RATIONALITY AND ARITHMETIC OF THE MODULI OF ABELIAN VARIETIES

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Abstract. We study the rationality properties of the moduli space $\mathcal{A}_g$ of principally polarised abelian g-folds over $\mathbb{Q}$ and apply the results to arithmetic questions. In particular we show that any principally polarised abelian threefold over $\mathbb{F}_p$ may be lifted to an abelian variety over $\mathbb{Q}$. This is a phenomenon of low dimension: assuming the Bombieri-Lang conjecture we also show that this is not the case for abelian varieties of dimension at least seven. About moduli spaces, we show that $\mathcal{A}_g$ is unirational over $\mathbb{Q}$ for $g \leq 5$ and stably rational for $g = 3$. This also allows us to make unconditional one of the results of Masser and Zannier about the existence of abelian varieties over $\mathbb{Q}$ that are not isogenous to Jacobians.

Contents

1. Introduction

Arithmetic properties of abelian varieties are strongly linked to the geometry of their moduli spaces. Here we study both the birational geometry over $\mathbb{Q}$ of the coarse moduli space $\mathcal{A}_g$ of principally polarised abelian varieties (ppavs), and lifting and other arithmetic properties of ppavs themselves.

1.1. **Lifting abelian varieties.** For any prime p and any elliptic curve E_p over $\mathbb{F}_p$, there exists an elliptic curve E over $\mathbb{Q}$ such that E_p is the reduction modulo p of E (we say that E is a *lift* of E_p to $\mathbb{Q}$). Indeed, one simply takes a suitable lift of the coefficients of E_p.

For higher dimensional abelian varieties the problem becomes more interesting. Firstly we at least need that the abelian variety A_p lifts from $\mathbb{F}_p$ to $\mathbb{Q}_p$; to guarantee this we assume that A_p is equipped with a principal polarisation [Oor, Cor. 2.4.2]. It is not too difficult to see that principally polarised abelian surfaces always lift to $\mathbb{Q}$ as the generic such surface is the Jacobian of a hyperelliptic curve (see §4.3). One of

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our first results is that lifting can be achieved in dimension three, and moreover for finitely many primes simultaneously.

For an abelian variety A over $\mathbb{Q}$ and a prime p, we denote by $A_{\mathbb{F}_p}$ the reduction modulo p of the Néron model of A.

Theorem 1.1. *Let S be a finite set of rational primes, and for each $p \in S$ fix A_p, a principally polarised abelian 3-fold over $\mathbb{F}_p$. Then there exists a principally polarised abelian 3-fold A over $\mathbb{Q}$ such that $A_p \cong A_{\mathbb{F}_p}$ for all $p \in S$ as principally polarised abelian varieties.*

This property should not hold in higher dimension.

Theorem 1.2. *Let $g \geq 7$ and assume the Bombieri–Lang conjecture. Then for all but finitely many primes p, there exists a principally polarised abelian g-fold A_p over $\mathbb{F}_p$ such that $A_p \ncong A_{\mathbb{F}_p}$ for any principally polarised abelian g-folds A over $\mathbb{Q}$.*

1.2. **Rationality properties of moduli spaces.** We achieve our arithmetic results through a consideration of the birational geometry of the coarse moduli space $\mathcal{A}_g$ of principally polarised abelian g-folds over $\mathbb{Q}$. This has been much studied over the complex numbers. It is known that $\mathcal{A}_g$ unirational over $\mathbb{C}$ for $g \leq 5$, and rational over $\mathbb{C}$ for $g \leq 3$. On the other hand, if $g \geq 7$ then $\mathcal{A}_g$ is of general type [Tai], and $\mathcal{A}_6$ has non-negative Kodaira dimension [DSMS]. Questions about the rationality of $\mathcal{A}_g$ over non-closed fields seem by contrast to have had little attention, and we address some of them here, over $\mathbb{Q}$.

We first consider unirationality and, incorporating previous results, obtain the following theorem.

Theorem 1.3. *The coarse moduli space $\mathcal{A}_g$ of principally polarised abelian varieties is unirational over $\mathbb{Q}$ if and only if $g \leq 5$.*

This result has motivation from the recent paper [MZ], in which Masser and Zannier prove various results on the existence of abelian varieties over $\overline{\mathbb{Q}}$ that are not isogenous to Jacobians. Some of their results also hold for abelian varieties over $\mathbb{Q}$, but for $g = 4$ and $g = 5$ that refinement is conditional on the unirationality of $\mathcal{A}_g$ over $\mathbb{Q}$.

From this and [MZ, Thm 1.5, Cor 1.6] we obtain the following immediate application, which is new for $g = 4, 5$ (see [MZ, Thm 1.5] for a stronger statement).

Corollary 1.4. *Let k be a number field. For $g = 4, 5$, there exists a principally polarised abelian g-fold over k that is Hodge generic and not isogenous to any Jacobian.*

We prove Theorem 1.3 by showing that the Prym moduli space $\mathcal{R}_{g+1}$ is unirational for $g = 4, 5$. This is sufficient to get unirationality of $\mathcal{A}_g$, as the Prym map $\mathcal{R}_{g+1} \to \mathcal{A}_g$ is dominant for $g \leq 5$ over any field of characteristic not equal to 2 [Bea, Thm. 6.5].

