Abelian surfaces with odd bilevel structure

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Abelian surfaces with weak bilevel structure were introduced by S. Mukai in [14]. There is a coarse moduli space, denoted $\mathcal{A}_t^{\rm bil}$, for abelian surfaces of type $(1,t)$ with weak bilevel structure. $\mathcal{A}_t^{\rm bil}$ is a Siegel modular threefold, and can be compactified in a standard way by Mumford's toroidal method [1]. We denote the toroidal compactification (in this situation also known as the Igusa compactification) by $\mathcal{A}_t^{\rm bil*}$. It is a projective variety over $\mathbb{C}$, and it is shown in [14] that $\mathcal{A}_t^{\rm bil*}$ is rational for $t \le 5$. In this paper we examine the Kodaira dimension $\kappa(\mathcal{A}_t^{\rm bil*})$ for larger t. Our main result is the following (Theorem VIII.1).

Theorem. *$\mathcal{A}_t^{\rm bil*}$ is of general type for t odd and $t \ge 17$.*

It follows from the theorem of L. Borisov [2] that $\mathcal{A}_t^{\rm bil*}$ is of general type for t sufficiently large. If $t = p$ is prime, then it follows from [7] and [12] that $\mathcal{A}_p^{\rm bil*}$ is of general type for $p \ge 37$. Our result provides an effective bound in the general case and a better bound in the case $t = p$. As far as we know, all previous explicit general type results (for instance [7, 12, 15, 8, 16]) have been for the cases $t = p$ or $t = p^2$ only.

It is for brevity that we assume t is odd. If t is even the combinatorial details are more complicated, especially when $t \equiv 2 \bmod 4$, but the method is still applicable. In fact the method is essentially that of [12], with some modifications.

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I Background

If A is an abelian surface with a polarisation H of type $(1,t)$, $t > 1$, then a *canonical level structure*, or simply *level structure*, is a symplectic isomorphism

$$\alpha : \mathbb{Z}_t^2 \longrightarrow K(H) = \{\mathbf{x} \in A \mid t_{\mathbf{x}}^*\mathcal{L} \cong \mathcal{L} \text{ if } c_1(\mathcal{L}) = H\}.$$

The moduli space $\mathcal{A}_t^{\rm lev}$ of abelian surfaces with a canonical level structure has been studied in detail in [11], chiefly in the case $t = p$.

A *colevel structure* on A is a level structure on the dual abelian surface $\hat{A}$: note that H induces a polarisation $\hat{H}$ on $\hat{A}$, also of type $(1,t)$. Alternatively, a colevel structure may be thought of as a symplectic isomorphism

$$\beta : \mathbb{Z}_t^2 \longrightarrow A[t]/K(H)$$

where $A[t]$ is the group of all t-torsion points of A. Obviously the moduli space $\mathcal{A}_t^{\mathrm{col}}$ of abelian surfaces of type $(1,t)$ with a colevel structure is isomorphic to $\mathcal{A}_t^{\mathrm{lev}}$, and each of them has a forgetful morphism ψ^{lev}, ψ^{col} to the moduli space $\mathcal{A}_t$ of abelian surfaces of type $(1,t)$. We define

$$\mathcal{A}_t^{\mathrm{bil}} = \mathcal{A}_t^{\mathrm{lev}} \times_{\mathcal{A}_t} \mathcal{A}_t^{\mathrm{col}}.$$

The forgetful map $\psi^{\mathrm{lev}} : \mathcal{A}_t^{\mathrm{lev}} \to \mathcal{A}_t$ is the quotient map under the action of $\mathrm{SL}(2,\mathbb{Z}_t)$ given by

$$\gamma : [(A,H,\alpha)] \mapsto [(A,H,\alpha\gamma)]$$

where $\gamma \in \mathrm{SL}(2,\mathbb{Z}_t)$ is viewed as a symplectic automorphism of $\mathbb{Z}_t^2$. The action is not effective, because (A,H,α) is isomorphic to $(A,H,-\alpha)$ via the isomorphism $\mathbf{x} \mapsto -\mathbf{x}$; so $-\mathbf{1}_2 \in \mathrm{SL}(2,\mathbb{Z}_t)$ acts trivially. Thus ψ^{lev} is a Galois morphism with Galois group $\mathrm{PSL}(2,\mathbb{Z}_t) = \mathrm{SL}(2,\mathbb{Z}_t)/\pm\mathbf{1}_2$.
A point of $\mathcal{A}_t^{\mathrm{bil}}$ thus corresponds to an equivalence class $[(A,H,\alpha,\beta)]$, where (A,H) is a polarised abelian surface of type $(1,t)$, α and β are level and colevel structures, and (A,H,α,β) is equivalent to (A',H',α',β') if there is an isomorphism $\rho : A \to A'$ such that $\rho^*H' = H$, $\rho\alpha = \alpha'$ and $\hat{\rho}^{-1}\beta = \beta'$. In particular, for general A, we have $(A,H,\alpha,\beta) \cong (A,H,-\alpha,-\beta)$ but $(A,H,\alpha,\beta) \not\cong (A,H,-\alpha,\beta)$. Another way to express this is to say that the wreath product $\mathbb{Z}_2 \wr \mathrm{PSL}(2,\mathbb{Z}_t)$, acts on $\mathcal{A}_t^{\mathrm{bil}}$ with quotient $\mathcal{A}_t$.

Theorem I.1 (Mukai [14]) *$\mathcal{A}_t^{\mathrm{bil}}$ is the quotient of the Siegel upper half-plane $\mathbb{H}_2$ by the group*

$$\Gamma_t^{\mathrm{bil}} = \Gamma_t^{\natural} \cup \zeta\Gamma_t^{\natural}$$

where

$$\Gamma_t^{\natural} = \left\{ \gamma \in \mathrm{Sp}(4,\mathbb{Z}) \mid \gamma - \mathbf{1}_4 \in \begin{pmatrix} t\mathbb{Z} & * & t\mathbb{Z} & t\mathbb{Z} \\ t\mathbb{Z} & t\mathbb{Z} & t\mathbb{Z} & t^2\mathbb{Z} \\ t\mathbb{Z} & * & t\mathbb{Z} & t\mathbb{Z} \\ * & * & * & t\mathbb{Z} \end{pmatrix} \right\}$$

and $\zeta = \mathrm{diag}(1,-1,1,-1)$, acting by fractional linear transformations.

Thus Γ_t^{bil} should be thought of as a subgroup of the paramodular group

$$\Gamma_t = \left\{ \gamma \in \mathrm{Sp}(4,\mathbb{Q}) \mid \gamma - \mathbf{1}_4 \in \begin{pmatrix} * & * & * & t\mathbb{Z} \\ t\mathbb{Z} & * & t\mathbb{Z} & t\mathbb{Z} \\ * & * & * & t\mathbb{Z} \\ * & \frac{1}{t}\mathbb{Z} & * & * \end{pmatrix} \right\}.$$

(The paramodular group is the group denoted $\Gamma^{\circ}_{1,t}$ in [11] and [5].)
For some purposes it is more convenient to work with the conjugate $\tilde{\Gamma}^{\rm bil}_t = R_t\Gamma^{\rm bil}_t R_{t^{-1}}$ of $\Gamma^{\rm bil}_t$ by $R_t = \mathrm{diag}(1,1,1,t)$, and with the corresponding conjugates $\tilde{\Gamma}^{\natural}_t$, $\tilde{\Gamma}^{\rm lev}_t$ etcetera. These groups have the advantage that they are subgroups of $\mathrm{Sp}(4,\mathbb{Z})$ rather than $\mathrm{Sp}(4,\mathbb{Q})$, and are defined by congruences mod t, not mod t^2, but their action on $\mathbb{H}_2$ is not the usual one by fractional linear transformations.
If E_i are elliptic curves and $(A,H) = \big(E_1\times E_2, c_1\big(\mathcal{O}_{E_1}(1)\boxtimes\mathcal{O}_{E_2}(t)\big)\big)$, we say that (A,H) is a product surface. In this case $K(H) = \{0_{E_1}\}\times E_2[t]$, so a level structure on A may be thought of as a full level-t structure on E_2. The automorphism $(\mathbf{x},\mathbf{y})\mapsto(\mathbf{x},-\mathbf{y})$ of $A = E_1\times E_2$ induces an isomorphism $(A,H,\alpha,\beta)\to(A,H,-\alpha,\beta)$ in this case, so a product surface with a weak bilevel structure still has an extra automorphism. The corresponding locus in the moduli space arises from the fixed locus of ζ in $\mathbb{H}_2$, and will be of great importance in this paper.
The geometry of $\mathcal{A}^{\rm bil*}_t$ shows many similarities with that of $\mathcal{A}^{\rm lev*}_t$, which was studied (in the case of t an odd prime) in the book [11]. In many cases where the proofs of intermediate results are very similar to those of corresponding results in [11] we omit the details and simply indicate the appropriate reference.

