The Moduli Space of Bilevel-6 Abelian Surfaces

G.K. Sankaran & J.G Spandaw

The moduli space $\mathcal{A}_t^{\rm bil}$ of $(1,t)$-polarised abelian surfaces with a weak bilevel structure was introduced by S. Mukai in [Mu]. Mukai showed that $\mathcal{A}_t^{\rm bil}$ is rational for $t = 2,3,4,5$. More generally, we may ask for birational invariants, such as Kodaira dimension, of a smooth model of a compactification of $\mathcal{A}_t^{\rm bil}$: since the choice of model does not affect birational invariants, we refer to the Kodaira dimension, etc., of $\mathcal{A}_t^{\rm bil}$.

From the description of $\mathcal{A}_t^{\rm bil}$ as a Siegel modular 3-fold $\Gamma_t^{\rm bil}\backslash\mathbb{H}_2$ and the fact that $\Gamma_t^{\rm bil} \subset \mathrm{Sp}(4,\mathbb{Z})$ it follows, by a result of L. Borisov [Bo], that $\kappa(\mathcal{A}_t^{\rm bil}) = 3$ for all sufficiently large t. For an effective result in this direction see [Sa]. In this note we shall prove an intermediate result for the case $t = 6$.

Theorem A. *The moduli space $\mathcal{A}_6^{\rm bil}$ has geometric genus $p_g(\mathcal{A}_6^{\rm bil}) \geq 3$ and Kodaira dimension $\kappa(\mathcal{A}_6^{\rm bil}) \geq 1$.*

The case $t = 6$ attracts attention for two reasons: it is the first case not covered by the results of [Mu]; and the image of the Humbert surface $\mathcal{H}_1(1)$ in $\mathcal{A}_t^{\rm bil}$, which in the cases $2 \leq t \leq 5$ is a quadric and plays an important role both in [Mu] and below, becomes an abelian surface (at least birationally) because the modular curve $X(6)$ has genus 1.

The method we use is that of Gritsenko, who proved a similar result for the moduli spaces of $(1,t)$-polarised abelian surfaces with canonical level structure for certain values of t: see [Gr], especially Corollary 2. We use some of the weight 3 modular forms constructed by Gritsenko and Nikulin as lifts of Jacobi forms in [GN] to produce canonical forms having effective, nonzero, divisors on a suitable projective model X_6 of $\mathcal{A}_6^{\rm bil}$. A similar method was used by Gritsenko and Hulek in [GH2] to give a new proof that the Barth–Nieto threefold is Calabi-Yau.

We also derive some information about divisors in X_6 and linear relations among them.

Acknowledgement: We are grateful to the DAAD and the British Council for financial assistance under ARC Project 313-ARC-XIII-99/45.

1. Compactification

According to [Mu], $\mathcal{A}_t^{\rm bil}$ is isomorphic to the quotient $\Gamma_t^{\rm bil}\backslash\mathbb{H}_2$, where $\mathbb{H}_2$ is the Siegel upper half-plane $\{Z \in M_{2\times 2}(\mathbb{C}) \mid Z = {}^\top Z,\ \mathrm{Im}\, Z > 0\}$ and $\Gamma_t^{\rm bil} = \Gamma_t^\natural \cup \zeta\Gamma_t^\natural \subset \mathrm{Sp}(4,\mathbb{Z})$ acts on $\mathbb{H}_2$ by fractional linear transformations. Here $\zeta = \mathrm{diag}(-1,1,-1,1)$ and, writing $\mathbf{I}_n$ for the $n \times n$ identity matrix,

$$\Gamma_t^\natural = \left\{ \gamma \in \mathrm{Sp}(4,\mathbb{Z}) \;\middle|\; \gamma - \mathbf{I}_4 \in \begin{pmatrix} t\mathbb{Z} & \mathbb{Z} & t\mathbb{Z} & t\mathbb{Z} \\ t\mathbb{Z} & t\mathbb{Z} & t\mathbb{Z} & t^2\mathbb{Z} \\ t\mathbb{Z} & \mathbb{Z} & t\mathbb{Z} & t\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t\mathbb{Z} \end{pmatrix} \right\}.$$

We define $H(\mathbb{Z})$ to be the Heisenberg group $\mathbb{Z} \rtimes \mathbb{Z}^2$ embedded in $\mathrm{Sp}(4,\mathbb{Z})$ as

$$H(\mathbb{Z}) = \left\{ [m,n;k] = \begin{pmatrix} 1 & m & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & n & 1 & 0 \\ n & k & -m & 1 \end{pmatrix} \;\middle|\; m,n,k \in \mathbb{Z} \right\}.$$

Lemma 1.1. *$\Gamma_6^\natural$ is neat; that is, if λ is an eigenvalue of some $\gamma \in \Gamma_6^\natural$ which is a root of unity, then $\lambda = 1$. Any torsion element of $\Gamma_6^{\rm bil}$ has order 2 and fixes a divisor in $\mathbb{H}_2$.*

Proof: Suppose that $\gamma \in \Gamma_6^\natural$: then the characteristic polynomial of γ is congruent to $(1-x)^4$ mod 6. If some $\gamma \in \Gamma_6^\natural$ has an eigenvalue λ which is a nontrivial root of unity, then we may assume that λ is a primitive pth root of unity for some prime p. The minimum polynomial $m_\lambda(x)$ of λ over $\mathbb{Z}$ divides the characteristic polynomial of γ; so $p = 2, 3$ or 5, since $\deg m_\lambda = p-1$. But then $m_\lambda(x) = 1+x$, $1+x+x^2$ or $1+x+x^2+x^3+x^4$. The second of these does not divide $(1-x)^4$ in $\mathbb{F}_2[x]$ and the other two do not divide $(1-x)^4$ in $\mathbb{F}_3[x]$.

