The Moduli Space of Bilevel-6 Abelian Surfaces

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The moduli space \( \mathcal{A}^{\text{bil}} \) of \((1, t)\)-polarised abelian surfaces with a weak bilevel structure was introduced by S. Mukai in [Mu]. Mukai showed that \( \mathcal{A}^{\text{bil}} \) is rational for \( t = 2, 3, 4, 5 \). More generally, we may ask for birational invariants, such as Kodaira dimension, of a smooth model of a compactification of \( \mathcal{A}^{\text{bil}} \); since the choice of model does not affect birational invariants, we refer to the Kodaira dimension, etc., of \( \mathcal{A}^{\text{bil}} \).

From the description of \( \mathcal{A}^{\text{bil}} \) as a Siegel modular 3-fold \( \Gamma^{\text{bil}} \backslash \mathbb{H}_2 \) and the fact that \( \Gamma^{\text{bil}} \subset \text{Sp}(4, \mathbb{Z}) \) it follows, by a result of L. Borisov [Bo], that \( \kappa(\mathcal{A}^{\text{bil}}) = 3 \) for all sufficiently large \( t \). For an effective result in this direction see [Sa]. In this note we shall prove an intermediate result for the case \( t = 6 \).

**Theorem A.** The moduli space \( \mathcal{A}^{\text{bil}} \) has geometric genus \( p_g(\mathcal{A}^{\text{bil}}) \geq 3 \) and Kodaira dimension \( \kappa(\mathcal{A}^{\text{bil}}) \geq 1 \).

The case \( t = 6 \) attracts attention for two reasons: it is the first case not covered by the results of [Mu]; and the image of the Humbert surface \( \mathcal{H}_1(1) \) in \( \mathcal{A}^{\text{bil}} \), which in the cases \( 2 \leq t \leq 5 \) is a quadric and plays an important role both in [Mu] and below, becomes an abelian surface (at least birationally) because the modular curve \( X(6) \) has genus 1.

The method we use is that of Gritsenko, who proved a similar result for the moduli spaces of \((1, t)\)-polarised abelian surfaces with canonical level structure for certain values of \( t \): see [Gr], especially Corollary 2. We use some of the weight 3 modular forms constructed by Gritsenko and Nikulin as lifts of Jacobi forms in [GN] to produce canonical forms having effective, nonzero, divisors on a suitable projective model \( X_6 \) of \( \mathcal{A}^{\text{bil}} \). A similar method was used by Gritsenko and Hulek in [GH2] to give a new proof that the Barth–Nieto threefold is Calabi-Yau.

We also derive some information about divisors in \( X_6 \) and linear relations among them.

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1. **Compactification**

According to [Mu], \( \mathcal{A}^{\text{bil}} \) is isomorphic to the quotient \( \Gamma^{\text{bil}} \backslash \mathbb{H}_2 \), where \( \mathbb{H}_2 \) is the Siegel upper half-plane \( \{ Z \in M_{2 \times 2}(\mathbb{C}) \mid Z = \overline{Z}, \text{Im} Z > 0 \} \) and \( \Gamma^{\text{bil}} = \Gamma_1 \cup (\Gamma_1 \cap \mathbb{Z}) \subset \text{Sp}(4, \mathbb{Z}) \) acts on \( \mathbb{H}_2 \) by fractional linear transformations. Here \( \zeta = \text{diag}(-1, 1, -1, 1) \) and, writing \( I_n \) for the \( n \times n \) identity matrix,

\[
\Gamma_1^{\text{bil}} = \left\{ \gamma \in \text{Sp}(4, \mathbb{Z}) \mid \gamma - I_4 \begin{pmatrix} t & 0 & 0 \\
 & t & t \\
 & 0 & t \\
 & 0 & 0 \
\end{pmatrix} \in \mathbb{Z} \begin{pmatrix} t & 0 & 0 \\
 & t & t \\
 & 0 & t \\
 & 0 & 0 \
\end{pmatrix} \right\}.
\]

We define \( H(\mathbb{Z}) \) to be the Heisenberg group \( \mathbb{Z} \times \mathbb{Z}^2 \) embedded in \( \text{Sp}(4, \mathbb{Z}) \) as

\[
H(\mathbb{Z}) = \left\{ [m, n; k] = \begin{pmatrix} 1 & m & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & n & 1 & 0 \\
0 & 0 & k & -m \\
\end{pmatrix} \mid m, n, k \in \mathbb{Z} \right\}.
\]
Lemma 1.1. $\Gamma_6^1$ is neat; that is, if $\lambda$ is an eigenvalue of some $\gamma \in \Gamma_6^1$ which is a root of unity, then $\lambda = 1$. Any torsion element of $\Gamma_6$ has order 2 and fixes a divisor in $\mathbb{H}_2$.

Proof: Suppose that $\gamma \in \Gamma_6^1$; then the characteristic polynomial of $\gamma$ is congruent to $(1 - x)^4$ mod 6. If some $\gamma \in \Gamma_6^1$ has an eigenvalue $\lambda$ which is a nontrivial root of unity, then we may assume that $\lambda$ is a primitive $p$th root of unity for some prime $p$. The minimum polynomial $m_\lambda(x)$ of $\lambda$ over $\mathbb{Z}$ divides the characteristic polynomial of $\gamma$; so $p = 2, 3$ or 5, since $\deg m_\lambda = p - 1$. But then $m_\lambda(x) = 1 + x + x^2$ or $1 + x + x^2 + x^3 + x^4$. The second of these does not divide $(1 - x)^4$ in $\mathbb{F}_2[x]$ and the other two do not divide $(1 - x)^4$ in $\mathbb{F}_3[x]$.

