1. (a) Define what is meant by a *noetherian* ring.

   (b) Show that if R is a noetherian ring then the polynomial ring $R[t]$ in one variable over R is also noetherian. You need not carry out the verification that any ideals you define are indeed ideals: it is enough to say that they are.

   (c) Suppose that K is an algebraically closed field and that $V \subset \mathbb{A}^n_K$ is an irreducible affine variety not contained in any bigger irreducible subvariety of $\mathbb{A}^n$. In other words, suppose that V is irreducible and that if W is an irreducible variety with $V \subseteq W \subseteq \mathbb{A}^n$, then $W = V$ or $W = \mathbb{A}^n$. Show that $I(V)$ is a principal ideal of $K[t_1, \ldots, t_n]$ (recall that an ideal is principal if it is generated by a single element). *[Hilbert's Nullstellensatz may be assumed.]*

2. (a) If $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ are irreducible projective varieties, define what is meant by a *rational map* $\phi\colon X \dashrightarrow Y$ and what is meant by a *morphism* $f\colon X \to Y$.

   (b) What does it mean to say that X and Y are *birationally equivalent*? What does it mean to say that X and Y are *isomorphic*?

   (c) The *Segre embedding*
$$\sigma\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$
is given by
$$\sigma\big((x_0 : x_1), (y_0 : y_1)\big) = (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1).$$
Show that σ is a morphism and that it is injective.

   (d) By considering the rational map
$$\phi\colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$$
given by
$$\phi\big((x_0 : x_1), (y_0 : y_1)\big) = (x_0y_1 : x_1y_0 : x_1y_1)$$
show that $\mathbb{P}^1 \times \mathbb{P}^1$ is birationally equivalent to $\mathbb{P}^2$. Find the domain and the image of ϕ.

   (e) Is $\mathbb{P}^1 \times \mathbb{P}^1$ isomorphic to $\mathbb{P}^2$? Justify your answer briefly.

3. (a) Suppose that $P_1, \ldots, P_8 \in \mathbb{P}^2$ are eight points, no three of which lie on a line and no six of which lie on a conic. Show that at most two independent cubics pass through $P_1, \ldots, P_8$.

(b) Suppose that $E \subset \mathbb{P}^2$ is a smooth plane cubic curve over a field K. Explain, either in words or by drawing a diagram, how to define a group law on the set of points of E whose coordinates lie in K, assuming that this set is non-empty.

(c) Take $K = \mathbb{F}_{31}$ and let E be given by the affine equation

$$y^2 = x^3 + 11x + 3.$$

If $P = (2, -8)$ and $Q = (16, 11)$, calculate $(P + P) + Q$ and $P + (P + Q)$ and show that they are equal.

[You may find it useful to know that $14 \times 20 \equiv 1 \mod 31$ *and that* $13 \times 12 \equiv 1 \mod 31$*. You should find that* $P + P = (4, 7)$ *and that* $P + Q = (15, -3)$*.]*

4. (a) Define the *tangent space* $T_P V$ to a hypersurface $V \subset \mathbb{A}^n$ in affine space at a point $P \in V$. What does it mean to say that P is a *singular point* of V?

(b) Show that if the ground field K is algebraically closed and of characteristic zero, then the set of non-singular points of V is non-empty.

[Hilbert's Nullstellensatz may be assumed.]

(c) Find the singular points of the Cayley sextic, which is the curve in $\mathbb{P}^2$ over $K = \mathbb{C}$ given by

$$4(x^2 + y^2 - xz)^3 = 27(x^2 + y^2)^2 z^2.$$