II Modular groups and modular forms

We first collect some facts about congruence subgroups in $\mathrm{SL}(2,\mathbb{Z})$ and some related combinatorial information. For $r\in\mathbb{N}$ we denote by $\Gamma_1(r)$ the principal congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$. We denote the modular curve $\Gamma_1(r)\backslash\mathbb{H}$ by $X^\circ(r)$, and the compactification obtained by adding the cusps by $X(r)$.
For $m,r\in\mathbb{N}$, define

$$\Phi_m(r) = \{\mathbf{a}\in\mathbb{Z}_r^m \mid \mathbf{a} \text{ is not a multiple of a zerodivisor in } \mathbb{Z}_r\},$$

that is, $\mathbf{a}\in\Phi_m(r)$ if and only if $\mathbf{a} = z\mathbf{a}'$ implies $z\in\mathbb{Z}_r^*$; and put $\phi_m(r) = \#\Phi_m(r)$. We also put $\overline{\Phi}_m(r) = \Phi_m(r)/\pm 1$.

Lemma II.1 *If the primes dividing r are $p_1 < p_2 < \ldots < p_n$ then*

$$\phi_m(r) = \sum_{i=0}^{n}(-1)^i \sum_{p_{j_1},\ldots,p_{j_i}} \left(r\prod_{k=1}^{i} p_{j_k}^{-1}\right)^m = r^m\prod_{p|r}(1-p^{-m}).$$

Proof. We first prove that $\phi_m(r)$ is a multiplicative function. First we suppose that $r = pq$, with $\gcd(p,q) = 1$. It is easy to see that $\mathbf{a}\in\Phi_m(r)$ if and only if $\mathbf{a}_p\in\Phi_m(p)$ and $\mathbf{a}_q\in\Phi_m(q)$, where $\mathbf{a}_p$ denotes the reduction of $\mathbf{a}$ mod p.

We divide $\mathbb{Z}_r^m$ into residue classes mod p: that is, we write $\mathbb{Z}_r^m$ as the disjoint union of subsets $S_{\mathbf{c}}$ for $\mathbf{c} \in \mathbb{Z}_p^m$, where $S_{\mathbf{c}} = \{\mathbf{a} \mid \mathbf{a}_p = \mathbf{c}\}$. There are $\phi_m(p)$ subsets $S_{\mathbf{c}}$ such that $\mathbf{r} \in \Phi_m(p)$.
The reduction mod q map $S_{\mathbf{c}} \to \mathbb{Z}_q^m$ is bijective, since it is the inverse of the injective map $\mathbf{b} \mapsto \mathbf{c} + p\mathbf{b} \in \mathbb{Z}_r^m$. Hence in each of the $\phi_m(p)$ subsets $S_{\mathbf{c}}$, $\mathbf{c} \in \Phi_m(p)$ there are $\phi_m(q)$ elements whose reduction mod q belongs to $\Phi_m(q)$. It follows that $\phi_m(r) = \phi_m(p)\phi_m(q)$.
Finally, we check that if $r = p^k$, p prime, then $\phi_m(r) = r^m(1 - p^{-m})$. If $\mathbf{a} \notin \Phi_m(r)$, then $\mathbf{a} = p\mathbf{a}'$ for a unique $\mathbf{a}' \in \mathbb{Z}_{r/p}^m$, so there are $(p^{k-1})^m$ such elements $\mathbf{a}$. □

Note that ϕ_1 is the Euler ϕ function, and $\Phi_1(r)$ is the set of non-zerodivisors of $\mathbb{Z}_r$.

Corollary II.2 *The order of* $\mathrm{SL}(2, \mathbb{Z}_t)$ *is given by*

$$|\,\mathrm{SL}(2, \mathbb{Z}_t)| = t\phi_2(t) = t^3 \prod_{p|t} (1 - p^{-2}).$$

Proof. (See also [18, §1.6].) If $A \in \mathrm{SL}(2, \mathbb{Z}_t)$, then $A_1 = (a_{11}, a_{12}) \in \Phi_2(t)$. So by Euclid's algorithm we can find $A_2' = (a_{21}', a_{22}')$ such that $\det \begin{pmatrix} A_1 \\ A_2' \end{pmatrix} = \gcd(a_{11}, a_{12}) = r$. Replacing A_2' by $A_2 = r^{-1}A_2'$, we get a matrix A with $\det(A) = 1$. Furthermore, if $B_j = \begin{pmatrix} A_1 \\ A_2 + jA_1 \end{pmatrix}$, $j = 0, \ldots, t-1$, then $\det(B_j) = \det(A) = 1$, and $B_j \neq B_{j'}$ if $j \neq j'$. So $|\,\mathrm{SL}(2, \mathbb{Z}_t)| = t\phi_2(t)$. □

For $r > 2$, put $\mu(r) = [\mathrm{PSL}(2, \mathbb{Z}) : \Gamma_1(r)]$. By Corollary II.2 we have

$$\mu(r) = r^3 \prod_{p|r} (1 - p^{-2}).$$

We need the following well-known lemma.

Lemma II.3 *If* $r > 2$ *then* $X(r)$ *has*

$$\nu(r) = \mu(r)/r = r^2 \prod_{p|r} (1 - p^{-2})$$

cusps and is a smooth complete curve of genus $g = 1 + \frac{\mu(r)}{12} - \frac{\nu(r)}{2}$.

Proof. See [18, pp. 23–24]. □

We denote $\mu(t)$ by μ and $\nu(t)$ by ν. Note that $\phi_2(1) = \nu(1) = 1$ and $\phi_2(r) = 2\nu(r)$ for $r > 2$.

Now we turn to subgroups of $\mathrm{Sp}(4,\mathbb{Q})$ and to modular forms. Denote by $\mathfrak{S}_n^*(\Gamma)$ the space of weight n cusp forms for $\Gamma \subseteq \mathrm{Sp}(4,\mathbb{Q})$. We need the groups $\bar{\Gamma}(1) = \mathrm{PSp}(4,\mathbb{Z})$ and, for $\ell \in \mathbb{N}$,

$$\Gamma(\ell) = \big\{\gamma \in \mathrm{Sp}(4,\mathbb{Z}) \mid \bar{\gamma} = \mathbf{1}_4 \in \mathrm{Sp}(4,\mathbb{Z}_\ell)\big\}.$$

If $t^2|\ell$ then $\Gamma(\ell) \lhd \Gamma_t^{\mathrm{bil}}$, because $\Gamma(\ell) \subseteq \Gamma_t^{\mathrm{bil}}$ and $\Gamma(\ell)$ is normal in $\Gamma(1) = \mathrm{Sp}(4,\mathbb{Z})$.
By a previous calculation [19] we know that

$$\dim \mathfrak{S}_n^*\big(\Gamma(\ell)\big) = \frac{n^3}{8640}\left[\bar{\Gamma}(1) : \Gamma(\ell)\right] + O(n^2)$$

(as long as $\ell > 2$ we can consider $\Gamma(\ell)$ as a subgroup of $\mathrm{PSp}(4,\mathbb{Z})$ rather than $\mathrm{Sp}(4,\mathbb{Z})$). A standard application of the Atiyah–Bott fixed-point theorem (see [9], or in this context [12]) gives

$$\dim \mathfrak{S}_n^*\big(\Gamma_t^{\mathrm{bil}}\big) = \frac{a}{\left[\Gamma_t^{\mathrm{bil}} : \Gamma(\ell)\right]} \dim \mathfrak{S}_n^*\big(\Gamma(\ell)\big) + O(n^2)$$

where a is the number of elements $\gamma \in \Gamma_t^{\mathrm{bil}}$ whose fixed locus in $\mathbb{H}_2$ has dimension 3. Thus a is the number of elements of Γ_t^{bil} that act trivially on $\mathbb{H}_2$. In $\mathrm{Sp}(4,\mathbb{Z})$ there are two such elements, $\pm\mathbf{1}_4$, but if $t > 2$ then $-\mathbf{1}_4 \not\in \Gamma_t^{\mathrm{bil}}$. So $a = 1$, and hence

$$\begin{aligned}
\dim \mathfrak{S}_n^*\big(\Gamma_t^{\mathrm{bil}}\big) &= \frac{1}{\left[\Gamma_t^{\mathrm{bil}} : \Gamma(\ell)\right]} \dim \mathfrak{S}_n^*\big(\Gamma(\ell)\big) + O(n^2) \\
&= \frac{n^3}{8640} \frac{\left[\bar{\Gamma}(1) : \Gamma(\ell)\right]}{\left[\Gamma_t^{\mathrm{bil}} : \Gamma(\ell)\right]} + O(n^2) \\
&= \frac{n^3}{8640}\left[\bar{\Gamma}(1) : \Gamma_t^{\mathrm{bil}}\right] + O(n^2). \qquad (1)
\end{aligned}$$

