Toroidal compactification: the generalised ball case

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Abstract

We give an introduction to toroidal compactification in two ways: first we give an overview of the origins and uses of the construction, and then we work out many of the details in the concrete case of quotients of the generalised ball $\mathbb{B}(a, b) = \mathrm{U}(a, b)/(U(a) \times U(b))$.

1 Introduction

This article has two aims. In the first part, we give an explanation in general terms of toroidal compactifications and some of their relatives, intended as an introduction to the subject but not as a detailed reference. In the second part, we illustrate this by describing in some detail one case of toroidal compactification, associated with classical bounded symmetric domains of type I (also known as generalised balls) and the Lie groups $\mathrm{U}(a, b)$.

Nothing in this article is truly new. The description given in Section 2 consists entirely of basic ideas that are familiar to the experts (to whom they will indeed appear naive), but our experience with students and with researchers not so close to the subject has taught us that it is not very easy to acquire an overview of the scope, significance and content of toroidal compactification, even though it has been in use for almost fifty years. Similarly, the remainder of the article is nothing more than a specialisation of the general theory to one particular case, which allows us to exhibit the workings of the construction in a very concrete, but limited, way.

The practical difficulty faced by a would-be user of toroidal compactification is that the standard book [AMRT] is simultaneously definitive and unapproachable. Other sources are available, but they are all written with a specific end in view and handle special cases: for example [Nam] is concerned with the type III case, associated with $\mathrm{Sp}(2g)$ and with moduli of abelian varieties, and [Sca] is associated with type IV and $\mathrm{SO}(2, n)$, more precisely with $\mathrm{SO}(2, 19)$ and the moduli of K3 surfaces. A still more specific case is worked out in detail in [HKW], but there the algebraic group is Sp(4) and the focus is on the details of a particular choice of compactification for a quotient by some very specific discrete groups.

We aim to provide details more concretely than is done in [AMRT], concentrating on the case of $\mathrm{U}(a,b)$, but still at a sufficient level of generality to allow the reader to adapt them to other cases as necessary. Thus we also handle a special case, but in a rather more elementary and, we hope, approachable way. We were motivated to write this introduction by the discovery that the limited account of the construction included in our paper [KS] was being found useful as a guide by research students, although that had not been our aim. For this we thank Nils Scheithauer and Maximilian Rössler.

2 Moduli spaces, locally symmetric varieties and compactifications

In this section we give a quick and rather informal explanation of how and why toroidal compactification arises in algebraic geometry. The most usual context is moduli problems: more specifically, coarse moduli spaces of polarised complex algebraic varieties. More sophisticated matters such as moduli stacks can also be incorporated, but we do not attempt to do that. In particular, as we work entirely over the complex numbers, we do not address arithmetic questions of any kind, although much has been written on the subject, starting perhaps with [FC].

2.1 Motivation for construction of toroidal compactifications

Let us start with the formula, given in Shimura's book [Shi], for the genus of a compactified modular curve $\overline{X}_{\overline{\Gamma}} \supset \overline{\Gamma}\backslash\mathbb{H}$:

$$g(\overline{X}_{\overline{\Gamma}}) = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{6} - \frac{\nu_\infty}{2}.$$

Here $\overline{\Gamma}$ is an arithmetic subgroup of $\overline{\Gamma}(1) = \mathrm{PSL}(2,\mathbb{Z})$, acting on the upper half-plane $\mathbb{H}$; the index of $\overline{\Gamma}$ in $\overline{\Gamma}(1)$ is μ; the numbers ν_2 and ν_3 count orbits of fixed points with stabilisers in $\overline{\Gamma}$ of orders 2 and 3 respectively; and ν_∞ is the number of cusps, which is the number of orbits of $\overline{\Gamma}$ on $\mathbb{P}^1_{\mathbb{Q}}$.

Shimura shows this by Hurwitz's Theorem applied to the map $\overline{X}_{\overline{\Gamma}} \to \overline{X}_{\overline{\Gamma}(1)} \cong \mathbb{P}^1$, and then interprets it via Riemann-Roch as a dimension formula for the space of modular forms. One could try to go in the other direction and estimate the number of modular forms first: in higher dimensional cases that is rather more natural, as analytic methods to find such estimates exist, whereas a canonical bundle formula for coverings, analogous to Hurwitz's Theorem, is much harder to describe if the branching is complicated.

Roughly speaking the contribution μ comes from the modular forms and there are correction terms for torsion and cusps: the "singularities" coming from 2- and 3-torsion, giving ν_2 and ν_3, and also implicitly (the choice to use $\overline{\Gamma} \subset \mathrm{PSL}(2,\mathbb{Z})$ rather than $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$) the "generic singularity" caused by $\pm I$; and then a correction for the compactification, measured by ν_∞.

$X_{\overline{\Gamma}}$ is typically the coarse moduli space of elliptic curves with some level structure; but we are parametrising the elliptic curves via their periods. In other words, we are using a Torelli theorem for genus 1 curves. A genus 1 curve has an infinite group of automorphisms so we shouldn't expect a decent moduli space of genus 1 curves. We get a coarse moduli space nevertheless because a genus 1 curve E is non-canonically isomorphic to its own Jacobian $JE = \mathrm{Pic}^0(E)$, which is an elliptic curve (i.e. it has a distinguished point $0 = [\mathcal{O}_E]$).

We are also using surjectivity of the period map. In this case that is trivial, because there is nothing else that the image could be; but we do rely on knowing what the period domain is (in this case $\mathbb{H}$) a priori.

Very roughly, $\mathbb{H}$ is a parameter space for Hodge structures on the cohomology of E. The curve E itself is determined uniquely by its Hodge structure up to Hodge isometry (this is the Torelli theorem); all Hodge structures occur (surjectivity of the period map) and the Hodge isometries are given by Γ, which we are also requiring to preserve some supplementary structure.

The other thing that we are using is that $\Gamma\backslash\mathbb{H}$ does have an algebraic compactification. This is slightly less trivial. It is only a complex manifold: a priori it could be something like a disc. To show that it is in fact algebraic we need to use elliptic modular forms.

Once we move to higher-dimensional cases we have in principle a further possible cause of trouble: we might find that we can compactify but that the resulting space is not projective. In fact this can really happen, but only with a bad choice of compactification. The spaces we are looking at will always be Moishezon manifolds, i.e. will always have plenty of meromorphic functions. Even so, if we make the wrong choices we can find that the resulting compact space fails to be Kähler, and thus fails to be projective; but good choices, yielding a projective compactification, always exist.

The higher-dimensional cases arise as soon as we look at the cohomology of algebraic varieties more complicated topologically than elliptic curves. In general the period map is neither injective nor surjective, but in several important cases it is surjective, and either is injective or is sufficiently well understood that the failure of injectivity (i.e. of Torelli) can be controlled. This happens, for example, in the case of hyperkähler manifolds. In situations of this type, one has a classifying space for Hodge structures, which is the target of the period map, and the (coarse) moduli space will be a quotient of this classifying space by a discrete group.

A *symmetric space* of non-compact type is a quotient $\mathcal{D} = G/K$ of a non-compact (linear) semisimple Lie group G, defined over $\mathbb{Q}$, by a maximal compact connected subgroup K of G acting by right translations (cf. [Hel, Theorem IV.3.3], [BJ, III.1.1]). If the centre of K is not discrete, then $\mathcal{D}$ carries a Hermitian structure and hence the structure of a complex manifold, in fact a Kähler manifold. In the kind of situation we have been describing, this will be the target of the period map. It is often convenient to work with reductive groups rather than semi-simple groups: for instance, later we shall work in $\mathrm{U}(a,b)$ rather than $\mathrm{SU}(a,b)$.

By a *lattice* in G we mean a discrete subgroup of G of finite covolume with respect to Haar measure. The groups we are interested in are algebraic groups defined over $\mathbb{Z}$, and it therefore makes sense to speak of the group $G(\mathbb{Z})$ of integer points. A lattice Γ is said to be *arithmetic* if $\Gamma \cap G(\mathbb{Z})$ is of finite index in both Γ and $G(\mathbb{Z})$. It is said to be *neat* if the subgroup of $\mathbb{C}^*$ generated by all eigenvalues of elements of Γ is torsion free.

Evidently, G or any subgroup of G acts from the left on $\mathcal{D} = G/K$. A *locally symmetric variety* is the quotient of a Hermitian symmetric space $\mathcal{D}$ by a lattice $\Gamma < G$. If $X = \Gamma\backslash\mathcal{D}$ is compact then Γ is said to be *cocompact* or *uniform*. Non-uniform lattices are very common, however, and it is this case that we are concerned with. The coarse moduli spaces that we have been considering will be (close to) locally symmetric varieties.

(At the risk of stating the obvious, let us mention here a common beginner's mistake, which is to confuse this construction with geometric invariant theory quotients. Those involve actions of algebraic groups or Lie groups, not discrete groups. Both constructions are used to describe moduli spaces, and both are associated with Mumford.)

In this situation, $\mathcal{D}$ is a open subset of a projective variety $\check{\mathcal{D}} \subset \mathbb{P}^n$ (called the *compact dual*) on which G acts, and we can lift this action to the corresponding affine space $\mathbb{A}^{n+1}$ and look at the affine cone $\mathcal{D}^\bullet$ on $\mathcal{D}$. Here "open" means open in the analytic topology, not the Zariski topology: the prototypical example of $\mathcal{D}$ is the upper half-plane in its compact dual $\check{\mathcal{D}} = \mathbb{P}^1$, which arises when $G = \mathrm{SL}_2$.

A *modular form* of weight k and character χ is a holomorphic function $F\colon \mathcal{D}^\bullet \to \mathbb{C}$ such that

$$F(tZ) = t^{-k}F(Z) \qquad \forall t \in \mathbb{C}^*$$

and

$$F(gZ) = \chi(g)F(Z) \qquad g \in \Gamma,$$

where $\chi\colon \Gamma \to \mathbb{C}^*$ is a character and k is an integer (sometimes, but not for us, a half-integer). In the case of elliptic modular forms, i.e. $\dim\mathcal{D} = 1$, an extra condition of holomorphicity at infinity is needed, but we shall hardly ever be concerned with that case. It is nevertheless an amusing exercise to see that this does indeed give the usual definition in terms of Möbius transformations in the case of elliptic modular forms.

It is a theorem of Baily and Borel [BB] that locally symmetric varieties are always algebraic varieties, not just complex analytic spaces. In fact the argument of [BB] treats only arithmetic lattices: the case of non-arithmetic lattices is dealt with by Mok [Mok].

To prove that a complex manifold is an algebraic variety we need to produce an ample line bundle. We use the bundle $\mathcal{L}$ on X whose local sections are modular forms for Γ of weight one (and character det), sometimes called the Hodge line bundle. In general it is only a $\mathbb{Q}$-line bundle, but that does not affect us for now. Then we take $X^* = \operatorname{Proj}\mathcal{M}_\Gamma$, where $\mathcal{M}_\Gamma = \bigoplus_k H^0(\mathcal{L}^{\otimes k})$ is the ring of all modular forms of all weights for Γ with character $\det^k$. The essential content of [BB] is that $\mathcal{L}$ is indeed ample on X, so that X is identified with a Zariski-open subset of X^*. The compactification X^* is known as the Baily-Borel or Satake-Baily-Borel compactification of X: for historical reasons, in the $\mathrm{Sp}(2g)$ case, in connection with the moduli of abelian varieties, X^* is often called the Satake compactification. (As the referee pointed out, this last usage, though standard, is potentially misleading as several other compactifications and compactification procedures are also due to Satake.)

A consequence of this is that modular forms know everything about the birational geometry of X. Getting them to tell us what they know is quite challenging. A first difficulty is that X itself will in general be singular (more precisely, the quotient map $\mathcal{D} \to X$ may be ramified): one avoids this if the lattice Γ is torsion free or, better still, neat (which also avoids certain singularities in the boundary, as we shall see shortly). However, although one can achieve neatness by replacing Γ by a finite index subgroup, this is often undesirable and one must instead work with the singularities or ramification and compute the consequences: the correction terms ν_2 and ν_3 in Shimura's formula are examples of this. In good cases one can prove that the singularities are canonical: this means, approximately, that they have no effect on top-degree differential forms.