So any torsion element of $\Gamma_6^{\rm bil}$ is of the form $\gamma = \zeta\gamma'$ for some $\gamma' \in \Gamma_6^\natural$; but then the characteristic polynomial is

$$\begin{aligned}\det(\gamma - x\mathbf{I}_4) &= \det(\zeta\gamma' - x\zeta^2)\\ &\equiv (1-x^2)(1+x^2) \bmod 6.\end{aligned}$$

From the classification of torsion elements of $\mathrm{Sp}(4,\mathbb{Z})$ and their characteristic polynomials [Ue], it follows that γ is conjugate in $\mathrm{Sp}(4,\mathbb{Z})$ to either ζ or $\zeta[0,1;0]$. Both these are elements of $\Gamma_6^{\rm bil}$ of order 2; their fixed loci in $\mathbb{H}_2$ are the divisors $\{\tau_2 = 0\}$ and $\{2\tau_2 + (\tau_2^2 - \tau_1\tau_3) = 0\}$ respectively (Humbert surfaces of discriminants 1 and 4). ■

In view of Lemma 1.1, the toroidal (Voronoi, or Igusa) compactification $(\mathcal{A}_6^\natural)^*$ of $\mathcal{A}_6^\natural = \Gamma_6^\natural\backslash\mathbb{H}_2$ is smooth, cf [SC], pp. 276–7. The action of ζ on $\mathcal{A}_6^\natural$ extends to $(\mathcal{A}_6^\natural)^*$, and the quotient X_6 is a compactification of $\mathcal{A}_6^{\rm bil}$ whose singularities are isolated ordinary double points or transverse A_1 singularities. Hence X_6 has canonical singularities. It agrees with the Voronoi compactification $(\mathcal{A}_6^{\rm bil})^*$ at least in codimension 1.

2. Modular forms and canonical forms

Gritsenko and Nikulin, in [GN], construct the weight 3 cusp forms

$$\begin{aligned}F_3 &= \mathrm{Lift}\left(\eta^5(\tau_1)\vartheta(\tau_1, 2\tau_2)\right) \in \mathfrak{M}_3^*\left(\Gamma_6^+, v_\eta^8 \times \mathrm{id}_H\right)\\ F_3' &= \mathrm{Lift}_{-1}\left(\eta^5(\tau_1)\vartheta(\tau_1, 2\tau_2)\right) \in \mathfrak{M}_3^*\left(\Gamma_6^+, v_\eta^{16} \times \mathrm{id}_H\right)\\ F_3'' &= \mathrm{Lift}\left(\eta^3(\tau_1)\vartheta(\tau_1, \tau_2)^2\vartheta(\tau_1, 2\tau_2)\right) \in \mathfrak{M}_3^*\left(\Gamma_6^+, v_\eta^{12} \times \mathrm{id}_H\right)\end{aligned}$$

for the extended paramodular group Γ_6^+, with character χ_D induced from the characters $v_\eta^D \times \mathrm{id}_H$ of the Jacobi group $\mathrm{SL}(2,\mathbb{Z}) \ltimes H(\mathbb{Z})$. Recall (see [GH1], [GN]: for compatibility with [Mu] and other sources we work with the transposes of the groups given in [GN]) that Γ_6^+ is the group generated by the paramodular group

$$\Gamma_6 = \left\{ \gamma \in \mathrm{Sp}(4,\mathbb{Q}) \;\middle|\; \gamma \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & 6\mathbb{Z} \\ 6\mathbb{Z} & \mathbb{Z} & 6\mathbb{Z} & 6\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & 6\mathbb{Z} \\ \mathbb{Z} & \frac{1}{6}\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}$$

and the extra involution

$$V_6 = \begin{pmatrix} 0 & 1/\sqrt{6} & 0 & 0 \\ \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} \\ 0 & 0 & 1/\sqrt{6} & 0 \end{pmatrix}.$$

Proposition 2.1. *All three of F_3, F_3' and F_3'' are cusp forms, without character, of weight 3 for Γ_6^{bil}.*

Proof: The character is induced from $v_\eta^D \times \mathrm{id}_H$ by the injective map $j : \mathrm{SL}(2,\mathbb{Z}) \ltimes H(\mathbb{Z}) \to \Gamma_6^+$ given by

$$j : \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, [m,n;k] \right) \mapsto \begin{pmatrix} a & m & c & 0 \\ 0 & 1 & 0 & 0 \\ b & n & d & 0 \\ n & k & -m & 1 \end{pmatrix}.$$

For $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ we define $j_1(\gamma) = j(\gamma, [0,0;0])$, putting γ in the first and third rows and columns in $\mathrm{Sp}(4,\mathbb{Z})$; and similarly $j_2(\gamma)$ puts it in the second and fourth.

The character $v_\eta^D \times \mathrm{id}_H$ is trivial on $H(\mathbb{Z})$. In the present cases, where $D = 8$, 16 or 12, v_η^D is trivial on $\pm\Gamma(6) = \pm \operatorname{Ker}(\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/6))$ by [GN], Lemma 1.2. Since $j(-\mathbf{I}_2, [0,0;0]) = \zeta$, we see that

$$\Gamma_6^{\mathrm{bil}} \cap j\big(\mathrm{SL}(2,\mathbb{Z}) \ltimes H(\mathbb{Z})\big) \subseteq j\big(\pm\Gamma(6) \ltimes H(\mathbb{Z})\big) \subseteq \operatorname{Ker}\chi_D$$

for $D = 8, 12, 16$. If $D = 8$ or 16 then, since V_6 and $I = j_1\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$ are in Γ_6^+ and have even order and the order of χ_D is 3, we know that $\chi_D(V_6) = \chi_D(I) = 1$. Therefore the element