The number $\left[\bar{\Gamma}(1) : \Gamma_t^{\mathrm{bil}}\right]$ is equal to the degree of the map $\mathcal{A}_t^{\mathrm{bil}} \to \mathcal{A}_1$ (actually there are two such maps of the same degree), where $\mathcal{A}_1$ is the moduli space of principally polarized abelian surfaces. Now

$$\begin{aligned}
\left[\bar{\Gamma}(1) : \Gamma_t^{\mathrm{bil}}\right] &= \tfrac{1}{2}\left[\bar{\Gamma}(1) : \Gamma_t^{\natural}\right] \\
&= \tfrac{1}{2}\left[\bar{\Gamma}(1) : \Gamma_t^{\mathrm{lev}}\right]\left[\Gamma_t^{\mathrm{lev}} : \Gamma_t^{\natural}\right].
\end{aligned}$$

We can see directly that $\Gamma_t^{\mathrm{lev}} \supset \Gamma_t^{\natural}$ since

$$\Gamma_t^{\mathrm{lev}} = \left\{ \gamma \in \mathrm{Sp}(4,\mathbb{Z}) \mid \gamma - \mathbf{1}_4 \in \begin{pmatrix} * & * & * & t\mathbb{Z} \\ t\mathbb{Z} & t\mathbb{Z} & t\mathbb{Z} & t^2\mathbb{Z} \\ * & * & * & t\mathbb{Z} \\ * & * & * & t\mathbb{Z} \end{pmatrix} \right\}.$$

Lemma II.4 *The map*

$$\varphi : \Gamma_t^{\mathrm{lev}} \longrightarrow \mathrm{SL}(2,\mathbb{Z}_t),\ A \mapsto \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}$$

is a surjective group homomorphism, and the kernel is $\Gamma_t^{\natural}$.

Proof. The surjectivity follows from the well-known fact that the redution mod t map $\mathrm{red}_t : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}_t)$ is surjective, and the rest is obvious. □

Lemma II.5 *For* $t > 2$*, the index* $[\bar{\Gamma}(1) : \Gamma_t^{\mathrm{lev}}]$ *is equal to* $t\phi_4(t)/2$.

Proof. The proof is almost the same as proof of [13, Lemma 0.5]. In place of the chain of groups $\Gamma_{1,p} < {}_0\Gamma_{1,p} < \Gamma' = \Gamma(1)$, we use the chain $\Gamma_t^{\mathrm{lev}} < {}_0\Gamma_{1,t} < \Gamma(1)$. Furthermore, we use the set $\Phi_4(t)$ where $\mathrm{SL}(4,\mathbb{Z}_t)$ acts. Note that $\mathrm{SL}(4,\mathbb{Z})$ still acts transitively on $\Phi_4(t)$, via

$$\begin{pmatrix} b_{11} & 0 & b_{12} & 0 \\ 0 & 1 & 0 & 0 \\ b_{21} & 0 & b_{22} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} B & 0 \\ 0 & {}^tB^{-1} \end{pmatrix},$$

for $B \in \mathrm{SL}(2,\mathbb{Z})$.

Following the same steps as in [13], and substituting $\phi_m(t)$ for $p^m - 1 = \phi_m(p)$, we then find that $[\,{}_0\Gamma_{1,t} : \Gamma_t^{\mathrm{lev}}] = t\phi_1(t)$ and $[\,{}_0\Gamma_{1,t} : \Gamma(1)| = \phi_4(t)/\phi_1(t)$, so $[\overline{\Gamma}(1) : \Gamma_t^{\mathrm{lev}}] = t\phi_4(t)/2$. □

Theorem II.6 *The number of cusp forms of weight* n *for* Γ_t^{bil} *(for* $t > 2$*) is given by*

$$\begin{aligned} \dim \mathfrak{S}_n^*(\Gamma_t^{\mathrm{bil}}) &= \frac{n^3}{34560} t^2 \phi_2(t)\phi_4(t) \\ &= \frac{n^3}{34560} t^8 \prod_{p|t} (1 - p^{-2})(1 - p^{-4}). \end{aligned}$$

Proof. Immediate from equation (1), Corollary II.2 and Lemma II.5. □

III Torsion in the modular group

We know that $\Gamma_t^{\mathrm{bil}} \subset \mathrm{Sp}(4,\mathbb{Z})$, and the conjugacy classes of torsion elements in $\mathrm{Sp}(4,\mathbb{Z})$ are known ([6, 20]). See [10] for a summary of the relevant information.

If $\gamma \in \Gamma_t^\natural$ then the reduction mod t of γ is

$$\bar{\gamma} = \begin{pmatrix} 1 & * & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & * & 1 & 0 \\ * & * & * & 1 \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{Z}_t),$$

so the characteristic polynomial $\chi(\bar{\gamma})$ is $(1-x)^4 \in \mathbb{Z}_t[x]$. On the other hand, if $\gamma \in \zeta\Gamma_t^\natural$ then

$$\bar{\gamma} = \zeta \begin{pmatrix} 1 & * & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & * & 1 & 0 \\ * & * & * & 1 \end{pmatrix} = \begin{pmatrix} 1 & * & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & * & 1 & 0 \\ * & * & * & -1 \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{Z}_t),$$

so $\chi(\bar{\gamma}) = (1-x)^2(1+x)^2 \in \mathbb{Z}_t[x]$.
The only classes in the list in [20], up to conjugacy, where the characteristic polynomials have this reduction mod t ($t > 2$) are I(1), where $\chi(\gamma) = (1-x)^4$, II(1)a and II(1)b. Class I(1) consists of the identity; class II(1)a includes ζ so this just gives us the conjugacy class of ζ. Class II(2)b is the $\mathrm{Sp}(4, \mathbb{Z})$-conjugacy class of ξ, where

$$\xi = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \in \Gamma_t^{\mathrm{bil}}.$$

Proposition III.1 *Every nontrivial element of finite order in Γ_t^{bil} (for $t > 2$) has order 2, and is conjugate to ζ or to ξ in Γ_t^{bil} if t is odd.*

Proof. It follows from the list in [20] that the only torsion for $t > 2$ is 2-torsion (this is still true if t is even). The 2-torsion of the group Γ_t^{lev} was studied by Brasch [3]. There are five types but only two of them occur for odd t. The representatives for these conjugacy classes given in [3] are (up to sign) ζ and ξ; so the assertion of the theorem is that the Γ_t^{bil}-conjugacy classes of ζ and ξ coincide with the intersections of their Γ_t^{lev}-conjugacy classes with Γ_t^{bil}. This is checked in [17, Proposition 3.2] for the case $t = 6$ (the relevant cases are called ζ_0 and ζ_3 there), but the proof is valid for all $t > 2$. □

We put

$$\mathcal{H}_1 = \left\{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix} \middle| \operatorname{Im}\tau_1 > 0,\ \operatorname{Im}\tau_3 > 0 \right\} \subset \mathbb{H}_2 \tag{2}$$

and

$$\mathcal{H}_2 = \left\{ \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \middle| 2\tau_2 + \tau_3 = 0 \right\} \subset \mathbb{H}_2. \tag{3}$$

These are the fixed loci of ζ and ξ respectively. We denote by H_1° and H_2° the images of $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{A}_t^{\mathrm{bil}}$, and by H_1 and H_2 their respective closures in $\mathcal{A}_t^{\mathrm{bil}*}$.