A more serious difficulty is that except in a few trivial cases the boundary $X^* \setminus X$ is usually of high codimension. A particular case of this is that the Satake compactification of the moduli space $\mathcal{A}_g$ of principally polarised abelian g-folds, which corresponds to the case $G = \mathrm{Sp}(2g)$, has as its largest boundary component a copy of $\mathcal{A}_{g-1}$, which is of codimension g: there is a stratification that decomposes $\mathcal{A}_g^* = \mathcal{A}_g \coprod \mathcal{A}_{g-1} \coprod \mathcal{A}_{g-2} \coprod \ldots \coprod \mathcal{A}_0$.

In almost all cases X^* has bad singularities at the boundary, though they are far from arbitrary: for example, it is pointed out in [Ale1] that they are log canonical. Moreover, if X is a moduli space, it is usually hard to attach much geometric meaning to the boundary points. In the case of $\mathcal{A}_g$ a boundary point in $\mathcal{A}_{g-1}$ corresponds to an abelian $(g-1)$-fold, and one may interpret this as saying that a curve passing through that point corresponds to a family of abelian varieties degenerating to a semi-abelian variety whose abelian part is that $(g-1)$-fold, but other information (the extension class), which depends on the curve chosen, is lost.

In such a situation one naturally wants to blow up the boundary, both to resolve the singularities and to distinguish different families approaching the boundary. At a general point of $\mathcal{A}_{g-1} \subset \mathcal{A}_g^*$ one blow-up does indeed achieve these aims, but even at a point of $\mathcal{A}_{g-2} \subset \mathcal{A}_{g-1}^*$ the situation is much more complicated and there are choices to be made.

2.2 Outline of the construction of toroidal compactification

One way to describe toroidal compactification is to say that it is a way of specifying a blow-up of X^* at the boundary: of course there are many possible blow-ups and correspondingly many possible toroidal compactifications. However, it is probably better to think of it in the first instance as a different approach to compactification, in which one adds a boundary divisor to X so as to obtain a much less singular compactification X_Σ.

In fact, the construction of X_Σ does not use modular forms and X^* at all. Only once the construction is complete does one show that there is a map $X_\Sigma \to X^*$, which is proper and, sometimes, projective, and use this to deduce the compactness and, sometimes, projectivity of X_Σ.

To explain how the process works we assume that G is a reductive group, whose quotient $G/Z(G)$ by its centre $Z(G)$ is simple. This is by no means necessary and indeed many interesting spaces, such as Hilbert modular varieties, arise from non-simple cases G, but the essentials are the same. As in Section 2.1 above, we think of the symmetric space $\mathcal{D}$ as an open subset (in the Euclidean topology) of the compact dual $\check{\mathcal{D}}$. The closure $\overline{\mathcal{D}}$ of $\mathcal{D}$ in $\check{\mathcal{D}}$ decomposes into boundary components, one of which is $\mathcal{D}$ itself: two points belong to the same boundary component if they can be joined by a complex curve. Each boundary component F corresponds to a real maximal parabolic subgroup $P \subset G$, the stabiliser of F, and is in fact itself a symmetric space, associated with a factor of the reductive part of P.

The Langlands decomposition of P represents $P = (U_P \rtimes V_P) \rtimes [G_{1,l}(P) \times G_{1,h}(P)]$ as a product of subgroups, where $G_{1,l}(P)$ and $G_{1,h}(P)$ are reductive Lie groups and $N_P = U_P \rtimes V_P$ is the 2-step nilpotent unipotent radical of P. More precisely, U_P is the centre of N_P, and $V_P = N_P/U_P$ and U_P are both abelian Lie groups, and thus isomorphic as groups to their own Lie algebras. This induces a diffeomorphic realisation (called *horospherical decomposition*) $\mathcal{D} = \mathrm{Lie}(U_P) \times \mathrm{Lie}(V_P) \times C_P \times F_P$ and a fibration $\pi_P \colon \mathcal{D} \to \mathrm{Lie}(V_P) \times F_P$ over the product of the complex vector space $\mathrm{Lie}(V_P)$ and the bounded symmetric domain $F_P = G_{1,h}(P)/[G_{1,h}(P) \cap K]$.

For toroidal compactification of $\Gamma\backslash\mathcal{D}$ we consider only the Γ-rational parabolic subgroups, which are the ones for which $\Gamma_P = P \cap \Gamma$ is a lattice in P. In that case, π_P descends to a fibration $\Gamma_P\backslash\mathcal{D} \to \Gamma_P\backslash(\mathrm{Lie}(V_P) \times F_P)$, and the fibres of this map appear as (Euclidean) open subsets in a torus (that is, a copy of $(\mathbb{C}^*)^r$ for some r), which is in fact the quotient of $U_P \otimes \mathbb{C}$ by $\Upsilon = \Gamma \cap U_P$. Thinking of $U_P \otimes \mathbb{C}$ as $U_P \oplus iU_P$ (as real vector spaces), the condition for a point in $\Upsilon\backslash(U_P\otimes\mathbb{C})$ to lie in $\Upsilon\backslash\mathcal{D}$ is that its imaginary part should lie in a certain cone C_P, which has a natural embedding in $i\,\mathrm{Lie}(U_P)$. In the case of the upper half-plane $\mathbb{H}$ (when F is the infinite point $i\infty$), the cone is the positive half-line $\mathbb{R}_{>0}$ and if $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ then Υ consists of the matrices $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ for $k \in \mathbb{Z}$.

Equivariant partial compactifications of $(\mathbb{C}^*)^r$ are well-known. These are toric varieties: references for them include [Ful, Oda, CLS]. We shall need a slightly more general version, treated in [Oda], in that we do want to allow non-Noetherian schemes rather than just varieties. We shall get back to the Noetherian world very soon, as we still need to take a quotient by an infinite group. In general, a toric variety is determined by some combinatorial data. This consists of a free abelian group N of rank r, a cone C in $N \otimes \mathbb{R}$, and a fan Δ, which is a decomposition of C into relatively open rational polyhedral cones satisfying some simple boundary compatibilities. With this data, one has a variety $\mathbb{TV}(\Delta)$ of dimension r that contains $T_N = (N \otimes \mathbb{C})/N \cong (\mathbb{C}^*)^r$ as a dense open subset. In other words, N is the cocharacter lattice of the torus: it is standard in toric geometry to use this notation, and to use M to denote the character lattice.

For us, the toric varieties will appear once we have made a choice of a rational boundary component and thus a parabolic subgroup P. The role of the free abelian group N (not to be confused with the unipotent radical N_P) will be played by Υ, the fan will be called Σ_P and the cone that it decomposes will be the cone C_P that defined $\mathcal{D}$ as above. We shall use Σ to denote the collection $\{\Sigma_P\}$ of all the Σ_P.

The compactification now proceeds as follows: for each rational maximal parabolic subgroup P (or rational boundary component F) we choose a fan Σ_P and we take a partial compactification $\overline{(\Upsilon\backslash\mathcal{D})}_P$ by taking the interior of the closure of $\Upsilon\backslash\mathcal{D} \subset T_\Upsilon$ in $\mathbb{TV}(\Sigma_P)$. This is really just $\Upsilon\backslash\mathcal{D}$ but with the boundary of $\mathbb{TV}(\Sigma_P)$ added on. Then the compactification X_Σ arises from taking quotients by Γ and identifying the different copies of $\Gamma\backslash\mathcal{D}$ that are contained in these various partial compactifications. For this to be possible the collection Σ must satisfy some conditions, which we now describe.

Two boundary components are said to be adjacent if one is contained in the closure of the other. In order to maintain this relationship when we pass to the quotients and identify, the collection $\Sigma = \{\Sigma_P\}$ must be chosen to satisfy some compatibility conditions. We also need to account for the action of Γ. Elements of Γ that do not preserve P will simply move us to a different boundary component, corresponding to a conjugate of P: we should choose $\Sigma = \{\Sigma_P\}$ compatibly with this. The normaliser of P, on the other hand, will act on $\mathbb{TV}(\Sigma_P)$ as long as we have chosen Σ_P to be Γ-invariant. A collection of fans that is compatible both with the action of Γ and with adjacencies among boundary components is called an admissible collection. Admissible collections do exist, but they are far from unique. They can also be chosen so that the variety constructed by the whole process is projective.

2.3 Properties of toroidal compactifications

From the above description, we see that local structure of the compactification is that each point may be regarded as having a neighbourhood that is contained in a quotient of a toric variety. Although the toric variety may, if we wish, be chosen to be smooth, by taking a suitable Σ, the action of the normaliser of P will in general have fixed points, so we cannot always ensure that X_Σ is smooth, even if $\Gamma\backslash\mathcal{D}$ is smooth. If Γ is neat, we can ensure this, and indeed that the quotient maps by the normalisers of P are unbranched.

The freedom to choose any admissible Σ is both a strength and a weakness of toroidal compactification. It makes the construction non-canonical in general: we cannot expect to have toroidal compactification functors, nor can we expect that a point of a toroidal boundary will have a geometric meaning extending the geometric meaning of the points of $\Gamma\backslash\mathcal{D}$ if it is a moduli space. On the other hand, we can sometimes choose a compactification for which we can assign such a meaning: for instance, this is done in [Ale2] for the second Voronoi compactification of $\mathcal{A}_g$. Alternatively, we could choose a compactification for which X_Σ has good properties viewed as a projective variety: for instance, this is done in [S-B] where it is shown that the first Voronoi compactification of $\mathcal{A}_g$ is the canonical model in the sense of birational geometry. Notice that [Ale2] and [S-B] involve different choices of toroidal compactification.

Because toroidal compactification is very concrete, and much information is encoded combinatorially in the data Σ, it has proved to be a very effective tool for studying the global geometry and topology of the moduli spaces. One of the first applications was the result of Tai [Tai], who interpreted modular forms as differential forms with poles at the boundary of a toroidal compactification and showed thereby that $\mathcal{A}_g$ is of general type for $g \geq 7$. The same idea has been used to prove general-type results for moduli of K3 surfaces [GHS1] and some hyperkähler manifolds (see [GHS2]), and for almost all locally symmetric varieties associated with $\mathrm{O}(2,n)$ [Ma]. The concreteness and combinatorial nature of toroidal compactification also allows computation of topological invariants: among many such works, we mention [GHT]. Finally, we should mention a further generalisation due to Looijenga [Loo1, Loo2], who gave compactifications or partial compactifications intermediate between toroidal and Baily-Borel, which have recently been related also to GIT compactifications in some specific cases [LO].

3 Background on Hermitian symmetric domains

Before we begin the more detailed description of toroidal compactification for generalised balls in Section 4, we collect here some basic notions about Hermitian symmetric domains and linear algebraic groups. As general references we mention [Hel, P-S, BJ, KW].

3.1 Bounded symmetric domains

Definition 1. *A* symmetric domain *is a connected complex manifold $\mathcal{D}$ such that every $z \in \mathcal{D}$ is an isolated fixed point of a holomorphic involution of $\mathcal{D}$. If $\mathcal{D}$ has a Hermitian structure we say that $\mathcal{D}$ is a* Hermitian symmetric space*; if $\mathcal{D}$ is a bounded open subset of some $\mathbb{C}^n$ we say that $\mathcal{D}$ is a* bounded symmetric domain*.*

If $\mathcal{D}$ is a Hermitian symmetric space we may identify $\mathcal{D}$ with G/K, where G is the identity component of the group of holomorphic isometries of $\mathcal{D}$ and K is the stabiliser of a point. Moreover, $\mathcal{D}$ is said to be of compact or of noncompact type according to whether G is compact or noncompact. By the Harish-Chandra embedding theorem, the bounded symmetric domains are precisely the Hermitian symmetric spaces of noncompact type [Hel, Theorem VIII.7.1]; according to [Hel, Theorem VIII.6.1(i)], the irreducible Hermitian symmetric spaces are the ones that arise when G is simple and K is a maximal compact subgroup of G with non-discrete centre. Equivalently, if we view $\mathcal{D} = \tilde{G}/\tilde{K}$ as a quotient of a non-compact reductive Lie group, then $\mathcal{D}$ is irreducible if the quotient $\tilde{G}/Z(\tilde{G})$ is simple.