So any torsion element of $\Gamma_6$ is of the form $\gamma = \zeta \gamma'$ for some $\gamma' \in \Gamma_6^1$; but then the characteristic polynomial is

$$\det(\gamma - xI_4) = \det(\zeta \gamma' - x^2)$$

$$\equiv (1 - x^2)(1 + x^2) \mod 6.$$  

From the classification of torsion elements of $\text{Sp}(4, \mathbb{Z})$ and their characteristic polynomials [Ue], it follows that $\gamma$ is conjugate in $\text{Sp}(4, \mathbb{Z})$ to either $\zeta$ or $\zeta[0, 1; 0]$. Both these are elements of $\Gamma_6$ of order 2; their fixed loci in $\mathbb{H}_2$ are the divisors $\{\tau_2 = 0\}$ and $\{2\tau_2 + (\tau_2^2 - \tau_1 \tau_3) = 0\}$ respectively (Humbert surfaces of discriminants 1 and 4). 

In view of Lemma 1.1, the toroidal (Voronoi, or Igusa) compactification $(\mathcal{A}_6^1)^*$ of $\mathcal{A}_6^1 = \Gamma_6^1 \backslash \mathbb{H}_2$ is smooth, cf [SC], pp. 276-7. The action of $\zeta$ on $\mathcal{A}_6^1$ extends to $(\mathcal{A}_6^1)^*$, and the quotient $X_6$ is a compactification of $\mathcal{A}_6^1$ whose singularities are isolated ordinary double points or transverse $A_1$ singularities. Hence $X_6$ has canonical singularities. It agrees with the Voronoi compactification $(\mathcal{A}_6^{[\Pi]}_6)^*$ at least in codimension 1.

2. Modular forms and canonical forms

Gritsenko and Nikulin, in [GN], construct the weight 3 cusps forms

$$F_3 = \text{Lift}(\eta(\tau_1) \eta(\tau_2)) \in \mathcal{M}_3^1(\Gamma_6^0, \nu_{10}^8 \times \text{id}_H)$$

$$F_3' = \text{Lift}_{-1}(\eta(\tau_1) \eta(\tau_2)) \in \mathcal{M}_3^1(\Gamma_6^0, \nu_{10}^{16} \times \text{id}_H)$$

$$F_3'' = \text{Lift}(\eta(\tau_1) \eta(\tau_2^2) \eta(\tau_1, \tau_2)) \in \mathcal{M}_3^1(\Gamma_6^+, \nu_{10}^{12} \times \text{id}_H)$$

for the extended paramodular group $\Gamma_6^+$, with character $\chi_D$ induced from the characters $\nu_{10}^D \times \text{id}_H$ of the Jacobi group $\text{SL}(2, \mathbb{Z}) \times H(\mathbb{Z})$. Recall (see [GH1], [GN]; for compatibility with [Mu] and other sources we work with the transposes of the groups given in [GN]) that $\Gamma_6^+$ is the group generated by the paramodular group

$$\Gamma_6 = \left\{ \gamma \in \text{Sp}(4, \mathbb{Q}) \mid \gamma \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}$$

and the extra involution

$$V_6 = \begin{pmatrix} 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 \\ \sqrt{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{6} \\ 0 & 1/\sqrt{6} & 0 & 0 & 0 \end{pmatrix}.$$
Proposition 2.1. All three of $F_3$, $F_3'$ and $F_3''$ are cusp forms, without character, of weight 3 for $\Gamma_0^{\text{bil}}$.

Proof: The character is induced from $\psi^0 \times \text{id}_H$ by the injective map $j : \text{SL}(2, \mathbb{Z}) \times H(\mathbb{Z}) \to \Gamma_0^{\text{bil}}$ given by

$$j : \left( \begin{array}{ccc} a & b & c \\ c & d & 0 \\ m & n & k \end{array} \right), [m,n;k]) \mapsto \left( \begin{array}{ccc} a & m & c \\ 0 & 1 & 0 \\ b & n & d \\ n & k & -m \end{array} \right).$$

For $\gamma \in \text{SL}(2, \mathbb{Z})$ we define $j_1(\gamma) = j(\gamma, [0,0,0])$, putting $\gamma$ in the first and third rows and columns in $\text{Sp}(4, \mathbb{Z})$; and similarly $j_2(\gamma)$ puts it in the second and fourth.

