University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES EXAMINATION

MA40188: ALGEBRAIC CURVES

Friday 15th May 2009, 16.30–18.30

No calculators may be brought in and used.

Full marks will be given for correct answers to THREE questions.
Only the best three answers will contribute towards the assessment.

1. (a) State the Nullstellensatz.

   (b) Define the *radical* $\sqrt{I}$ of an ideal I in a ring R. Show that $\sqrt{I}$ is an ideal in R.

   (c) Suppose that K is a algebraically closed field, I is an ideal of $K[x_1, \ldots, x_n]$ and V is an affine variety in $\mathbb{A}^n$. Define the variety $V(I)$ and the ideal $I(V)$. Give an example to show that $I\big(V(I)\big) \neq I$ in general.

   (d) With notation as in part (c) and assuming the Nullstellensatz, show that $I\big(V(I)\big) = \sqrt{I}$.

2. Let E be the projective curve in $\mathbb{P}^2$, over a field K whose characteristic is not 2 or 3, given in affine coordinates by

$$y^2 = x^3 + ax + b.$$

Assume that a and b have been chosen so that E is non-singular. Let $P = (p, q)$ be a point of E.

   (a) Explain very briefly how the group law on E is defined. What are the coordinates of the point $-P$?

   (b) Write down the equation of the tangent line ℓ_P to E at P.

   (c) By considering where ℓ_P meets E again, show that the x-coordinate of $-2P$ is

$$\frac{(p^2 - a)^2 - 8pb}{4q^2}.$$

   (d) Now let K be the field $\mathbb{F}_{23}$ with 23 elements and let $a = 9$, $b = 19$. Suppose that P is the point $(1, 11)$. Show that P is in E. Calculate the x-coordinates of $-2P$ and of $4P$ and hence show that $5P = 0$.

3. Suppose that V and W are projective varieties over $\mathbb{C}$.

(a) Say what is meant by a *rational map* $\phi\colon V \dashrightarrow W$.

(b) Say what it means for V and W to be *birationally equivalent*.

(c) Say what it means for V to be *rational*.

(d) Say what it means for a point $P \in V$ to be a *singular point* of V.

(e) Find the singular points of the curve C in $\mathbb{P}^2$ given by

$$x^2(x-y)(x+y)z + x^5 + 3y^5 = 0.$$

(f) By projecting from a singular point, show that C is rational.

4. (a) Explain carefully what is meant by a *map* $\phi\colon V \to W$ between two affine varieties over an algebraically closed field K. Define the corresponding map $\phi^*\colon K[W] \to K[V]$. What does it mean to say that ϕ is an *isomorphism*? What property does ϕ^* have in this case?

(b) Let K be an algebraically closed field of characteristic $p > 0$. The Frobenius map $\Phi\colon K \to K$ is given by $\Phi(r) = r^p$ for all $r \in K$. Show that if $a,\ b \in K$ and $a^p = b$, then the polynomials $x^p - b$ and $(x-a)^p$ are equal.

[You may use without proof the fact that the binomial coefficient $\binom{p}{r}$ is divisible by p if p is a prime and $0 < r < p$.]

(c) Deduce from part (b) that is Φ is bijective.

(d) Say why the Frobenius map may also be thought of as a map of algebraic varieties $\Phi\colon \mathbb{A}^1_K \to \mathbb{A}^1_K$. What is Φ^*? Is $\Phi\colon \mathbb{A}^1_K \to \mathbb{A}^1_K$ an isomorphism of varieties? Justify your answer.