$$J_6 = IV_6IV_6 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 & 0 \end{pmatrix} \in \Gamma_6^+$$

is in $\operatorname{Ker}\chi_D$. If $D = 12$ then $\chi_{12}(J_6) = \chi_{12}(IV_6)^2 = 1$ so again $J_6 \in \operatorname{Ker}\chi_D$. Now we proceed as in [Gr], Lemma 2.2, and show that the group generated by $j\big(\Gamma(6) \ltimes H(\mathbb{Z})\big)$ and J_6 includes $\Gamma_6^\natural$. To see this, we work with the conjugate groups $\tilde\Gamma_6^\natural = \nu_6(\Gamma_6^\natural)$ and $\tilde\Gamma_6 = \nu_6(\Gamma_6)$, where ν_6 denotes conjugation by $R_6 = \mathrm{diag}(1,1,1,6)$. Note that $\nu_6(J_6) = R_6 J_6 R_6^{-1} = \begin{pmatrix} 0 & -\mathbf{I}_2 \\ \mathbf{I}_2 & 0 \end{pmatrix}$. If $\tilde\gamma \in \tilde\Gamma_6^\natural$ then its second row $\tilde\gamma_{2*}$ is $(0,1,0,0) \bmod 6$. Suppose first that $\tilde\gamma_{22} = 1$ and put

$$\tilde\beta = \nu_6\big(J_6[\tilde\gamma_{21}/6, \tilde\gamma_{23}/6; \tilde\gamma_{24}/6]J_6^{-1}\big) = \begin{pmatrix} 1 & 0 & 0 & \tilde\gamma_{23}/6 \\ \tilde\gamma_{21} & 1 & \tilde\gamma_{23} & \tilde\gamma_{24} \\ 0 & 0 & 1 & \tilde\gamma_{23}/6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now $(0,1,0,0)\tilde\beta = \tilde\gamma_{2*}$ so the second row of $\tilde\gamma\tilde\beta^{-1} \in \tilde\Gamma_6^\natural$ is $(0,1,0,0)$. Such a matrix is in $\nu_6\left(j\big(\Gamma(6) \ltimes H(\mathbb{Z})\big)\right)$.

It remains to reduce to the case $\tilde\gamma_{22} = 1$. Certainly the vector $\tilde\gamma_{2*}$ is primitive, since $\det\tilde\gamma = 1$, and since $\tilde\gamma \in \tilde\Gamma_6^\natural$ we have $\gcd(6, \tilde\gamma_{21}, \tilde\gamma_{23}) = 6$. In the proof of [FS], Satz 2.1 it is shown that there are integers λ, μ such that $\tilde\gamma' = \tilde\gamma\nu_6\left([\mu,0;0]J_6[0,\lambda;0]J_6^{-1}\right)$ has $\gcd(\tilde\gamma'_{21}, \tilde\gamma'_{23}) = 6$, so the second row of $\tilde\gamma'$ is $(6x_1, 6x_2+1, 6x_3, 6x_4)$ with $\gcd(x_1, x_3) = 1$. But then the $(2,2)$-entry of $\tilde\gamma'\nu_6([m,n;0])$ is $6(mx_1 + nx_3 + x_2) + 1$ which is equal to 1 if we choose m and n suitably. ■

Proposition 2.2. *The differential forms $\tilde\omega = F_3\, d\tau_1 \wedge d\tau_2 \wedge d\tau_3$, $\tilde\omega' = F_3'\, d\tau_1 \wedge d\tau_2 \wedge d\tau_3$ and $\tilde\omega'' = F_3''\, d\tau_1 \wedge d\tau_2 \wedge d\tau_3$ give rise to canonical forms $\omega, \omega', \omega'' \in H^0(K_{X_6})$.*

Proof: By Proposition 2.1, $\tilde\omega$, $\tilde\omega'$ and $\tilde\omega''$ are all Γ_6^{bil}-invariant, so they give rise to forms ω, ω', ω'' on $\mathcal{A}_6^{\mathrm{bil}}$. Since F_3, F_3' and F_3'' are cusp forms, if any of ω, ω' and ω'' are holomorphic on $\mathcal{A}_6^{\mathrm{bil}}$ they extend holomorphically to the cusps of $(\mathcal{A}_6^{\mathrm{bil}})^*$. Since X_6 agrees with $(\mathcal{A}_6^{\mathrm{bil}})^*$ in codimension 1 and has canonical singularities it follows that these forms can be thought of as 3-forms on X_6 holomorphic at infinity. We need to check that ω, ω' and ω'' are holomorphic everywhere. But this is a well-known result of Freitag([Fr], Satz II.2.6). ■

3. Divisors in the moduli spaces.

In this section we shall describe the canonical divisors $\mathrm{Div}_{X_6}(\omega)$, $\mathrm{Div}_{X_6}(\omega')$ and $\mathrm{Div}_{X_6}(\omega'')$ in X_6 and give some detail about the branching locus in X_6 arising from torsion in Γ_6^{bil}.

Γ_6^{bil} is a subgroup both of the paramodular group Γ_6 and of Γ_6^+. Hence there is a finite morphism $\sigma : \mathcal{A}_6^{\mathrm{bil}} \to \mathcal{A}_6^+$. We denote the projection map $\mathbb{H}_2 \to \mathcal{A}_6^{\mathrm{bil}}$ by π_6^{bil} and similarly π_6, π_6^+, etc.