The character $\psi^0 \times \text{id}_H$ is trivial on $H(\mathbb{Z})$. In the present cases, where $D = 8, 16$ or $12$, $\psi^0$ is trivial on $\pm \Gamma(6) = \pm \text{Ker}(\text{SL}(2, \mathbb{Z}) \to \text{SL}(2, \mathbb{Z}/6))$ by [GN], Lemma 1.2. Since $j([-I_2,[0,0,0]) = \zeta$, we see that

$$\Gamma_0^{\text{bil}} \cap j(\text{SL}(2, \mathbb{Z}) \times H(\mathbb{Z})) \subseteq j(\pm \Gamma(6) \times H(\mathbb{Z})) \subseteq \text{Ker} \chi_D$$

for $D = 8, 12, 16$. If $D = 8$ or $16$ then, since $V_0$ and $I = j_1 \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ are in $\Gamma_0^{\text{bil}}$ and have even order and the order of $\chi_D$ is 3, we know that $\chi_D(V_0) = \chi_D(I) = 1$. Therefore the element

$$J_0 = IV_0I^{-1}_0 = \left( \begin{array}{ccc} 0 & 0 & -1 \\ 0 & 0 & -6 \\ 1 & 0 & 0 \end{array} \right) \in \Gamma_0^{\text{bil}}$$

is in $\text{Ker} \chi_D$. If $D = 12$ then $\chi_{12}(J_0) = \chi_{12}(IV_0I^{-1}_0) = 1$ so again $J_0 \in \text{Ker} \chi_D$. Now we proceed as in [Gr], Lemma 2.2, and show that the group generated by $j(\Gamma(6) \times H(\mathbb{Z}))$ and $J_0$ includes $\Gamma_0^{\text{bil}}$. To see this, we work with the conjugate groups $\tilde{\Gamma}_0^{\text{bil}} = \nu_6(\Gamma_0^{\text{bil}})$ and $\tilde{\Gamma}_0 = \nu_6(\Gamma_0)$, where $\nu_6$ denotes conjugation by $R_0 = \text{diag}(1,1,1,6)$. Note that $\nu_6(J_0) = R_0J_0R_0^{-1} = \left( \begin{array}{ccc} 0 & -I_2 \\ I_2 & 0 \end{array} \right)$. If $\tilde{\gamma} \in \tilde{\Gamma}_0^{\text{bil}}$ then its second row $\tilde{\gamma}_{22}$ is $(0,1,0,0)$ mod $6$. Suppose first that $\tilde{\gamma}_{22} = 1$ and put

$$\tilde{\beta} = \nu_6(J_0[\tilde{\gamma}_{21}/6, \tilde{\gamma}_{23}/6, \tilde{\gamma}_{24}/6]J_0^{-1}) = \left( \begin{array}{ccc} 1 & 0 & 0 \\ \tilde{\gamma}_{21} & 1 & \tilde{\gamma}_{23}/6 \\ 0 & 0 & 1 \end{array} \right).$$

Now $(0,1,0,0)\tilde{\beta} = \tilde{\gamma}_{22}$, so the second row of $\tilde{\gamma} \tilde{\beta}^{-1} \in \tilde{\Gamma}_0^{\text{bil}}$ is $(0,1,0,0)$. Such a matrix is in $\nu_6(j(\Gamma(6) \times H(\mathbb{Z})))$.

It remains to reduce to the case $\tilde{\gamma}_{22} = 1$. Certainly the vector $\tilde{\gamma}_{21}$ is primitive, since det $\tilde{\gamma} = 1$, and since $\tilde{\gamma} \in \tilde{\Gamma}_0^{\text{bil}}$ we have gcd(6, $\tilde{\gamma}_{21}$, $\tilde{\gamma}_{23}$) = 6. In the proof of [FS], Satz 2.1 it is shown that there are integers $\lambda$, $\mu$ such that $\tilde{\gamma}' = \gamma_6$ $\left( \begin{array}{cc} \mu & 0 \\ 0 & \lambda \\ 0 \\ 0 \end{array} \right)$ has gcd($\gamma_{21}'$, $\gamma_{23}'$) = 6, so the second row of $\tilde{\gamma}'$ is $(6x_1, 6x_2, 1, 6x_3, 6x_4)$ with gcd($x_1$, $x_3$) = 1. But then the $(2, 2)$-entry of $\tilde{\gamma}' \nu_6([m,n,0])$ is $6(mx_1 + nx_3 + x_2) + 1$ which is equal to 1 if we choose $m$ and $n$ suitably. ■

Proposition 2.2. The differential forms $\tilde{\omega} = F_3^0 d\tau_1 \wedge d\tau_2 \wedge d\tau_3$, $\tilde{\omega}' = F_3^0 d\tau_1 \wedge d\tau_2 \wedge d\tau_3$ and $\tilde{\omega}'' = F_3^0 d\tau_1 \wedge d\tau_2 \wedge d\tau_3$ give rise to canonical forms $\omega, \omega', \omega'' \in H^0(K_X\nu)$.  

Proof: By Proposition 2.1, $\tilde{\omega}$, $\tilde{\omega}'$ and $\tilde{\omega}''$ are all $\Gamma_0^{\text{bil}}$-invariant, so they give rise to forms $\omega$, $\omega'$, $\omega''$ on $A_0^{\text{bil}}$. Since $F_3$, $F_3'$ and $F_3''$ are cusp forms, if any of $\omega$, $\omega'$ and $\omega''$ are holomorphic on $A_0^{\text{bil}}$ they extend holomorphically to the cusps of $(A_0^{\text{bil}})^*$. Since $X_0$ agrees with $(A_0^{\text{bil}})^*$ in codimension 1 and has canonical singularities it follows that these forms can be thought of as 3-forms on $X_0$ holomorphic at infinity. We need to check that $\omega$, $\omega'$ and $\omega''$ are holomorphic everywhere. But this is a well-known result of Freitag([Fr], Satz II.2.6). ■
3. Divisors in the moduli spaces.