The bounded symmetric domains are therefore classified as part of the classification of Riemannian symmetric spaces: see [Hel, X.6.2, Table V] and [Hel, X.6.3]. There are four classical types and two exceptional bounded symmetric domains. One of the exceptional ones is associated with E_6 and has (complex) dimension 16; the other is associated with E_7 and has dimension 27. The classical types are:

I_{ab} $\mathrm{SU}(a,b)/\,\mathrm{S}(\mathrm{U}(a) \times \mathrm{U}(b)) = \mathbb{B}(a,b)$ (the generalised balls), of dimension ab;

II_n $\mathrm{SO}^*(2n)/\,\mathrm{U}(n)$, of dimension $n(n-1)/2$;

III_n $\mathrm{Sp}(n,\mathbb{R})/\,\mathrm{U}(n) = \mathbb{H}_n$ (the Siegel upper half-spaces), of dimension $n(n+1)/2$;

IV_n $\mathrm{SO}_0(2,n)/(\mathrm{SO}(2) \times \mathrm{SO}(n))$, of dimension n.

Here $\mathrm{SO}^*(2n)$ is the subgroup of $\mathrm{SO}(2n,\mathbb{C})$ that preserves a skew-Hermitian form.

From the point of view of moduli problems in algebraic geometry the most prominent of these are III_n which is the period domain for polarised abelian varieties of dimension n; and IV_n, in particular IV_{19} which is the period domain for polarised K3 surfaces.

3.2 Langlands decomposition

Let $\mathcal{D} = G/K$ be an irreducible bounded symmetric domain, represented as a quotient of a non-compact reductive Lie group G, and let Q be a maximal parabolic subgroup of G. We make use of the refined Langlands decomposition of Q, which parallels the

Iwasawa decomposition of G itself. Let N_Q be the unipotent radical of Q and let L_Q be a Levi subgroup of Q, i.e. a reductive complement of N_Q in $Q = N_Q \rtimes L_Q$.

We can decompose further the unipotent radical N_Q of Q. Since N_Q is a 2-step nilpotent group, i.e. $[[N_Q, N_Q], N_Q] = 0$, the centre U_Q of N_Q coincides with the commutator subgroup $[N_Q, N_Q]$. We may identify U_Q with its Lie algebra $\mathrm{Lie}(U_Q) \cong \mathbb{R}^m$, for $m = \dim_{\mathbb{R}} U_Q$. The quotient $V_Q = N_Q/U_Q$ is also an abelian group, naturally isomorphic to $\mathbb{C}^n$ (see for instance [BJ, (III.7.10)]) and $N_Q = U_Q \rtimes V_Q$ is a semi-direct product of U_Q and V_Q.

The reductive group L_Q is a product $L_Q = G_{1,l}(Q) \times G_{1,h}(Q)$ of reductive Lie groups with simple quotients $G'_{1,l}(Q) = G_{1,l}(Q)/Z(G_{1,l}(Q))$ and $G'_{1,h}(Q) = G_{1,h}(Q)/Z(G_{1,h}(Q))$ by the corresponding centres. The centre $Z(G)$ of G coincides with the ineffective kernel of the G-action on $\mathcal{D}$, and $Z(G)$ is contained in L_Q. There is a decomposition $Z(L_Q) = Z(G) \times A_Q$ for a 1-dimensional real torus $A_Q \cong (\mathbb{R}_{>0}, .)$, called the *split component* of Q. Moreover, A_Q is a subgroup of $G_{1,l}(Q)$, and we have $Z(G_{1,l}(Q)) = [Z(G) \cap G_{1,l}(Q)] \times A_Q$ and $Z(G_{1,h}(Q) = Z(G) \cap G_{1,h}(Q)$. The $G_{1,h}(Q)$-orbit of the origin is an irreducible Hermitian symmetric space

$$D_{1,h}(Q) = G_{1,h}(Q)/[G_{1,h}(Q) \cap K] = G'_{1,h}(Q)/[G'_{1,h}(Q) \cap K]$$

of non-compact type, called the *boundary component* of $\mathcal{D}$, associated with Q [BJ, (III.7.8)]. The orbit

$$D'_{1,l}(Q) = G'_{1,l}(Q)/(G'_{1,l}(Q) \cap K)$$

of the simple part $G'_{1,l}(Q)$ of $G_{1,l}(Q)$ is an irreducible Riemannian symmetric space of non-compact type. The split component A_Q has trivial intersection with K, and the $G_{1,l}(Q)$-orbit

$$C_Q = G_{1,l}(Q)/(G_{1,l}(Q) \cap K) = (A_Q \times G'_{1,l}(Q))/(G'_{1,l}(Q) \cap K) = A_Q D'_{1,l}(Q)$$

of the origin coincides with the A_Q-orbit of $D'_{1,l}(Q)$. Note that C_Q is an open strongly convex homogeneous cone with base $D'_{1,l}(Q)$. Altogether, the L_Q-orbit of the origin is

$$L_Q/(L_Q \cap K) = C_Q \times D_{1,h}(Q).$$

Definition 2. *The decomposition*

$$Q = (U_Q \rtimes V_Q) \rtimes [Z(G) \times A_Q \times G'_{1,l}(Q) \times G'_{1,h}(Q)] \tag{1}$$

of a maximal parabolic subgroup Q of G is called the refined Langlands decomposition.

Note also that $G = QK$ and $Q \cap K = L_Q \cap K$, so that the refined Langlands decomposition (1) induces a diffeomorphic refined horospherical decomposition

$$\mathcal{D} = Q/(Q \cap K) = N_Q \times [L_Q/(L_Q \cap K)] = U_Q \times V_Q \times C_Q \times D_{1,h}(Q)$$

of the Hermitian symmetric space $\mathcal{D}$ of non-compact type.

4 Generalised ball quotients

We intend to describe the procedure leading to the construction of toroidal compactifications for the case of $\mathrm{U}(a,b)/\mathrm{U}(a)\times\mathrm{U}(b)$, the generalised ball $\mathbb{B}(a,b)$ (see Definition 3). The quotients $\Gamma\backslash\mathbb{B}(a,b)$ are known as *generalised ball quotients* or *Picard modular varieties*. The space $\mathbb{B}(a,b)$ is naturally embedded in the Siegel upper half-space $\mathbb{H}_{a+b}=\mathrm{Sp}(a+b,\mathbb{R})/\mathrm{U}(a+b)$ and may be thought of as a period domain for abelian $(a+b)$-folds with a type (a,b) involution [vanG].

Definition 3. *Fix a Hermitian form χ on $\mathbb{C}^{a+b}$ of signature (a,b): for convenience we always assume that $a\le b$. The* generalised ball *of signature (a,b) is the set $\mathbb{B}(a,b)$ of a-dimensional subspaces $S\subset\mathbb{C}^{a+b}$ such that the restriction $\chi|_S$ is positive definite.*

Note that $\mathbb{B}(1,b)$ is the usual ball in $\mathbb{C}^b$. The generalised ball is a classical (type I) irreducible Hermitian symmetric space of non-compact type. The associated reductive group is the indefinite unitary group $G=\mathrm{GL}_{\mathbb{C}}(\mathbb{C}^{a+b},\chi)=\mathrm{U}(a,b)$, which acts transitively on $\mathbb{B}(a,b)$ with stabiliser $\mathrm{U}(a)\times\mathrm{U}(b)$. (We could instead work with the simple group $\mathrm{SU}(a,b)$ and the stabiliser $\mathrm{S}(\mathrm{U}(a)\times\mathrm{U}(b))$ but working with $\mathrm{U}(a,b)$ simplifies the notation somewhat.)

From here on, we regard χ (and hence a and b) as fixed. Terms such as isotropic, orthogonal, etc. are to be understood as meaning "with respect to χ". Note that a coincides with the real rank of G, also denoted $\mathrm{rank}(G)$ or $\mathrm{rank}\,\mathbb{B}(a,b)$.

4.1 Langlands decomposition of maximal parabolics

In this section we describe the Langlands decomposition of maximal parabolic subgroups in detail.

The maximal parabolic subgroups of $G=\mathrm{U}(a,b)$ are the stabilisers $P=\mathrm{Stab}_G(E)$ of the isotropic subspaces $E\subset\mathbb{C}^{a+b}$. Any such E is of dimension $s=\dim_{\mathbb{C}}(E)\le a=\mathrm{rank}(G)$ and has an isotropic conjugate space $\check{E}\subset\mathbb{C}^{a+b}$, with $E\cap\check{E}=\{0\}$ and $\dim_{\mathbb{C}}(\check{E})=s$, such that the restriction $\chi|_{E\oplus\check{E}}$ is a non-degenerate form of signature (s,s). The restriction χ_1 of χ to the orthogonal complement F of $E\oplus\check{E}$ in $\mathbb{C}^{a+b}$ is a non-degenerate Hermitian form of signature $(a-s,b-s)$. Let us put $t=\dim_{\mathbb{C}}(F)=(a-s)+(b-s)$.

Our first aim is to describe the components of the Langlands decomposition of $P=\mathrm{Stab}_G(E)$. We want to do this in matrix terms, so we first choose a suitable basis for our calculations.

Definition 4. *A basis $\{e_1,\ldots,e_s,f_1,\ldots,f_{a-s},f'_1,\ldots,f'_{b-s},\check{e}_1,\ldots,\check{e}_s\}$ of $\mathbb{C}^{a+b}$ is said to be E-*adapted *if $\mathbf{e}=\{e_1,\ldots,e_s\}$ and $\check{\mathbf{e}}=\{\check{e}_1,\ldots,\check{e}_s\}$ are bases of E and $\check{E}$ respectively, such that*

$$\chi(e_i,\check{e}_j)=\chi(\check{e}_j,e_i)=\delta_{ij},$$

and $\mathbf{f}=\{f_1,\ldots,f_{a-s},f'_1,\ldots,f'_{b-s}\}$ is a basis for F such that

$$\chi(f_i,f_j)=\delta_{ij}=-\chi(f'_i,f'_j)\text{ and }\chi(f_i,f'_j)=0.$$

Because χ is nondegenerate, we can extend an arbitrary $\mathbb{C}$-basis $\mathbf{e} = \{e_1, \dots, e_s\}$ of E to an E-adapted basis $\{\mathbf{e}, \mathbf{f}, \check{\mathbf{e}}\}$ with respect to which the matrices of χ and χ_1 are

$$\chi = \begin{pmatrix} 0 & 0 & I_s \\ 0 & \chi_1 & 0 \\ I_s & 0 & 0 \end{pmatrix} \text{ and } \chi_1 = \begin{pmatrix} I_{a-s} & 0 \\ 0 & -I_{b-s} \end{pmatrix}. \tag{2}$$

We begin with the unipotent part of P, described with respect to an E-adapted basis.

Proposition 5. *The unipotent radical N_P of P has Lie algebra*

$$\mathrm{Lie}(N_P) = \left\{ \begin{pmatrix} 0 & -\overline{\mu}^\top \chi_1 & \lambda \\ 0 & 0 & \mu \\ 0 & 0 & 0 \end{pmatrix} \,\Big|\, \lambda \in M_{s\times s}(\mathbb{C}),\, \mu \in M_{t\times s}(\mathbb{C}), \overline{\lambda}^\top = -\lambda \right\}$$

with respect to $\mathbf{e}, \mathbf{f}, \check{\mathbf{e}}$. *The Lie group N_P is*

$$N_P = \left\{ \begin{pmatrix} I_s & -\overline{\mu}^\top \chi_1 & \nu \\ 0 & I_t & \mu \\ 0 & 0 & I_s \end{pmatrix} \,\Big|\, \nu \in M_{s\times s}(\mathbb{C}),\, \mu \in M_{t\times s}(\mathbb{C}),\, \nu + \overline{\nu}^\top + \overline{\mu}^\top \chi_1 \mu = 0 \right\}$$

and the exponential map $\exp\colon \mathrm{Lie}(N_P) \to N_P$ *and its inverse are given by*

$$\exp \begin{pmatrix} 0 & -\overline{\mu}^\top \chi_1 & \lambda \\ 0 & 0 & \mu \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I_s & -\overline{\mu}^\top \chi_1 & \lambda - \frac{1}{2}\overline{\mu}^\top \chi_1 \mu \\ 0 & I_t & \mu \\ 0 & 0 & I_s \end{pmatrix}$$

and

$$\exp^{-1} \begin{pmatrix} I_s & -\overline{\mu}^\top \chi_1 & \nu \\ 0 & I_t & \mu \\ 0 & 0 & I_s \end{pmatrix} = \begin{pmatrix} 0 & -\overline{\mu}^\top \chi_1 & \nu + \frac{1}{2}\overline{\mu}^\top \chi_1 \mu \\ 0 & 0 & \mu \\ 0 & 0 & 0 \end{pmatrix}.$$

Fundamental to the whole construction is the centre of the unipotent radical. If the parabolic subgroup P of G is maximal then N_P is a 2-step nilpotent group, i.e., $[[N_P, N_P], N_P] = 0$. The centre U_P of N_P coincides with the commutator subgroup $[N_P, N_P]$. Note that U_P and V_P are isomorphic as groups to their respective Lie algebras, so we may write $N_P = U_P \rtimes V_P$ but also, often more conveniently, $N_P = \mathrm{Lie}(U_P) \rtimes \mathrm{Lie}(V_P)$ as groups.