For deeper rationality properties, it is known that $\mathcal{A}_2$ is rational over any field, by work of Igusa [Igu]. The rationality of $\mathcal{A}_3$ over $\mathbb{C}$ was first proven by Katsylo [Kat] (see also Böhning's exposition [Böh]). It seems possible that this result could also hold over $\mathbb{Q}$; however Katsylo's proof is notoriously delicate and technical. We content ourselves with the following weaker statement, which has a much simpler proof and is sufficient for arithmetic applications.

Theorem 1.5. *The coarse moduli space $\mathcal{A}_3$ is stably rational over $\mathbb{Q}$.*

1.3. **Weak approximation for algebraic stacks.** Theorem 1.5 is the key geometric input for the proof of Theorem 1.1. Namely, one can interpret Theorem 1.1 as a version of *weak approximation* for the moduli stack $\mathscr{A}_3$ of principally polarised abelian threefolds. Recall that a smooth variety X over a number field k is said to satisfy *weak approximation* if $X(k)$ is dense in $\prod_v X(k_v)$ where the product is over all places v of k. We prove a version of this for the stack $\mathscr{A}_3$ (see §4 for definitions for stacks).

Theorem 1.6. *The stack $\mathscr{A}_3$ satisfies weak approximation over any number field k.*

The classical weak approximation theorem implies that rational varieties satisfy weak approximation, and a fibration argument extends this to stably rational varieties. However Theorem 1.5 does not immediately imply Theorem 1.6; indeed there are algebraic stacks that fail weak approximation but whose coarse moduli space satisfies weak approximation (see Example 4.8). To exclude this possibility for $\mathscr{A}_3$ one needs to be careful with twists. Theorem 1.1 is then an application of this weak approximation result.

1.4. **Questions.** Our results raise the following, to which we do not know the answer.

Question 1.7. Do $\mathcal{A}_4$ and $\mathcal{A}_5$ satisfy weak weak approximation (i.e. weak approximation away from a finite set of places)?

Standard conjectures in arithmetic geometry, together with the unirationality from Theorem 1.3, would imply a positive answer to Question 1.7. This in turn would give a version of Theorem 1.6 for $\mathscr{A}_5$, and a version of Theorem 1.1 for $g = 5$ away from finitely many primes. The case of $\mathscr{A}_4$ is less clear as here the generic gerbe is non-neutral (see Lemma 4.11).

Question 1.8. Can one prove an unconditional version of Theorem 1.2?

Conventions. We denote by $\mathscr{A}_g$ the moduli stack of principally polarised abelian g-folds over $\mathbb{Z}$ and by $\mathcal{A}_g$ its coarse moduli space. The stack $\mathscr{A}_g$ is smooth over $\mathbb{Z}$ [Oor, Thm. 2.4.1]. We sometimes abuse notation and also denote by $\mathscr{A}_g$ the base change of the stack to some field, which will be clear from the context.

For a group scheme G we denote by BG the associated classifying stack. Regarding gerbes, we use the conventions of [Ols, Ch. 12]. If G is an abelian group scheme, we say that a G-gerbe is neutral if it has a section; this is equivalent to being isomorphic to BG.

For an algebraic stack $\mathscr{X}$ and a scheme S, we abuse notation and denote by $\mathscr{X}(S)$ the *set* of isomorphism classes of S-points of $\mathscr{X}$ (rather than the groupoid).

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2. Background on moduli

In this preliminary section we outline the known results over the complex numbers and make some remarks about certain moduli spaces associated with curves, which we shall use as auxiliaries.

Let k be a field. Recall that a variety X over k is called *rational* (resp. *unirational*) if it admits a birational (resp. dominant rational) map from a projective space. It is called *stably rational* if $X \times \mathbb{P}^n$ is rational for some n.

2.1. **Known rationality results.** The coarse moduli spaces $\mathcal{A}_g$ have been much studied over the complex numbers from a birational point of view. With the exception of the case $g = 6$, the broad picture remains as described in [HS], to which we refer for more details.

2.1.1. *$g = 2$.* Igusa [Igu, Thm. 5] showed that $\mathcal{A}_2 = \mathcal{M}_2$ is rational over any field, and there are numerous results on the moduli space for abelian surfaces with non-principal polarisations or level structures.

2.1.2. *$g = 3$.* It is easy to see that $\mathcal{A}_3$ is unirational over $\mathbb{C}$. Katsylo [Kat] showed that $\mathcal{M}_3$ (and hence $\mathcal{A}_3$, by Torelli) is rational over $\mathbb{C}$: we discuss this in §2.2 below.

2.1.3. *$g = 4$.* The first proof that $\mathcal{A}_4$ is unirational over $\mathbb{C}$ was given by Clemens [Cle], using intermediate Jacobians. Other proofs were subsequently given by Verra [Ver1] and by Izadi, Lo Giudice and the second author [ILS]. Of these, the proof in [ILS], which uses the moduli space of Prym curves (see §2.3 below), seems the easiest to adapt to non-closed fields.

2.1.4. *$g = 5$.* The first proofs that $\mathcal{A}_5$ is unirational over $\mathbb{C}$ were given by Donagi [Don] and by Mori and Mukai [MM]. There is a different proof in [Ver2] and a more recent one by Farkas and Verra [FV]. All of these use Prym curves. We found the argument in [FV] the easiest to adapt for our purpose.

2.1.5. *$g \geq 6$.* The spaces $\mathcal{A}_g$ are non-rational for $g > 5$, at least in characteristic zero. Tai [Tai], building on earlier work of Mumford and of Freitag, showed that $\mathcal{A}_g$ is of general type for $g \geq 7$. The case of $\mathcal{A}_6$ remained completely mysterious until 2020, when Dittman, Salvati Manni, and Scheithauer [DSMS] showed that the second plurigenus is positive: thus the Kodaira dimension is non-negative.