In this section we shall describe the canonical divisors \( \Div_{X_0}(\omega) \), \( \Div_{X_0}(\omega') \) and \( \Div_{X_0}(\omega'') \) in \( X_0 \) and give some detail about the branching locus in \( X_0 \) arising from torsion in \( \Gamma^\text{bil}_0 \).

\( \Gamma^\text{bil}_0 \) is a subgroup both of the paramodular group \( \Gamma_0 \) and of \( \Gamma^\text{bil}_0 \). Hence there is a finite morphism \( \sigma : \mathcal{A}^\text{bil}_0 \to \mathcal{A}^\text{bil}_6 \). We denote the projection map \( \mathbb{H}_2 \to \mathcal{A}^\text{bil}_0 \) by \( \pi^\text{bil}_0 \) and similarly \( \pi_0, \pi_0^+ \), etc.

For discriminant \( \Delta = 1, 4 \) we put

\[
\mathcal{H}_\Delta(k) = \left\{ \left( \frac{\tau_1}{\tau_2}, \frac{\tau_2}{\tau_3} \right) \in \mathbb{H}_2 \right\} = 0
\]

where \( k \in \mathbb{Z} \) is chosen so that \( \frac{1}{\pi^2}(k^2 - \Delta) \in \mathbb{Z} \). The irreducible components of the Humbert surfaces \( H_1 \) and \( H_4 \) of discriminants 1 and 4 in \( \mathcal{A}_0 \) are \( \pi_0(H_1(k)) \) and \( \pi_0(H_4(k)) \) for \( 0 \leq k < 6 \); the statements of [vdG], Theorem IX.2.4 and of [GH1], Corollary 3.3 are wrong because \( \mathcal{H}_\Delta(-k) \) is \( \Gamma_1 \)-equivalent to \( \mathcal{H}_\Delta(k) \). Nevertheless the irreducible components of the Humbert surfaces of discriminants 1 and 4 in \( \mathcal{A}^+_0 \) are as stated in [GN], namely \( \pi^+_0(H_1(1)) \) and \( \pi^+_0(H_1(5)) \) for discriminant 1 and \( \pi^+_0(H_4(1)) \) for discriminant 4.

The calculation of the divisors uses the product expansion of the modular forms \( F_3, F_3^\prime \) and \( F_3^\prime \prime \) given in [GN]. We have chosen to work with the transposes of the matrices given in [GN], so we have to write

\[
q = e^{2\pi i r}, \quad r = e^{2\pi i \tau_2/6}, \quad s = e^{2\pi i \tau_3/36}
\]

for these expansions to be correct. This is because \( \Gamma_1 = \text{diag}(1, t, 1, t^{-1}) \Gamma_1 \text{diag}(1, t^{-1}, 1, t) \) (for any \( t \in \mathbb{N} \)), and \( \text{diag}(1, t, 1, t^{-1}) : (\tau_1, \tau_2, \tau_3) \to (\tau_1, t\tau_2, t^2\tau_3) \). A similar correction is needed in [GH2].

By [GN], equations (4.12)-(4.14), correcting a minor misprint, we have

\[
F_3 = \text{Exp-Lift}(5\phi^2_{0,3} - 4\phi_{0,4} \phi_{0,4}) = \text{Exp-Lift}(\phi_3)
\]
\[
F_3^\prime = \text{Exp-Lift}(\phi^2_{0,3}) = \text{Exp-Lift}(\phi_3^\prime)
\]
\[
F_3^\prime\prime = \text{Exp-Lift}(3\phi^2_{0,3} - 2\phi_{0,4} \phi_{0,4}) = \text{Exp-Lift}(\phi_3^\prime\prime)
\]

(\( \phi_3, \phi_3^\prime \) and \( \phi_3^\prime\prime \) are defined by these formulae.)

By [GN], Example 2.3 and Lemma 2.5, we have

\[
\phi_{0,2} = (r^{\pm1} + 4) + q(r^{\pm3} - 8r^{\pm2} - r^{\pm1} + 16) + O(q^2)
\]
\[
\phi_{0,3} = (r^{\pm1} + 2) + q(-2r^{\pm3} - 2r^{\pm2} + 2r^{\pm1} + 4) + O(q^2)
\]
\[
\phi_{0,4} = (r^{\pm1} + 1) + q(-r^{\pm4} - r^{\pm3} + r^{\pm1} + 2) + O(q^2)
\]

where the notation \( r^{\pm k} \) means \( r^k + r^{-k} \).

**Proposition 3.1.** The divisors in \( \mathbb{H}_2 \) of the cusp forms are

\[
\Div(F_3) = (\pi^+_0)^{-1} (\pi^+_0 (H_1(1) + 5H_1(5) + H_4(1)))
\]
\[
\Div(F_3^\prime) = (\pi^+_0)^{-1} (\pi^+_0 (5H_1(1) + H_1(5) + H_4(1)))
\]
\[
\Div(F_3^\prime\prime) = (\pi^+_0)^{-1} (\pi^+_0 (3H_1(1) + 3H_1(5) + H_4(1)))
\]

**Remark.** This corrects the coefficients given in [GN], Example 4.6: for instance, it is easy to see, by considering the effect of an element of order 2 fixing an Humbert surface, that the coefficients of \( H_1(1) \), \( H_1(5) \) and \( H_4(1) \) must be odd.

