University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES EXAMINATION

MA40188: ALGEBRAIC CURVES

May 2012

Candidates may use university-supplied calculators.

Full marks will be given for correct answers to THREE questions.
Only the best three answers will contribute towards the assessment.

1. (a) $\mathbb{A}^n_K = K^n$; $\mathbb{P}^n = \mathbb{A}^{n+1}/K^*$ with K^* acting by coordinatewise multiplication. [2, book]

   (b) $(x_0 : \ldots : x_n) = (\lambda x_0 : \ldots : \lambda x_n)$ for any x_i not all zero and any $\lambda \in K^*$. [2, book]

   (c) $\mathbb{A}^n_K$ has q^n points; $\mathbb{P}^n_K$ has $\frac{q^{n+1}-1}{q-1}$ because the action is free. [2, on examples sheet]

   (d) f is a homogeneous polynomial of degree d if $f(\lambda a_0, \ldots \lambda a_n) = \lambda^d f(a_0, \ldots, a_n)$ for all $a_i \in K$, $\lambda \in K^*$. [1, book]

   (e) $I \subset K[X_0, \ldots, X_n]$ is a homogeneous ideal if it is generated by homogeneous polynomials. [2, book]

   (f) An affine variety is defined by the conditions $P \in V$ iff $f(P) = 0$ for all $f \in I$, some ideal $I \subset K[X_1, \ldots, X_n]$. A projective variety is defined by $f(P) = 0$ for all $f \in I$, some homogeneous ideal $I \subset K[X_0, \ldots, X_n]$. This makes sense because for homogeneous polynomials, $f(\lambda x_0, \ldots, \lambda x_n) = 0$ iff $f(x_0, \ldots, x_n)$ if $\lambda \neq 0$. [3, book]

   (g) If $I(V)$ is generated by $f_1, \ldots, f_k$ then W corresponds to the homogeneous ideal generated by the homgenisations g_i of f_i wrt X_0. If $F \in K[X_1, \ldots, X_n]$ is of degree d, write $F = \sum_{r \leq d} F_r$ with F_r homogeneous of degree r: then the homogenisation of F wrt X_0 is $G = \sum_{r \leq d} X_0^{d-r} F_r$. [4, book]

   (h) The projective closure is given by

$$X_1^3 + X_1^2 X_2 + X_1 X_2^2 + X_0 X_2^2 = 0.$$

   The points at infinity are $(0 : x_1 : x_2)$, where $x_1^3 + x_1^2 x_2 + x_1 x_2^2 = 0$: that means $x_1 = 0$, or $x_1 = 1$ and $1 + x_2 + x_2^2 = 0$, so the points are $(0 : 0 : 1)$ and $(0 : 1 : e^{\pm 2\pi i/3})$. [4, unseen]

2. (a) $\sqrt{I} = \{f \in A \mid \exists k \in \mathbb{N}\, f^k \in I\}$. It is an ideal because if $f^k \in I$ and $g^l \in I$ and $a, b \in A$ then $(af+bg)^{k+l} = \sum_{0\le r\le k+l} \binom{k+l}{r} a^{k+l-r} b^r f^{k+l-r} g^r$, and each term is in I because if $r \ge l$ then $g^r \in I$ and if $r < l$ then $k+l-r > k$ so $f^{k+l-r} \in I$. [3, book]

(b) If $K = \overline{K}$ and $V(I) = \emptyset \subset \mathbb{A}^n_K$ then $1 \in I$. [1, book]

(c) Suppose $f \in A$. Consider $B = A[Y] = K[X_1, \ldots, X_n, Y]$ and the ideal $I^+ := IB + (yf-1)B$. Notice that $Q = (x_1, \ldots, x_n, y) \in V(I^+)$ iff $P = (x_1, \ldots, x_n) \in V(I)$ and, in addition, $y = 1/f(P)$: in particular $f(P) \neq 0$.