For $\lambda \in M_{s\times s}(\mathbb{C})$ and $\mu \in M_{t\times s}(\mathbb{C})$ we put

$$u(\lambda) = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } v(\mu) = \begin{pmatrix} 0 & -\overline{\mu}^\top \chi_1 & 0 \\ 0 & 0 & \mu \\ 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

Proposition 6. *The centre of N_P agrees with the commutator subgroup $U_P = [N_P, N_P]$ of N_P and has Lie algebra*

$$\mathrm{Lie}(U_P) = \{u(\lambda) \mid \lambda \in M_{s\times s}(\mathbb{C}), \overline{\lambda}^\top = -\lambda\}. \tag{4}$$

The quotient $V_P = N_P/U_P$ is an abelian Lie group with Lie algebra

$$\mathrm{Lie}(V_P) = \mathrm{Lie}(N_P)/\,\mathrm{Lie}(U_P) = \{v(\mu) + \mathrm{Lie}(U_P) \mid \mu \in M_{t\times s}(\mathbb{C})\}. \tag{5}$$

In particular Proposition 6 shows that $\mathrm{Lie}(V_P)$ is isomorphic to the space of $\mathbb{C}$-linear maps $\mathrm{Hom}(F, \check{E})$ and $\mathrm{Lie}(U_P)$ is isomorphic to the space $\mathrm{Hom}^{\chi}(E, \check{E})$ of infinitesimally χ-Hermitian $\mathbb{C}$-linear maps $E \to \check{E}$.

Our convention is that the matrices of linear transformations are multiplied on the left with the coordinates of the vectors from $\mathbb{C}^{a+b}$.

Having described the unipotent part, we move on to the reductive part L_P of P. It is a product $L_P = G_{1,l}(P) \times G_{1,h}(P)$ of reductive groups $G_{1,l}(P)$ and $G_{1,h}(P)$ of noncompact type, where $G_{1,h}(P)$ is the subgroup corresponding to a Hermitian symmetric space [BJ, (III.7.8)]. Note that L_P is a quotient of P, but the extension is split so we may view it as a subgroup, the Levi subgroup, defined up to conjugacy. In this case the Lie algebras do not help us and we work directly with the Lie groups. For $\xi \in \mathrm{GL}(s, \mathbb{C})$ and $\zeta \in \mathrm{GL}(t, \mathbb{C})$, we put

$$g(\xi) = \begin{pmatrix} \xi & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & (\overline{\xi}^{\top})^{-1} \end{pmatrix} \text{ and } h(\zeta) = \begin{pmatrix} I_s & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & I_s \end{pmatrix}. \tag{6}$$

Proposition 7. *The Levi subgroup of P is the direct product $L_P = G_{1,l}(P) \times G_{1.h}(P)$, where*

$$G_{1,l}(P) = \{g(\xi) \mid \xi \in \mathrm{GL}(s, \mathbb{C})\} \cong \mathrm{GL}(s, \mathbb{C}), \tag{7}$$

and

$$G_{1,h}(P) = \{h(\zeta) \mid \zeta \in \mathrm{GL}(t, \mathbb{C}),\, \overline{\zeta}^{\top} \chi_1 \zeta = \chi_1\} = \mathrm{U}(F, \chi_1) \cong \mathrm{U}(a - s, b - s). \tag{8}$$

The centre of L_P is

$$Z(L_P) = \left\{ \begin{pmatrix} pe^{i\varphi} I_s & 0 & 0 \\ 0 & e^{i\psi} I_t & 0 \\ 0 & 0 & p^{-1} e^{i\varphi} I_s \end{pmatrix} \,\Big|\, \varphi, \psi \in \mathbb{R},\, p \in \mathbb{R}_{>0} \right\}.$$

The proof of these propositions is not difficult, but we want to write out some of the details. With respect to an E-adapted basis of $\mathbb{C}^{a+b}$, the stabiliser $P = \mathrm{Stab}_G(E)$ of E in $G = \mathrm{U}(\mathbb{C}^{a+b}, \chi) \cong \mathrm{U}(a, b)$ consists of the matrices

$$p = \begin{pmatrix} \xi & \mu' & \lambda \\ 0 & \zeta & \mu \\ 0 & \nu & \xi' \end{pmatrix} \quad \text{with} \quad \overline{p}^{\top} \chi p = \chi \quad \text{for} \quad \xi \in \mathrm{GL}(s, \mathbb{C}), \quad \lambda, \xi' \in M_{s\times s}(\mathbb{C}),$$

$$\mu', \nu \in M_{s\times t}(\mathbb{C}), \quad \mu \in M_{t\times s}(\mathbb{C}), \quad \zeta \in M_{t\times t}(\mathbb{C}).$$

In other words,

$$p = \begin{pmatrix} \xi & -\xi \overline{\mu}^{\top} \chi_1 \zeta & \lambda \\ 0 & \zeta & \mu \\ 0 & 0 & (\overline{\xi}^{\top})^{-1} \end{pmatrix} \tag{9}$$

with $\overline{\zeta}^{\top} \chi_1 \zeta = \chi_1$ and $(\xi^{-1}\lambda) + \overline{(\xi^{-1}\lambda)}^{\top} + \overline{\mu}^{\top} \chi_1 \mu = 0$.

Now the homomorphism

$$p \longmapsto \begin{pmatrix} \xi & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & (\overline{\xi}^{\top})^{-1} \end{pmatrix}$$

has kernel N_P and image L_P, which immediately gives the descriptions of N_P and L_P as well as $Z(L_P)$ above. Similarly the Lie algebra $\mathrm{Lie}(P)$ consists of the matrices

$$Y = \begin{pmatrix} \xi & \mu' & \lambda \\ 0 & \zeta & \mu \\ 0 & \nu & \xi' \end{pmatrix} \text{ with } \overline{Y}^{\top}\chi + \chi Y = 0 \text{ for } \xi,\, \lambda,\, \xi' \in M_{s\times s}(\mathbb{C}),$$
$$\mu',\, \nu \in M_{s\times t}(\mathbb{C}), \mu \in M_{t\times s}(\mathbb{C}), \text{ and } \zeta \in M_{t\times t}(\mathbb{C}).$$

For convenience, given $\xi,\, \lambda \in M_{s\times s}(\mathbb{C})$, $\mu \in M_{t\times s}(\mathbb{C})$ and $\zeta \in M_{t\times t}(\mathbb{C})$ we set

$$Y(\xi,\zeta,\lambda,\mu) = \begin{pmatrix} \xi & -\overline{\mu}^{\top}\chi_1 & \lambda \\ 0 & \zeta & \mu \\ 0 & 0 & -\overline{\xi}^{\top} \end{pmatrix}. \tag{10}$$

Then a straightforward verification shows that

$$\mathrm{Lie}(P) = \{Y(\xi,\zeta,\lambda,\mu) \mid \overline{\zeta}^{\top}\chi_1 + \chi_1\zeta = 0,\ \overline{\lambda}^{\top} = -\lambda\}. \tag{11}$$

Everything now follows by simple calculations once we observe that $X = Y(0,0,\lambda,\mu) \in \mathrm{Lie}(N_P)$ satisfies

$$\exp(X) = I_{a+b} + X + \frac{1}{2}X^2 = \begin{pmatrix} I_s & -\overline{\mu}^{\top}\chi_1 & \lambda - \frac{1}{2}\overline{\mu}^{\top}\chi_1\mu \\ 0 & I_t & \mu \\ 0 & 0 & I_s \end{pmatrix}$$

with

$$\left(\lambda - \frac{1}{2}\overline{\mu}^{\top}\chi_1\mu\right) + \overline{\left(\lambda - \frac{1}{2}\overline{\mu}^{\top}\chi_1\mu\right)}^{\top} + \overline{\mu}^{\top}\chi_1\mu = 0.$$

We also want to write down the group law in P. It is convenient to use the group isomorphism between an abelian Lie group and its Lie algebra and write the unipotent radical in term of Lie algebras: namely, we have group isomorphisms

$$\begin{aligned} P \cong N_P \rtimes L_P &= (U_P \rtimes V_P) \rtimes [G_{1,l}(P) \times G_{1,h}(P)] \\ &\cong [\mathrm{Lie}(U_P) \rtimes \mathrm{Lie}(V_P)] \rtimes [G_{1,l}(P) \times G_{1,h}(P)] \end{aligned}$$

and we use the last formulation. Then the group law is given by

$$(u(\lambda_1), v(\mu_1), g(\xi_1), h(\zeta_1))(u(\lambda_2), v(\mu_2), g(\xi_2), h(\zeta_2)) = (u(\lambda), v(\mu), g(\xi_1\xi_2), h(\zeta_1\zeta_2)) \tag{12}$$

where

$$\lambda = \lambda_1 + \xi_1\lambda_2\overline{\xi_1}^{\top} + \frac{1}{2}(\xi_1\overline{\mu_2}^{\top}\overline{\zeta_1}^{\top}\chi_1\mu_1 - \overline{\mu_1}^{\top}\chi_1\zeta_1\mu_2\overline{\xi_1}^{\top}), \tag{13}$$

and $\mu = \mu_1 + \zeta_1\mu_2\overline{\xi_1}^{\top}$.

Note that an isotropic subspace $E \subset \mathbb{C}^{a+b}$ does not determine uniquely its conjugate $\check{E}$. Likewise, $P = \mathrm{Stab}_G(E)$ determines E only up to translation by an element of P. On the other hand, P is uniquely determined by the boundary component

$$D_{1,h}(P) = \mathrm{U}(F,\chi_1)/(\mathrm{U}(F,\chi_1) \cap K).$$

Suppose that $\{\alpha, \beta\} = \{\alpha_1, \ldots, \alpha_a, \beta_1, \ldots, \beta_b\}$ is an orthonormal basis of $\mathbb{C}^{a+b}$: that is, one such that the matrix of χ is $\begin{pmatrix} I_a & 0 \\ 0 & -I_b \end{pmatrix}$. Such a basis determines a maximal compact subgroup $K_{\alpha,\beta} = \mathrm{U}(\mathrm{Span}_{\mathbb{C}}(\alpha)) \times \mathrm{U}(\mathrm{Span}_{\mathbb{C}}(\beta)) \cong \mathrm{U}(a) \times \mathrm{U}(b)$ of $G \cong \mathrm{U}(a, b)$, and thus a base point in $\mathbb{B}(a, b)$. We fix a maximal compact subgroup K and an orthonormal basis $\{\alpha, \beta\}$ such that $K_{\alpha,\beta} = K$.

Now $D_{1,h}(P)$ determines the subspace $F \subset \mathbb{C}^{a+b}$ and, therefore, its χ-orthogonal complement $E \oplus \check{E}$. If $E_1 \subset E \oplus \check{E}$ is a maximal isotropic subspace then $\dim_{\mathbb{C}}(E_1) = s = \dim_{\mathbb{C}}(E)$ and there exists $g \in G_{1,l}(P)$ with $E_1 = g(E)$. The corresponding maximal parabolic subgroup

$$\mathrm{Stab}_G(E_1) = \mathrm{Stab}_G(g(E)) = g\,\mathrm{Stab}_G(E) g^{-1} = gPg^{-1} = P$$

coincides with $P = \mathrm{Stab}_G(E)$.