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Proof: Write \( \phi_3 = \sum f(n,l)q^n r^l \), and similarly for \( \phi'_3 \) and \( \phi''_3 \). By [GN], Theorem 2.1, the coefficient of \( \pi^+_6(\mathcal{H}_\Delta(b)) \) in \( \mathcal{A}_6^+ \) is

\[
m_{\Delta,b} = \sum_{d>0} f(d^2a,db)
\]

where \( b^2 - 24a = \Delta \). So to calculate \( m_{1,1} \) we may take \( b = 1 \) and \( a = 0 \), so \( m_{1,1} = \sum_{d>0} f(0,d) \). From the formulae above, \( \phi_3 = (r^{\pm 2} + 6) + O(q) \), so \( m_{1,1} = f(0,2) = 1 \). Similarly we have \( \phi'_3 = (r^{\pm 2} + 4r^{\pm 1} + 6) \) so \( m'_{1,1} = 5 \) and \( \phi''_3 = (r^{\pm 2} + 2r^{\pm 1} + 6) \) so \( m''_{1,1} = 3 \).

To calculate the coefficients of \( \pi^+_6(\mathcal{H}_4(1)) \) we note that \( \mathcal{H}_4(1) \) is \( \Gamma^+_6 \)-equivalent to \( \mathcal{H}_4(2) \), so we may as well work with that and calculate \( m_{4,2} \). For this purpose we can take \( b = 2 \) and \( a = 0 \); so \( m_{4,2} = \sum_{d>0} f(0,2d) = 1 \), and \( m'_{4,2} = m''_{4,2} = 1 \) also.

To calculate \( m_{1,5} \) we take \( b = 5 \) and \( a = 1 \), so \( m_{1,5} = \sum_{d>0} f(d^2 , 5d) \). The Fourier coefficient \( f(n,l) \) depends only on \( 24n - l^2 \) and on the residue class of \( l \mod 12 \) (see [GN]); that is, in our case, on \( d^2 \) and on \( d \mod 12 \). If \( d \not\equiv \pm 1 \mod 6 \) then \( 5d \not\equiv d \mod 12 \), so \( f(d^2 , 5d) = f(0, \pm d) \) which is zero unless \( d = \pm 2 \) or \( d = 0 \). Since we are only interested in \( d > 0 \) the only contribution for \( d \not\equiv \pm 1 \mod 6 \) arises from \( d = 2 \), when \( f(4,10) = f(0,-2) = 1 \). If \( d \equiv \pm 5 \mod 12 \) then \( f(d^2 , 5d) = f(d^2, \pm 1) = 1 \) which vanishes because \( f(n,l) = 0 \) for \( n < 0 \). If \( d \equiv \pm 1 \mod 12 \) then \( f(d^2 , 5d) = f(d^2 , 5d) = f(\frac{d^2+25}{12} , \pm 5) \) which vanishes except possibly when \( d = 1 \). So \( m_{1,5} = 1 + f(1,5) \) and from the expansions of \( \phi_{0,2} , \phi_{0,3} \) and \( \phi_{0,4} \) we calculate \( f(1,5) = 4 \). Similarly \( m'_{1,5} = 1 + f'(1,5) = 1 \) and \( m''_{1,5} = 1 + f''(1,5) = 3 \). ●

Brasher [Br] has studied the branch locus of \( \pi^+_6: \mathbb{H}_6 \to \mathcal{A}^+_6 \) for all \( t \) for \( t \equiv 2 \mod 4 \) the divisorial part has five irreducible components. They are \( \pi^+_6(\mathcal{H}_t) \) for \( 0 \leq t \leq 4 \), where \( \mathcal{H}_t \subset \mathbb{H}_6 \) is the fixed locus of \( \zeta \) and

\[
\zeta_0 = \zeta, \quad \zeta_1 = \zeta^{[6,0;0]}, \quad \zeta_2 = \begin{pmatrix} -7 & 4 & 0 & 0 \\ -12 & 7 & 0 & 0 \\ 0 & 0 & -7 & -12 \\ 0 & 0 & 4 & 7 \end{pmatrix},
\]

\[
\zeta_3 = \zeta^{[1,0;0]}, \quad \zeta_4 = \begin{pmatrix} -1 & -1 & 0 & 6 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.
\]

These are all elements of \( \Gamma^+_6 \). Their fixed loci are

\[
\mathcal{H}_{\zeta_0} = \{ \tau_2 = 0 \}, \quad \mathcal{H}_{\zeta_1} = \{ 6\tau_1 - 2\tau_2 = 0 \}, \quad \mathcal{H}_{\zeta_2} = \{ 6\tau_1 - 7\tau_2 + 2\tau_3 = 0 \},
\]

\[
\mathcal{H}_{\zeta_3} = \{ 2\tau_2 + \tau_3 = 0 \}, \quad \mathcal{H}_{\zeta_4} = \{ 2\tau_2 + \tau_3 - 6 = 0 \},
\]