Lemma III.2 *H_i° is irreducible for $i = 1, 2$.*

Proof. This follows at once from Proposition III.1 together with equations (2) and (3). □

The abelian surfaces corresponding to points in H_1° and H_2° are, respectively, product surfaces and bielliptic abelian surfaces, as described in [13] for the case t prime.
We define the subgroup $\Gamma(2t, 2t)$ of $\Gamma(t) \times \Gamma(t)$ by

$$\Gamma(2t, 2t) = \{(M, N) \in \Gamma(t) \times \Gamma(t) \mid M \equiv {}^\top N^{-1} \mod 2\}$$

Lemma III.3 *H_1° is isomorphic to $X^\circ(t) \times X^\circ(t)$, and H_2° is isomorphic to $\Gamma(2t, 2t)\backslash\mathbb{H} \times \mathbb{H}$.*

Proof. Identical to the proofs of the corresponding results [11, Lemma I.5.43] and [11, Lemma I.5.45]. The level-t structure now occurs in both factors, whereas in [11] there is level-1 structure in the first factor and level-p structure in the second. In [11] the level p is assumed to be an odd prime but this fact is not used at that stage: p odd suffices, so we may replace p by t. Thereafter one simply replaces all the groups with their intersection with Γ_t^{bil}, which imposes a level-t structure in the first factor and causes it to behave exactly like the second factor. □

Lemma III.4 *H_1° and H_2° are disjoint.*

Proof. The stabiliser of any point of $\mathbb{H}_2$ in Γ_t^{bil} is cyclic (of order 2), since $\Gamma_t^\natural$ is torsion-free and therefore has no fixed points. A point of $\mathcal{H}_1 \cap \mathcal{H}_2$ would be the image of a point of $\mathbb{H}_2$ stabilised by the subgroup generated by ζ and ξ, which is not cyclic. □

IV Boundary divisors

We begin by counting the boundary divisors. These correspond to $\tilde{\Gamma}_t^{\mathrm{bil}}$-orbits of lines in $\mathbb{Q}^4$: we identify a line by its unique (up to sign) primitive generator $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \mathbb{Z}^4$ with $\mathrm{hcf}(v_1, v_2, v_3, v_4) = 1$. We denote the reduction of $\mathbf{v}$ mod t by $\bar{\mathbf{v}} = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4) \in \mathbb{Z}_t^4$. To fix things we shall say, arbitrarily, that $\mathbf{v}$ is positive if the first non-zero entry $\bar{v}_i$ of $\bar{\mathbf{v}}$ satisfies $\bar{v}_i \in \{1, \ldots, (t-1)/2\}$ (remember that we have assumed that t is odd). Then each line has a unique positive primitive generator.
If $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \mathbb{Z}^4$, we define the t-divisor to be $r = \mathrm{hcf}(t, v_1, v_3)$.

Proposition IV.1 *The lines $\mathbb{Q}\mathbf{v}$ and $\mathbb{Q}\mathbf{w}$ spanned by positive primitive vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^4$ are in the same $\tilde{\Gamma}_t^{\mathrm{bil}}$-orbit if and only if $(\bar{v}_1, \bar{v}_3) = (\bar{w}_1, \bar{w}_3)$ (in particular $\mathbf{v}$ and $\mathbf{w}$ have the same t-divisor, r), and $(v_2, v_4) \equiv \pm(w_2, w_4)$ mod r.*

Proof. Note that if $\Gamma(t)$ is the principal congruence subgroup of level t in $\mathrm{Sp}(4, \mathbb{Z})$ then $\Gamma(t) \lhd \tilde{\Gamma}_t^{\natural}$ and the quotient is

$$\tilde{\Gamma}_t^{\natural}(t) = \left\{ \begin{pmatrix} 1 & k & 0 & k' \\ 0 & 1 & 0 & 0 \\ 0 & l & 1 & l' \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{Z}_t) \right\} \cong \mathbb{Z}_t^4.$$

We claim that two primitive vectors $\mathbf{v}$ and $\mathbf{w}$ are equivalent modulo $\Gamma(t)$ if and only if $\bar{v} = \bar{w}$. It is obvious that $\Gamma(t)$ preserves the residue classes mod t. Conversely, suppose that $\bar{v} = \bar{w}$. Then we can find $\gamma \in \mathrm{Sp}(4, \mathbb{Z})$ such that $\gamma\mathbf{v} = (1, 0, 0, 0)$ (the corresponding geometric fact is that the moduli space $\mathcal{A}_2$ of principally polarised abelian surfaces has only one rank 1 cusp). Since $\Gamma(t) \lhd \mathrm{Sp}(4, \mathbb{Z})$ this means that in order to prove the claim we may assume $\mathbf{v} = (1, 0, 0, 0)$. Then we proceed exactly as in the proof of [5, Lemma 3.3], taking $p = 1$ and $q = t$ (the assumptions that p and q are prime are not used at that point).
The group $\tilde{\Gamma}_t^{\natural}(t)$ acts on the set $(\mathbb{Z}_t^4)^{\times}$ of non-zero elements of $\mathbb{Z}_t^4$ by $\bar{v}_2 \mapsto \bar{v}_2 + k\bar{v}_1 + l\bar{v}_3$ and $\bar{v}_4 \mapsto \bar{v}_4 + k'\bar{v}_1 + l'\bar{v}_3$: so $\bar{\mathbf{v}}$ is equivalent to $\bar{\mathbf{w}}$ if and only if $(\bar{v}_1, \bar{v}_3) = (\bar{w}_1, \bar{w}_3)$, so they have the same t-divisor, and $\bar{v}_2 \in \bar{w}_2 + \mathbb{Z}_t r$ and $\bar{v}_4 \in \bar{w}_4 + \mathbb{Z}_t r$. These are therefore the conditions for primitive vectors $\mathbf{v}$ and $\mathbf{w}$ to be equivalent under $\tilde{\Gamma}_t^{\natural}$. For equivalence under $\tilde{\Gamma}_t^{\mathrm{bil}}$, we get the extra element ζ which makes (v_1, v_2, v_3, v_4) equivalent to $(v_1, -v_2, v_3, -v_4)$. Since we are interested in orbits of lines, not primitive generators, we may restrict ourselves to positive generators $\mathbf{v}$. □

The irreducible components of the boundary divisor of $\mathcal{A}_t^{\mathrm{bil}*}$ correspond to the Γ_t^{bil}-orbits (or equivalently to $\tilde{\Gamma}_t^{\mathrm{bil}}$-orbits) of lines in $\mathbb{Q}^4$. We denote the boundary component corresponding to $\mathbb{Q}\mathbf{v}$ by $D_{\mathbf{v}}$. We shall be chiefly interested in the cases $r = t$ and $r = 1$. We refer to these as the standard components. They are represented by vectors $(0, a, 0, b)$ and $(a, 0, b, 0)$ respectively, in both cases with $\mathrm{hcf}(a, b) = 1$, $0 \le a \le (t-1)/2$ and $0 \le b < t$. Note that there are ν of each of these.

Corollary IV.2 *If t is odd then the number of irreducible boundary divisors of $\mathcal{A}_t^{\mathrm{bil}*}$ with t-divisor r is $\#\overline{\Phi}_2(h)\#\overline{\Phi}_2(r)$, where $h = t/r$. For $r \neq 1$, t, this is equal to $\frac{1}{4}\phi_2(h)\phi_2(r)$.*

Proof. See above for the standard cases. In general, the $\Gamma_t^{\natural}$-orbit of a primitive vector $\mathbf{v}$ is determined by the classes of $(v_1/r, v_3/r)$ in $\Phi_2(h)$ and of

$(\bar{v}_2, \bar{v}_4) \in \Phi_2(r)$. The extra element ζ and the freedom to multiply $\mathbf{v}$ by $-1 \in \mathbb{Q}$ allow us to multiply either of these classes by -1 and the choices therefore lie in $\bar{\Phi}_2(h)$ and $\overline{\Phi}_2(r)$. □

V Jacobi forms

In this section we shall describe the behaviour of a modular form $F \in \mathfrak{S}^*_{3n}(\Gamma_t^{\mathrm{bil}})$ near a boundary divisor $D_{\mathbf{v}}$. The standard boundary divisors are best treated separately, since it is in those cases only that the torsion plays a role: on the other hand, the standard boundary divisors occur for all t and their behaviour is not much dependent on the factorisation of t.

We assume at first, then, that $D_{\mathbf{v}}$ is a nonstandard boundary divisor. Since all the divisors of given t-divisor are equivalent under the action of $\mathbb{Z}_2 \wr \mathrm{SL}(2, \mathbb{Z}_t)$, (because the t-divisor is the only invariant of a boundary divisor of $\mathcal{A}_t$: see [5]) it will be enough to calculate the number of conditions imposed by one divisor of each type. That is to say, we only need consider boundary components in $\mathcal{A}_t^*$.