2.2. **Moduli of curves.** In view of Katsylo's result, it is natural to ask whether $\mathcal{A}_3$ is rational over $\mathbb{Q}$. What Katsylo proves directly, however, is that $\mathcal{M}_3$ is rational (over $\mathbb{C}$), and the wider context into which the proof naturally fits is the moduli of curves rather than of abelian varieties. Katsylo's proof is one of many rationality proofs for moduli spaces of curves: for some examples, see [Ver3]. The main tool for many of these is classical invariant theory. Few of them are written with much attention to fields of definition. Shepherd-Barron's proof [S-B] that $\mathcal{M}_6$ is rational over $\mathbb{Q}$ is an exception. In fact, Katsylo's proof for $\mathcal{M}_3$ is among the most complicated of these arguments.

2.3. **Prym curves.** A Prym curve (of genus g) over a scheme S is a pair (D, C) where C is a smooth proper scheme over S whose fibres are smooth projective geometrically integral curves of genus g, and $D \to C$ is a finite étale morphism of degree 2 whose fibres over S are geometrically integral. We denote by $\mathscr{R}_g$ the moduli stack of Prym curves of genus g and by $\mathcal{R}_g$ its coarse moduli space.

An explicit construction of this stack over $\mathbb{Z}[1/2]$ can be found in the proof of [Bea, Thm. 6.5], which also constructs a morphism (the Prym map) $\mathscr{R}_g \to \mathscr{A}_{g-1}$ of stacks over $\mathbb{Z}[1/2]$. Moreover, $\mathscr{R}_g \to \mathscr{A}_{g-1}$ is dominant for $g \leq 6$, by [Bea, Lem. 6.5.2].

3. Rationality results

3.1. **Stable rationality of moduli of hypersurfaces.** We will prove that $\mathcal{A}_3$ is stably rational by showing that the moduli of plane quartic curves is stably rational. Our argument is sufficiently robust that it also works for other moduli of hypersurfaces. Our result is as follows.

Theorem 3.1. *Let $d, n \geq 2$ with $(d, n) \neq (3, 2)$. Let k be a field and $\mathcal{H}_{d,n}$ denote the Hilbert scheme of hypersurfaces of degree d in $\mathbb{P}^n$. The group* PGL_{n+1} *acts on $\mathcal{H}_{d,n}$ in a natural way via linear change of variables. Assume that* $\mathrm{char}(k)$ *does not divide $n+1$ and that* $\gcd(d, n+1) = 1$. *Then $\mathcal{H}_{d,n}/\mathrm{PGL}_{n+1}$ is stably rational.*

Proof. The case $d = 2$ is classical so we assume $d > 2$. Let $V_{d,n} = \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ be the vector space of forms of degree d in $(n+1)$ variables. The group SL_{n+1} acts on $V_{d,n}$ in a natural way via linear change of variables. The coprimality conditions ensure that the central copy of $\mu_{n+1} \subset \mathrm{SL}_{n+1}$ acts faithfully on $V_{d,n}$. Moreover the induced action of SL_{n+1} on $\mathcal{H}_{d,n}$ factors through the $\mathrm{SL}_{n+1}/\mu_{n+1} = \mathrm{PGL}_{n+1}$-action. We obtain the commutative diagram

$$\begin{tikzcd}
V_{d,n} \setminus \{0\} \arrow[r] \arrow[d] & \mathcal{H}_{d,n} \arrow[d] \\
(V_{d,n} \setminus \{0\})/\mathrm{SL}_{n+1} \arrow[r] & \mathcal{H}_{d,n}/\mathrm{PGL}_{n+1} .
\end{tikzcd}$$

The action of SL_{n+1} on $V_{d,n}$ is generically free: indeed μ_{n+1} acts faithfully and the action of PGL_{n+1} on $\mathcal{H}_{d,n}$ is generically free as there exist hypersurfaces of degree d with trivial linear automorphism group [Poo1]. We conclude that the generic fibre of the left-hand map is an SL_{n+1}-torsor. But such a torsor is necessary trivial as $\mathrm{H}^1(k, \mathrm{SL}_{n+1}) = 0$ for any field k [PR, Lem. 2.3]. We conclude that $V_{d,n}$ is birational to $\mathrm{SL}_{n+1} \times (V_{d,n}/\mathrm{SL}_{n+1})$.

However, SL_{n+1} is a split reductive group, and thus rational. To see this (well-known) fact, note that a Borel subgroup B has an opposite B' such that $B \cap B'$ is a (split) maximal torus. The product $B \cdot R_u(B')$, the big cell in the Bruhat decomposition, is then rational as it is the product of copies of $\mathbb{A}^1$ and $\mathbb{A}^1 \setminus \{0\}$: see [Mil, pp. 98–99]. Thus $V_{d,n}/\mathrm{SL}_{n+1}$ is stably rational.

Next, the top arrow in the diagram is a $\mathbb{G}_\mathrm{m}$-torsor. The bottom arrow is thus a $\mathbb{G}_\mathrm{m}/\mu_{n+1} = \mathbb{G}_\mathrm{m}$-torsor (here the map $\mu_{n+1} \to \mathbb{G}_\mathrm{m}$ is the one induced by the SL_{n+1}-action and not necessarily the standard embedding). By Hilbert's Theorem 90 the generic fibre is the trivial torsor: thus we see that $\mathcal{H}_{d,n}/\operatorname{PGL}_{n+1}$ is stably birational to $V_{d,n}/\operatorname{SL}_{n+1}$, which we already proved is stably rational. $\square$

3.2. **Abelian 3-folds.** Let k be a field of characteristic not equal to 3.

