## GROUPS AND RINGS (MA22017)

## SOLUTIONS TO PROBLEM SHEET 8

**1 W** Suppose that $R$ is a nontrivial ring (i.e. $0_R \neq 1_R$). Prove the assertion made in lectures that the map $i\colon X \to F_X$ in the definition of a free module is automatically injective.

(a) Suppose that $i$ is not injective, so that there exist $x, x' \in X$ with $x \neq x'$ but $i(x_0) = i(x_1)$. Construct an $R$-module $M$ (you could take $M = R$) and a map of sets $f\colon X \to M$ such that $f(x) \neq f(x')$.

(b) Show that no linear map $\varphi\colon F_X \to M$, as required by the definition of free module, can exist.

**Solution:**

*(a) Take $M = R$ and define $f(x) = 1_R$ and $f(y) = 0_R$ if $y \in X$ and $y \neq X$.*

*(b) If $\varphi$ exists then by the universal property $f = \varphi \circ i$ so $1_R = f(x) = \varphi(i(x)) = \varphi(i(x')) = f(x') = 0_R$, a contradiction.*

**2 H** Choose an integral domain $R$ (your choice) and give an example of a free $R$-module, an example of a torsion $R$-module and an example of an $R$-module that is neither free nor torsion.

**Solution:** *Just take $R = \mathbb{Z}$ and look at, respectively, $\mathbb{Z}$, $\mathbb{Z}/2$ and $\mathbb{Z} \oplus \mathbb{Z}/2$. Note that neither of the last two is free because free modules are isomorphic to $R^n$ and therefore a free module over an integral domain cannot have any torsion elements because $R^n$ hasn't any.*

**3 H** Explain why $2\mathbb{Z}$ cannot be a direct summand of $\mathbb{Z}$ as a $\mathbb{Z}$-module.

**Solution:** *There are several ways to say this. One answer is that $2\mathbb{Z}$ is of finite index (namely 2) in $\mathbb{Z}$, considered as an abelian group (which is all that a $\mathbb{Z}$-module is anyway), so if $\mathbb{Z} = 2\mathbb{Z} \oplus Q$ then $Q$ must be finite, but $\mathbb{Z}$ has no nontrivial finite abelian subgroups.*

**4 E** Show that if $X$ and $Y$ have the same cardinality then any free module on $X$ is isomorphic to any free module on $Y$, as follows.

Let $a\colon X \to Y$ be a bijection and let $b\colon Y \to X$ be its inverse. Suppose that $F_X$ is free on $X$ with respect to the map $i\colon X \to F_X$ and $F_Y$ is free on $Y$ with respect to the map $j\colon Y \to F_Y$.

(a) In the diagram that arises from the fact that $F_X$ is free, put $M = F_Y$ and hence construct a linear map $\alpha\colon F_X \to F_Y$.

(b) Similarly construct a linear map $\beta\colon F_Y \to F_X$.

(c) In the diagram that arises from the fact that $F_X$ is free, put $M = F_X$ and hence deduce that there is a *unique* linear map $\varphi\colon F_X \to F_X$ with a certain property. Show that both $\mathrm{id}_{F_X}$ and $\beta\alpha$ have this property.

(d) Do the same thing with $F_Y$, and deduce that $\alpha$ and $\beta$ are mutually inverse maps and hence (by definition of isomorphism) they are isomorphisms.

**Solution:** *It is best to draw the diagrams.*

*(a) $\alpha$ exists by the definition of free module, and it has the property that $j = \alpha \circ i$.*

*(b) Putting $M = F_X$ we get $\beta\colon F_Y \to F_X$ with $i = \beta \circ j$*

*(c) The definition asserts the existence of a unique linear map $\varphi\colon F_X \to F_X$ such that $i = \varphi \circ i$. Clearly one may take $\varphi = \mathrm{id}_{F_X}$. However, we may also take $\varphi = \beta\alpha$, because $\beta(\alpha(i(x))) = \beta(j(x)) = i(x)$, where the first equality is by (a) and the second by (b). But $\varphi$ is unique, hence $\beta\alpha = \mathrm{id}_{F_X}$.*

*(d) Simple swapping $X$ and $Y$, $i$ and $j$, and $\alpha$ and $\beta$ shows that $\alpha\beta = \mathrm{id}_{F_Y}$. This, with (c), is exactly the definition of isomorphism.*

GKS, 28/3/25