**DEPARTMENT OF MATHEMATICAL SCIENCES** University of Bath

Outline solution to Examination Questions

| Unit Code MA 40188 | Unit Title ALGEBRAIC CURVES | |
|---|---|---|
| Academic Year 2013/14 | Examiner ALASTAIR CRAW | |
| Semester 1 | Question No. 1 | Page 1 of 2 |

| Part | | Mark |
|---|---|---|
| [DEFN] | (a) $\mathbb{V}(J) = \{p \in \mathbb{A}^n \mid f(p) = 0 \ \forall f \in J\}$ and $\mathbb{I}(X) = \{f \in \mathbb{k}[x_1, \dots x_n] \mid f(p) = 0 \ \forall p \in X\}$ | 2. |
| [DEFN] | (b) $X$ is irreducible if there does not exist a decomposition $X = X_1 \cup X_2$ where $X_1, X_2 \subsetneq X$ are proper algebraic subsets. | 2. |
| [BOOKWORK] | ($\Leftarrow$) Suppose $X = X_1 \cup X_2$ is a reducible decomposition. Since $X_i \subsetneq X$ $\exists$ $f_i \in \mathbb{I}(X_i) \setminus \mathbb{I}(X)$ for $i = 1, 2$. Now $0 = f_1(p) f_2(p) = (f_1 f_2)(p)$ $\forall p \in X$, so $f_1 f_2 \in \mathbb{I}(X)$ and yet $f_i \notin \mathbb{I}(X)$ for $i = 1, 2$, ie $\mathbb{I}(X)$ is not prime. ($\Rightarrow$) Suppose $\mathbb{I}(X)$ not prime: $\exists$ $f_1 \cdot f_2 \in \mathbb{I}(X)$ with $f_i \notin \mathbb{I}(X)$ for $i = 1, 2$. Then $J_i := \langle \mathbb{I}(X), f_i \rangle \supsetneq \mathbb{I}(X)$ for $i = 1, 2$ defines $X_i = \mathbb{V}(J_i)$ a proper algebraic subset of $X$ satisfying $$X_1 \cup X_2 = \mathbb{V}(J_1) \cup \mathbb{V}(J_2) = \mathbb{V}(J_1 \cdot J_2) = \mathbb{V}(\mathbb{I}(X)) = X.$$ Therefore $X$ is reducible. | 5. |
| [UNSEEN BUT SIMILAR] | (c) (i) In the expression for $f$, replace $x^2 \mapsto [(x^2 - y^3) + y^3]$ and $y^2 \mapsto [(y^2 - z^3) + z^3]$ throughout. Expand the square brackets using binomial theorem and gather expressions involving $(x^2 - y^3)$ and $(y^2 - z^3)$ to obtain $h_1, h_2, h_3 \in \mathbb{k}[x, y, z]$ s.t $$f = h_1 (x^2 - y^3) + h_2 (y^2 - z^3) + h_3$$ where each term of $h_3$ is of form $c x^\alpha y^\beta z^\gamma$ for $c \in \mathbb{k}$ and $0 \leq \alpha, \beta \leq 1$. Separate $h_3$ into four polynomials according to values of $\alpha, \beta$ and set $g = h_1(x^2 - y^3) + h_2(y^2 - z^3)$ to get $$f = g + a + bx + cy + dxy \qquad a, b, c, d \in \mathbb{k}[z]$$ $g \in J$. | 4. |

| | Total | |
|---|---|---|

**DEPARTMENT OF MATHEMATICAL SCIENCES** University of Bath

Outline solution to Examination Questions

| | | |
|---|---|---|
| Unit Code MA 40188 | Unit Title ALGEBRAIC CURVES. | |
| Academic Year 2013/14 | Examiner ALASTAIR CRAW | |
| Semester 1 | Question No. 1 | Page 2 of 2 |

