In all questions, $\mathbb{k}$ is an algebraically closed field of characteristic 0.

1. (a) Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set. Define its ideal $\mathbb{I}(X)$ and coordinate ring $\mathbb{k}[X]$. What does it mean to say X is irreducible? If X is irreducible, what property does $\mathbb{I}(X)$ have? [4]

   (b) Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set and $I \subseteq \mathbb{k}[x_1, \cdots, x_n]$ a radical ideal. Prove that $X = \mathbb{V}(I)$ if and only if $I = \mathbb{I}(X)$. [4]

   (c) For any $a_1, \cdots, a_n \in \mathbb{k}$, prove that the ideal $I = (x_1 - a_1, \cdots, x_n - a_n) \subseteq \mathbb{k}[x_1, \cdots, x_n]$ is a maximal ideal. [4]

   (d) Write down explicit polynomial maps to prove that the affine algebraic set $C = \mathbb{V}(y - x^2, z - x^3) \subseteq \mathbb{A}^3$ is isomorphic to $\mathbb{A}^1$. [4]

   (e) Assume the degree of the polynomial $f(x) \in \mathbb{k}[x]$ is odd. Prove that the affine algebraic set $\mathbb{V}(y^2 - f(x)) \subseteq \mathbb{A}^2$ is an affine variety. [4]

2. (a) Define the projective space $\mathbb{P}^n$ and a standard affine chart of $\mathbb{P}^n$. What does it mean to say an ideal $I \subseteq \mathbb{k}[z_0, \cdots, z_n]$ is homogeneous? If I is homogeneous, what is the projective algebraic set $\mathbb{V}(I)$? [4]

   (b) For any subset $X \subseteq \mathbb{P}^n$, define its ideal $\mathbb{I}(X)$. Prove that the ideal $\mathbb{I}(X) \subseteq \mathbb{k}[z_0, \cdots, z_n]$ is homogeneous. (You do not need to prove it is an ideal.) [4]

   (c) Explain why $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by $\varphi([x : y : z]) = [yz : zx : xy]$ is a rational map. Is it dominant? Fully explain your answer. [4]

   (d) Prove that the projective algebraic set $C = \mathbb{V}(z_0 z_2 - z_1^2) \subseteq \mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$. [4]

   (e) Prove that the ideal $I = (x + y^2, y^2 + z^3, z^3 + x) \subseteq \mathbb{k}[x, y, z]$ is a homogeneous ideal. What is the projective algebraic set $\mathbb{V}(I) \subseteq \mathbb{P}^2$? [4]

3. (a) Let $X \subseteq \mathbb{P}^n$ be a projective variety. Define the function field of X and a rational function on X. Let $\varphi : X \dashrightarrow Y$ be a dominant rational map between two projective varieties and g a rational function on Y. Define the pullback of g along φ. [4]

(b) Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set. Define the projective closure of X and points at infinity. Find the projective closure and points at infinity for the affine algebraic set $\mathbb{V}((x^2+y^2-1)^3 - x^2y^3) \subseteq \mathbb{A}^2$. [4]

(c) Let $X = \mathbb{V}(f) \subseteq \mathbb{A}^n$ be an affine hypersurface defined by a non-constant irreducible polynomial $f \in \Bbbk[x_1, \cdots, x_n]$. Prove that the set of non-singular points in X is non-empty. [4]

(d) Consider the affine hypersurface $X = \mathbb{V}(f) \subseteq \mathbb{A}^2$ defined by the irreducible polynomial $f = (x^2+y^2)^3 - 4x^2y^2$. Find all singular points on X. Show all your calculation. [4]

(e) Prove that the projective closure of the affine variety $X = \mathbb{V}(y^2 - (x^3+ax+b)) \subseteq \mathbb{A}^2$ is non-singular if and only if $4a^3 + 27b^2 \neq 0$. [4]

4. (a) Let $C \subseteq \mathbb{P}^2$ be a non-singular cubic curve and $O \in C$ a point on C. State the group law on C with O as the identity element. [4]

(b) Consider the group law on the non-singular cubic curve $C = \mathbb{V}(y^2z - x^3 + 4xz^2 - z^3) \subseteq \mathbb{P}^2$ with $O = [0 : 1 : 0]$ as the identity element. Let $A = [2 : 1 : 1]$ and $B = [-2 : -1 : 1]$ be two points on C. Find $A + B$ and $-B$. [4]

(c) Prove that the cuspidal curve $C = \mathbb{V}(y^2z - x^3) \subseteq \mathbb{P}^2$ is rational. [4]

(d) Let L be a line in $\mathbb{P}^2$ and D a plane curve of degree d. Suppose that L is not a component of D. Prove that $L \cap D$ contains at most d distinct points. Explain briefly why, when counting with multiplicities, L and D meet in precisely d points. [4]

(e) Let $C \subseteq \mathbb{P}^2$ be a non-singular cubic curve and O an inflection point on C. Consider the group law on C with O as the identity element. A point $P \in C$ is called a 3-torsion point if $P + P + P = O$. Prove that P is a 3-torsion point if and only if P is an inflection point. [4]