4.2 Compatibility with integral structure

Next we need to verify that the Langlands decomposition respects the integral structure. In general we want to work with an arithmetic lattice Γ, but for the sake of definiteness we will state the result for the special case $\Gamma = G^{\mathbb{Z}}$, where we denote by $\mathbb{I}$ the Gaussian integers $\mathbb{Z} + i\mathbb{Z}$ and define $G^{\mathbb{Z}} = G \cap \mathrm{GL}(a + b, \mathbb{I})$.

Proposition 8. *Suppose that $E \subset \mathbb{C}^{a+b}$ is a $G^{\mathbb{Z}}$-rational χ-isotropic subspace, $\{\mathbf{e}, \mathbf{f}, \check{\mathbf{e}}\} \subset \mathbb{I}^{a+b}$ is an integral E-adapted basis of $\mathbb{C}^{a+b}$, and $P = \mathrm{Stab}_G(E)$ is the associated maximal parabolic subgroup of $G = \mathrm{U}(\mathbb{C}^{a+b}, \chi)$. Then $G^{\mathbb{Z}} = G \cap \mathrm{GL}(a + b, \mathbb{I})$ is the arithmetic lattice preserving $\mathrm{Span}_{\mathbb{Z}}\{\mathbf{e}, \mathbf{f}, \check{\mathbf{e}}\}$, and*

$$P \cap G^{\mathbb{Z}} \cong [\mathrm{Lie}(U_P)^{\mathbb{Z}} \rtimes \mathrm{Lie}(V_P)^{\mathbb{Z}}] \rtimes [G_{1,l}(P)^{\mathbb{Z}} \times G_{1,h}(P)^{\mathbb{Z}}]$$

where $\mathrm{Lie}(U_P)^{\mathbb{Z}} = \mathrm{Lie}(U_P) \cap M_{(a+b)\times(a+b)}(\mathbb{I})$ and $G_{1,l}(P)^{\mathbb{Z}} = G_{1,l}(P) \cap \mathrm{GL}(a + b, \mathbb{I})$, and similarly for $\mathrm{Lie}(V_P)^{\mathbb{Z}}$ and $G_{1,h}(P)^{\mathbb{Z}}$.

This is a straightforward verification. Notice that Proposition 8 is written in terms of the Lie algebras of U_P and V_P, but if $n(\lambda, \mu) \in \mathrm{Lie}(N_P) \cap M_{(a+b)\times(a+b)}(\mathbb{I})$ we do not have $\exp(n(\lambda, \mu)) \in M_{(a+b)\times(a+b)}(\mathbb{I})$ in general because of the $\frac{1}{2}$ in the exponential (see Proposition 5).

4.3 Horospherical decomposition

Horospherical decomposition is the name given to the decomposition of the symmetric space (in this case, $\mathbb{B}(a, b)$) induced by the Langlands decomposition of a maximal parabolic subgroup.

The choice of an orthonormal basis $\{\alpha, \beta\}$ and a maximal compact subgroup $K = K_{\alpha,\beta}$ of $G = \mathrm{U}(a, b)$ determines a maximal flag $E_1 \subset \ldots \subset E_s \subset \ldots \subset E_r$ of isotropic subspaces of $\mathbb{C}^{a+b}$, and our purpose is to describe the horospherical decompositions of $\mathbb{B}(a, b)$ with respect to $P_s = \mathrm{Stab}_G(E_s)$ by means of convenient matrix realisations.

Lemma 9. *Given a maximal compact subgroup $K < G = \mathrm{U}(a,b)$ and an orthonormal basis $\{\alpha, \beta\}$ with $K_{\alpha,\beta} = K$, define*

$$e_i = \frac{1}{\sqrt{2}}(\alpha_i + \beta_i) \quad \textit{and} \quad \check{e}_i = \frac{1}{\sqrt{2}}(\alpha_i - \beta_i), \quad \textit{for} \quad 1 \leq i \leq a = \operatorname{rank} \mathbb{B}(a,b).$$

For $1 \leq s \leq a$, put $E_s = \mathrm{Span}_{\mathbb{C}}\{e_i \mid 1 \leq i \leq s\}$ and $\check{E}_s = \mathrm{Span}_{\mathbb{C}}\{\check{e}_i \mid 1 \leq i \leq s\}$. Then $E_1 \subset \ldots \subset E_s \subset \ldots \subset E_a$ is a maximal flag of isotropic subspaces of $\mathbb{C}^{a+b}$, the space $\check{E}_s$ is conjugate to E_s and, for a fixed s, if we also set $f_j = \alpha_{s+j}$ and $f'_j = \beta_{s+j}$, the basis

$$\{\mathbf{e}, \mathbf{f}, \check{\mathbf{e}}\}_{E_s} = \{e_1, \ldots, e_s, f_1, \ldots, f_{a-s}, f'_1, \ldots, f'_{b-s}, \check{e}_1, \ldots, \check{e}_b\} \tag{14}$$

is an E_s-adapted basis of $\mathbb{C}^{a+b}$.

This is a straightforward verification: the factor $\frac{1}{\sqrt{2}}$ is unimportant but will simplify notation later.

We keep the notation of Lemma 9 and recall the definition of the matrices $Y(\xi, \zeta, \lambda, \mu)$ from (10).

Proposition 10. *The maximal parabolic subgroup $P = \mathrm{Stab}_G(E_s)$ has Levi subgroup $L_P = G_{1,l}(P) \times G_{1,h}(P)$ satisfying*

$$\mathrm{Lie}(P \cap K) = \mathrm{Lie}(L_P \cap K) = \mathrm{Lie}(G_{1,l}(P) \cap K) \times \mathrm{Lie}(G_{1,h}(P) \cap K).$$

With respect to the basis $\{\mathbf{e}, \mathbf{f}, \check{\mathbf{e}}\}_{E_s}$, these factors are the matrix Lie algebras

$$\begin{aligned} \mathrm{Lie}(G_{1,l}(P) \cap K) &= \mathrm{Lie}(G_{1,l}(P)) \cap \mathrm{Lie}(K) \\ &= \left\{ \begin{pmatrix} \xi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi \end{pmatrix} \,\middle|\, \xi \in M_{s \times s}(\mathbb{C}),\, \overline{\xi}^{\top} = -\xi \right\} \\ &= \{Y(\xi, 0, 0, 0) \mid \overline{\xi}^{\top} = -\xi\} \end{aligned} \tag{15}$$

and

$$\begin{aligned} \mathrm{Lie}(G_{1,h}(P) \cap K) &= \mathrm{Lie}(G_{1,h}(P)) \cap \mathrm{Lie}(K) \\ &= \left\{ Y(0, \zeta, 0, 0) \,\middle|\, \zeta = \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{pmatrix}, \overline{\zeta_j}^{\top} = -\zeta_j \right\}, \end{aligned} \tag{16}$$

where $\zeta_1 \in M_{(a-s)\times(a-s)}(\mathbb{C})$, and $\zeta_2 \in M_{(b-s)\times(b-s)}(\mathbb{C})$.

Corresponding to the Langlands decomposition of any maximal parabolic subgroup P we have the horospherical decomposition of $\mathbb{B}(a,b)$. In the first place

$$\mathbb{B}(a,b) = PK/K \cong P/(P \cap K)$$

and since N_P has no nontrivial compact subgroups that gives

$$\begin{aligned} \mathbb{B}(a,b) &= (N_P \times L_P)/(L_P \cap K) \\ &\cong U_P \times V_P \times [G_{1,l}(P)/(G_{1,l}(P) \cap K)] \times [G_{1,h}(P)/(G_{1,h}(P) \cap K)]\,. \end{aligned}$$

Proposition 11. *If $P = \mathrm{Stab}_G(E_s)$, then*

$$\mathbb{B}(a,b) = U_P \times V_P \times C_P \times D_{1,h}(P).$$

The factor C_P may be identified with the the cone of positive definite Hermitian $(s \times s)$-matrices. In the coordinates given by the basis $(\mathbf{e}, \mathbf{f}, \check{\mathbf{e}})_{E_s}$ in equation (14), using the notation from (6),

$$\begin{aligned} C_P &= \{g(\xi) \mid \xi \in M_{s\times s}(\mathbb{C}),\ \overline{\xi}^\top = \xi,\ \xi > 0\} \\ &= \{g(c\overline{c}^\top) \mid c \in \mathrm{GL}(s,\mathbb{C})\}, \end{aligned} \tag{17}$$

whose real tangent space at any point $o \in C_P$ is

$$T_o^{\mathbb{R}} C_P = \left\{ Y(\xi_o, 0, 0, 0) = \begin{pmatrix} \xi_o & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\xi_o \end{pmatrix} \,\middle|\, \xi_o \in M_{s\times s}(\mathbb{C}),\ \overline{\xi_o}^\top = \xi_o \right\}. \tag{18}$$

The other factor is the generalised ball

$$D_{1,h}(P) = G_{1,h}(P)/(G_{1,h}(P) \cap K) \cong \mathrm{U}(a-s, b-s)/\,\mathrm{U}(a-s) \times \mathrm{U}(b-s) = \mathbb{B}(a-s, b-s),$$

whose real tangent space at the origin is

$$T_o^{\mathbb{R}} D_{1,h}(P) = \left\{ Y(0, \zeta, 0, 0) \,\middle|\, \zeta = \begin{pmatrix} 0 & \zeta_{12} \\ \overline{\zeta_{12}}^\top & 0 \end{pmatrix},\ \zeta_{12} \in M_{(a-s)\times(b-s)}(\mathbb{C}) \right\}. \tag{19}$$

The proof of Proposition 10 is a straightforward computation: one has to write down the transition matrices between the two bases $\{\alpha, \beta\}$ and $\{\mathbf{e}, \mathbf{f}, \check{\mathbf{e}}\}$, and use the fact that $\mathrm{Lie}(K)$ consists (with respect to $\{\alpha, \beta\}$) of the matrices $\begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix}$, with $Y_1 \in M_{a\times a}(\mathbb{C})$, $Y_2 \in M_{b\times b}(\mathbb{C})$, and $\overline{Y_j}^\top = -Y_j$.

The proof of Proposition 11 is only slightly less elementary. The open strongly convex homogeneous cone

$$C_P = G_{1,l}(P)/(G_{1,l}(P) \cap K)$$

is a Cartan-Hadamard manifold and can be identified with the image of its tangent space $T_o^{\mathbb{R}} C_P$ under $\exp\colon T_o^{\mathbb{R}} C_P \to C_P$. However, $T_o^{\mathbb{R}} C_P = (\mathrm{Lie}(G_{1,l}(P) \cap K))^\perp$ is the orthogonal complement of $\mathrm{Lie}(G_{1,l}(P) \cap K)$ in $\mathrm{Lie}\, G_{1,l}(P)$ with respect to the Killing form $\langle X, Y \rangle = \mathrm{Tr}[X, Y]$. This, together with (15) and the description of $\mathrm{Lie}(K)$ above, gives (18) immediately and implies that the exponential map of C_P coincides with the exponential of a matrix. Applying exp to the matrices in (18) gives (17): one has $\exp(\xi_o) = \xi$.

Notice in particular that $T_o^{\mathbb{R}} C_P$ is isomorphic to the space $\mathrm{Hom}^\chi(\check{E}_s, \check{E}_s)$ of infinitesimally χ-Hermitian $\mathbb{C}$-linear operators on $\check{E}_s$.

Similarly, the generalised ball

$$D_{1,h}(P) = G_{1,h}(P)/G_{1,h}(P) \cap K$$

is the image of $\exp\colon T_o^{\mathbb{R}}D_{1,h}(P) \to D_{1,h}(P)$. Using $\overline{\zeta}^\top\chi_1 + \chi_1\zeta = 0$ (from (11)) and writing $\zeta = \begin{pmatrix} \zeta_1 & \zeta_{12} \\ \zeta_{21} & \zeta_2 \end{pmatrix}$, with $\zeta_1 \in M_{(a-s)\times(a-s)}(\mathbb{C})$ and $\zeta_2 \in M_{(b-s)\times(b-s)}(\mathbb{C})$, we obtain $\overline{\zeta_j}^\top = -\zeta_j$ and $\overline{\zeta_{12}}^\top = \zeta_{12}$. Computing the Killing form, we find that the subspace $T_o^{\mathbb{R}}D_{1,h}(P) = \mathrm{Lie}(G_{1,h}(P)\cap K)^\perp$ of $\mathrm{Lie}\,G_{1,h}(P)$ is given by $\zeta_1 = \zeta_2 = 0$, yielding (19).