| Part | | Mark |
|---|---|---|
| (c) (ii) | $\phi(x^2-y^3) = \phi(x^2)-\phi(y^3) = t^{18}-t^{18}=0$, $\phi(y^2-z^3) = \phi(y^2)-\phi(z^3) = t^{12}-t^{12}=0$, so $J \subseteq \mathrm{Ker}(\phi)$. For opposite inclusion, let $f \in \mathrm{Ker}(\phi)$. By part (i) write $$f = g + a + bx + cy + dxy.$$ Then $$\begin{aligned} 0 &= \phi(f) \\ &= \phi(g) + a(t^4) + b(t^4)t^9 + c(t^4)t^6 + d(t^4)t^{15} \\ &= a(t^4) + b(t^4)t^9 + c(t^4)t^6 + d(t^4)t^{15} \quad \text{as } g \in J \end{aligned}$$ Notice $a(t^4)$, $b(t^4)t^9$, $c(t^4)t^6$, $d(t^4)t^{15}$ involve only terms in which the exponent of $t$ is $0, 1, 2, 3$ mod 4 respectively, so each of these polynomials has all coefficients equal 0 by comparing with left hand side of the above. Thus $a=b=c=d=0$, leaving $f \in J$ as required. ie $J = \mathrm{Ker}(\phi)$. | 4 |
| (c) (iii) | $J = \mathrm{Ker}(\phi)$ is prime because $\dfrac{\Bbbk[x,y,z]}{\mathrm{Ker}(\phi)} \cong \mathrm{Im}(\phi) \cong \Bbbk[t^9, t^6, t^4]$ is a subring of an integral domain, hence an integral domain. Since $\Bbbk$ algebraically closed, $\mathbb{I}(\mathbb{V}(J)) = J$ is prime, so $\mathbb{V}(J)$ is irreducible by part (b). | 3 |

[UNSEEN BUT SIMILAR]

[IDENTICAL TO HOMEWORK]

Total

**DEPARTMENT OF MATHEMATICAL SCIENCES** University of Bath

Outline solution to Examination Questions

| | | | |
|---|---|---|---|
| Unit Code MA40188 | Unit Title ALGEBRAIC CURVES | | |
| Academic Year 2013/14 | Examiner ALASTAIR CRAW | | |
| Semester 1 | Question No. 2 | Page 1 | of 2 |

Part | Mark

[EASY BOOKWORK] [DEF'N]

(a) $\Bbbk[X] = \{ f : X \to \Bbbk \mid \exists\, F \in \Bbbk[x_1, \dots, x_n] \text{ with } f = F|_X \}$ 1

(b) The restriction map $\mathrm{res}|_X : \Bbbk[x_1,\dots,x_n] \longrightarrow \Bbbk[X]$ is a surjective ring homomorphism. The kernel is $\{F \in \Bbbk[x_1,\dots,x_n] : F(p) = 0\ \forall p \in X\} = \mathbb{I}(X)$ so the statement follows from the first isomorphism theorem. 3

[DEF'N]

(c) $\phi : X \to Y$ is polynomial if $\exists\, f_1, \dots, f_m \in \Bbbk[X]$ s.t. $\phi(p) = (f_1(p), \dots, f_m(p))$ or, equivalently, if $\exists\, F_1, \dots, F_m \in \Bbbk[x_1, \dots, x_n]$ s.t. $\phi(p) = (F_1(p), \dots, F_m(p))$. 3
The pullback is $\phi^* : \Bbbk[Y] \to \Bbbk[X] : g \longmapsto g \circ \phi$.