In view of this we may take $\mathbf{v} = (0, 0, r, 1)$ for some $r|t$ with $1 < r < t$. We write $(0, 0, 0, 1) = \mathbf{v}_{(0,1)}$ (for consistency with [11]) and we put $h = t/r$. Since we want to work with Γ_t^{bil} rather than $\tilde{\Gamma}_t^{\mathrm{bil}}$ (so as to use fractional linear transformations) we must consider the lines $\mathbb{Q}\mathbf{v}R_t = \mathbb{Q}\mathbf{v}'$, where $\mathbf{v}' = (0, 0, 1, h)$, and $\mathbb{Q}\mathbf{v}_{(0,1)}R_t = \mathbb{Q}\mathbf{v}_{(0,1)}$.

Note that $\mathbf{v}'Q_r = \mathbf{v}_{(0,1)}$, where

$$Q_r = \begin{pmatrix} 1 & 1 & 0 & 0 \\ h-1 & h & 0 & 0 \\ 0 & 0 & h & 1-h \\ 0 & 0 & -1 & 1 \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{Z}).$$

Proposition V.1 *If $\mathbf{v}$ has t-divisor $r \neq t$, 1, and $F \in \mathfrak{S}^*_k(\Gamma_t^{\mathrm{bil}})$ is a cusp form of weight k, then there are coordinates $\tau_i^{\mathbf{v}}$ such that F has a Fourier expansion near $D_{\mathbf{v}}$ as*

$$F = \sum_{w \geq 0} \theta_w^{\mathbf{v}}(\tau_1^{\mathbf{v}}, \tau_2^{\mathbf{v}}) \exp 2\pi i w \tau_3^{\mathbf{v}}/rt.$$

Proof. As usual (cf. [11]) we write $\mathcal{P}'_{\mathbf{v}}$ for the stabiliser of $\mathbf{v}'$ in $\mathrm{Sp}(4, \mathbb{R})$, so $\mathcal{P}'_{\mathbf{v}} = Q_r^{-1}\mathcal{P}_{\mathbf{v}_{(0,1)}}Q_r$. We take $P'_{\mathbf{v}} = \mathcal{P}'_{\mathbf{v}} \cap \Gamma_t^{\mathrm{bil}}$: this group determines the structure of $\mathcal{A}_t^{\mathrm{bil}*}$ near $D_{\mathbf{v}}$. It is shown in [11, Proposition I.3.87] that $\mathcal{P}_{\mathbf{v}_{(0,1)}}$ is generated by $g_1(\gamma)$ for $\gamma \in \mathrm{SL}(2, \mathbb{R})$, $g_2 = \zeta$, $g_3(m, n)$ and $g_4(s)$ for m, n, $s \in \mathbb{R}$, where

$$g_1(\gamma) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and g_3 and g_4 are given by

$$g_3(m,n) = \begin{pmatrix} 1 & 0 & 0 & n \\ m & 1 & n & 0 \\ 0 & 1 & 0 & -m \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad g_4(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So $P'_{\mathbf{v}}$ includes the subgroup generated by all elements of the form $Q_r^{-1} g_i Q_r$ with $a, b, c, d, m, n, s \in \mathbb{Z}$ which lie in Γ_t^{bil}. In particular it includes the lattice $\{Q_r^{-1} g_4(rts) Q_r \mid s \in \mathbb{Z}\}$. If we take $Z^{\mathbf{v}} = Q_r^{-1}(Z)$ for $Z = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}$ then we obtain

$$Z^{\mathbf{v}} = \begin{pmatrix} h^2\tau_1 - 2h\tau_2 + \tau_3 & -h(h-1)\tau_1 + (2h-1)\tau_2 - \tau_3 \\ -h(h-1)\tau_1 + (2h-1)\tau_2 - \tau_3 & (h-1)^2\tau_1 - 2(h-1)\tau_2 + \tau_3 \end{pmatrix}.$$

One easily checks that

$$Q_r^{-1} g_4(rt) Q_r : Z^{\mathbf{v}} = \begin{pmatrix} \tau_1^{\mathbf{v}} & \tau_2^{\mathbf{v}} \\ \tau_2^{\mathbf{v}} & \tau_3^{\mathbf{v}} \end{pmatrix} \longmapsto \begin{pmatrix} \tau_1^{\mathbf{v}} & \tau_2^{\mathbf{v}} \\ \tau_2^{\mathbf{v}} & \tau_3^{\mathbf{v}} + rt \end{pmatrix}$$

and this proves the result. □

We define a subgroup $\Gamma(t,r)$ of $\mathrm{SL}(2,\mathbb{Z})$ by

$$\Gamma(t,r) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \equiv d \equiv 1 \bmod t,\ b \equiv 0 \bmod t^2,\ c \equiv 0 \bmod r \right\}.$$

Lemma V.2 *If $D_{\mathbf{v}}$ is nonstandard then $P'_{\mathbf{v}}$ is torsion-free.*

Proof. The only torsion in Γ_t^{bil} is 2-torsion and a simple calculation shows that if $\mathbf{1}_4 \neq g \in \mathcal{P}_{\mathbf{v}_{(0,1)}}$ and $g^2 = \mathbf{1}_4$, then $Q_r^{-1} g Q_r \notin \Gamma_t^{\mathrm{bil}}$ for $r \neq 1,\ t$. □

Proposition V.3 *If $D_{\mathbf{v}}$ is nonstandard and $F \in \mathfrak{S}_k^*(\Gamma_t^{\mathrm{bil}})$ then $\theta_w^{\mathbf{v}}(r\tau_1^{\mathbf{v}}, t\tau_2^{\mathbf{v}})$ is a Jacobi form of weight k and index w for $\Gamma(t,r)$.*

Proof. By direct calculation we find that $Q_r^{-1} g_1(\gamma) Q_r \in \Gamma_t^{\mathrm{bil}}$ if $\gamma \in \Gamma(t,r)$ and $Q_r^{-1} g_3(rm, tn) Q_r \in \Gamma_t^{\mathrm{bil}}$ for $m, n \in \mathbb{Z}$. Using these two elements, another elementary calculation verifies that the transformation laws for Jacobi forms given in [4] are satisfied, since

$$Q_r^{-1} g_3(rm, tn) Q_r : Z^{\mathbf{v}} \longmapsto \begin{pmatrix} \tau_1^{\mathbf{v}} & \tau_2^{\mathbf{v}} + rm\tau_1^{\mathbf{v}} + tn \\ \tau_2^{\mathbf{v}} + rm\tau_1^{\mathbf{v}} + tn & \tau_3^{\mathbf{v}} + 2rm\tau_2^{\mathbf{v}} + r^2m^2\tau_1^{\mathbf{v}} \end{pmatrix}$$

and

$$Q_r^{-1} g_1(\gamma) Q_r : Z^{\mathbf{v}} \longmapsto \begin{pmatrix} \gamma(\tau_1^{\mathbf{v}}) & \tau_2^{\mathbf{v}}/(c\tau_1^{\mathbf{v}} + d) \\ \tau_2^{\mathbf{v}}/(c\tau_1^{\mathbf{v}} + d) & \tau_3^{\mathbf{v}} - c\tau_2^{\mathbf{v}}/(c\tau_1^{\mathbf{v}} + d). \end{pmatrix}$$

□

Lemma V.4 *The index of* $\Gamma(t,r)$ *in* $\Gamma(1)$ *is equal to* $rt\phi_2(t)$ *for* $r \neq 1,\ t$.