For discriminant $\Delta = 1, 4$ we put

$$\mathcal{H}_\Delta(k) = \left\{ \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathbb{H}_2 \;\middle|\; \tfrac{1}{24}(k^2 - \Delta)\tau_1 + k\tau_2 + 6\tau_3 \right\} = 0$$

where $k \in \mathbb{Z}$ is chosen so that $\frac{1}{24}(k^2 - \Delta) \in \mathbb{Z}$. The irreducible components of the Humbert surfaces H_1 and H_4 of discriminants 1 and 4 in $\mathcal{A}_6$ are $\pi_6(\mathcal{H}_1(k))$ and $\pi_6(\mathcal{H}_4(k))$ for $0 \le k < 6$: the statements of [vdG], Theorem IX.2.4 and of [GH1], Corollary 3.3 are wrong because $\mathcal{H}_\Delta(-k)$ is Γ_t-equivalent to $\mathcal{H}_\Delta(k)$. Nevertheless the irreducible components of the Humbert surfaces of discriminants 1 and 4 in $\mathcal{A}_6^+$ are as stated in [GN], namely $\pi_6^+(\mathcal{H}_1(1))$ and $\pi_6^+(\mathcal{H}_1(5))$ for discriminant 1 and $\pi_6^+(\mathcal{H}_4(1))$ for discriminant 4.

The calculation of the divisors uses the product expansion of the modular forms F_3, F_3' and F_3'' given in [GN]. We have chosen to work with the transposes of the matrices given in [GN], so we have to write $q = e^{2\pi i\tau_1}$, $r = e^{2\pi i\tau_2/6}$ and $s = e^{2\pi i\tau_3/36}$ for these expansions to be correct. This is because ${}^\top\Gamma_t = \mathrm{diag}(1,t,1,t^{-1})\Gamma_t\,\mathrm{diag}(1,t^{-1},1,t)$ (for any $t \in \mathbb{N}$), and $\mathrm{diag}(1,t,1,t^{-1}) : (\tau_1,\tau_2,\tau_3) \to (\tau_1, t\tau_2, t^2\tau_3)$. A similar correction is needed in [GH2].

By [GN], equations (4.12)–(4.14), correcting a minor misprint, we have

$$\begin{aligned} F_3 &= \mathrm{Exp-Lift}(5\phi_{0,3}^2 - 4\phi_{0,2}\phi_{0,4}) = \mathrm{Exp-Lift}(\phi_3) \\ F_3' &= \mathrm{Exp-Lift}(\phi_{0,3}^2) = \mathrm{Exp-Lift}(\phi_3') \\ F_3'' &= \mathrm{Exp-Lift}(3\phi_{0,3}^2 - 2\phi_{0,2}\phi_{0,4}) = \mathrm{Exp-Lift}(\phi_3''). \end{aligned}$$

(ϕ_3, ϕ_3' and ϕ_3'' are defined by these formulae.)

By [GN], Example 2.3 and Lemma 2.5, we have

$$\begin{aligned} \phi_{0,2} &= (r^{\pm1} + 4) + q(r^{\pm3} - 8r^{\pm2} - r^{\pm1} + 16) + O(q^2) \\ \phi_{0,3} &= (r^{\pm1} + 2) + q(-2r^{\pm3} - 2r^{\pm2} + 2r^{\pm1} + 4) + O(q^2) \\ \phi_{0,4} &= (r^{\pm1} + 1) + q(-r^{\pm4} - r^{\pm3} + r^{\pm1} + 2) + O(q^2), \end{aligned}$$

where the notation $r^{\pm k}$ means $r^k + r^{-k}$.

Proposition 3.1. *The divisors in* $\mathbb{H}_2$ *of the cusp forms are*

$$\begin{aligned} \mathrm{Div}(F_3) &= (\pi_6^+)^{-1}\left(\pi_6^+\left(\mathcal{H}_1(1) + 5\mathcal{H}_1(5) + \mathcal{H}_4(1)\right)\right), \\ \mathrm{Div}(F_3') &= (\pi_6^+)^{-1}\left(\pi_6^+\left(5\mathcal{H}_1(1) + \mathcal{H}_1(5) + \mathcal{H}_4(1)\right)\right), \\ \mathrm{Div}(F_3'') &= (\pi_6^+)^{-1}\left(\pi_6^+\left(3\mathcal{H}_1(1) + 3\mathcal{H}_1(5) + \mathcal{H}_4(1)\right)\right). \end{aligned}$$

Remark. This corrects the coefficients given in [GN], Example 4.6: for instance, it is easy to see, by considering the effect of an element of order 2 fixing an Humbert surface, that the coefficients of $\mathcal{H}_1(1)$, $\mathcal{H}_1(5)$ and $\mathcal{H}_4(1)$ must be odd.

Proof: Write $\phi_3 = \sum f(n,l)q^n r^l$, and similarly for ϕ_3' and ϕ_3''. By [GN], Theorem 2.1, the coefficient of $\pi_6^+\big(\mathcal{H}_\Delta(b)\big)$ in $\mathcal{A}_6^+$ is

$$m_{\Delta,b} = \sum_{d>0} f(d^2a, db)$$

where $b^2 - 24a = \Delta$. So to calculate $m_{1,1}$ we may take $b = 1$ and $a = 0$, so $m_{1,1} = \sum_{d>0} f(0,d)$. From the formulae above, $\phi_3 = (r^{\pm 2} + 6) + O(q)$, so $m_{1,1} = f(0,2) = 1$. Similarly we have $\phi_3' = (r^{\pm 2} + 4r^{\pm 1} + 6)$ so $m_{1,1}' = 5$ and $\phi_3'' = (r^{\pm 2} + 2r^{\pm 1} + 6)$ so $m_{1,1}'' = 3$.

