Exam feedback on MA40188 Algebraic Curves

There were sixty candidates. Fifty-nine of them managed the cover sheet correctly. The sixtieth managed, depite at least three years' experience with this thing, to stick the sticky bit down wrong and cover the box I am supposed to write the mark in. She knows who she is. I know who she is, too, because if you do that the bit with your name written on is not sealed.
This minor glitch apart, the only real problem was caused by me. I got the section lettering wrong in Q1, so that there were sections called (a), (b), (c), (d), (f), (e) and (f) again. Nobody seemed confused by this, although some of pointed it out quite firmly.

Q1. All but one candidate attempted this question and most did reasonably well on it: however, few people reached the very end and nobody actually scored full marks. There was very little trouble with (a), (b) and (c) (well, there shouldn't have been), and not much with (d), although there were a few people who wrote nonsense at that stage. In the part that should have been called (e), a good few of you stated the Nullstellensatz first and then showed the inclusion by observing that it is part of the Nullstellensatz. That doesn't show anything. All you've done is state a stronger theorem. The Nullstellensatz itself can be stated in various ways and I accepted any of them. The better students mentioned at this point that $\mathbb{K}$ needs to be algebraically closed, but as that was a standing assumption in the lectures I did not dock a mark from those who didn't bother. Most people could identify the variety defined in (what should have been) part (f) but many of you did not bother to observe that at least one of x, y, z has to vanish, which is essential. The last part, let us call it (g), stopped most people, and the few who did do it completely had all dropped marks elsewhere. But it's not difficult. Some of you couldn't show, or didn't bother to show, that the polynomial you are given is in $\mathbb{I}(Y)$, but mostly you gave wrong proofs or none that it is not in J. Many simply asserted that it isn't. Some thought that J consists of linear combinations of the generators with coefficients from $\mathbb{K}$ rather than from $\mathbb{K}[x, y, z]$, forgetting what an ideal is. The commonest mistake was to say that the coefficient of xyz must be 0 because it has degree 3 and the other generator has degree 2: but you are allowed to multiply the other generator by $-z$, for example. Others thought it relevant that this polynomial vanishes at $(1, 1, 1)$ and the generators of J don't, but so what? More happily, a couple of people found other correct methods, bypassing the hint altogether. Some of you tried to prove that J is not radical, which would be enough and indeed can be got to work, but nobody got the details right.

Q2. There are two possible answers to (a), depending on which you think is the definition and which the immediate consequence, and most people correctly gave one or the other. Only a few people gave the standard wrong answer to (b), which was encouraging: you know that elements of homogeneous ideals don't have to be homogeneous themselves. The explanations in (c) were a bit less successful, although most people knew what to say, but the last part cost many people a mark. Either they forgot they were dealing with projective varieties and just said a version of the (usual, affine) Nullstellensatz or, more often, they remembered about the irrelevant ideal but related it to I rather than $\sqrt{I}$. What if I is the ideal in $\mathbb{K}[x, y]$ generated by x^2 and y^2? Part (d) was where things started to get a bit harder. The definition of $\mathbb{I}(X)$ does not make it obvious even that it is an ideal, so you should check that. There is a trick to show that it is homogeneous, which about half of you remembered, and an easier one to show that it is radical, which most of you remembered. Those who were still going at that point had little trouble with (e), but (f) defeated almost everybody. The main reason why is that you don't think of using the freedom to pick homogeneous coordinates that you like. Don't like the factor that comes with x? Divide everything by it! Some of you guessed where you were going and some didn't, but most bogged down in calculations. My solution is six lines long.

Q3. This time things started to go wrong at once. A significant minority of you couldn't do (a), or didn't know what is required. To show that something is an affine variety you have to show that it's given by some equations: what are they? On the other hand (b), which is harder, was done well. Nearly everybody dropped a mark in (c), because if you are going to say that Φ^* is composition with Φ (which it is), you have to have a map to compose it with, so you have to say why an element of $\mathbb{K}(X)$ can be regarded as a (rational) map, or at least say that it can. A good many had fallen over before that, though, by ignoring the word "projective" and talking about fields of fractions, which is the affine case. Part (d), on the other hand, was just free marks. Most of you knew how to homogenise the equation in (e), though a few wrote obviously inhomogeneous things. Some of you made life harder for yourselves by multiplying out the bracket.

This does no harm, but why bother? Most of you could find the points at infinity. A few were spooked by the square root of -1, which you should have stopped worrying about at the time you stopped worrying about Gruffalos, and some happily wrote about $(0 : 0 : 0)$ as if it meant something. The main trouble, though, was in choosing what to differentiate. You have to dehomogenise first! Many of you differentiated the homogeneous equation: this is wrong for the conceptual reason that singularity is local so it can't depend on what happens at infinity, which is what you need homogeneity for.

Q4. There were very few answers indeed to this, and half of those were fragments by people who clearly had a few minutes to spare at the end, having answered the other three questions. The handful who did make a serious attempt at the question did it rather well, so there are no common mistakes to describe.