Again, $T_o^{\mathbb{R}}D_{1,h}(P)$ to $D_{1,h}(P)$ can be identified with the space $\mathrm{Hom}^\chi(F_s, F_s)$ of infinitesimally χ-Hermitian $\mathbb{C}$-linear operators on F_s.

So far we have worked only with isotropic subspaces E_s arising from the filtrations associated with a choice of K and $\{\alpha, \beta\}$. There is, however, no practical loss of generality in doing so, because G acts transitively on the set of isotropic subspaces of dimension s. Thus, if E is another such subspace, the associated maximal parabolic subgroup $P = \mathrm{Stab}_G(E)$ of G is conjugate to some $P_s = \mathrm{Stab}_G(E_s)$, and P inherits a Langlands decomposition and induces a horospherical decomposition of $\mathbb{B}(a,b)$, as in Proposition 11.

4.4 Siegel domain decomposition

The horospherical decomposition of Section 4.3 is a diffeomorphism, but there is a complex structure to be considered as well. By (4) we have

$$\begin{aligned} i\,\mathrm{Lie}(U_P) &= \{u(i\lambda) = Y(0,0,i\lambda,0) \mid \lambda \in M_{s\times s}(\mathbb{C}), \overline{\lambda}^\top = -\lambda\} \\ &= \{u(\xi_o) = Y(0,0,\xi_o,0) \mid \xi_o \in M_{s\times s}(\mathbb{C}), \overline{\xi_o}^\top = \xi_o\}. \end{aligned}$$

So $i\,\mathrm{Lie}(U_P)$ is parametrised by the Hermitian $(s\times s)$-matrices and there is a canonical isomorphism of $\mathbb{R}$-linear spaces

$$\kappa_P\colon T_o^{\mathbb{R}}C_P \longrightarrow i\,\mathrm{Lie}(U_P),$$

given by $\kappa_P(Y(\xi_o,0,0,0)) = Y(0,0,\xi_o,0)$. It transforms the open strongly convex homogeneous cone $C_P \subset T_o^{\mathbb{R}}C_P$ into an open strongly convex homogeneous cone $\kappa_P(C_P) \subset i\,\mathrm{Lie}(U_P)$ and realises

$$\mathfrak{S}_P = \mathrm{Lie}(U_P) + \kappa_P(C_P) \subset \mathrm{Lie}(U_P) + i\,\mathrm{Lie}(U_P) = \mathrm{Lie}(U_P)\otimes_{\mathbb{R}}\mathbb{C}$$

as a Siegel domain of first kind. After application of the Lie group isomorphism $V_P \cong (\mathrm{Lie}(V_P), +)$, we obtain a *diffeomorphic* realisation $D^{[P]}$ of $\mathbb{B}(a,b)$, which is the product

$$D^{[P]} = \mathfrak{S}_P \times \mathfrak{V}_P, \tag{20}$$

where $\mathfrak{V}_P = \mathrm{Lie}(V_P)\times D_{1,h}(P)$.

On the other hand, we also have the classical holomorphic Siegel domain realisation $D^{(P)}$ of of $\mathbb{B}(a,b)$ associated with P, which is a *holomorphic* fibration

$$\pi_P\colon D^{(P)} \longrightarrow \mathfrak{V}_P = \mathrm{Lie}(V_P)\times D_{1,h}(P) \tag{21}$$

by Siegel domains $\pi_P^{-1}(v,z)$ of the first kind (see [P-S]), each diffeomorphic to the central fibre $\pi_P^{-1}(0,o) = U_P + iC_P = \mathfrak{S}_P$. This is described in [BJ, Proposition III.7.12]: for any point $z \in D_{1,h}(P)$ there is a Hermitian vector-valued quadratic form

$$\mathbf{h}_z\colon\ \mathrm{Lie}(V_P)\times\mathrm{Lie}(V_P) \longrightarrow \mathrm{Lie}(U_P)^{\mathbb{C}} = \mathrm{Lie}(U_P)\otimes_{\mathbb{R}}\mathbb{C}, \tag{22}$$

which depends real-analytically on z. (There is a slight inaccuracy in [BJ, Proposition III.7.12], where it is said that $\mathbf{h}_z$ depends holomorphically on z. In [P-S, Section 1.3], the Siegel domains of third kind are defined by Hermitian vector-valued forms $\mathbf{h}_z$, which depend real-analytically on z. See also [Wolf, Section 2] or [AMRT, Section 4.3(iv)].)

If $\overline{\kappa_P(C_P)} \subset i\,\mathrm{Lie}(U_P)$ is the closure of $\kappa_P(C_P)$ in $i\,\mathrm{Lie}(U_P)$, then $i\mathbf{h}_z(v,v) \in \overline{\kappa_P(C_P)}$ for all $v \in \mathrm{Lie}(V_P)$.

The fibres of the holomorphic projection π_P are

$$\pi_P^{-1}(v,z) = \{u_1 + iu_2 \in \mathrm{Lie}(U_P)^{\mathbb{C}} \mid iu_2 - i\mathbf{h}_z(v,v) \in \kappa_P(C_P)\}. \tag{23}$$

For any fixed $b = (v,z) \in \mathfrak{B}_P$ the map

$$\delta_P^b \colon \mathfrak{S}_P = \pi_P^{-1}(o) \longrightarrow \pi_p^{-1}(v,z), \tag{24}$$

given by $\delta_P^b(u_1 + iu_2) = u_1 + i(u_2 + \mathbf{h}_z(v,v))$, for $u_1 \in \mathrm{Lie}(U_P)$ and $iu_2 \in \kappa_P(C_P)$, is a biholomorphism. By dropping the requirement $iu_2 \in \kappa_P(C_P)$, we get a biholomorphism $\delta_P^b\colon \mathrm{Lie}(U_P)^{\mathbb{C}} \to \mathrm{Lie}(U_P)^{\mathbb{C}}$, which depends diffeomorphically on $v \in \mathrm{Lie}(V_P)$ and on $z \in D_{1,h}(P)$. Hence these fibrewise maps $\{\delta_P^b\}_{b \in \mathfrak{B}_P}$ are the restrictions of a diffeomorphism

$$\delta_P \colon [\mathrm{Lie}(U_P) + \kappa_P(C_P)] \times \mathrm{Lie}(V_P) \times D_{1,h}(P) \longrightarrow D \subset \mathrm{Lie}(U_P)^{\mathbb{C}} \times \mathrm{Lie}(V_P) \times D_{1,h}(P),$$

given by

$$\delta_P(\sigma, v, z) = (\delta_P^b(\sigma), v, z) = (\sigma + i\mathbf{h}_z(v,v), v, z)$$

depending holomorphically on $\sigma \in \mathfrak{S}_P$.

5 A partial compactification at a cusp

Next we concentrate on a single boundary component and build the partial compactification there.

5.1 Maximal parabolic subgroups and toric varieties

Let $P = \mathrm{Stab}_G(E_s)$ be a maximal parabolic subgroup of G and let

$$\{\mathbf{e}, \mathbf{f}, \check{\mathbf{e}}\} = \{e_1, \ldots, e_s, f_1, \ldots, f_{a-s}, f_1', \ldots, f_{b-s}', \check{e}_1, \ldots, \check{e}_s\}$$

be an E_s-adapted basis of $\mathbb{C}^{a+b}$, and Γ be an arithmetic lattice in $G = \mathrm{U}(\mathbb{C}^{a+b}, \chi)$.

Recall that the exponential map

$$\exp\colon (\mathrm{Lie}(U_P), +) \longrightarrow U_P$$

of the abelian Lie group U_P is a group isomorphism and denote by

$$\Upsilon_P := \exp^{-1}(U_P \cap \Gamma)$$

the preimage of the lattice $U_P \cap \Gamma$ of U_P. The quotient

$$\mathbb{T}(P) = \Upsilon_P \backslash \mathrm{Lie}(U_P)^{\mathbb{C}}$$

is an algebraic torus over $\mathbb{C}$.

A closed, strongly convex polyhedral cone τ in $\mathrm{Lie}(U_P) \cong (\mathbb{R}^{s^2}, +)$ is a subset of the form $\tau = \mathbb{R}_{\geq 0}u_1 + \ldots + \mathbb{R}_{\geq 0}u_d$ for some $u_i \in \mathrm{Lie}(U_P)$. One says that τ is Υ_P-rational, or just *rational*, if all the u_i can be chosen to belong to Υ_P.

A collection $\Sigma(P)$ of closed, strongly convex polyhedral cones is a *fan* if any face of a cone $\tau \in \Sigma(P)$ belongs to $\Sigma(P)$, and $\tau_1 \cap \tau_2 \in \Sigma(P)$ if $\tau_i \in \Sigma(P)$. A fan $\Sigma(P)$ is *rational* if all of its cones are rational.

A Υ_P-rational fan $\Sigma(P)$ is Γ-*admissible* if the cone

$$\kappa_P(C_P) \subseteq \bigcup_{\sigma \in \Sigma(P)} i\sigma$$

and the lattice $\Gamma \cap G_{1,l}(P) < G_{1,l}(P)$ acts on $\Sigma(P)$ with finitely many orbits.

Let $\Sigma(P)$ be a Γ-admissible fan. For any $\tau \in \Sigma(P)$ we define $\mathbb{T}(\tau) = \Upsilon_P \backslash (\tau_{\mathbb{C}} + \Upsilon_P)$; this is the subgroup of $\mathbb{T}(P)$ generated by the complex span $\tau_{\mathbb{C}} = \mathrm{Span}_{\mathbb{C}}(\tau)$ of τ.

The quotient group $\mathbb{T}(P/\tau) = \mathbb{T}(P)/\mathbb{T}(\tau)$ still has a (left) action of $\mathbb{T}(P)$ coming from the multiplication action of $\mathbb{T}(P)$ on itself, and the quotient map $\mathbb{T}(P) \to \mathbb{T}(P/\tau)$ is equivariant. In this situation the fan $\Sigma(P)$ and the lattice Υ_P determine a toric variety $\mathbb{TV}(\Sigma(P))$ (which, as a set, is $\coprod_{\sigma \in \Sigma(P)} \mathbb{T}(P/\sigma)$), on which $\mathbb{T}(P)$ acts with an open dense orbit.

There is a vast literature on toric varieties. Two standard references are [CLS] and [Ful], but they deal with finite fans (so the resulting toric varieties are Noetherian schemes). The fan $\Sigma(P)$ will not be finite, only admissible. For the slightly greater generality that we need, we refer to [Oda], where infinite fans are allowed. In the notation of [Oda], the toric variety $\mathbb{TV}(\Sigma(P))$ is the object denoted by $T_{\Upsilon_P}\,\mathrm{emb}(\Sigma(P))$.

In order to define the partial completion $Z_{\Sigma(P)} = (\Upsilon_P \backslash \mathcal{D})_{\Sigma(P)}$ at the cusp associated with P, we use the holomorphic Siegel domain realisation of D arising from P and the canonical projection

$$\pi_P \colon D \longrightarrow \mathfrak{B}_P = \mathrm{Lie}(V_P) \times D_{1,h}(P).$$

Bearing in mind that $\Upsilon_P \backslash \mathfrak{S}_P \subset \mathbb{T}(P) \subset \mathbb{TV}(\Sigma(P))$ since $\mathfrak{S}_P = \mathrm{Lie}(U_P) + \kappa_P(C_P)$, one takes $Y_{\Sigma(P)} = \mathrm{Int}(\overline{\Upsilon_P \backslash \mathfrak{S}_P})$, the interior of the closure of $\Upsilon_P \backslash \mathfrak{S}_P$ inside $\mathbb{TV}(\Sigma(P))$.

In terms of the holomorphic Siegel domain realisation $\mathcal{D}^{(P)}$ of $\mathcal{D}$, the partial completion of $\Upsilon_P \backslash \mathcal{D}$ at P is

$$Z_{\Sigma(P)} = \coprod_{(v,z) \in \mathcal{B}_P} \mathrm{Int}(\overline{\Upsilon_P \backslash (\mathfrak{S}_P + i\mathbf{h}_z(v,v))}).$$

Note that Υ_P acts on the fibres of π_P, as they are translates in an imaginary direction of $\pi_P^{-1}(o) = \mathfrak{S}_P$. So $Z_{\Sigma(P)}$ is a family of spaces $Y_{\Sigma(P)}$, partially compactifying the fibration of $\Upsilon_P \backslash \mathcal{D}$ over $\mathfrak{B}_P$ fibrewise.