To calculate the coefficients of $\pi_6^+\big(\mathcal{H}_4(1)\big)$ we note that $\mathcal{H}_4(1)$ is Γ_6^+-equivalent to $\mathcal{H}_4(2)$, so we may as well work with that and calculate $m_{4,2}$. For this purpose we can take $b = 2$ and $a = 0$; so $m_{4,2} = \sum_{d>0} f(0,2d) = 1$, and $m_{4,2}' = m_{4,2}'' = 1$ also.

To calculate $m_{1,5}$ we take $b = 5$ and $a = 1$, so $m_{1,5} = \sum_{d>0} f(d^2, 5d)$. The Fourier coefficient $f(n,l)$ depends only on $24n - l^2$ and on the residue class of l mod 12 (see [GN]); that is, in our case, on d^2 and on d mod 12. If $d \not\equiv \pm 1$ mod 6 then $5d \equiv \pm d$ mod 12, so $f(d^2, 5d) = f(0, \pm d)$ which is zero unless $d = \pm 2$ or $d = 0$. Since we are only interested in $d > 0$ the only contribution for $d \not\equiv \pm 1$ mod 6 arises from $d = 2$, when $f(4,10) = f(0,-2) = 1$. If $d \equiv \pm 5$ mod 12 then $f(d^2, 5d) = f(\frac{-d^2+1}{24}, \pm 1)$ which vanishes because $f(n,l) = 0$ for $n < 0$. If $d \equiv \pm 1$ mod 12 then $f(d^2, 5d) = f(\frac{-d^2+25}{24}, \pm 5)$ which vanishes except possibly when $d = 1$. So $m_{1,5} = 1 + f(1,5)$ and from the expansions of $\phi_{0,2}$, $\phi_{0,3}$ and $\phi_{0,4}$ we calculate $f(1,5) = 4$. Similarly $m_{1,5}' = 1 + f'(1,5) = 1$ and $m_{1,5}'' = 1 + f''(1,5) = 3$. ∎

Brasch [Br] has studied the branch locus of $\pi_t^{\mathrm{lev}} : \mathbb{H}_2 \to \mathcal{A}_t^{\mathrm{lev}}$ for all t: for $t \equiv 2$ mod 4 the divisorial part has five irreducible components. They are $\pi_6^{\mathrm{lev}}(\mathcal{H}_{\zeta_i})$ for $0 \le i \le 4$, where $\mathcal{H}_{\zeta_i} \subset \mathbb{H}_2$ is the fixed locus of ζ_i and

$$\zeta_0 = \zeta, \qquad \zeta_1 = \zeta^\top[-6,0;0], \qquad \zeta_2 = \begin{pmatrix} -7 & 4 & 0 & 0 \\ -12 & 7 & 0 & 0 \\ 0 & 0 & -7 & -12 \\ 0 & 0 & 4 & 7 \end{pmatrix},$$

$$\zeta_3 = \zeta[1,0;0], \qquad \zeta_4 = \begin{pmatrix} -1 & -1 & 0 & 6 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

These are all elements of Γ_6^{bil}. Their fixed loci are

$$\mathcal{H}_{\zeta_0} = \{\tau_2 = 0\}, \qquad \mathcal{H}_{\zeta_1} = \{6\tau_1 - 2\tau_2 = 0\}, \qquad \mathcal{H}_{\zeta_2} = \{6\tau_1 - 7\tau_2 + 2\tau_3 = 0\},$$

$$\mathcal{H}_{\zeta_3} = \{2\tau_2 + \tau_3 = 0\}, \qquad \mathcal{H}_{\zeta_4} = \{2\tau_2 + \tau_3 - 6 = 0\},$$

of discriminants $1,4,1,4,4$ respectively. Thus three of the components have discriminant 4 and therefore map to $\pi_6^+\mathcal{H}_4(1) \subset \mathcal{A}_6^+$ (they correspond to bielliptic abelian surfaces). $\mathcal{H}_{\zeta_0} = \mathcal{H}_1(1)$ corresponds to product surfaces $E \times E'$ with polarisation given by $\mathcal{O}_E(1) \boxtimes \mathcal{O}_{E'}(6)$, and $\mathcal{H}_{\zeta_2}$ maps to $\pi_6^+\big(\mathcal{H}_1(5)\big)$, corresponding to abelian surfaces $E \times E'$ with polarisation $\mathcal{O}_E(2) \boxtimes \mathcal{O}_{E'}(3)$.

Proposition 3.2. *The branch locus of $\pi_6^{\rm bil}:\mathbb{H}_2\to\mathcal{A}_6^{\rm bil}$ has seven irreducible components, each with branching of order 2. They are $\pi_6^{\rm bil}(\mathcal{H}_{\zeta_i})$ and two other components $\pi_6^{\rm bil}(\mathcal{H}_{\zeta_1'})$, $\pi_6^{\rm bil}(\mathcal{H}_{\zeta_1''})$, which are equivalent to $\pi_6^{\rm bil}(\mathcal{H}_{\zeta_1})$ in $\mathcal{A}_6^{\rm lev}$.*

Proof: It follows from Lemma 1.1 that the branch locus consists of divisors only and that the branching is of order 2.

Write $G=\Gamma_6^{\rm lev}\rhd H=\Gamma_6^{\rm bil}$ and let G act on $\Omega=G/H\cong\mathrm{PSL}(2,\mathbb{Z}/6)$. By [Br], Corollary 1.3, the number of irreducible divisors in $\mathcal{A}_6^{\rm bil}$ mapping to $\pi_6^{\rm lev}(\mathcal{H}_{\zeta_i})$, which is equal to the number of H-conjugacy classes in the G-conjugacy class of ζ_i, is $|G:H.C_G(\zeta_i)|$. (If $\xi\in G$ for some group G then $C_G(\xi)$ denotes the centraliser of ξ in G.) Moreover, for fixed i, these divisors are permuted transitively by Ω so they all have the same branching behaviour: $\pi_6^{\rm bil}$ is branched of order 2 above each one.