What we want to do is find out when this set $(f \neq 0) \subset V(I)$ is empty: that happens when $f = 0$ everywhere on $V(I)$, i.e. when $f \in I(V(I))$. So suppose $f(P) = 0$ for all $P \in V(I)$: that means that $V(I^+) = \emptyset$, since the map $P \mapsto (P, 1/f(P)$ gives a (set-theoretic) bijections between $(f \neq 0) \cap V(I)$ and $V(I^+)$. By the Nullstellensatz, that implies that $1 \in I^+$, and because I^+ is generated by I and $yf - 1$ we can find polynomials $g_0, g_1, \ldots g_k \in B$ such that

$$g_0(Yf-1) + g_1 f_1 + \cdots + g_k f_k = 1,$$

where $f_1, \ldots, f_k$ are generators for the ideal I.

This equation is an identity, so we may write $1/f$ instead of Y and it will still hold: that is

$$\sum_{i=1}^{k} g_i\big(X_1, \ldots, X_n, 1/f(X_1, \ldots, X_n)\big) f_i(X_1, \ldots, X_n) = 1$$

(since the g_0 term is now zero). The left-hand side here is a rational function, but the denominator is some power of f (namely, f^N where N is the maximum of the degrees of the g_i in Y): in other words,

$$g_i\big(X_1, \ldots, X_n, 1/f(X_1, \ldots, X_n)\big) = h_i(X_1, \ldots, X_n)/\big(f(X_1, \ldots, X_n)\big)^N$$

for some polynomials h_i. If we multiply through by f^N we get

$$\sum_{i=1}^{k} h_i(X_1, \ldots, X_n, 1) f_i(X_1, \ldots, X_n) = f(X_1, \ldots, X_n)^N$$

so $f \in \sqrt{I}$ as claimed. [7, book]

(d) $K[V] = A/I(V)$ and $K(V)$ is the field of fractions of $K[V]$. We say $f \in K(V)$ is regular at $P \in V$ if there exist $g, h \in A$ such that $(g+I)/(h+I) = f \in K(V)$ and $h(P) \neq 0$. [4, book]

(e) Let $J \subset K[V]$ be the ideal of denominators of f, i.e. $h \in J$ if $f = g/h$ for some $g \in K[V]$, or $h = 0$. If f is regular at P then $P \notin V(I+J)$: so if f is regular at every $P \in V$ then $1 \in I + J$ so $1 + I \in J$, i.e. $f \in K[V]$. [3, on sheet]

(f) From the equation, $x/y = x + y - 1$ so this is regular everywhere. [2, unseen]

3. (a) An affine (projective) hypersurface is given by the vanishing of a single (homogeneous) polynomial. [3, book]

(b) If $V = (f = 0)$ then V is singular at $P \in V$ iff $\frac{\partial f}{\partial x_i}(P) = 0$ for all i. [2, book]

(c) By the Nullstellensatz, if not then $\frac{\partial f}{\partial x_i} \in \sqrt{I(V)}$ which is generated by f. So $f | \frac{\partial f}{\partial x_i}$, which is impossible in characteristic zero because the x_i-degree of the derivative is less than the degree of f: so all the derivatives are zero, so f is a constant and $V = \emptyset$. In characteristic p it could happen that $\frac{\partial f}{\partial x_i} \equiv 0$ for all i even though $f \not\equiv 0$; but then $f \in K[X_1^p, \ldots, X_n^p]$, and if $f = \sum_m a_m \prod_i X_i^{m_i p}$ then $f = g^p$ where $g = \sum_m a_m^{1/p} \prod_i X_i^{m_i}$: since $K = \overline{K}$ these coefficients exist, so f is not irreducible. [6, book]

(d) W is singular at $Q \in W$ if the affine hypersurface $W \cap U_j$ is singular at Q, where $U_j \cong \mathbb{A}^n_K$ is an affine piece containing Q. [3, book]