The rational maps $\mathcal{H}_{4,2}/\operatorname{PGL}_3 \dashrightarrow \mathcal{M}_3 \dashrightarrow \mathcal{A}_3$ over k are birational over $\overline{k}$ (this follows from Torelli and a dimension count) and hence birational over k. Therefore $\mathcal{A}_3$ is stably rational over k by Theorem 3.1. This is sufficient for Theorem 1.5.

3.3. **Abelian 4-folds.** Let k be a field of characteristic 0. We prove the $g = 4$ case of Theorem 1.3.

For the following result we closely follow [ILS], which proves the analogous result over $\mathbb{C}$, but a few modifications are required. The main difference is that in [ILS], the authors work with certain quartic surfaces, and then pass to the double covers of $\mathbb{P}^3$ ramified along the quartic surface. Some care is needed over non-algebraically closed fields where there may be many such covers given by quadratic twists.

Theorem 3.2. *The coarse moduli space $\mathcal{R}_5$ of étale double covers of a genus 5 curve is unirational over k. Hence $\mathcal{A}_4$ is unirational over k.*

In the rest of this section, we prove the first part of this: the second part follows since the Pyrm map $\mathcal{R}_5 \to \mathcal{A}_4$ is dominant and defined over k.

Fix five points $P_1, \dots, P_5 \in \mathbb{P}^3(k)$ in linear general position. Without loss of generality $P_1 = (0 : 0 : 0 : 1)$. We consider the space $\mathcal{Q}' \subset \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(4))$ of quartic forms F in four variables such that the associated quartic surface $X := \{x \in \mathbb{P}^4 \mid F(x) = 0\}$ has exactly six ordinary double points over the algebraic closure, at least five of which are $P_1, \dots, P_5$. This is a quasi-affine variety defined over k. Moreover, if F is defined over k then necessarily the sixth double point is also defined over k. The key result is as follows.

Lemma 3.3. *The space $\mathcal{Q}'$ is geometrically irreducible and unirational.*

Proof. We follow [ILS, Prop. 2.1], which proves unirationality over the algebraic closure (our $\mathcal{Q}'$ is the affine cone over the $\mathcal{Q}$ appearing in [ILS]). The space of quartic polynomials F whose associated quartic surface has at least five double points at $P_1, \dots, P_5$ is a vector space which we denote by V. For the sixth double point, consider the scheme

$$B_0 := \{(F, P_0) \in V \times \mathbb{P}^3 \mid F(P_0) = \partial F/\partial x_0(P_0) = \cdots = \partial F/\partial x_3(P_0) = 0\}.$$

The projection onto $\mathbb{P}^3$ is surjective and the fibres are vector spaces (not just affine spaces: there is a section, the zero section). It follows that there is a dense open $U \subset \mathbb{P}^3$ over which B_0 becomes isomorphic to $U \times \mathbb{A}^n$ for some n, and therefore B_0 is rational.

The other projection is a map $B_0 \to V$ whose image contains $\mathcal{Q}'$ as a dense open subset. As B_0 is rational, we see that $\mathcal{Q}'$ is unirational. $\square$

There is a natural map $\varrho \colon \mathcal{Q}' \longrightarrow \mathcal{R}_5$ defined in the following way [ILS, §1]. For $F \in \mathcal{Q}'$, let X be the associated quartic surface and

$$\Lambda_X := \{y^2 = F(x)\} \subset \mathbb{P}(1, 1, 1, 1, 2)$$

the associated quartic double solid. Let W_X be the blow-up of Λ_X at $(0:0:0:1:0)$. Then composing the blow-up map with the natural projection to $\mathbb{P}^2$ yields a morphism $f\colon W_X \to \mathbb{P}^2$. This is a conic bundle morphism whose non-smooth locus $C_X \subset \mathbb{P}^2$ is a plane sextic curve whose singular points are exactly the images of the singular points of X (see [ILS, Prop. 1.5, 1.6]). Thus if F is general, C_X has five ordinary double points, so its normalisation $\widetilde{C}_X$ has genus 5.

Next let $S = W_X \times_{\mathbb{P}^2} \widetilde{C}_X$ and let $\widetilde{S}$ be the normalisation of S. We then apply Stein factorisation to the induced map $\widetilde{S} \to \widetilde{C}_X$ to obtain a finite morphism $\Gamma_F \to \widetilde{C}_X$, which by [ILS, Prop. 1.8] is a geometrically connected finite étale cover of degree 2. Thus the pair $(\widetilde{C}_X, \Gamma_F)$ gives a well-defined element of $\mathcal{R}_5$. All these constructions are natural, so we define the morphism ϱ by $\varrho(F) = (\tilde{C}_X, \Gamma_F)$.

To show that $\mathcal{R}_5$ is unirational, we note from [ILS, Cor. 3.3] that $\mathcal{Q}' \to \mathcal{R}_5$ is dominant over $\overline{k}$, hence it is dominant over k, whence $\mathcal{R}_5$ is unirational by Lemma 3.3. As $\mathcal{R}_5 \to \mathcal{A}_4$ is dominant, $\mathcal{A}_4$ is also unirational. □

3.4. **Abelian 5-folds.** Let k be a field of characteristic 0. We prove the $g = 5$ case of Theorem 1.3.