$|G:H.C_G(\zeta_i)|=|G/H:C_G(\zeta_i)/\big(H\cap C_G(\zeta_i)\big)|$, which is the index of the image of $C_G(\zeta_i)$ in Ω. For $i=0,1,2,3$ the centraliser $C_{\mathrm{Sp}(4,\mathbb{Q})}(\zeta_i)$ is described in [Br], Lemma 2.1, and $C_G(\zeta_i)=C_{\mathrm{Sp}(4,\mathbb{Q})}(\zeta_i)\cap G$.

For ζ_0, if $\gamma\in\mathrm{PSL}(2,\mathbb{Z}/6)\cong\Omega$ and $\tilde\gamma\in\mathrm{SL}(2,\mathbb{Z})$ is some lift of γ then $j(\tilde\gamma,[0,0;0])\in C_G(\zeta_0)$ so the index is 1.

For ζ_1, if $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{PSL}(2,\mathbb{Z}/6)$ and b is even then

$$\begin{pmatrix}\tilde a&0&\tilde b&3\tilde b\\3(\tilde a-1)&1&3\tilde b&0\\\tilde c&0&\tilde d&3(\tilde d-1)\\0&0&0&1\end{pmatrix}\in C_G(\zeta_1)$$

for a lift $\tilde\gamma$; and this is a necessary condition for such an element to exist since if $\beta=\beta_{ij}\in C_G(\zeta_1)$ then $3\beta_{13}\equiv 0 \bmod 6$. So $C_G(\zeta_1)/\big(C_G(\zeta_1)\cap H\big)\subset\mathrm{PSL}(2,\mathbb{Z}/6)$ is the reduction mod 6 of ${}^\top\Gamma_0(2)$, i.e. the preimage of $\left\{\begin{pmatrix}a&0\\c&d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z}/2)\right\}$, which is of index 3 because it is the stabiliser of $(1,0)$ when $\mathrm{SL}(2,\mathbb{Z}/2)$ acts as the symmetric group on the nonzero vectors in $\mathbb{F}_2^2$.

For ζ_2, any two elements $\gamma,\gamma^*\in\mathrm{SL}(2,\mathbb{Q})$ determine an element $\beta(\gamma,\gamma^*)\in C_{\mathrm{Sp}(4,\mathbb{Q})}$ (see [Br], Lemma 2.1 and the preceding discussion), namely

$$\beta(\gamma,\gamma^*)=\begin{pmatrix}4\gamma_{11}-3\gamma_{11}^*&-2\gamma_{11}+2\gamma_{11}^*&4\gamma_{12}+\gamma_{12}^*&6\gamma_{12}+2\gamma_{12}^*\\6\gamma_{11}-6\gamma_{11}^*&-3\gamma_{11}+4\gamma_{11}^*&6\gamma_{12}+2\gamma_{12}^*&9\gamma_{12}+4\gamma_{12}^*\\4\gamma_{21}+9\gamma_{21}^*&-2\gamma_{21}-6\gamma_{21}^*&4\gamma_{22}-3\gamma_{22}^*&6\gamma_{22}-6\gamma_{22}^*\\-2\gamma_{21}-6\gamma_{21}^*&\gamma_{21}+4\gamma_{21}^*&-2\gamma_{22}+2\gamma_{22}^*&-3\gamma_{22}+4\gamma_{22}^*\\.&&&\end{pmatrix}$$

In particular we choose

$$\beta=\beta\left(\begin{pmatrix}3&4\\2&3\end{pmatrix},\begin{pmatrix}10&9\\11&10\end{pmatrix}\right)=\begin{pmatrix}-18&14&25&42\\-42&31&42&72\\107&-70&-18&-42\\-70&46&-14&31\end{pmatrix}$$

and

$$\beta'=\beta\left(\begin{pmatrix}11&4\\8&3\end{pmatrix},\begin{pmatrix}7&9\\3&4\end{pmatrix}\right)=\begin{pmatrix}23&-8&25&42\\24&-5&42&72\\59&-34&0&-6\\-34&20&-6&7\end{pmatrix}.$$

β and β' both belong to $\Gamma_6^{\rm lev}$, and their images in $\mathrm{PSL}(2,\mathbb{Z}/6)$ are $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. These two elements generate $\mathrm{PSL}(2,\mathbb{Z}/6)$ because their lifts generate $\mathrm{SL}(2,\mathbb{Z})$, so the index we want is 1.

For ζ_3, as for ζ_0, $j(\tilde{\gamma},[0,0;0]) \in C_G(\zeta_3)$ so the index is 1.

For ζ_4, note that $\zeta_4 = {}^\top[0,0;6]\zeta_3({}^\top[0,0;6])^{-1}$ so $C_{\mathrm{Sp}(4,\mathbb{Q})}(\zeta_4) = {}^\top[0,0;6]C_{\mathrm{Sp}(4,\mathbb{Q})}(\zeta_3)({}^\top[0,0;6])^{-1}$. It happens that ${}^\top[0,0;6]j(\tilde{\gamma},[0,0;0])({}^\top[0,0;6])^{-1} = j(\tilde{\gamma},[0,0;0])$, so again the index is 1. ■

Next we look at the boundary divisors of X_6. These correspond to 1-dimensional subspaces of $\mathbb{Q}^4$ up to the action of $\Gamma_6^{\rm bil}$. We may think of such a space as being given by a unique, up to sign, primitive vector $\mathbf{v} = (v_1,v_2,v_3,v_4) \in \mathbb{Z}^4$. It is shown in [FS], Satz 2.1, that the Γ_6-orbit of $\mathbf{v}$ is determined by $r = \gcd(6,v_1,v_3)$, so $\mathcal{A}_6$ has four corank 1 cusps (or boundary divisors in the toroidal compactification). However, the cusps $r=1$ and $r=6$ are interchanged by V_6, as are the cusps $r=2$ and $r=3$, so $\mathcal{A}_6^+$ has just two corank 1 cusps. Since F_3, F_3' and F_3'' are modular forms (with character) for Γ_6^+, the order of vanishing of any of them at a cusp of X_6 given by $\mathbf{v}$ depends only on which cusp of $\mathcal{A}_6^+$ it lies over, i.e. on whether r is or is not a proper divisor of 6.