[HARDER BOOKWORK]

(d) For $\pi : \Bbbk[y_1, \dots, y_m] \longrightarrow \Bbbk[Y] : F \to F|_Y$ set $f_j = \alpha(\pi(y_j))$ for $1 \le j \le m$. Let $F_j \in \Bbbk[x_1, \dots, x_n]$ be any lift of $f_j$ from $\Bbbk[X]$, ie $F_j(p) = f_j(p)\ \forall p \in X$. Define $\Bbbk$-algebra homo $\tilde{\alpha}$ so that diagram commutes:

$$\begin{tikzcd}
\Bbbk[y_1, \dots, y_m] \arrow[r, "\tilde{\alpha}"] \arrow[d, "\pi"'] & \Bbbk[x_1, \dots, x_n] \arrow[d] \\
\Bbbk[Y] \arrow[r, "\alpha"] & \Bbbk[X]
\end{tikzcd}$$

ie $\tilde{\alpha}(y_j) = F_j \quad \forall 1 \le j \le m$.

Notice that $\tilde{\alpha}(\mathbb{I}(Y)) \subseteq \mathbb{I}(X)$. Define $\Phi : \mathbb{A}^n \to \mathbb{A}^m$ by $\Phi(p) = (F_1(p), \dots, F_m(p))$, so $\Phi^*(y_j) = F_j = \tilde{\alpha}(y_j)$ giving $\Phi^* = \tilde{\alpha}$. 8
For $G \in \mathbb{I}(Y)$ and $p \in X$

$$\begin{aligned} 0 &= \Phi^*(G)(p) \qquad \text{as } \Phi^*(\mathbb{I}(Y)) = \tilde{\alpha}(\mathbb{I}(Y)) \subseteq \mathbb{I}(X) \\ &= G(\Phi(p)) \end{aligned}$$

So $\Phi(p) \in \mathbb{V}(\mathbb{I}(Y)) = Y$. Thus $\phi := \Phi|_X : X \to Y$ satisfies $\phi^* = \alpha$ by construction.

Total

DEPARTMENT OF MATHEMATICAL SCIENCES University of Bath

Outline solution to Examination Questions

| | |
|---|---|
| Unit Code MA40188 | Unit Title ALGEBRAIC CURVES |
| Academic Year 2013/14 | Examiner ALASTAIR CRAW |
| Semester 1 | Question No. 2 / Page 2 of 2 |

Part | Mark

[UNSEEN]

(e)(i) The map $\phi_1$ is not an isomorphism because it's not even a bijection: $\phi(1) = (0,0) = \phi(-1)$. — ~~2~~

(ii) The map $\phi_2$ is an isomorphism. One approach is to write down the inverse map: set $\psi_2$ to be the restriction to $C_2$ of the polynomial map — 5

$$\Psi_2 : \mathbb{A}^2 \longrightarrow \mathbb{A}^1 \;:\; \psi_2(x,y) = x$$

Then — ~~3~~

$$(\psi_2 \circ \phi_2)(t) = \psi_2(t, t^5) = t$$

and

$$\begin{aligned}(\phi_2 \circ \psi_2)(p_1,p_2) &= \phi_2(p_1)\\ &= (p_1, p_1^5) \qquad \text{as } p_1^5 = p_2 \text{ for } (p_1,p_2)\in C_2\\ &= (p_1, p_2)\end{aligned}$$

as required.

Total

**DEPARTMENT OF MATHEMATICAL SCIENCES** University of Bath

Outline solution to Examination Questions

| | |
|---|---|
| Unit Code MA40188 | Unit Title ALGEBRAIC CURVES |
| Academic Year 2013/14 | Examiner ALASTAIR CRAW |
| Semester 1 | Question No. 3 — Page 1 of 2 |

Part | Mark

[DEF'N]

(a) $T_pX = \mathbb{V}(g) \subseteq \mathbb{A}^n$ for $g = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)\cdot(x_i - p_i)$ 2

$= \{q \in \mathbb{A}^n : \sum_{i=1}^n \partial f/\partial x_i(p)\cdot(q_i - p_i) = 0\}$.