of discriminants 1, 4, 1, 4, 4 respectively. Thus three of the components have discriminant 4 and therefore map to \( \pi^+_6 \mathcal{H}_4(1) \subset \mathcal{A}^+_4 \) (they correspond to bielliptic abelian surfaces). \( \mathcal{H}_{\zeta_0} = \mathcal{H}_4(1) \) corresponds to product surfaces \( E \times E' \) with polarisation given by \( O_E(1) \boxtimes O_{E'}(6) \), and \( \mathcal{H}_{\zeta_2} \) maps to \( \pi^+_6(\mathcal{H}_4(5)) \), corresponding to abelian surfaces \( E \times E' \) with polarisation \( O_E(2) \boxtimes O_{E'}(3) \).
Proposition 3.2. The branch locus of $\pi_{6}^{\text{bil}} : \mathbb{H}_{2} \to \mathcal{A}_{6}^{\text{bil}}$ has seven irreducible components, each with branching of order 2. They are $\pi_{6}^{\text{bil}}(H_{C_{i}})$ and two other components $\pi_{6}^{\text{bil}}(H_{C_{j}}), \pi_{6}^{\text{bil}}(H_{G_{v}})$, which are equivalent to $\pi_{6}^{\text{bil}}(H_{C_{i}})$ in $\mathcal{A}_{6}^{\text{ev}}$.

Proof: It follows from Lemma 1.1 that the branch locus consists of divisors only and that the branching is of order 2.

Write $G = \Gamma_{6}^{\text{ev}} \rhd H = \Gamma_{6}^{\text{bil}}$ and let $G$ act on $\Omega = G/H \cong \text{PSL}(2,\mathbb{Z}/6)$. By [Br], Corollary 1.3, the number of irreducible divisors in $\mathcal{A}_{6}^{\text{bil}}$ mapping to $\pi_{6}^{\text{ev}}(H_{C_{i}})$, which is equal to the number of $H$-conjugacy classes in the $G$-conjugacy class of $\zeta_{i}$, is $|G : H.C_{G}(\zeta_{i})|$. (If $\xi \in G$ for some group $G$ then $C_{G}(\xi)$ denotes the centraliser of $\xi$ in $G$.) Moreover, for fixed $i$, these divisors are permuted transitively by $\Omega$ so they all have the same branching behaviour: $\pi_{6}^{\text{bil}}$ is branched of order 2 above each one.

$|G : H.C_{G}(\zeta_{i})| = |G/H : C_{G}(\zeta_{i})/(H \cap C_{G}(\zeta_{i}))|$, which is the index of the image of $C_{G}(\zeta_{i})$ in $\Omega$. For $i = 0, 1, 2, 3$ the centraliser $C_{\text{Sp}(4,\mathbb{Q})}(\zeta_{i})$ is described in [Br], Lemma 2.1, and $C_{G}(\zeta_{i}) = C_{\text{Sp}(4,\mathbb{Q})}(\zeta_{i}) \cap G$.

For $\zeta_{0}$, if $\gamma \in \text{PSL}(2,\mathbb{Z}/6) \cong \Omega$ and $\tilde{\gamma} \in \text{SL}(2,\mathbb{Z})$ is some lift of $\gamma$ then $j(\tilde{\gamma},[0,0;0]) \in C_{G}(\tilde{\zeta})$ so the index is 1.

For $\zeta_{1}$, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2,\mathbb{Z}/6)$ and $b$ is even then

$$\begin{pmatrix} \tilde{a} & 0 & \tilde{b} \\ 3(\tilde{a} - 1) & 1 & 3\tilde{b} \\ c & 0 & d \\ 0 & 0 & 1 \end{pmatrix} \in C_{G}(\zeta_{1})$$

for a lift $\tilde{\gamma}$; and this is a necessary condition for such an element to exist since if $\beta = \beta_{ij} \in C_{G}(\zeta_{1})$ then $3\beta_{13} \equiv 0 \mod 6$. So $C_{G}(\zeta_{1})/(C_{G}(\zeta_{1}) \cap H) \subset \text{PSL}(2,\mathbb{Z}/6)$ is the reduction mod 6 of $\Gamma_{0}(2)$, i.e. the preimage of $\left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \text{SL}(2,\mathbb{Z}/2) \right\}$, which is of index 3 because it is the stabiliser of $(1,0)$ when $\text{SL}(2,\mathbb{Z}/2)$ acts as the symmetric group on the nonzero vectors in $\mathbb{F}_{2}$.

For $\zeta_{2}$, any two elements $\gamma, \gamma^{*} \in \text{SL}(2,\mathbb{Q})$ determine an element $\beta(\gamma, \gamma^{*}) \in C_{\text{Sp}(4,\mathbb{Q})}$ (see [Br], Lemma 2.1 and the preceding discussion), namely

$$\beta(\gamma, \gamma^{*}) = \begin{pmatrix} 4\gamma_{11} - 3\gamma_{11}^{*} & -2\gamma_{11} + 2\gamma_{11}^{*} & 4\gamma_{12} + \gamma_{12}^{*} & 6\gamma_{12} + 2\gamma_{12}^{*} \\ 6\gamma_{11} - 6\gamma_{11}^{*} & -3\gamma_{11} + 4\gamma_{11}^{*} & 4\gamma_{12} + 2\gamma_{12}^{*} & 9\gamma_{12} + 4\gamma_{12}^{*} \\ 4\gamma_{21} + 9\gamma_{21}^{*} & -2\gamma_{21} - 6\gamma_{21}^{*} & 4\gamma_{22} - 3\gamma_{22}^{*} & 6\gamma_{22} - 6\gamma_{22}^{*} \\ -2\gamma_{21} - 6\gamma_{21}^{*} & \gamma_{21} + 4\gamma_{21}^{*} & -2\gamma_{22} + 2\gamma_{22}^{*} & -3\gamma_{22} + 4\gamma_{22}^{*} \end{pmatrix}$$

