ALGEBRAIC CURVES (MA30188): 2003 EXAM

Retyped hastily after the original was lost. No guarantees, and no solutions either.

1. Describe the group law on a smooth plane cubic $E \subset \mathbb{P}^2$ with a given point $O \in E$ as origin.
Let $E \subset \mathbb{P}^2$ be the curve over $\mathbb{C}$ given by

$$y^2 z = x(x+3z)(x+27z)$$

and take O to be the point $(-9\colon -18\sqrt{3}\colon 1)$. What is the tangent line to E at O? Show that it meets E again at $(0:0:1)$.
There are exactly four points $A \in E$ such that $A + A = O$. Find them.

2. What does it mean to say that a commutativ ring is Noetherian?
State and prove Hilbert's Basis Theorem.
If $\mathbb{Z}_{100}$ denotes the ring of integers modulo 100, is $\mathbb{Z}_{100}[X]$ a Noetherian ring? Give reasons for your answer.

3. Define the tangent space $T_P V$ to a hypersurface $V \subseteq \mathbb{A}^n$ in affine space at a point $P \in V$. What does it mean to say that P is a singular point of V?
Show that if the ground field k is algebraically closed then the set of non-singular points of V is non-empty. [Hilbert's Nullstellensatz may be assumed.]
Find the singular points of the astroid, which is the curve in $\mathbb{A}^2$ over $k = \mathbb{C}$ given by

$$(x^2 + y^2 + 1)^3 + 27x^2y^2 = 0.$$

4. What is meant by a rational map between projective algebraic varieties? What is meant by a morphism of algebraic varieties? What does it man to say that two varieties are birationally equivalent? What does it mean to say that they are isomorphic?
Show that the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by

$$(x : y : z) \to (yz : zx : xy)$$

has a rational inverse map.
Show that the projective curves with equations

$$y^2 z - x(x-z)(x-2z) = 0$$

and

$$x^3 z - y^2 z^2 + 3xy^2 z - 2x^2 y^2$$

are birationally equivalent.