Proof. Consider the chain of groups

$$\Gamma(1) = \mathrm{SL}(2,\mathbb{Z}) > \Gamma_0(t) > \Gamma_0(t)(r) > \Gamma(t,r)$$

and the normal subgroup $\Gamma_1(t) \lhd \Gamma_0(t)$, where

$$\begin{aligned}
\Gamma_0(t) &= \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \mid \begin{array}{l} a \equiv d \equiv 1 \bmod t, \\ b \equiv 0 \bmod t \end{array} \right\}, \\
\Gamma_1(t) &= \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \mid \begin{array}{l} a \equiv d \equiv 1 \bmod t, \\ b \equiv c \equiv 0 \bmod t \end{array} \right\}, \\
\Gamma_0(t)(h) &= \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \mid \begin{array}{l} a \equiv d \equiv 1 \bmod t, \\ b \equiv 0 \bmod t,\ c \equiv 0 \bmod h \end{array} \right\}.
\end{aligned}$$

Thus $\Gamma_0(t)(r)$ is the kernel of reduction mod r in $\Gamma_0(t)$. By Corollary II.2, $[\Gamma(1) : \Gamma_1(t)] = t\phi_2(t)$. By the exact sequence

$$0 \longrightarrow \Gamma_1(t) \longrightarrow \Gamma_0(t) \longrightarrow \left\{ \begin{pmatrix} 1 & 0 \\ \bar{c} & 1 \end{pmatrix} \mid \bar{c} \in \mathbb{Z}_t \right\} \cong \mathbb{Z}_t \longrightarrow 0$$

we have $[\Gamma_0(t) : \Gamma_1(t)] = t$, and similarly

$$0 \longrightarrow \Gamma_0(t)(r) \longrightarrow \Gamma_0(t) \longrightarrow \left\{ \begin{pmatrix} 1 & 0 \\ \bar{c} & 1 \end{pmatrix} \mid \bar{c} \in \mathbb{Z}_r \right\} \cong \mathbb{Z}_r \longrightarrow 0$$

gives $[\Gamma_0(t) : \Gamma_0(t)(r)] = r$.
To calculate $[\Gamma(t)(r) : \Gamma(t,r)]$ we let $\Gamma_0(t)(r)$ act on $\mathbb{Z}_t \times \mathbb{Z}_{t^2}$ by multiplication on the right, i.e. by $\gamma : (x,y) \to (ax+cy, bx+dy)$. The stabiliser of $(1,0) \in \mathbb{Z}_t \times \mathbb{Z}_{t^2}$ is then $\{\bar{\gamma} \in \Gamma_0(t)(r) \mid a \equiv 1 \mod t, b \equiv 0 \mod t^2\}$, which is $\Gamma(t,r)$. On the other hand the orbit of $(1,0) \in \mathbb{Z}_t \times \mathbb{Z}_{t^2}$ is $\left\{ (\bar{a},\bar{b}) \in \mathbb{Z}_t \times \mathbb{Z}_{t^2} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(t)(r) \right\}$: that is, the set of possible first rows of a matrix in $\Gamma_0(t)(r)$ taken mod t in the first column and mod t^2 in the second. This is evidently equal to $\{(1, tb') \mid b' \in \mathbb{Z}_t\}$, and hence of size t. Thus $[\Gamma(t)(r) : \Gamma(t,r)] = t$, which completes the proof. □

The standard case is only slightly different, but now there is torsion.

Proposition V.5 *If* $D_{\mathbf{v}}$ *is standard and* $F \in \mathfrak{S}^*_k(\Gamma^{\mathrm{bil}}_t)$ *then* $\theta^{\mathbf{v}}_w(r\tau^{\mathbf{v}}_1, t\tau^{\mathbf{v}}_2)$ *is a Jacobi form of weight* k *and index* w *for a group* $\Gamma'(t,r)$, *which contains* $\Gamma(t,r)$ *as a subgroup of index* 2.

Proof. Although the standard boundary components are most obviously given by $(0,0,0,1)$ for $r = t$ and $(0,0,1,0)$ for $r = 1$, we choose to take advantage of the calculations that we have already performed by working instead with $(0,0,t,1)$ and $(0,0,1,1)$. Lemma V.3 is still true, but we also have $Q_t^{-1}\zeta Q_t \in \Gamma^{\mathrm{bil}}_t$ and $Q_1^{-1}(-\zeta)Q_1 \in \Gamma^{\mathrm{bil}}_t$. These give rise to the stated extra invariance. □

Lemma V.6 *The dimension of the space $J_{3k,w}\big(\Gamma'(t,r)\big)$ of Jacobi forms of weight $3k$ and index w for $\Gamma'(t,r)$ is given as a polynomial in k and w by*

$$\dim J_{3k,w}\big(\Gamma'(t,r)\big) = \delta rt\nu\left(\tfrac{kw}{2}+\tfrac{w^2}{6}\right) + \text{linear terms}$$

where $\delta = \frac{1}{2}$ if $r=1$ or $r=t$ and $\delta = 1$ otherwise.

Proof. By [4, Theorem 3.4] we have

$$\dim J_{3k,w}\big(\Gamma'(t,r)\big) \le \sum_{i=0}^{2w} \dim \mathfrak{S}_{3k+i}\big(\Gamma'(t,r)\big). \tag{4}$$

Since $\Gamma'(t,r)$ is torsion-free, the corresponding modular curve has genus $1+\frac{\mu(t,r)}{12}-\frac{\nu(t,r)}{2}$, where $\mu(t,r)$ is the index of $\Gamma'(t,r)$ in $\mathrm{PSL}(2,\mathbb{Z})$ and $\nu(t,r)$ is the number of cusps (see [18, Proposition 1.40]). Hence by [18, Theorem 2.23] the space of modular forms satisfies

$$\begin{aligned}\dim \mathfrak{S}_k\big(\Gamma'(t,r)\big) &= k\left(\tfrac{\mu(t,r)}{12}-\tfrac{\nu(t,r)}{2}\right)+\tfrac{k}{2}\nu(t,r)+O(1)\\ &= \frac{k\mu(t,r)}{12}+O(1) \end{aligned}\tag{5}$$

as a polynomial in k. By Lemma V.4 we have $\mu(t,r)=\frac{1}{2}rt\phi_2(t)=rt\nu$ for the nonstandard cases, $\mu(t,1)=\frac{1}{2}t\nu$ and $\mu(t,t)=\frac{1}{2}t^2\nu$. Now the result follows from equations (5) and (4). □

If $F\in\mathfrak{S}^*_{3k}(\Gamma_t^{\mathrm{bil}})$ then $F.(d\tau_1\wedge d\tau_2\wedge d\tau_3)^{\otimes k}$ extends over the component $D_{\mathbf{v}}$ if and only if $\theta_w^{\mathbf{v}}=0$ for all $w<k$: see [1, Chapter IV, Theorem 1]. Hence the obstruction $\Omega_{\mathbf{v}}$ coming from the boundary component $D_{\mathbf{v}}$ is

$$\Omega_{\mathbf{v}} = \sum_{w=0}^{k-1}\dim J_{3k,w}\left(\Gamma'(t,r)\right) \tag{6}$$

where $\Gamma'(t,r)=\Gamma(t,r)$ if $D_{\mathbf{v}}$ is nonstandard.
By Corollary IV.2 the total obstruction from the boundary is

$$\Omega_\infty = \sum_{r|t}\#\overline{\Phi}(h)\#\overline{\Phi}(r)\sum_{w=0}^{k-1}\dim J_{3k,w}\big(\Gamma'(t,r)\big),$$

and we may assume that k is even.

Corollary V.7 *The obstruction coming from the boundary is*

$$\Omega_\infty \le \left(\sum_{r|t}\delta rt\nu\#\overline{\Phi}(h)\#\overline{\Phi}(r)\right)\tfrac{11}{36}k^3+O(k^2).$$

Proof. Summing the expression in Lemma V.6 for $0\le w<k$, as required by equation (6) gives the coefficient of $\frac{11}{36}$ and the rest comes directly from Lemma V.6 and Corollary IV.2. □

VI Intersection numbers

We need to know the degrees of the normal bundles of the curves that generate $\operatorname{Pic} H_1$ and $\operatorname{Pic} H_2$. For this we first need to describe the surfaces H_1 and H_2. The statements and the proofs are very similar to the corresponding results for the case of $\mathcal{A}_p^{\mathrm{lev}}$, given in [11] and [12]. Therefore we simply refer to those sources for proofs, pointing out such differences as there are.

Proposition VI.1 *H_1 is isomorphic to $X(t) \times X(t)$.*

Proof. Identical to [11, I.5.53]. □

Proposition VI.2 *H_2 is the minimal resolution of a surface $\bar{H}_2$ which is given by two $\mathrm{SL}(2, \mathbb{Z}_2)$-covering maps*

$$X(2t) \times X(2t) \longrightarrow \bar{H}_2 \longrightarrow X(t) \times X(t).$$

The singularities that are resolved are ν^2 ordinary double points, one over each point $(\alpha, \beta) \in X(t) \times X(t)$ for which α and β are cusps.