We proceed as in §3.3, this time following [FV, §1], which proves the analogous result over $\mathbb{C}$. This time the method extends to k with little difficulty.

Theorem 3.4. *The coarse moduli space $\mathcal{R}_6$ of étale double covers of a genus 6 curve is unirational over k. Hence $\mathcal{A}_5$ is unirational over k.*

Pick four points $O_1, \dots, O_4 \in \mathbb{P}^2(k)$ in linear general position and let $P_i = (O_i, O_i) \in \mathbb{P}^2 \times \mathbb{P}^2$. Consider the linear system $\mathbb{P}^{15}$ of hypersurfaces of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ that have ordinary double points at P_1, P_2, P_3, P_4. For such a threefold X, projecting onto the first factor $\mathbb{P}^2$ induces a conic bundle morphism $X \to \mathbb{P}^2$. The discriminant is a sextic curve C_X whose singularities are generically the image of the singularities of X [FV, Prop. 1.2]. Thus for general X there are four nodes, so the normalisation $\widetilde{C}_X$ has genus 6. Moreover, applying Stein factorisation to the singular locus as in §3.3, we obtain a geometrically connected étale double cover $\Gamma_X \to \widetilde{C}_X$. This constructs a rational map $\mathbb{P}^{15} \dashrightarrow \mathcal{R}_6$ over k. By [FV, Thm. 1.4] this map is dominant, and hence $\mathcal{R}_6$ is unirational. Again $\mathcal{R}_6 \to \mathcal{A}_5$ is dominant, and thus $\mathcal{A}_5$ is also unirational. □

Combining the results of this section with the results from §2.1 completes the proof of Theorem 1.3. □

4. Weak approximation

4.1. **k_v-points of a stack.** Let k be a number field and v a place of k. For finitely presented algebraic stacks $\mathscr{X}$ over k_v, Christensen [Chr, §5] gave a topology on $\mathscr{X}(k_v)$, extending the v-adic topology on the usual k_v-points of schemes. It has the following properties:

(1) Any morphism of stacks over k_v induces a continuous map on k_v-points [Chr, Thm. 9.0.3].

(2) Any smooth morphism of stacks over k_v induces an open map on k_v-points [Chr, Thm. 11.0.4].

The topology is unique because Christensen proves in [Chr, Thm. 7.0.7] that any k_v-point of an algebraic stack is the image of a k_v-point under a smooth morphism from some scheme.

The following corresponds to the well-known fact that a non-empty v-adic open subset of a smooth irreducible scheme is Zariski dense, which is an application of the implicit function theorem.

Lemma 4.1. *Let $\mathscr{X}$ be a smooth finitely presented irreducible algebraic stack over k_v and let $W \subseteq \mathscr{X}(k_v)$ be non-empty and open in the v-adic sense above. Then W is dense in $\mathscr{X}$.*

Proof. Let $f\colon Z \to \mathscr{X}$ be a smooth morphism from a finitely presented irreducible scheme such that $f(Z(k_v)) \cap W \neq \varnothing$; this exists by [Chr, Thm 7.0.7]. Then $f^{-1}(W)$ is a non-empty v-adic open set, so it is dense in Z as Z is a scheme. However both $Z \to \mathscr{X}$ and $Z(k_v) \to \mathscr{X}(k_v)$ are open and continuous. Thus $f(Z)$ is dense in $\mathscr{X}$ and $f(f^{-1}(W))$ is dense in $f(Z)$, and the result easily follows. $\square$

We require the following version of Hensel's lemma for algebraic stacks.

Lemma 4.2. *Let R be a complete noetherian local ring, or an excellent Henselian discrete valuation ring, with maximal ideal $\mathfrak{m}$. Let $\mathscr{X}$ be a smooth algebraic stack over R. Then for all $n \in \mathbb{N}$ the natural map*

$$\mathscr{X}(R) \longrightarrow \mathscr{X}(R/\mathfrak{m}^n)$$

is surjective.

Proof. Recall that smooth means formally smooth and locally of finite presentation. Formal smoothness (see [SP, Tag 0DNV]) implies for any $n \in \mathbb{N}$ and any 1-commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} R/\mathfrak{m}^n & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} R/\mathfrak{m}^{n+1} & \longrightarrow & \operatorname{Spec} R \end{array}$$

there exists a diagonal arrow making the diagram 2-commutative. We conclude that $\mathscr{X}(R/\mathfrak{m}^{n+1}) \to \mathscr{X}(R/\mathfrak{m}^n)$ is surjective for all $n \in \mathbb{N}$, and it follows that $\lim_i \mathscr{X}(R/\mathfrak{m}^{n+i}) \to \mathscr{X}(R/\mathfrak{m}^n)$ is surjective. By effectivity of formal objects [SP, Lem. 98.9.5, Tag 07X3] we have $\lim_i \mathscr{X}(R/\mathfrak{m}^{n+i}) = \mathscr{X}(\widehat{R})$, where $\widehat{R}$ denotes the completion of R. Thus if R is complete we are done. Otherwise, Artin approximation [Art, Thm 1.12], applied to the functor given by the isomorphism classes of objects of $\mathscr{X}$, shows that the composition $\mathscr{X}(R) \to \mathscr{X}(\widehat{R}) \to \mathscr{X}(R/\mathfrak{m}^n)$ is surjective for any n, as required. $\square$

4.2. **Weak approximation.**

Definition 4.3. We say that a finitely presented algebraic stack $\mathscr{X}$ over k satisfies *weak approximation* if the natural map $\mathscr{X}^{\mathrm{sm}}(k) \to \prod_v \mathscr{X}^{\mathrm{sm}}(k_v)$ has dense image. (Here $\mathscr{X}^{\mathrm{sm}}$ denotes the smooth locus of $\mathscr{X}$.)