[HOMEWORK]

(b) We've given $g \in \mathbb{I}(\mathbb{V}(f))$. The Nullstellensatz implies that $g \in \mathrm{rad}\langle f\rangle$. Irreducibility of $f$ implies $\langle f\rangle$ is prime and hence radical, so $g \in \langle f\rangle$. This means $\exists\, h \in \mathbb{C}[x_1,\dots,x_n]$ with $g = hf$ 4

[BOOKWORK]

(c) Point $p \in X$ is singular if $\partial f/\partial x_i(p) = 0$ for $1 \le i \le n$.

Proving that the nonsingular locus is nonempty and Zariski-open is equivalent to proving that the singular locus is a proper, Zariski-closed subset of $X = \mathbb{V}(f)$. Now

$$p \in X \text{ singular} \iff p \in \mathbb{V}(f, \partial f/\partial x_1, \dots, \partial f/\partial x_n)$$
$$\iff p \in \mathbb{V}(f) \cap \mathbb{V}(\partial f/\partial x_1) \cap \dots \cap \mathbb{V}(\partial f/\partial x_n) \quad \circledast$$ 7

which is Zariski-closed, so we need only prove it's proper in $\mathbb{V}(f)$. Suppose otherwise. Then by $\circledast$ each ~~pt of $\mathbb{V}(f)$~~ $\partial f/\partial x_i$ vanishes at each point of $\mathbb{V}(f)$. By (b) above, $\exists\, h_i \in \mathbb{C}[x_1,\dots,x_n]$ satisfying $\partial f/\partial x_i = h_i f$ for $1 \le i \le n$. Viewed as a polynomial in $x_i$, degree of $\partial f/\partial x_i <$ degree of $f$, forcing $\partial f/\partial x_i = 0$. This is true for $1 \le i \le n$, so no $x_i$ appears in $f$ giving $f \in \mathbb{C}$, a contradiction.

[UNSEEN EASY]

(d) (i) $\partial f/\partial x = -3x^2 - 2x = -x(3x+2)$ so singular points

$\partial f/\partial y = 2y$ when $(x,y) \in \{(0,0), (-\frac{2}{3}, 0)\}$ 2

not on $\mathbb{V}(f)$

Total

# DEPARTMENT OF MATHEMATICAL SCIENCES

University of Bath

Outline solution to Examination Questions

| Unit Code MA 40188 | Unit Title ALGEBRAIC CURVES | |
|---|---|---|
| Academic Year 2013/14 | Examiner ALASTAIR CRAW | |
| Semester 1 | Question No. 3 | Page 2 of 2 |

Part | Mark

(d)(i) ctd hence $X$ singular only at $(0,0)$.

(ii) Need $\partial g/\partial z = 0$ ie $2z=0$ so $z=0$.

Need $\partial g/\partial y = 0$ ie $2x^2y=0$ so $x=0$ or $y=0$.

Two cases:

- $x=z=0$ ; locus $\mathbb{V}(x,z)$ contained in $\mathbb{V}(g)\cap\mathbb{V}(\partial g/\partial x)$ so $Y$ is singular along $\mathbb{V}(x,z)\subseteq Y$.
- $z=y=0$ and $x\neq 0$, then setting $g(x,0,0)=(x-1)^2x^2=0$ for $x\neq 0$ forces $x=1$, and then note $(1,0,0)\in\mathbb{V}(\partial g/\partial x)$ so $Y$ also singular at $(1,0,0)\in Y$

hence $Y$ singular along the union $\mathbb{V}(x,z)\cup\{(1,0,0)\}$.

5

[UNSEEN, HARDER]