In particular we choose

$$\beta = \beta \left( \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 10 & 9 \\ 11 & 10 \end{pmatrix} \right) = \begin{pmatrix} -18 & 14 & 25 & 42 \\ -42 & 31 & 42 & 72 \\ 107 & -70 & -18 & -42 \\ -70 & 46 & -14 & 31 \end{pmatrix}$$

and

$$\beta' = \beta' \left( \begin{pmatrix} 11 & 4 \\ 8 & 3 \end{pmatrix}, \begin{pmatrix} 7 & 9 \\ 3 & 4 \end{pmatrix} \right) = \begin{pmatrix} 23 & -8 & 25 & 42 \\ 24 & -5 & 42 & 72 \\ 59 & -34 & 0 & -6 \\ -34 & 20 & -6 & 7 \end{pmatrix}.$$
\(\beta\) and \(\beta'\) both belong to \(\Gamma_6^\text{lev}\), and their images in \(\text{PSL}(2,\mathbb{Z}/6)\) are \(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\) and \(\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\). These two elements generate \(\text{PSL}(2,\mathbb{Z}/6)\) because their lifts generate \(\text{SL}(2,\mathbb{Z})\), so the index we want is 1.

For \(\zeta_3\), as for \(\zeta_5\), \(j(\zeta_3, [0, 0, 0]) \in C_G(\zeta_3)\) so the index is 1.

For \(\zeta_4\), note that \(\zeta_4 = \tau(0, 0, 0)\) so \(C_{\text{Sp}(4,\mathbb{Q})}(\zeta_4) = \tau(0, 0, 0)\text{Sp}(4,\mathbb{Q})(\zeta_4)(\tau(0, 0, 0))^{-1}\). It happens that \(\tau(0, 0, 0)j(\zeta_4, [0, 0, 0])\tau(0, 0, 0)^{-1} = j(\zeta_4, [0, 0, 0])\), so again the index is 1. ■

Next we look at the boundary divisors of \(X_6\). These correspond to 1-dimensional subspaces of \(Q^4\) up to the action of \(\Gamma_6^\text{bil}\). We may think of such a space as being given by a unique, up to sign, primitive vector \(v = (v_1, v_2, v_3, v_4) \in \mathbb{Z}^4\). It is shown in [FS, Satz 2.1], that the \(\Gamma_6\)-orbit of \(v\) is determined by \(r = \gcd(6, v_1, v_3)\), so \(A_6\) has four corank 1 cusps (or boundary divisors in the toroidal compactification). However, the cusps \(r = 1\) and \(r = 6\) are exchanged by \(V_6\), as are the cusps \(r = 2\) and \(r = 3\), so \(A_6^+\) has just two corank 1 cusps. Since \(F_3, F_3'\) and \(F_3''\) are modular forms (with character) for \(\Gamma_6^+\), the order of vanishing of any of them at a cusp of \(X_6\) given by \(v\) depends only on which cusp of \(A_6^+\) it lies over, i.e. on whether \(r\) is or is not a proper divisor of 6.

We write \(D_1\) for the divisor in \(X_6\) which is the sum of all the boundary components with \(r = 1\) or \(r = 6\), and \(D_2\) for the sum of all the components with \(r = 2\) or \(r = 3\). By modifying the argument of [FS, Satz 2.1] as in [Sa], it can be shown that \(D_1\) has 28 irreducible components and \(D_2\) has 12, but we shall not make any use of this.

**Theorem 3.3.** The divisors of \(\omega, \omega'\) and \(\omega''\) in \(X_6\) are

\[
\begin{align*}
\text{Div}_{X_6}(\omega) &= 4\pi_6^{\text{bil}}(H_\zeta_3) + D_1 + D_2, \\
\text{Div}_{X_6}(\omega') &= 4\pi_6^{\text{bil}}(H_\zeta_3) + 3(D_1 + D_2), \\
\text{Div}_{X_6}(\omega'') &= 2\pi_6^{\text{bil}}(H_\zeta_3) + 2\pi_6^{\text{bil}}(H_\zeta_3) + 2(D_1 + D_2).
\end{align*}
\]

**Proof:** If \(\pi_6^{\text{bil}}\) is branched along the irreducible divisors \(B_\alpha\) with ramification index \(e_\alpha\), then \(d\tau_1 \wedge d\tau_3 \wedge d\tau_3\) acquires poles of order \(e_\alpha/2\) along \(B_\alpha\). So by Proposition 3.1

\[
\begin{align*}
\text{Div}_{X_6}(\omega) &= \sigma^{-1}\pi_6^+(H_1(1) + 5H_1(5) + H_4(1)) - \frac{1}{2}\sum e_\alpha B_\alpha + D, \\
\text{Div}_{X_6}(\omega') &= \sigma^{-1}\pi_6^+(5H_1(1) + H_1(5) + H_4(1)) - \frac{1}{2}\sum e_\alpha B_\alpha + D', \\
\text{Div}_{X_6}(\omega'') &= \sigma^{-1}\pi_6^+(3H_1(1) + 3H_1(5) + H_4(1)) - \frac{1}{2}\sum e_\alpha B_\alpha + D'',
\end{align*}
\]

where \(D, D'\) and \(D''\) are effective divisors supported on the boundary \(X_6 \setminus A_6^{\text{bil}}\). The form of the branch locus part of the divisors follows now from Proposition 3.2 and the discriminants of \(H_\zeta_3\).