Note that, unlike the case of varieties, the map $\mathscr{X}(k) \to \prod_v \mathscr{X}(k_v)$ need not be injective (its injectivity is related to triviality of various Tate–Shafarevich sets). Nevertheless we will sometimes abuse notation and use $\mathscr{X}(k)$ also to denote the image of this map, for instance in Lemma 4.10.

Lemma 4.4. *$\mathscr{X}$ satisfies weak approximation if and only if either $\prod_v \mathscr{X}^{\mathrm{sm}}(k_v)$ is empty or, for any finite set S of places of k and for any non-empty open subset $W \subseteq \prod_{v\in S} \mathscr{X}^{\mathrm{sm}}(k_v)$, there exists an element of $\mathscr{X}(k)$ whose images lies in W.*

Proof. Follows immediately from the definition. □

A rational map $f\colon \mathscr{X}_1 \dashrightarrow \mathscr{X}_2$ of finitely presented algebraic stacks is called *birational* if there exist dense open substacks $\mathscr{U}_i \subset \mathscr{X}_i$ such that f is defined on $\mathscr{U}_1$, $f(\mathscr{U}_1) \subseteq \mathscr{U}_2$, and $f|_{\mathscr{U}_1}\colon \mathscr{U}_1 \to \mathscr{U}_2$ is an isomorphism. If such a rational map exists, we say that $\mathscr{X}_1$ and $\mathscr{X}_2$ are *birationally equivalent*.

Weak approximation for smooth stacks is a birationally invariant property.

Lemma 4.5. *Let $\mathscr{X}$ and $\mathscr{Y}$ be birationally equivalent smooth irreducible algebraic stacks over k. Then $\mathscr{X}$ satisfies weak approximation if and only if $\mathscr{Y}$ does.*

Proof. It suffices to prove the result in the case where $\mathscr{X} \to \mathscr{Y}$ is an open immersion. Weak approximation for $\mathscr{Y}$ implies weak approximation for $\mathscr{X}$ as a special case of Lemma 4.6 below. On the other hand, assume that $\mathscr{X}$ satisfies weak approximation. If $\prod_v \mathscr{Y}(k_v) = \varnothing$, there is nothing to prove. Otherwise, let $W \subseteq \prod_{v\in S} \mathscr{Y}(k_v)$ as in Lemma 4.4. It suffices to show that $W \cap \mathscr{X} \neq \varnothing$. However this is Lemma 4.1 applied to $\mathscr{Y}$. □

In particular if $\mathscr{X}$ admits an open dense substack U that is isomorphic to a scheme, then weak approximation for $\mathscr{X}$ is equivalent to weak approximation for the scheme U, and the definition does not offer anything new. Therefore to get interesting new problems in general one should consider stacks with non-trivial generic stabilisers; such stacks typically admit open substacks that are gerbes over a scheme. In such cases one may hope to prove weak approximation using the following fibration result, which is a stacky version of [C-TG, Prop. 1.1].

Lemma 4.6. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a smooth morphism of smooth irreducible algebraic stacks over k. Assume that $\mathscr{Y}$ satisfies weak approximation and that the fibre over every rational point of $\mathscr{Y}$ is everywhere locally soluble and satisfies weak approximation. Then $\mathscr{X}$ satisfies weak approximation.*

Proof. If $\prod_v \mathscr{X}(k_v) = \varnothing$, there is nothing to prove. Otherwise let $W \subseteq \prod_{v\in S} \mathscr{X}(k_v)$ as in Lemma 4.4. As f is smooth the image $f(W) \subseteq \prod_{v\in S} \mathscr{Y}(k_v)$ is open. So let $y \in \mathscr{Y}(k)$ with image in $f(W)$. Then $f^{-1}(y) \cap W \neq \varnothing$ and $f^{-1}(y)$ is everywhere locally soluble and satisfies weak approximation, and thus $f^{-1}(y) \cap W$ contains a rational point, as required. □

For neutral affine gerbes, weak approximation is equivalent to a statement in Galois cohomology.

Lemma 4.7. *Let G be a finite type affine group scheme over k. Then BG satisfies weak approximation if and only if the natural map*

$$\mathrm{H}^1(k, G) \longrightarrow \prod_{v \in S} \mathrm{H}^1(k_v, G)$$

is surjective for all finite sets of places S of k.

Proof. Recall that for any field extension $k \subset L$ we have $BG(L) = \mathrm{H}^1(L, G)$, since both sets classify G_L torsors over L. The set $\mathrm{H}^1(k_v, G)$ is finite [PR, Thm. 6.14] and the induced topology is simply the discrete topology. Therefore it suffices to note that a subset of a product of discrete sets is dense if and only if it surjects onto any finite collection of factors. □

We emphasise that for a general algebraic stack $\mathscr{X}$ over $\mathbb{Q}$ it can happen that its coarse moduli space $\mathcal{X}$ satisfies weak approximation but $\mathscr{X}$ itself does not.