(e) Start with $z = 1$: then we have $x^3(x+1) - 2x^2y - 2y^3 = 0$, $4x^3 + 3x^2 - 4xy = 0$, and $2x^2 - 6y^2 = 0$. One solution to all of these is $x = y = 0$, i.e. the point $(0 : 0 : 1)$. Otherwise the first equation gives $x \neq 0$. The third equation gives $y = \pm x/\sqrt{3}$ and substituting in the first equation gives $x = -1 \pm 8/3\sqrt{3}$ (since $x \neq 0$) while the second gives $x = -3/4 \pm 1/\sqrt{3}$. As these do not agree there are no more singular points with $z = 1$. On the other hand, if $z = 0$ then the only point of the curve is $(0 : 1 : 0)$, so let us look at the affine piece $y = 1$. There we have $\partial f/\partial z = x^3 - 2x^2 - 2$ which does not vanish when $x = z = 0$. So $(0 : 0 : 1)$ is the only singular point. [6, unseen]

4. (a) W is rational if there are mutually inverse dominating rational maps $\phi\colon W \dashrightarrow \mathbb{P}^1_K$ and $\psi\colon \mathbb{P}^1_K \dashrightarrow W$ defined over K. [2, book]

(b) $K[t]$ is a UFD. To show that $C = (y^2 = x(x-1)(x-a))$ is not rational for $a \neq 0, 1$ we show that $K(C) \not\cong K(t)$. If $K(C) \cong K(t)$, then there exist $f, g \in K(t)$ such that $f^2 = g(g-1)(g-a)$. We may assume $K = \overline{K}$: we claim that then $f, g \in K$. Suppose that $f = p/q$ and $g = r/s$, where $p, q, r, s \in K[t]$ and p, q are coprime and r, s are coprime. Then

$$p^2s^3 = q^2r(r-s)(r-as).$$

Hence, by coprimality, $q^2|s^3$ and similarly $s^3|q^2$, since s does not divide $r(r-s)(r-as)$. Hence, $q^2 = \alpha s^3$ for some $\alpha \in K$, so $p^2 = \alpha r(r-s)(r-as)$. Now, $\alpha s = (q/s)^2$ is a square in $K[t]$, and so are βr, $\gamma(r-s)$ and $\delta(r-as)$ for some $\beta, \gamma, \delta \in K$. Now consider this situation: $r, s \in K[t]$ and four different linear combinations of r and s are all squares. This forces r and s to be constant polynomials. It may be assumed (replacing r and s by $ar+bs$ and $cr+ds$ with $ad-bc = 1$ if necessary) that r, s, $r-s$ and $r-\mu s$ are squares, so write $r = u^2$ and $s = v^2$. Given such a pair (r, s), define the size of the pair to be $\max\{\deg r, \deg s\}$. Suppose (r, s) is of least possible size (not zero). Notice that $\max\{\deg u, \deg v\} < \max\{\deg r, \deg s\}$. Moreover, because K is an algebraically closed field of characteristic not 2,

$$r - s = u^2 - v^2 = (u+v)(u-v), \quad r - \mu s = u^2 - \mu v^2 = (u + \sqrt{\mu}v)(u - \sqrt{\mu}v)$$

and since $r-s$, $r-\mu s$ are squares, so are $u+v$, $u-v$, $u+\sqrt{\mu}v$ and $u-\sqrt{\mu}v$: but this contradicts the minimality. [9, book]

(c) The map $t \mapsto (t^2-1, t^3-t)$ is a rational map with inverse $(x, y) \mapsto y/x \in \mathbb{A}^1 \subset \mathbb{P}^1$. [3, book]

(d) The projection gives $x = t^2$ and $y = t + t^2$. So $y = t + tx$ and therefore $t = y/(1+x)$: hence the image is given by $x = y^2/(1+x)^2$, i.e. $y^2 = x(1+x)^2$ which is a nodal cubic (not the same one, but by the assumption in part (c) that doesn't matter), and the inverse map is given by $(x, y) \mapsto (\frac{y}{1+x}, (\frac{y}{1+x})^2, (\frac{y}{1+x})^3)$. [6, unseen]