We write D_1 for the divisor in X_6 which is the sum of all the boundary components with $r=1$ or $r=6$, and D_2 for the sum of all the components with $r=2$ or $r=3$. By modifying the argument of [FS, Satz 2.1] as in [Sa], it can be shown that D_1 has 28 irreducible components and D_2 has 12, but we shall not make any use of this.

Theorem 3.3. *The divisors of ω, ω' and ω'' in X_6 are*

$$\begin{aligned}
\mathrm{Div}_{X_6}(\omega) &= 4\pi_6^{\rm bil}(\mathcal{H}_{\zeta_2}) + D_1 + D_2,\\
\mathrm{Div}_{X_6}(\omega') &= 4\pi_6^{\rm bil}(\mathcal{H}_{\zeta_0}) + 3(D_1 + D_2),\\
\mathrm{Div}_{X_6}(\omega'') &= 2\pi_6^{\rm bil}(\mathcal{H}_{\zeta_0}) + 2\pi_6^{\rm bil}(\mathcal{H}_{\zeta_2}) + 2(D_1 + D_2).
\end{aligned}$$

Proof: If $\pi_6^{\rm bil}$ is branched along the irreducible divisors B_α with ramification index e_α, then $d\tau_1\wedge d\tau_3\wedge d\tau_3$ acquires poles of order $e_\alpha/2$ along B_α. So by Proposition 3.1

$$\begin{aligned}
\mathrm{Div}_{X_6}(\omega) &= \sigma^{-1}\pi_6^+\big(\mathcal{H}_1(1) + 5\mathcal{H}_1(5) + \mathcal{H}_4(1)\big) - \frac{1}{2}\sum e_\alpha B_\alpha + D,\\
\mathrm{Div}_{X_6}(\omega') &= \sigma^{-1}\pi_6^+\big(5\mathcal{H}_1(1) + \mathcal{H}_1(5) + \mathcal{H}_4(1)\big) - \frac{1}{2}\sum e_\alpha B_\alpha + D',\\
\mathrm{Div}_{X_6}(\omega'') &= \sigma^{-1}\pi_6^+\big(3\mathcal{H}_1(1) + 3\mathcal{H}_1(5) + \mathcal{H}_4(1)\big) - \frac{1}{2}\sum e_\alpha B_\alpha + D'',
\end{aligned}$$

where D, D' and D'' are effective divisors supported on the boundary $X_6\setminus\mathcal{A}_6^{\rm bil}$. The form of the branch locus part of the divisors follows now from Proposition 3.2 and the discriminants of $\mathcal{H}_{\zeta_i}$.

It remains to calculate the vanishing orders of the forms at each boundary divisor. For each form, we need only consider two boundary components, one from D_1 and one from D_2. We use the components $D(\mathbf{v}_1)$, $D(\mathbf{v}_2)$ corresponding to $\mathbf{v}_1 = (0,0,1,0)$ and $\mathbf{v}_2 = (0,0,2,1)$. The first step in constructing the toroidal compactification near $D(\mathbf{v}_1)$ is to take a quotient by the lattice $P'_{\mathbf{v}_1}(\Gamma_6^{\rm bil})$ (see for instance [GH2], pp.925–926 or for a full explanation [HKW], Section I.3D). As in [HKW], Proposition I.3.98, $P'_{\mathbf{v}_1}(\Gamma_6^{\rm bil})$ is generated by $j_1\left(\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}\right)$; so a local equation for $D(\mathbf{v}_1)$ at a general point is $t_1 = 0$, where $t_1 = e^{2\pi i\tau_1/6} = q^{1/6}$. Using the values of $f(0,l)$ calculated above and the Fourier expansion given in [GN], Theorem 2.1, we see that the expansions of F_3, F_3' and F_3'' begin $q^{1/3}rs^2$, $q^{2/3}r^3s^4$ and $q^{1/2}r^2s^3$ respectively, so their orders of vanishing along D_1 are 2, 4 and 3. The form $d\tau_1\wedge d\tau_2\wedge d\tau_3$ contributes a simple pole at the boundary so the coefficients of D_1 in the divisors of ω, ω' and ω'' are 1, 3 and 2.

We put

$$\theta = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z}),$$

so that $\mathbf{v}_2 = \mathbf{v}_1\theta$. Then $\mathcal{P}_{\mathbf{v}_2} = \theta^{-1}\mathcal{P}_{\mathbf{v}_1}\theta$ (where, as in [HKW], $\mathcal{P}_{\mathbf{v}}$ denotes the stabiliser of $\mathbf{v}$ in $\mathrm{Sp}(4,\mathbb{Q})$), and from this one readily calculates that

$$P'_{\mathbf{v}_2}(\Gamma_6^{\mathrm{bil}}) = \left\{ \begin{pmatrix} 1 & 0 & 4n & 2n \\ 0 & 1 & 2n & n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, n \equiv 0 \bmod 36 \right\}.$$

So the cusp D_2 is given by $t_2 = 0$, where $t_2 = e^{2\pi i(\tau_1/144+\tau_2/72+\tau_3/36)} = q^{1/144}r^{1/12}s$. The number of times this term divides the expressions for F_3, F_3' and F_3'' is in fact equal to the power of s that occurs, namely 2, 4 and 3 respectively; so we get the same orders of vanishing along D_2 as along D_1. ■

This calculation shows directly (without appealing to Freitag's result in [Fr]) that ω, ω' and ω'' are all holomorphic.