If $D^{[P]} = \mathfrak{S}_P \times \mathrm{Lie}(V_P) \times D_{1,h}(P)$ is the diffeomorphic Siegel domain realisation, then

$$\Upsilon_P \backslash D^{[P]} = [\Upsilon_P \backslash \mathfrak{S}_P] \times \mathrm{Lie}(V_P) \times D_{1,h}(P)$$

and

$$Z_{\Sigma(P)} \cong \mathrm{Int}(\overline{\Upsilon_P \backslash \mathfrak{S}_P}) \times \mathrm{Lie}(V_P) \times D_{1,h}(P) = Y_{\Sigma(P)} \times \mathrm{Lie}(V_P) \times D_{1,h}(P).$$

5.2 Group actions

The action of G on $\mathbb{C}^{a+b}$ induces a transitive action on the set of isotropic subspaces of any fixed dimension s. Thus, for an arbitrary s-dimensional isotropic subspace $E \subset \mathbb{C}^{a+b}$ there exists $g \in G = \mathrm{U}(a,b)$ with $E = g(E_s)$ and $P = \mathrm{Stab}_G(E) = g\,\mathrm{Stab}_G(E_s)g^{-1}$.

We are now concerned with the Γ-action, and in practice we shall be only concerned with rational isotropic subspaces. Corollary 12 does not address the question of transitivity, but simply observes that the constructions we have made so far are Γ-invariant.

Corollary 12. *We use the notation of Lemma 9. Then Γ acts on the set of isotropic subspaces of $\mathbb{C}^{a+b}$, and if $E = g_s(E_s) \subset \mathbb{C}^{a+b}$ with $g_s \in \Gamma$ is an isotropic subspace in the Γ-orbit of E_s then the diffeomorphism*

$$\widetilde{\eta^s_P} \colon D^{[P_s]} \longrightarrow D^{[P]}$$

given by

$$\widetilde{\eta^s_P}(u_s + \kappa_{P_s}(c_s), v_s, z_s) = (g_s u_s g_s^{-1} + \kappa_P(g_s c_s g_s^{-1}), g_s v_s g_s^{-1}, g_s z_s g_s^{-1})$$

induces a diffeomorphism

$$\eta^s_P \colon \mathbb{TV}(\Sigma(P_s)) \times \mathrm{Lie}(V_{P_s}) \times D_{1,h}(P_s) \longrightarrow \mathbb{TV}(g_s\Sigma(P_s)g_s^{-1})^{g_s\Gamma g_s^{-1}} \times \mathrm{Lie}(V_P) \times D_{1,h}(P)$$

which restricts to a diffeomorphism $\eta^s_P \colon Z_{\Sigma(P_s)} \to Z_{\Sigma(P)}$.

The proof is immediate.

Next, we describe the action of P on the associated holomorphic Siegel domain realisation of $\mathbb{B}(a,b)$ and the action of $\Gamma \cap P$ on $\Upsilon_P\backslash\mathbb{B}(a,b)$, on the toric variety $\mathbb{TV}(\Sigma(P))$ and on the closure $\overline{\Upsilon_P\backslash\mathbb{B}(a,b)}$ of $\Upsilon_P\backslash\mathbb{B}(a,b)$ in $\mathbb{TV}(\Sigma(P))$.

In the notation from Lemma 9, let $P = \mathrm{Stab}_G(E_s)$ be a standard maximal parabolic K-adapted subgroup of G. According to (21) and (23), we may write the associated holomorphic Siegel domain realisation of $\mathcal{D} = \mathbb{B}(a,b)$ as

$$\begin{aligned} D^{(P)} &= \big\{d(\lambda,\mu,\xi,z) = \big(u(\lambda) + \kappa_P(g(\xi)) + i\mathbf{h}_z\big(v(\mu),v(\mu)\big), v(\mu), z\big)\big\} && (25)\\ &\subset \mathrm{Lie}(U_P)^{\mathbb{C}} \times \mathrm{Lie}(V_P) \times D_{1,h}(P) && (26)\end{aligned}$$

where $(v(\mu),z) \in \mathfrak{B}_P = \mathrm{Lie}\,V_P \times D_{1,h}(P)$ and $u(\lambda) \in \mathrm{Lie}(U_P)$, and $g(\xi) \in C_P$ for some $\xi = \overline{\xi}^\top \in M_{s\times s}(\mathbb{C})$.

Corollary 13. *In the coordinates given by (25), the action of P on $D^{(P)}$ is as follows: if*

$$p = ((u(\lambda_0), v(\mu_0), g(\xi_0), h(\zeta_0)) \in [\mathrm{Lie}(U_P) \rtimes \mathrm{Lie}(V_P)] \rtimes [G_{1,l}(P) \times G_{1,h}(P)] \cong P$$

as in Proposition 6 and Proposition 7, then

$$p \colon d(\lambda,\mu,\xi,z) \longrightarrow d(\lambda',\mu',\xi',z')$$

where

$$\lambda' = \lambda_0 + \xi_0\lambda\overline{\xi_0}^\top + \frac{1}{2}(\xi_0\overline{\mu}^\top\overline{\zeta_0}^\top\chi_1\mu_0 - \overline{\mu_0}^\top\chi_1\zeta_0\mu\overline{\xi_0}^\top), \quad \xi' = \xi_0\xi\overline{\xi_0}^\top, \qquad (27)$$

and

$$\mu' = \mu_0 + \zeta_0 \mu \overline{\xi_0}^\top, \quad z' = \zeta_0 z. \tag{28}$$

Moreover, if $p \in \Gamma \cap P$ then this action descends to an action on $\Upsilon_P \backslash \mathbb{B}(a,b)$, and if furthermore $\tau \in \Sigma(P)$ for a Γ-admissible fan $\Sigma(P)$ then it descends to an action on $\mathbb{T}(P/\tau) = \mathrm{Lie}(U_P)^{\mathbb{C}} / \operatorname{Span}_{\mathbb{C}}(\tau) + \Upsilon_P$.

Proof. The action of P on the holomorphic Siegel domain realisation $D^{(P)} = P/P \cap K$ arises from the group multiplication in P as given in (13). The only thing to be taken care of is the action of $\xi_0 \in \mathrm{GL}(s,\mathbb{C})$ on $g(\xi) \in C_P$ by the rule $g(\xi) \mapsto g(\xi_0 \xi \overline{\xi_0}^\top)$ and the action of $\zeta_0 \in \mathrm{U}(F,\chi_1)$ on $z \in D_{1,h}(P) = \mathrm{U}(F,\chi_1)/K \cap \mathrm{U}(F,\chi_1)$ by $z \mapsto \zeta_0 z$: the latter is immediate and the former follows from (17). □

6 Assembling the toroidal compactifications

Now we consider the identifications that we need to make among the partial compactifications defined in Section 5.

6.1 Two adjacent boundary components

In the situation of Lemma 9, we choose $1 \le s < q \le r$. We denote the corresponding parabolic subgroups by P_s and P_q, and we define $E_q^s = \operatorname{Span}_{\mathbb{C}}\{e_i \mid s < i \le q\}$ and $\check{E}_q^s = \operatorname{Span}_{\mathbb{C}}\{\check{e}_i \mid s < i \le q\}$, so that $E_q = E_s \oplus E_q^s$.

Then if we consider the subspace $F_s = \operatorname{Span}_{\mathbb{C}}\{\alpha_i,\, \beta_i \mid i > s\}$ we have a decomposition $F_s = E_q^s \oplus F_q \oplus \check{E}_q^s$, in which E_q^s is a χ-isotropic subspace of F_s. It thus corresponds to a maximal parabolic subgroup $P_q^s = \operatorname{Stab}_{G_{1,h}(P_s)}(E_q^s)$ of $G_{1,h}(P_s) = \mathrm{U}(F_s, \chi|_{F_s})$. We also write $\mathfrak{S}_q^s = \mathfrak{S}_{P_q^s}$ and similarly $\mathfrak{S}_s$ and $\mathfrak{S}_q$, and $V_q^s = V_{P_q^s}$ and similarly V_s and V_q.

We now have two ways to decompose $\mathbb{B}(a,b)$, as in (20), into a product one of whose factors is $D_{1,h}(P_q)$. One is to use (20) directly, with $P = P_q$, and the other is to use (20) twice, first with $P = P_s$ and then to decompose $D_{1,h}(P_s)$ as well. To each of these there corresponds also a holomorphic fibration as in (21), and we can glue the partial compactifications $Z_{\Sigma(P_q)}$ and $Z_{\Sigma(P_s)}$ by identifying these two fibrations.

Proposition 14. *With notation as above, we have diffeomorphic decompositions*

$$D_{1,h}(P_s) \cong \mathfrak{S}_q^s \times \mathrm{Lie}(V_q^s) \times D_{1,h}(P_q), \tag{29}$$

$$\mathrm{Lie}(V_s) \cong \mathrm{Hom}(E_q^s \oplus \check{E}_q^s, \check{E}_s) \times \mathrm{Hom}(F_q, \check{E}_s) \tag{30}$$

$$\mathfrak{S}_q \cong \mathfrak{S}_s \times \mathfrak{S}_q^s \times \mathrm{Hom}(E_q^s \oplus \check{E}_q^s, \check{E}_s) \tag{31}$$

$$\mathrm{Lie}(V_q) \cong \mathrm{Lie}(V_q^s) \times \mathrm{Hom}(F_q, \check{E}_s). \tag{32}$$

Proof. The decomposition (29) comes from the diffeomorphic Siegel domain decomposition (20), which gives

$$D_{1,h}(P_s)^{[P_q^s]} = \mathfrak{S}_q^s \times \mathrm{Lie}(V_q^s) \times D_{1,h}(P_q^s).$$

The χ-orthogonal complement of $E_q^s \oplus \check{E}_q^s$ to $F_s = E_q^s \oplus F_q \oplus \check{E}_q^s$ is F_q, so that $G_{1,h}(P_q^s) = \mathrm{U}(F_q, \chi|_{F_q}) = G_{1,h}(P_q)$. Hence

$$D_{1,h}(P_q^s) = G_{1,h}(P_q^s)/G_{1,h}(P_q^s) \cap K = G_{1,h}(P_q)/G_{1,h}(P_q) \cap K = D_{1,h}(P_q),$$

which gives (29).

The decompositions (30) and (32) come from Proposition 6. Namely, $\mathrm{Lie}(V_s) \cong \mathrm{Hom}(E_q^s \oplus F_q \oplus \check{E}_q^s, \check{E}_s) = \mathrm{Hom}(E_q^s \oplus \check{E}_q^s, \check{E}_s) \times \mathrm{Hom}(F_q, \check{E}_s)$, and $\mathrm{Lie}(V_q) \cong \mathrm{Hom}(F_q, \check{E}_s \oplus \check{E}_q^s) = \mathrm{Lie}(V_q^s) \times \mathrm{Hom}(F_q, \check{E}_s)$.

We write $v_s(\mu)$ for $v(\mu)$ in (3), as we are dealing with V_s, and decompose $\mu = \mu_1 \oplus \mu_2 \oplus \mu_1'$ with $\mu_1 \in M_{(q-s)\times s}(\mathbb{C}) \cong \mathrm{Hom}(E_q^s, \check{E}_s)$, and $\mu_1' \in M_{(q-s)\times s}(\mathbb{C}) \cong \mathrm{Hom}(\check{E}_q^s, \check{E}_s)$ and $\mu_2 \in M_{(a+b-2q)\times s}(\mathbb{C}) \cong \mathrm{Hom}(F_q, \check{E}_s)$, where the isomorphisms come from the choice of basis that we have already made.