It remains to calculate the vanishing orders of the forms at each boundary divisor. For each form, we need only consider two boundary components, one from \(D_1\) and one from \(D_2\). We use the components \(D(v_1), D(v_2)\) corresponding to \(v_1 = (0, 0, 1, 0)\) and \(v_2 = (0, 0, 2, 1)\). The first step in constructing the toroidal compactification near \(D(v_1)\) is to take a quotient by the lattice \(P_{v_1}(\Gamma_6^{\text{bil}})\) (see for instance [GH2], pp 925–926 or for a full explanation [HKW, Section 1.3D]). As in [HKW], Proposition 1.3.98, \(P_{v_1}(\Gamma_6^{\text{bil}})\) is generated by \(j_1\left(\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}\right)\); so a local equation for \(D(v_1)\) at a general point is \(t_1 = 0\), where \(t_1 = e^{2\pi i t_1/3} = q^{1/6}\). Using the values of \(f(0, l)\) calculated above and the Fourier expansion given in [GN], Theorem 2.1, we see that the
expansions of $F_3$, $F_3'$ and $F_3''$ begin $q^{1/3}r^2$, $q^2/3r^3s^4$ and $q^{1/3}r^2s^3$ respectively, so their orders of vanishing along $D_1$ are 2, 4 and 3. The form $d\tau_1\wedge d\tau_2\wedge d\tau_3$ contributes a simple pole at the boundary so the coefficients of $D_1$ in the divisors of $\omega$, $\omega'$ and $\omega''$ are 1, 3 and 2.

We put

$$
\theta = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \in \text{Sp}(4, \mathbb{Z}),
$$

so that $v_2 = v_1\theta$. Then $P_{v_2} = \theta^{-1}P_{v_1}\theta$ (where, as in [HKW], $P_v$ denotes the stabiliser of $v$ in $\text{Sp}(4, \mathbb{Q})$), and from this one readily calculates that

$$
P'_{v_2}(\Gamma_0^{\text{bil}}) = \left\{ \begin{pmatrix} 1 & 0 & 4n & 2n \\ 0 & 1 & 2n & n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid n \equiv 0 \mod 36 \right\}.
$$

So the cusp $D_2$ is given by $t_2 = 0$, where $t_2 = e^{2\pi i (\tau_1 / 144 + \tau_2 / 72 + \tau_3 / 36)} = q^{1/144}r^{1/12}s$. The number of times this term divides the expressions for $F_3$, $F_3'$ and $F_3''$ is in fact equal to the power of $s$ that occurs, namely 2, 4 and 3 respectively; so we get the same orders of vanishing along $D_2$ as along $D_1$. ■

This calculation shows directly (without appealing to Freitag's result in [Fr]) that $\omega$, $\omega'$ and $\omega''$ are all holomorphic.

Remark. Notice that $\text{Div}_{X_6}(\omega) + \text{Div}_{X_6}(\omega') = 2\text{Div}_{X_6}(\omega'')$, reflecting the fact (easily seen from [GN]) that $F_3F_3' = (F_3'')^2$.

Theorem A now follows at once from the following observation.

**Proposition 3.4.** $\omega$, $\omega'$ and $\omega''$ are linearly independent elements of $H^0(K_{X_6})$.

**Proof:** Suppose that $\lambda\omega + \lambda'\omega' + \lambda''\omega'' = 0$. At a general point of $\pi_0^{\text{bil}}(H_{\mathcal{Q}})$, $\omega', \omega''$ vanish but $\omega$ does not. Therefore $\lambda = 0$. Similarly $\lambda' = 0$, considering a general point of $\pi_0^{\text{bil}}(H_{\mathcal{Q}})$. Finally, $\lambda'' \neq 0$ because $F_3''$ is not identically zero. ■

We want to remark that $\kappa(A_0^{\text{bil}}) \geq 1$ can be deduced from the existence of $\omega'$ alone. The divisor $\text{Div}_{X_6}(\omega')$ is effective and $\pi_0^{\text{bil}}(H_{\mathcal{Q}}) \subset \text{Supp}\text{Div}_{X_6}(\omega')$. Since $X_6$ has canonical singularities, $K$ is effective on any smooth model of $X_6$, and hence also on any minimal model $X_6'$ of $X_6$. Any surfaces contracted by the birational map $X_6 \rightarrow X_6'$ must be birationally ruled. But $\pi_0^{\text{bil}}(H_{\mathcal{Q}})$ is not birationally ruled: it is isomorphic to $\text{Pic}(X(6)) \times \text{Pic}(X(6))$, since $H_{\mathcal{Q}}$ is isomorphic to $\mathbb{H} \times \mathbb{H}$ and is preserved by the subgroup $\Gamma(6) \times \Gamma(6)$ embedded in $\Gamma_0^{\text{bil}}(j_1, j_2)$. Thus its closure is birationally an abelian surface, since $X(6)$ has genus 1. So the canonical divisor of $X_6'$ is effective and nontrivial; so, by abundance, some multiple of it moves and therefore $\kappa(A_0^{\text{bil}}) \geq 1$. 

8
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