Example 4.8. Take $\mathscr{X} = B\mathbb{Z}/8\mathbb{Z}$. Then the associated coarse moduli space is just $\operatorname{Spec} \mathbb{Q}$, which trivially satisfies weak approximation. But $B\mathbb{Z}/8\mathbb{Z}$ fails weak approximation (the famous example of Wang [Wang]): there is no $\mathbb{Z}/8\mathbb{Z}$-extension of $\mathbb{Q}$ that realises the unique unramified $\mathbb{Z}/8\mathbb{Z}$-extension of $\mathbb{Q}_2$.

For $G = \mu_2$, however, which is the case relevant to us, there is no such problem.

Lemma 4.9. *If $\mu_n \subset k$, then $B\mu_n$ satisfies weak approximation.*

Proof. By Kummer theory we have $\mathrm{H}^1(k, \mu_n) = k^\times / k^{\times n}$, and similarly for k_v. It thus suffices to apply Lemma 4.7 and note that weak approximation for k implies that $k^\times / k^{\times n} \to \prod_{v \in S} k_v^\times / k_v^{\times n}$ is surjective. □

From Hensel's lemma we obtain the following application of weak approximation.

Lemma 4.10. *Let $\mathscr{X}$ be a smooth irreducible finitely presented algebraic stack over k with $\prod_v \mathscr{X}(k_v) \neq \varnothing$ that satisfies weak approximation, and S a finite collection of non-zero prime ideals of k. Let $\mathscr{X}_{\mathcal{O}_k}$ be a model of $\mathscr{X}$ over $\mathcal{O}_k$ that is smooth over all elements of S. Then the map*

$$\mathscr{X}(k) \cap \prod_{\mathfrak{p} \in S} \mathscr{X}_{\mathcal{O}_k}(\mathcal{O}_\mathfrak{p}) \longrightarrow \prod_{\mathfrak{p} \in S} \mathscr{X}_{\mathcal{O}_k}(\mathbb{F}_\mathfrak{p})$$

is surjective.

Proof. We first note that the map $\prod_{\mathfrak{p} \in S} \mathscr{X}_{\mathcal{O}_k}(\mathcal{O}_\mathfrak{p}) \to \prod_{\mathfrak{p} \in S} \mathscr{X}_{\mathcal{O}_k}(\mathbb{F}_\mathfrak{p})$ is surjective for all $\mathfrak{p} \in S$: this follows from Hensel's lemma for stacks, Lemma 4.2. Moreover the map $\mathscr{X}_{\mathcal{O}_k}(\mathcal{O}_\mathfrak{p}) \to \mathscr{X}_{\mathcal{O}_k}(\mathbb{F}_\mathfrak{p})$ is continuous as $\mathcal{O}_\mathfrak{p} \to \mathbb{F}_\mathfrak{p}$ is continuous [Chr, Prop. 9.0.4]. As $\mathscr{X}_{\mathcal{O}_k}(\mathbb{F}_\mathfrak{p})$ is discrete, we find that the fibre of a point is an open subset of $\mathscr{X}(k_\mathfrak{p})$. Lemma 4.4 thus implies that it contains a rational point, as required. □

4.3. **Weak approximation for $\mathscr{A}_2$.** Before turning to $\mathscr{A}_3$, we briefly consider the simpler case of $\mathscr{A}_2$, where weak approximation holds. To see this, we first note that the natural map $\mathscr{M}_2 \to \mathscr{A}_2$ is birational. Indeed the induced map on coarse moduli spaces is birational, and furthermore the map on generic stabilisers is an isomorphism: this is the strong Torelli theorem given (by Serre) in [LS, Théorème 3].

Therefore by Lemma 4.5 it suffices to prove that $\mathscr{M}_2$ satisfies weak approximation. However an element of $\mathscr{M}_2(k_v)$ is represented by a hyperelliptic curve over k_v, and one can approximate this to an arbitrary precision by a hyperelliptic curve over k by simply approximating the coefficients (similarly for any finite collection of places).

A similar proof also shows that $\mathscr{A}_1$ satisfies weak approximation.

4.4. **Weak approximation for $\mathscr{A}_3$.** By Theorem 1.5 we know that $\mathcal{A}_3$ is stably rational over $\mathbb{Q}$, and hence satisfies weak approximation [C-TG, Prop. 1.2].

Let us briefly consider how one would try to use this to prove that $\mathscr{A}_3$ satisfies weak approximation, and some of the subtleties that arise. Let $h\colon \mathscr{A}_3 \to \mathcal{A}_3$ be the coarse moduli map. Let k be a number field and S a finite set of places of k. For $v \in S$ let $A_v \in \mathscr{A}_3(k_v)$. Then there exists a k-rational point $a \in \mathcal{A}_3(k)$ arbitrarily close to $a_v := h(A_v)$. The first issue that arises is that there is no guarantee that the fibre $h^{-1}(a)$ contains a rational point; in classical parlance k is a field of moduli for a, but there is no guarantee that k is a field of definition for a. Assuming that we overcome this issue and find $A \in \mathscr{A}_3(k)$ with $h(A) = a$, the second issue is as follows: we have that $h(A)$ is close to each $h(A_v)$, but this does not guarantee that A is close to the A_v; in classical parlance this means that we can only guarantee that A is close to some Galois twist of the A_v, and not our original A_v.

To overcome these issues we need to understand both the generic field of definition as well as the generic Galois twist. This is achieved by the following result of Shimura [Shi] reformulated in stacky language. (See [BV] for further applications of stacks to field of moduli questions.)