Proof. Similar to [11, Proposition I.5.55] and the discussion before [12, Proposition 4.21]. $X(2)$ and $X(2p)$ are both replaced by $X(2t)$ and $X(1)$ and $X(p)$ by $X(t)$. Since $t > 3$ there are no elliptic fixed points and hence no other singularities in this case. □

Proposition VI.3 *H_1° and H_2° meet the standard boundary components $D_{\mathbf{v}}$ transversally in irreducible curves $C_{\mathbf{v}} \cong X^\circ(t)$ and $C'_{\mathbf{v}} \cong X^\circ(2t)$ respectively. $D_{\mathbf{v}}$ is isomorphic to the (open) Kummer modular surface $K^\circ(t)$, $C_{\mathbf{v}}$ is the zero section and $C'_{\mathbf{v}}$ is the 3-section given by the 2-torsion points of the universal elliptic curve over $X(t)$.*

Proof. This is essentially the same as [11, Proposition I.5.49], slightly simpler in fact. We may work with $\mathbf{v} = (0, 0, 1, 0)$ and copy the proof for the central boundary component in $\mathcal{A}_p^{\mathrm{lev}}$, replacing p by t (again the fact that p is prime is not used). □

We do not claim that the closure of $D_{\mathbf{v}}$ is the Kummer modular surface $K(t)$. They are, however, isomorphic near H_1 and H_2. We remark that H_1 and H_2 do not meet the nonstandard boundary divisors, because of Lemma V.2.

Proposition VI.4 *$\mathcal{A}_t^{\mathrm{bil}*}$ is smooth near H_1 and H_2.*

Proof. Certainly $\mathcal{A}_t^{\mathrm{bil}}$ is smooth since the only torsion in Γ_t^{bil} is 2-torsion fixing a divisor in $\mathbb{H}_2$. There can in principle be singularities at infinity, but such singularities must lie on corank 2 boundary components not meeting H_1 nor H_2 (again this follows from Lemma V.2). □

Corollary VI.5 *H_1 does not meet H_2.*

Proof. Since $\mathcal{A}_t^{\mathrm{bil}*}$ and the divisors H_1 and H_2 are smooth at the relevant points, the intersection must either be empty or contain a curve. However, the intersection also lies in the corank 2 boundary components. These components consist entirely of rational curves, and if $t > 5$ then $H_1 \cong X(t) \times X(t)$ contains no rational curves. Hence $H_1 \cap H_2 = \emptyset$.
With a little more work one can check that this is still true for $t \leq 5$, but we are in any case not concerned with that. □

Proposition VI.6 *The Picard group* $\operatorname{Pic} H_1$ *is generated by the classes of* $\Sigma_1 = \bar{C}_{0010}$ *and* $\Psi_1 = \bar{C}_{0001}$. *The intersection numbers are* $\Sigma_1^2 = \Psi_1^2 = 0$, $\Sigma_1.\Psi_1 = 1$ *and* $\Sigma_1.H_1 = \Psi_1.H_1 = -\mu/6$.

Proof. As in [12, Proposition 4.18] (but one has to use the alternative indicated in the remark that follows). □

Proposition VI.7 *The Picard group* $\operatorname{Pic} H_2$ *is generated by the classes of* Σ_2 *and* Ψ_2, *which are the inverse images of general fibres of the two projections in* $X(t) \times X(t)$, *and of the exceptional curves* $R_{\alpha\beta}$ *of the resolution* $H_2 \to \bar{H}_2$. *The intersection numbers in* H_2 *are* $\Sigma_2^2 = \Psi_2^2 = \Sigma_2.R_{\alpha\beta} = \Psi_2.R_{\alpha\beta} = 0$, $R_{\alpha\beta}.R_{\alpha'\beta'} = -2\delta_{\alpha\alpha'}\delta_{\beta\beta'}$ *and* $\Sigma_2.\Psi_2 = 6$. *In* $\mathcal{A}_t^{\mathrm{bil}*}$ *we have* $\Sigma_2.H_2 = \Psi_2.H_2 = -\mu$ *and* $R_{\alpha\beta}.H_2 = -4$.

Proof. The same as the proofs of [12, Proposition 4.21] and [12, Lemma 4.24]. The curves $R'_{(a,b)}$ from [12] arise from elliptic fixed points so they are absent here. □

Notice that Σ_2 and Ψ_2 are also images of the general fibres in $X(2t) \times X(2t)$ and are themselves isomorphic to $X(2t)$.

VII Branch locus

The closure of the branch locus of the map $\mathbb{H}_2 \to \mathcal{A}_t^{\mathrm{bil}}$ is $H_1 \cup H_2$ and modular forms of weight $3k$ (for k even) give rise to k-fold differential forms with poles of order $k/2$ along H_1 and H_2. We have to calculate the number of conditions imposed by these poles.

Proposition VII.1 *The obstruction from H_1 to extending modular forms of weight $3k$ to k-fold holomorphic differential forms is*

$$\Omega_1 \leq \nu^2\Big(\tfrac{1}{2} - \tfrac{7t}{24} + t^2\big(\tfrac{1}{24} + \tfrac{1}{864}\big)\Big)k^3 + O(k^2).$$

Proof. If F is a modular form of weight $3k$ for k even, vanishing to sufficiently high order at infinity, and $\omega = d\tau_1 \wedge d\tau_2 \wedge d\tau_3$, then $F\omega^{\otimes k}$ determines a section of $kK + \frac{k}{2}H_1 + \frac{k}{2}H_2$, where K denotes the canonical sheaf of $\mathcal{A}_t^{\mathrm{bil}*}$. From

$$0 \longrightarrow \mathcal{O}(-H_1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{H_1} \longrightarrow 0$$

we get, for $0 \leq j < k/2$

$$\begin{aligned} 0 \quad \longrightarrow \quad & H^0\big(kK + (\tfrac{k}{2} - j - 1)H_1 + \tfrac{k}{2}H_2\big) \longrightarrow H^0\big(kK + (\tfrac{k}{2} - j)H_1 + \tfrac{k}{2}H_2\big) \\ \longrightarrow \quad & H^0\big((kK + (\tfrac{k}{2} - j)H_1 + \tfrac{k}{2}H_2)|_{H_1}\big) \end{aligned}$$

so

$$\begin{aligned} h^0\big(kK + (\tfrac{k}{2} - j)H_1 + \tfrac{k}{2}H_2\big) \quad \leq \quad & h^0\big(kK + (\tfrac{k}{2} - j - 1)H_1 + \tfrac{k}{2}H_2\big) \\ & + h^0\big((kK + (\tfrac{k}{2} - j)H_1 + \tfrac{k}{2}H_2)|_{H_1}\big). \end{aligned}$$

Note that, by Lemma VI.5, $H_2|_{H_1} = 0$. Therefore

$$h^0\big(kK + \tfrac{k}{2}H_2\big) \geq h^0\big(kK + \tfrac{k}{2}H_1 + \tfrac{k}{2}H_2\big) + \sum_{j=0}^{k/2-1} h^0\big((kK + (\tfrac{k}{2} - j)H_1)|_{H_1}\big),$$

so

$$\Omega_1 \leq \sum_{j=0}^{k/2-1} h^0\big((kK + (\tfrac{k}{2} - j)H_1)|_{H_1}\big) = \sum_{j=0}^{k/2-1} h^0\big(kK_{H_1} - (\tfrac{k}{2} + j)H_1|_{H_1}\big). \quad (7)$$

By Lemma VI.6, K_{H_1} and $H_1|_{H_1}$ are both multiples of $\Sigma_1 + \Phi_1$, and any positive multiple of $\Sigma_1 + \Psi_1$ is ample. Suppose $H_1|_{H_1} = a_1(\Sigma_1 + \Psi_1)$ and $K_{H_1} = b_1(\Sigma_1 + \Psi_1)$. Then

$$-\frac{\mu}{6} = \Sigma_1.H_1 = a\Sigma_1.(\Sigma_1 + \Psi_1) = a_1$$

and

$$\frac{\mu}{6} - \nu = 2g(\Sigma_1) - 2 = (K_{H_1} + \Sigma_1).\Sigma_1 = K_{H_1}.\Sigma_1 = b_1$$

Hence, using equation (7)

$$\begin{aligned} \Omega_1 \quad \leq \quad & \sum_{j=0}^{k/2-1} h^0\big((\tfrac{k\mu}{6} - k\nu + \tfrac{k\mu}{12} + \tfrac{j\mu}{6})(\Sigma_1 + \Psi_1)\big) \\ = \quad & \sum_{j=0}^{k/2-1} h^0\big((\tfrac{kt\nu}{4} - k\nu + \tfrac{jt\nu}{6})(\Sigma_1 + \Psi_1)\big). \end{aligned}$$

Since $t \geq 7$ (we know from [14] that $\mathcal{A}_t^{\rm bil*}$ is rational for $t \leq 5$), we have $\frac{kt\nu}{4} - k\nu + \frac{jt\nu}{6} - \frac{t\nu}{6} + \nu > 0$ for all j and hence $(\frac{kt\nu}{4} - k\nu + \frac{jt\nu}{6})(\Sigma_1 + \Psi_1) - K_{H_1}$ is ample. So by vanishing we have

$$\begin{aligned}
\Omega_1 &\leq \sum_{j=0}^{k/2-1} \tfrac{1}{2}(\tfrac{kt\nu}{4} - k\nu + \tfrac{jt\nu}{6})^2(\Sigma_1 + \Psi_1)^2 + O(k^2) \\
&= \sum_{j=0}^{k/2-1} (\tfrac{kt\nu}{4} - k\nu + \tfrac{jt\nu}{6})^2 + O(k^2) \\
&= \nu^2\big(\tfrac{1}{2} - \tfrac{7t}{24} + t^2(\tfrac{1}{24} + \tfrac{1}{864})\big)k^3 + O(k^2).
\end{aligned}$$

□

Next we carry out the same calculation for H_2.

