GROUPS AND RINGS (MA22017)

SOLUTIONS TO PROBLEM SHEET 9

1 W Write each of the following $\mathbb{Z}$-modules in the form $\prod_j \mathbb{Z}/p_j^{k_j}\mathbb{Z}$ with p_j primes.

(a) $\mathbb{Z}/12\mathbb{Z}$

(b) $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

(c) $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$

Solution:

(a) This is $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ by CRT.

(b) This is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, again by CRT.

(c) This is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Note that the version in the question is not in either of the standard forms.

2 W Write each of the following $\mathbb{Z}$-modules in the form $\prod_i \mathbb{Z}/a_i\mathbb{Z}$ with $a_i | a_{i+1}$.

(a) $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$

(b) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

(c) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$

Solution:

(a) $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$

(b) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/24\mathbb{Z}$

(c) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} = \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/36\mathbb{Z}$

3 H Find all $\mathbb{Z}$-modules of order $520 = 8 \times 5 \times 13$, expressed as $\prod_i \mathbb{Z}/a_i\mathbb{Z}$ with $a_i | a_{i+1}$. Rewrite each of them in the form $\prod_j \mathbb{Z}/p_j^{k_j}\mathbb{Z}$ with p_j primes.
Solution: *One factor: $\mathbb{Z}/520\mathbb{Z}$, which is $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/13\mathbb{Z}$ by CRT.*
Two factors: the first factor can only be $\mathbb{Z}/2\mathbb{Z}$, because if it is $\mathbb{Z}/a\mathbb{Z}$ then $ar|520$ and that is only true for $a = 2$. So the only possibility is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/260\mathbb{Z}$ which is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/13\mathbb{Z}$.
Three factors: again the first factor must be $\mathbb{Z}/2\mathbb{Z}$, and then we have the same problem for 260 with two factors. The first of those two factors must be $\mathbb{Z}/2\mathbb{Z}$, again because 4 is the only square that divides 260, so we are left with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/130\mathbb{Z}$ which is $(\mathbb{Z}/2\mathbb{Z})^3 \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/13\mathbb{Z}$.

Four or more factors is impossible because no fourth power divides 520.
4 H Find all abelian groups of order 360.
Solution: $360 = 8 \times 9 \times 5$. *The highest power of a prime that divides* 360 *is a cube so we may have up to three factors.*
One factor: $\mathbb{Z}/360\mathbb{Z}$.
Two factors: the first factor could in principle be 2 *or* 3 *or* 6, *as the squares of those numbers (and no others) divide* 360. *So the possibilities are* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/180\mathbb{Z}$ *or* $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/120\mathbb{Z}$ *or* $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/60\mathbb{Z}$.
Three factors: 2 *is the only possibility as* 8 *is the only cube dividing* 360, *and then we are left with finding two-factor groups of order* 180. *The next number must be even and its square must divide* 180, *so it is* 2 *or* 6, *so the possibilities are* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/90\mathbb{Z}$ *or* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/30\mathbb{Z}$.

GKS, 15/4/25