Total

**DEPARTMENT OF MATHEMATICAL SCIENCES** University of Bath
Outline solution to Examination Questions

| | | | |
|---|---|---|---|
| Unit Code MA 40188 | Unit Title ALGEBRAIC CURVES. | | |
| Academic Year 2013 /14 | Examiner ALASTAIR CRAW | | |
| Semester 1 | Question No. 4 | Page 1 | of 2 |

| Part | | Mark |
|---|---|---|
| (a) | $\mathbb{P}^2 = \mathbb{A}^3 \setminus \{0\} / \sim$ with $(p_0,p_1,p_2) \sim (q_0,q_1,q_2)$ iff $\exists\, \lambda \in \Bbbk \setminus \{0\}$ such that $q_i = \lambda p_i$ for $0 \leq i \leq 2$. | 2 |
| | $\mathbb{P}^2 \supseteq U_0 = \{[p_0 : p_1 : p_2] \in \mathbb{P}^2 \mid p_0 \neq 0\} \cong \{(p_1/p_0, p_2/p_0) \in \mathbb{A}^2\} = \mathbb{A}^2$ | |
| | The complement $\mathbb{P}^2 \setminus U_0 = \{[0 : p_1 : p_2] \in \mathbb{P}^2\} \cong \mathbb{P}^1$ records the asymptotic directions of unbounded curves in $\mathbb{A}^2$, e.g. $y = x \pm 1$ parallel in $\mathbb{A}^2$ are $\frac{p_2}{p_0} = \frac{p_1}{p_0} \pm 1$ in $U_0$ extend to $p_2 = p_1 \pm p_0$ in $\mathbb{P}^2$. The 'line at $\infty$' has $p_0 = 0$, i.e. $p_2 = p_1$: both lines intersect line at $\infty$ at $[0:1:1] \in \mathbb{P}^2$ which records the asymptotic direction '$y = x$' of both lines | 5 |
| (b) | The equation $f(x,y,z) = 0$ defining $C$ is a homogeneous cubic in the homogeneous coordinates $x,y,z$ on $\mathbb{P}^2$. For the line $L = \mathbb{P}^1$ with homogeneous coords $u,v$, the restriction of $f$ to $L$ is a homogeneous cubic in $u,v$ which splits (over a possibly larger field) as $\prod_{1 \leq i \leq 3} (\alpha_i u - \beta_i v)$. The roots of these factors are the points of intersection $L \cap C$. A multiple point of intersection corresponds to $L$ tangent to $C$ at $p$: either $L$ meets $C$ at $p$ with multiplicity 2, so $L$ meets $C$ at one other point; or $L$ meets $C$ at an inflection pt. | 5 |
| (c) | The group law is given in terms of intersecting $C$ with different lines: given $p,q \in C$, the line $L = pq$ will | |

| Total | |
|---|---|

[DEF'N] ⟶ [BOOKWORK] (a)

[BOOKWORK] (b)

**DEPARTMENT OF MATHEMATICAL SCIENCES** University of Bath
Outline solution to Examination Questions

| | | | |
|---|---|---|---|
| Unit Code MA 40188 | Unit Title ALGEBRAIC CURVES | | |
| Academic Year 2013/14 | Examiner ALASTAIR CRAW | | |
| Semester 1 | Question No. 4 | Page 2 of 2 | |

Part | Mark

← [UNSEEN] → [HOMEWORK] ← → [BOOKWORK] →

(c) ctd typically meet $C$ in a third point $r$; the addition in the group law with origin $O$ is given by joining this point $r$ to $O$ and defining $p+q$ to be the third point of intersection. Also

- inverse $p \mapsto -p$ is easy because the equation $y^2 = f_3(x)$ means lines through $O$ are vertical lines, so inverse is reflection in $x$-axis. 5
- to define $p+p$, replace the line $pq$ from the above by $T_pC$ the tangent line (well-defined as $C$ nonsingular). It intersects $C$ either in $2p+q$ or $3p$ according to pure mult 2 tangency or inflection as in (b) above.

(d) (i) $2p = 0$ means $T_pC$ passes through $O$, ie is vertical. This happens at the 3 roots of $f(x)$. 2

(ii) Tangent line to $C$ at $p = (2,4)$ is $y = 2x$ so it passes through $q = (0,0)$. Since $x=0$ is a root of $f(x) = x^3+4x$, the point $q$ has order 2 in the group law. Thus 3

$$2p = q \quad \therefore \quad 4p = 2q = 0.$$

Total