Lemma 4.11. *Let $g \in \mathbb{N}$ and $h\colon \mathscr{A}_g \to \mathcal{A}_g$ the coarse moduli map. There exists a dense open subset $V \subset \mathcal{A}_g$ such that $U := h^{-1}(V) \to V$ is a μ_2-gerbe. This gerbe is neutral if and only if g is odd.*

Proof. It is well known that the generic ppav has automorphism group μ_2; this implies that the generic fibre is a μ_2-gerbe and one finds V by spreading out, cf. [Poo2, §3.2].

It remains to show that the generic gerbe is neutral if and only if g is odd. Let K be the function field of $\mathcal{A}_g$. Then the generic gerbe is neutral if and only if the generic fibre has a K-point, which means exactly that the generic ppav has a model over K. The main result of [Shi] says this happens if and only if g is odd, as required. □

Proof of Theorem 1.6. By Lemma 4.11 we know that $\mathscr{A}_3$ is birational to a neutral μ_2-gerbe over $\mathcal{A}_3$. The latter satisfies weak approximation, and the fibres have rational points and satisfy weak approximation by Lemma 4.9. Thus the result follows from Lemmas 4.5 and 4.6. □

Remark 4.12. Our proof makes essential use of the fact that $g = 3$ is odd. In fact, the proof as written does not apply in the apparently easier case of $g = 2$ from §4.3, where Shimura showed that the generic hyperelliptic curve of even genus does not descend to its field of moduli [Shi, Thm. 3]. Then one might wonder whether, on the contrary, the simple proof for $g = 2$ can be adapted to the case of $g = 3$. This does not seem possible either since a key difference is that the map $\mathscr{M}_3 \to \mathscr{A}_3$ is no longer birational, despite inducing a birational map on the coarse moduli spaces, because the generic stabilisers no longer agree. The following gives an arithmetic manifestation of this phenomenon.

Lemma 4.13. *Let k be a field and C a smooth quartic curve over k. Then for each separable quadratic extension $k \subset L$ there exists a principally polarised abelian threefold A over k such that $A_L \cong J(C)_L$ but A is not in the image of map $\mathscr{M}_3(k) \to \mathscr{A}_3(k)$.*

Proof. The strong Torelli theorem [LS, Théorème 3] mentioned in §4.3 above shows that $\operatorname{Aut} J(C) = \operatorname{Aut} C \oplus \mathbb{Z}/2\mathbb{Z}$. As $\mathrm{H}^1(k, \mathbb{Z}/2\mathbb{Z})$ parametrises separable quadratic extensions of k, we see that for each such extension $k \subset L$ there is a quadratic twist A of $J(C)$ by L. Such a twist cannot arise from any twist of C, thus A is not the Jacobian of any curve of genus 3 over k. □

Proof of Theorem 1.1. Immediate from Theorem 1.6 and Lemma 4.10.

Proof of Theorem 1.2. As $g \geq 7$ the coarse moduli space has general type. Let p be a prime. To prove the result we will compare the cardinalities of the two sets

$$I_1 = \operatorname{Im}(\mathscr{A}_g(\mathbb{F}_p) \longrightarrow \mathcal{A}_g(\mathbb{F}_p)), \quad I_2 = \operatorname{Im}(\mathscr{A}_g(\mathbb{Q}) \cap \mathscr{A}_g(\mathbb{Z}_p) \longrightarrow \mathcal{A}_g(\mathbb{F}_p)).$$

The set I_1 is simply the collection of elements of $\mathcal{A}_g(\mathbb{F}_p)$ that arise from some ppav over $\mathbb{F}_p$. The set I_2 is the collection of elements of $\mathcal{A}_g(\mathbb{F}_p)$ that arise as the reduction modulo p of a ppav over $\mathbb{Q}$ with good reduction at p (note that $I_2 \subseteq I_1$). We will show that I_1 has more elements than I_2 for all sufficiently large p.

Firstly the Lang-Weil estimates [LW] imply that $|\mathcal{A}_g(\mathbb{F}_p)| \sim p^m$ as $p \to \infty$, where $m = g(g+1)/2 = \dim \mathcal{A}_g$. By Lemma 4.11, spreading out, and Lang–Weil, we have

$$\#\{a \in \mathcal{A}_g(\mathbb{F}_p) \mid h^{-1}(a) \text{ is not a } \mu_2\text{-gerbe}\} = O(p^{m-1}).$$

However a μ_2-gerbe over $\mathbb{F}_p$ is necessarily neutral for $p \neq 2$: indeed μ_2-gerbes are classified by $\mathrm{H}^2(\mathbb{F}_p, \mu_2)$ [Ols, Thm 12.2.8]. But Kummer theory implies that $\mathrm{H}^2(\mathbb{F}_p, \mu_2) = (\operatorname{Br} \mathbb{F}_p)[2]$, which is trivial as $\operatorname{Br} \mathbb{F}_p = 0$. We therefore deduce that $|I_1| \sim p^m$ as $p \to \infty$

For I_2, as $\mathcal{A}_g$ has general type, the Bombieri–Lang conjecture [Poo2, Conj. 9.5.11] predicts that $\mathcal{A}_g(\mathbb{Q})$ is not Zariski dense. Thus the Lang–Weil estimates imply that $|I_2| = O(p^{m-1})$. Altogether we deduce that for all sufficiently large primes p we have $|I_1| > |I_2|$, which concludes the proof. □

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