Remark. Notice that $\mathrm{Div}_{X_6}(\omega) + \mathrm{Div}_{X_6}(\omega') = 2\mathrm{Div}_{X_6}(\omega'')$, reflecting the fact (easily seen from [GN]) that $F_3F_3' = (F_3'')^2$.

Theorem A now follows at once from the following observation.

Proposition 3.4. *ω, ω' and ω'' are linearly independent elements of $H^0(K_{X_6})$.*

Proof: Suppose that $\lambda\omega + \lambda'\omega' + \lambda''\omega'' = 0$. At a general point of $\pi_6^{\mathrm{bil}}(\mathcal{H}_{\zeta_0})$, ω' and ω'' vanish but ω does not. Therefore $\lambda = 0$. Similarly $\lambda' = 0$, considering a general point of $\pi_6^{\mathrm{bil}}(\mathcal{H}_{\zeta_2})$. Finally, $\lambda'' \neq 0$ because F_3'' is not identically zero. ■

We want to remark that $\kappa(\mathcal{A}_6^{\mathrm{bil}}) \geq 1$ can be deduced from the existence of ω' alone. The divisor $\mathrm{Div}_{X_6}(\omega')$ is effective and $\pi_6^{\mathrm{bil}}(\mathcal{H}_\zeta) \subset \mathrm{Supp}\,\mathrm{Div}_{X_6}(\omega')$. Since X_6 has canonical singularities, K is effective on any smooth model of X_6, and hence also on any minimal model X_6' of X_6. Any surfaces contracted by the birational map $X_6 \dashrightarrow X_6'$ must be birationally ruled. But $\pi_6^{\mathrm{bil}}(\mathcal{H}_\zeta)$ is not birationally ruled: it is isomorphic to $X(6)\times X(6)$, since $\mathcal{H}_\zeta$ is isomorphic to $\mathbb{H}\times\mathbb{H}$ and is preserved by the subgroup $\Gamma(6)\times\Gamma(6)$ embedded in Γ_6^{bil} by (j_1, j_2). Thus its closure is birationally an abelian surface, since $X(6)$ has genus 1. So the canonical divisor of X_6' is effective and nontrivial; so, by abundance, some multiple of it moves and therefore $\kappa(\mathcal{A}_6^{\mathrm{bil}}) \geq 1$.

References

[SC] A. Ash, D. Mumford, M. Rapoport & Y. Tai, *Smooth Compactification of Locally Symmetric Varieties*, Math. Sci. Press, Brookline, Mass., 1975.

[Br] H.-J. Brasch, *Branch points in moduli spaces of certain abelian surfaces*, in: *Abelian Varieties, Egloffstein 1993*, de Gruyter, Berlin 1995, 27–54.

[Bo] L.A. Borisov, *A finiteness theorem for subgroups of* $\mathrm{Sp}(4,\mathbb{Z})$, Algebraic geometry, 9. J. Math. Sci. (New York) **94** (1999), 1073–1099.

[FS] M. Friedland & G.K. Sankaran, *Das Titsgebäude von Siegelschen Modulgruppen vom Geschlecht 2*, Math. Abh. Sem. Univ. Hamburg **71** (2001), 49–68.

[Fr] E. Freitag, *Siegelsche Modulfunktionen*, Grundlehren der mathematischen Wissenschaften Bd. **254**, Springer-Verlag, Berlin, 1983.

[vdG] G. van der Geer, *Hilbert Modular Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete Bd. **16**, Springer-Verlag, Berlin-New York, 1988.

[Gr] V. Gritsenko, *Irrationality of the moduli spaces of polarized abelian surfaces*, in: *Abelian Varieties, Egloffstein 1993*, de Gruyter, Berlin 1995, 63–84.

[GH1] V. Gritsenko & K. Hulek, *Minimal Siegel modular threefolds*, Math. Proc. Cambridge Philos. Soc. **123** (1998), 461–485.

[GH2] V. Gritsenko & K. Hulek, *The modular form of the Barth-Nieto quintic*, Internat. Math. Res. Notices **17** (1999), 915–937.

[GN] V. Gritsenko & V. Nikulin, *Automorphic forms and Lorentzian Kac-Moody Algebras II*, Internat. J. Math. **9** (1998), 201–275.

[HKW] K. Hulek, C. Kahn & S. Weintraub, *Moduli spaces of abelian surfaces: Compactification, degenerations and theta functions*, de Gruyter 1993.

[Mu] S. Mukai, *Moduli of abelian surfaces, and regular polyhedral groups*, in: *Moduli of algebraic varieties and the monster, Sapporo 1999* (I. Nakamura, Ed.), 68–74.

[Sa] G.K. Sankaran, *Abelian surfaces with odd bilevel structure*, preprint 2002, `math.AG/0209192`.

[Ue] K. Ueno, *On fibre spaces of normally polarized abelian varieties of dimension* 2. *II. Singular fibres of the first kind*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **19** (1972), 163–199.

G.K. Sankaran
Department of Mathematical Sciences
University of Bath
Bath BA2 7AY
ENGLAND

`gks@maths.bath.ac.uk`

J.G. Spandaw
Institut für Mathematik
Universität Hannover
Postfach 6009
D-30060 Hannover
GERMANY

`spandaw@math.uni-hannover.de`