Similarly, we write $v_q(\widetilde{\mu})$ and $\widetilde{\mu} = \mu_0 \oplus \mu_2$, with $\mu_0 \in M_{(a+b-2q)\times(q-s)}(\mathbb{C}) \cong \mathrm{Lie}(V_q^s)$ and $\mu_2 \in M_{(a+b-2q)\times s}(\mathbb{C}) \cong \mathrm{Hom}(F_q, \check{E}_s)$. In terms of matrices, if $\chi_1 = \chi|_{F_q}$ then one has

$$v_s(\mu) = \begin{pmatrix} 0 & -\overline{\mu_1'}^\top & \overline{\mu_2}^\top \chi_1 & -\overline{\mu_1}^\top & 0 \\ 0 & 0 & 0 & 0 & \mu_1 \\ 0 & 0 & 0 & 0 & \mu_2 \\ 0 & 0 & 0 & 0 & \mu_1' \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } v_q(\widetilde{\mu}) = \begin{pmatrix} 0 & 0 & -\overline{\mu_2}^\top \chi_1 & 0 & 0 \\ 0 & 0 & -\overline{\mu_0}^\top \chi_1 & 0 & 0 \\ 0 & 0 & 0 & \mu_0 & \mu_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the blocks given by $E_s \oplus E_q^s \oplus F_q \oplus \check{E}_q^s \oplus \check{E}_s$ for $v_s(\mu)$ (thus reordering the basis so that $\check{e}_{s+1}, \ldots, \check{e}_q$ precede $\check{e}_1, \ldots, \check{e}_s$, and thereby changing the matrix χ) and with respect to the blocks given by $E_s \oplus E_q^s \oplus F_q \oplus \check{E}_s \oplus \check{E}_q^s$ for $v_q(\widetilde{\mu})$.

For (31) it is more convenient to use the basis in its original order, so that χ has matrix as in (2). Then we describe $\mathrm{Lie}(U_q)$ using Proposition 6, and decompose the skew-Hermitian matrix $\lambda \in M_{q\times q}^{\mathrm{skHerm}}(\mathbb{C}) = \mathrm{Hom}^\chi(E_s \oplus E_q^s, \check{E}_s \oplus \check{E}_q^s)$ as $\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$. Then the matrices $\lambda_{11} \in M_{s\times s}^{\mathrm{skHerm}}(\mathbb{C}) = \mathrm{Hom}^\chi(E_s, E_s) \cong \mathrm{Lie}(U_s)$ and $\lambda_{22} \in M_{(q-s)\times(q-s)}^{\mathrm{skHerm}}(\mathbb{C}) = \mathrm{Hom}^\chi(E_q^s, \check{E}_q^s) \cong \mathrm{Lie}(U_q^s)$ are both skew-Hermitian and

$$\lambda_{21} = -\overline{\lambda_{12}}^\top \in M_{(q-s)\times s}(\mathbb{C}) = \mathrm{Hom}(E_q^s, \check{E}_s) \cong [\mathrm{Hom}(E_s, \check{E}_q^s) + \mathrm{Hom}(E_q^s, \check{E}_s)]^\chi.$$

Hence, using Proposition 6 again, we have a decomposition $\mathrm{Lie}(U_q) \cong \mathrm{Lie}(U_s) \times \mathrm{Lie}(U_q^s) \times \mathrm{Hom}(E_q^s, \check{E}_s)$.

Instead of decomposing $\kappa_{P_q}(C_q)$, we decompose the real tangent space $T_o^{\mathbb{R}} C_q$ to C_q at the origin. This is sufficient since C_q is a Cartan-Hadamard manifold and the exponential map $\exp\colon T_o^{\mathbb{R}} C_q \to C_q$ is a global diffeomorphism, as is $\kappa_{P_q}\colon C_q \to \kappa_{P_q}(C_q)$. But Proposition 11 identifies $T_o^{\mathbb{R}} C_q$ with the space $M_{q\times q}^{\mathrm{Herm}}(\mathbb{C}) = \mathrm{Hom}^\chi(\check{E}_s \oplus \check{E}_q^s, \check{E}_s \oplus \check{E}_q^s)$ of $q \times q$ Hermitian matrices ξ_o by the map $\xi_o \mapsto Y(\xi_0, 0, 0, 0)$ as in (18), and we decompose that as $\xi_o = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}$, with $\xi_{11} \in M_{s\times s}^{\mathrm{Herm}}(\mathbb{C}) = \mathrm{Hom}^\chi(\check{E}_s, \check{E}_s) \cong T_o^{\mathbb{R}} C_s$ and $\xi_{22} \in M_{(q-s)\times(q-s)}^{\mathrm{Herm}}(\mathbb{C}) = \mathrm{Hom}^\chi(\check{E}_q^s, \check{E}_q^s) \cong T_o^{\mathbb{R}} C_q^s$ both Hermitian and

$$\xi_{21} = \overline{\xi_{12}}^\top \in M_{(q-s)\times s}(\mathbb{C}) \cong \mathrm{Hom}(\check{E}_q^s, \check{E}_s) \cong [\mathrm{Hom}(\check{E}_q^s, \check{E}_s) + \mathrm{Hom}(\check{E}_s, \check{E}_q^s)]^\chi.$$

Now we have a decomposition

$$T_o^{\mathbb{R}} C_q \cong T_o^{\mathbb{R}} C_s \times T_o^{\mathbb{R}} C_q^s \times \mathrm{Hom}(\check{E}_q^s, \check{E}_s).$$

Since $\mathfrak{S}_q = \mathrm{Lie}(U_q) \times \kappa_{P_q}(C_q)$ and similarly for the other Siegel domains, we have $\mathfrak{S}_q \cong \mathfrak{S}_s \times \mathfrak{S}_q^s \times \mathrm{Hom}(E_q^s, \check{E}_s) \times \mathrm{Hom}(\check{E}_q^s, \check{E}_s)$ □

These diffeomorphisms together with (20) give decompositions of $\mathbb{B}(a,b)$:

$$\begin{aligned}
\mathbb{B}(a,b) \cong D^{[P_s]} &= \mathfrak{S}_s \times \mathrm{Lie}(V_s) \times D_{1,h}(P_s) \\
&\cong \mathfrak{S}_s \times \mathrm{Hom}(E_q^s \oplus \check{E}_q^s, \check{E}_s) \times \mathrm{Hom}(F_q, \check{E}_s) \times \mathfrak{S}_q^s \times \mathrm{Lie}(V_q^s) \times D_{1,h}(P_q); \\
\mathbb{B}(a,b) \cong D^{[P_q]} &= \mathfrak{S}_q \times \mathrm{Lie}(V_q) \times D_{1,h}(P_q) \\
&\cong \mathfrak{S}_s \times \mathfrak{S}_q^s \times \mathrm{Hom}(E_q^s \oplus \check{E}_q^s, \check{E}_s) \times \mathrm{Hom}(F_q, \check{E}_s) \times \mathrm{Lie}(V_q^s) \times D_{1,h}(P_q).
\end{aligned}$$

The second products are isomorphic to the first but do not coincide identically.

The first of these comes from (29) and (30), the second from (31) and (32).

Before we use these identifications they must be modified so as to become holomorphic fibrations as in (21). This is straightforward but we need to fix some notation.

If	$\sigma_q \in \mathfrak{S}_q$	we write	$\sigma_q = (\sigma_s, \sigma_q^s, \lambda_{21} \oplus \xi_{21})$;
If	$z_s \in D_{1,h}(P_s)$	we write	$z_s = (\sigma_q^s + i\mathbf{h}_{z_q}(v_q^s, v_q^s), v_q^s, z_q)$;
If	$v_s \in \mathrm{Lie}(V_s)$	we write	$v_s = \big(v_s(\mu_1 \oplus 0 \oplus \mu_1'), v_s(0 \oplus \mu_2 \oplus 0)\big) + \mathrm{Lie}(U_s)$;
If	$v_q \in \mathrm{Lie}(V_q)$	we write	$v_q = \big((v_q(\mu_0 \oplus 0), v_q(0 \oplus \mu_2)\big) + \mathrm{Lie}(U_q)$.

Corollary 15. *There is a holomorphic map* $\widetilde{\mu}_q^s \colon D^{(P_s)} \to D^{(P_q)}$*, given by*

$$\widetilde{\mu}_q^s(\sigma_s + i\mathbf{h}_{z_s}(v_s, v_s), v_s, z_s) = (\sigma_q + i\mathbf{h}_{z_q}(v_q, v_q), v_q, z_q).$$

with $\lambda_{21} = \mu_1$ *and* $\xi_{21} = -\mu_1'$*. It descends to a holomorphic map*

$$\mu_q^s \colon \Upsilon_s \backslash D^{(P_s)} \to \Upsilon_q \backslash D^{(P_q)},$$

and μ_q^s *extends to* $\mu_q^s \colon Z_{\Sigma(P_s)} \to Z_{\Sigma(P_q)}$.

6.2 Toroidal compactification

From Corollary 15 it immediately follows that if for some $g_1,\ g_2 \in G = \mathrm{U}(a,b)$ and $1 \le s < q \le a = \operatorname{rank} \mathbb{B}(a,b)$ we have $g_1(E_s) \subset g_2(E_q)$, then, writing P_1 and P_2 for the corresponding stabilisers, there is a holomorphic map $\mu_{P_2}^{P_1} \colon (\Gamma \cap U_{P_1}) \backslash D^{(P_1)} \to (\Gamma \cap U_{P_2}) \backslash D^{(P_2)}$ with a holomorphic extension $\mu_{P_2}^{P_1} \colon Z_{\Sigma(P_1)} \to Z_{\Sigma(P_2)}$.

Let Γ be a lattice of G and let Υ be the subgroup of Γ generated by $\Upsilon_P = \Gamma \cap U_P$ as P runs through the set $\mathrm{MPar}(\Gamma)$ all the Γ-rational maximal parabolic subgroups. Let Γ_0 be a normal subgroup of Γ, containing Υ. (In almost all cases one takes $\Gamma_0 = \Gamma$.)

Theorem 16. *If* Σ *is* Γ*-admissible then there exists a holomorphic equivalence relation* $\sim_{\Gamma_0}$ *and a complex analytic variety, the* toroidal compactification*,*

$$(\Gamma_0 \backslash \mathbb{B}(a,b))_\Sigma = \left(\coprod_{P \in \mathrm{MPar}(\Gamma)} Z_{\Sigma(P)} \right) / \sim_{\Gamma_0}$$

containing $\Gamma_0\backslash\mathbb{B}(a,b)$ *as an open dense subvariety. If* $\Gamma_0 = \Gamma$ *then* $(\Gamma_0\backslash\mathbb{B}(a,b))_\Sigma$ *is compact.*

This is a very mild generalisation of the construction in [AMRT], in that we allow some subgroups of Γ of infinite index as long as the index is finite in the centres of the unipotent radicals of the parabolic subgroups, so that the toric construction is still available. For general such Γ_0 the varieties obtained may be viewed as non-Noetherian schemes or as complex analytic spaces, but are not algebraic varieties. This extension was used in [San] to construct universal covers of some toroidal compactifications, but the idea is already implicit in [Oda] and some of the references there.

Here is the definition of the equivalence relation $\sim_{\Gamma_0}$, taken directly from [AMRT]. If z_1 and z_2 belong to the disjoint union above, then for $i = 1, 2$ there exist $g_i \in G$ and integers q_i with $1 \leq q_i \leq a$ such that $P_i = \mathrm{Stab}(g_i(E_{q_i})) \in \mathrm{MPar}(\Gamma)$. In that case $z_1 \sim_{\Gamma_0} z_2$ if and only if there exist an element $\gamma \in \Gamma_0$, a maximal parabolic subgroup $P = \mathrm{Stab}_G(g(E_s)) \in \mathrm{MPar}(\Gamma)$ and a point $z \in Z_{\Sigma(P)}$, such that $g(E_s) \subseteq g_1(E_{q_1})$ and $g(E_s) \subseteq \gamma g_2(E_{q_2})$, and $\mu^P_{P_1}(z) = z_1$, and $\mu^P_{\gamma P_2 \gamma^{-1}}(z) = \gamma z_2$.

If $a = \mathrm{rank}\,\mathbb{B}(a,b) \geq 2$ then by a result of Margulis, any lattice Γ of $G = \mathrm{U}(a,b)$ is arithmetic. It is shown in [AMRT] that under these circumstances $(\Gamma\backslash\mathbb{B}(a,b))_\Sigma$ is compact (and Σ may be chosen so as to ensure that it is a projective algebraic variety), containing $\Gamma\backslash\mathbb{B}(a,b)$ as an open dense subset (with respect to the analytic topology). If $\mathrm{rank}\,\mathbb{B}(a,b) = a = 1$ there are non-arithmetic lattices Γ of $\mathrm{U}(1,b)$, but the same results hold [Mok] even in this case.

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