of general type if \( \mathcal{A}_t^{\text{bl}} \) is of general type. \( \square \)

**References**


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