Proposition VII.2 *The obstruction from H_2 is*

$$\Omega_2 \leq \nu^2\big((\tfrac{1}{2} + \tfrac{1}{72})t^2 - (\tfrac{1}{4} + \tfrac{1}{24})t - \tfrac{7}{3} + \tfrac{1}{24}\big)k^3 + O(k^2).$$

Proof. By the same argument as above (equation (7)) the obstruction is

$$\Omega_2 \leq \sum_{j=0}^{k/2-1} h^0\big(kK_{H_2} - (\tfrac{k}{2} + j)H_2|_{H_2}\big).$$

In this case $H_2|_{H_2} = a_2(\Sigma_2 + \Psi_2) + c_2R$, where $R = \sum_{\alpha,\beta} R_{\alpha\beta}$ is the sum of all the exceptional curves of $H_2 \to \bar{H}_2$, and $K_{H_2} = b_2(\Sigma_2 + \Psi_2) + d_2R$. Since $\Sigma_2 \cong X(2t)$ we have by [18, 1.6.4]

$$2g(\Sigma_2) - 2 = \tfrac{1}{3}(t-3)\nu(2t) = \mu - \tfrac{\nu}{2}.$$

Hence

$$-\mu = \Sigma_2.H_2 = a_2\Sigma_2^2 + a_2\Sigma_2.\Psi_2 + c_2\Sigma_2.R = 6a_2$$

so $a_2 = -\mu/6$, and

$$-4\nu^2 = R.H_2 = a_2\Sigma_2.R + a_2\Psi_2.R + c_2R^2 = -2\nu^2c_2$$

so $c_2 = 2$. Therefore

$$H_2|_{H_2} = -\tfrac{\mu}{6}(\Sigma_2 + \Psi_2) + 2R.$$

Similarly

$$\mu - \tfrac{\nu}{2} = (K_{H_2} + \Sigma_2).\Sigma_2 = 6b_2$$

so $b_2 = \mu/6 - \nu/12$, and $0 = R.K_{H_2} = d_2R^2$ so $d_2 = 0$. Hence

$$K_{H_2} = \tfrac{1}{6}(\mu - \tfrac{\nu}{2})(\Sigma_2 + \Psi_2).$$

Moreover $L_j = (k-1)K_{H_2} - (\frac{k}{2}+j)H_2|_{H_2}$ is ample, as is easily checked using the Nakai criterion and the fact that the cone of effective curves on H_2 is spanned by $R_{\alpha\beta}$ and by the non-exceptional components of the fibres of the two maps $H_2 \to X(t)$. These components are $\Sigma_\alpha \equiv \Sigma_2 - \sum_\beta R_{\alpha\beta}$ and $\Psi_\beta \equiv \Psi_2 - \sum_\alpha R_{\alpha\beta}$, and it is simple to check that L_j^2, $L_j.\Sigma_\alpha = L_j.\Psi_\beta$ and $L_j.R_{\alpha\beta}$ are all positive for the relevant values of j, k and t. Therefore

$$\begin{aligned}
\Omega_2 &\leq \sum_{j=0}^{k/2-1} \tfrac{1}{2}\big(kK_{H_2} - (\tfrac{k}{2}+j)H_2|_{H_2}\big)^2 \\
&= \sum_{j=0}^{k/2-1} \tfrac{1}{2}\big(\nu(\tfrac{kt}{4} - \tfrac{k}{12} + \tfrac{jt}{6})(\Sigma_2+\Psi_2) + (k+2j)R\big)^2 \\
&= \nu^2k^3\big(t^2(\tfrac{3}{8}+\tfrac{1}{8}+\tfrac{1}{72}) - t(\tfrac{1}{4}+\tfrac{1}{24}) + \tfrac{1}{24} - 2 - \tfrac{1}{3}\big) + O(k^2)
\end{aligned}$$

since $(\Sigma_2+\Psi_2)^2 = 12$. □

VIII Final calculation

In this section we assemble the results of the previous sections into a proof of the main theorem.

Theorem VIII.1 *$\mathcal{A}_t^{\mathrm{bil}*}$ is of general type for t odd and $t \geq 17$.*

Proof. We put $n = 3k$ in Theorem II.6, and use $\phi_2(t) = 2\nu$ and the fact that

$$\phi_4(t) = t^4 \prod_{p|t}(1-p^{-4}) = t^2\phi_2(t)\prod_{p|t}(1+p^{-2}).$$

This gives the expression

$$\dim \mathfrak{S}_n^*(\Gamma_t^{\mathrm{bil}}) = \frac{k^3\nu^2}{320} t^4 \prod_{p|t}(1+p^{-2}) + O(k^2).$$

From Proposition VII.1 and Proposition VII.2 we have

$$\begin{aligned}
\Omega_1 &= k^3\nu^2\left(\tfrac{37}{864}t^2 - \tfrac{7}{24}t + \tfrac{1}{2}\right) + O(k^2), \\
\Omega_2 &= k^3\nu^2\left(\tfrac{37}{72}t^2 - \tfrac{7}{24}t - \tfrac{55}{24}\right) + O(k^2)
\end{aligned}$$

and from Corollary V.7 and Corollary IV.2

$$\Omega_\infty = k^3\nu^2 \sum_{r|t} \tfrac{11}{36r} t^2 \prod_{p|(r,h)} (1-p^{-2}) + O(k^2).$$

since $\phi_2(r)\phi_2(h) = t^2 \prod_{p|(r,h)}(1-p^{-2})$.

It follows that $\mathcal{A}_t^{\mathrm{bil}*}$ is of general type, for odd t, provided

$$\frac{1}{320}\prod_{p|t}(1+p^{-2})t^4 - \frac{481}{864}t^2 + \frac{7}{12}t + \frac{43}{24} - \sum_{r|t}\frac{11}{36r}t^2 \prod_{p|(r,h)}(1-p^{-2}) > 0. \quad (8)$$

This is simple to check: since either $r = 1$ or $r \geq 3$, and since the sum of the divisors of t is less than $t/2$, the last term can be replaced by $-\frac{11}{36}t^2 - \frac{11}{108}t^3$ and the t and constant terms, and the the $p^{-2}t^4$ term, can be discarded as they are positive. The resulting expression is a quadratic in t whose larger root is less than 40, so we need only consider odd $t \leq 39$. We deal with primes, products of two primes and prime powers separately. In the case of primes, the expression on the left-hand side of the inequality (8) becomes $\frac{1}{320}t^4 - \frac{7433}{8640}t^2 + \frac{5}{18}t + \frac{43}{24}$, which is positive for $t \geq 17$. The expression in the case of $t = pq$ is positive if $t \geq 21$. For $t = p^2$ we get an expression which is negative for $t = 9$ but positive for $t = 25$, and for $t = p^3$ the expression is positive. □

One can say something even for t even, though not if t is a power of 2.

Corollary VIII.2 $\mathcal{A}_t^{\mathrm{bil}*}$ *is of general type unless* $t = 2^a b$ *with* b *odd and* $b < 17$.

Proof. $\mathcal{A}_{nt}^{\mathrm{bil}}$ covers $\mathcal{A}_t^{\mathrm{bil}}$ for any n, and therefore $\mathcal{A}_{nt}^{\mathrm{bil}*}$ is of general type if $\mathcal{A}_t^{\mathrm{bil}*}$ is of general type. □

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